4. (a) Condition that a node at time t has degree smaller than k

$$k_i(t) = m\ln\left(e\,\frac{m_0+t-1}{m_0+t_i-1}\right) < k \longleftrightarrow \frac{m_0+t-1}{m_0+t_i-1} < e^{\frac{k}{m}-1}$$

$$\longleftrightarrow (m_0+t-1)\,e^{1-\frac{k}{m}} < m_0+t_i-1$$

$$\Longleftrightarrow (m_0+t-1)e^{1-\frac{k}{m}} - m_0+1 < t_i$$

# of nodes that have degree smaller than k at time t

$$\underbrace{N(t)}_{=t} - (m_0+t-1)e^{1-\frac{k}{m}} - m_0+1 = N(t)\left(1-e^{1-\frac{k}{m}}\right) \quad \text{for } t \gg m_0$$

$$\therefore P_{cum}(k) = 1 - e^{1-\frac{k}{m}} = 1 - e\cdot e^{-\frac{k}{m}}$$

$$P_k = \frac{\partial}{\partial k}P_{cum}(k) = \frac{e}{m}e^{-\frac{k}{m}}$$

2. $q_k = A k p_k \qquad p_k = (\gamma-1)\, k_{min}^{\gamma-1}\, k^{-\gamma}$

(a) $\int_{k_{min}}^{k_{max}} q_k\, dk = 1 = A\int_{k_{min}}^{k_{max}} k\, p_k\, dk = A\langle k\rangle$

Since $\langle k\rangle = (\gamma-1)\, k_{min}^{\gamma-1}\, \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{2-\gamma}$,

$$A = \frac{1}{\langle k\rangle} = \frac{2-\gamma}{\gamma-1}\left(k_{min}^{\gamma-1}\left(k_{max}^{2-\gamma} - k_{min}^{2-\gamma}\right)\right)^{-1}$$

(b) $\overline{\langle k\rangle} = \int_{k_{min}}^{k_{max}} k\, q_k\, dk = A\int_{k_{min}}^{k_{max}} k^2 p_k\, dk = A\langle k^2\rangle = \frac{1}{\langle k\rangle}\langle k^2\rangle$

$$= \frac{2-\gamma}{\cancel{\gamma-1}}\ \frac{1}{\cancel{k_{min}^{\gamma-1}}}\ \frac{1}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}\ \cancel{(\gamma-1)\, k_{min}^{\gamma-1}}\ \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{3-\gamma}$$

$$= \frac{2-\gamma}{3-\gamma}\ \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}$$

(c) $\langle k\rangle = (\gamma-1)\, k_{min}^{\gamma-1}\, \frac{k_{max}^{2-\gamma} - k_{min}^{2-\gamma}}{2-\gamma} = 1.3 \times 1 \times \frac{1000^{-0.3} - 1^{-0.3}}{-0.3} = 3.78$

$$\overline{\langle k\rangle} = \frac{\langle k^2\rangle}{\langle k\rangle} = \frac{1}{\langle k\rangle}(\gamma-1)\, k_{min}^{\gamma-1}\, \frac{k_{max}^{3-\gamma} - k_{min}^{3-\gamma}}{3-\gamma}$$

$$= \frac{1}{3.78}\ 1.3 \times 1 \times \frac{1000^{0.7} - 1^{0.7}}{0.7} = 61.36$$

$\overline{\langle k\rangle} \gg \langle k\rangle$

(d) $\overline{\langle k\rangle} = \frac{\langle k^2\rangle}{\langle k\rangle} = \frac{\langle k\rangle^2 + \sigma^2}{\langle k\rangle} = \langle k\rangle + \frac{\sigma^2}{\langle k\rangle} > \langle k\rangle$

$\therefore$ Average degree of neighbors of random nodes is bigger than Average degree of random nodes (Friendship paradox)