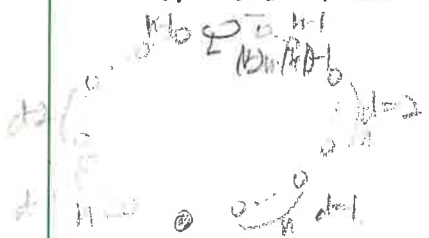
1. Beads-on-a string network

N node $\longrightarrow$ with $2n$ degrees

(a) Is the network is small world?

Fix n and think about how distance changes as N increase

For one node i, i has $2n$ neighbors for left and right
$2n$ nodes for distance 2 for left and right



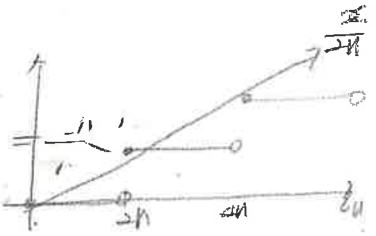
$\frac{2n}{N-2nk}$ $k-1$ $\quad k \quad k+1$

$N-2nk$, $-1 < 2n$ $\therefore k$ is quotient for $N-1$ divided by $2n$ $\therefore k = \left[\frac{N-1}{2n}\right]$
$N-1-2nk$ rest

Average distance for target node i is

$$\langle d_i \rangle = \frac{1}{N-1}\left(\sum_{j=1}^{k} 2nj + (N-1-2nk)(k+1)\right) = \frac{1}{N-1}\left(2n\frac{k(k+1)}{2} + (N-1-2nk)(k+1)\right)$$

$$= \frac{1}{N-1}(N-1-nk)(k+1) = k+1 - \frac{1}{N-1}nk(k+1)$$



Since all nodes are in the equivalent situation

$\langle d \rangle = \langle d_i \rangle \times N \times \frac{1}{N} = \langle d_i \rangle$

$\left[\frac{N-1}{2n}\right]$ is a step function, but it will increase as 1 when $N-1$ increasing to $2n$.

$k = \left[\frac{N-1}{2n}\right] \approx \frac{N-1}{2n}$, $\left(0 \le \frac{N-1}{2n} - \left[\frac{N-1}{2n}\right] < 1\right) \rightarrow N$이 커질수록 1 이하 오차는 무시가능)

Then, $\langle d \rangle = 1 + k - n\frac{k(k+1)}{N-1} \simeq \frac{N-1}{2n} - \frac{\cancel{N}}{\cancel{N-1}}\frac{N-1}{2n}\frac{N-1+2n}{2n} = \frac{N-1}{2n} - \frac{N-1}{4n} + \frac{1}{2}$

$= \frac{N-1}{4n} + \frac{1}{2} \propto N$ $\quad\therefore$ not small world

(b) $N=301$ $n=5$ $\quad k = \left[\frac{301-1}{2\cdot 5}\right] = \left[\frac{300}{10}\right] = 30$ $\quad N-1-2nk = 0$

$\langle d \rangle = \frac{1}{N(N-1)} N \times \sum_{k=1}^{30} 2nk = \frac{10}{300} \times \frac{30 \times 31}{2} = \frac{31}{2} = 15.5$

For $n \le [N/3]$, local clustering coefficient $\quad C_i = \frac{2L_i}{k(k-1)} =$

Let the target node as 0.

And including left and right neighbors $-n, -(n-1), \cdots, -1, 1, \cdots, n$

node index

$-n \longrightarrow -(n-1), -(n-2), \cdots, -1$ links to its right
$-(n-1) \longrightarrow -(n-2), \cdots, -1, 1$
$\vdots$
$-1 \longrightarrow 1, \cdots, n-1$ $\quad \longrightarrow n-1$ $\quad \therefore L_i = n(n-1) + (n-1) + (n-2) + \cdots$
$= n(n-1) + \frac{1}{2}n(n-1)$
$= \frac{3}{2}n(n-1)$

$1 \longrightarrow 2, \cdots, n \quad \longrightarrow n-1$ $\quad \because n \le [N/3] \longrightarrow -n$ node does not link to $+n$ and vice versa.
$2 \longrightarrow 3, \cdots, n \quad \longrightarrow n-2$
$n-1 \longrightarrow n \quad\quad 1$

$\frac{3}{2}\ \frac{\frac{k}{2}-1}{k-1}$

1.(b). $k_i = 2n$. $L_i = \frac{3}{2}n(n-1)$ for $n \le [N/3]$

$$C_i = \frac{2L_i}{k_i(k_i-1)} = \frac{1}{n(2n-1)}\frac{3}{2}n(n-1) = \frac{3}{2}\frac{n-1}{2n-1} = \frac{3}{2}\frac{k/2-1}{k-1} = \boxed{\frac{3}{4}\frac{k-2}{k-1}}$$

Since all nodes are equivalent, $\langle C\rangle = \frac{NC_i}{N} = \frac{3}{2}\frac{n-1}{2n-1} = \frac{3}{4}\frac{k-2}{k-1}$

$$\therefore \langle C\rangle = \frac{3}{2}\times\frac{5-1}{2\times5-1} = \frac{3}{2}\times\frac{4}{9} = \frac{2}{3}$$

(c) Process in (b) $\langle C\rangle = \frac{3}{2}\frac{n-1}{2n-1}$

(d) $n > [N/3] \to$ left ~~ka~~ node and right nodes can be linked.

Let $d = N-1-2n < n$ by $n>[N/3]$, $r = n-d = n-(N-1-2n) = 3n-(N-1) < n$.

We have indexing like (b) (except 0) for neighbors of target node 0.

node index → link to right — # of links for right

d: # of nodes which is not neighbor of target node.

$-n \to -(n-1), \dots, -1$ — $n-1$

$-(n-1) \to -(n-2), \dots, -1, 1$ — $n-1$

⋮

$-1 \to 1, \dots, n-1$ — $n-1$

n nodes

$+n \to -n, \dots, -(n-(r-1))$ — r

$+(n-1) \to n, -n, \dots, -(n-(r-2))$ — r

$+(n-2) \to n-1, n, -n, \dots, -(n-(r-3))$ — r

r node

$+(n-r) \to n-r+1, \dots, n$ — r

$+(n-r-1) \to n-r, \dots, n$ — $r+1$

⋮

$+1 \to 2, 3, \dots, n$ — $n-1$

n-r node

$\frac{n}{2}(n+1)$

$n-r = (N-1)-2n$

$n-1+r = 4n-N$

$$L_i = n(n-1) + r^2 + \sum_{k=r}^{n-1} k = n(n-1) + (3n-(N-1))^2 + \frac{n-r}{2}(n-1+r)$$
$$= n^2 - n + 9n^2 - 6n(N-1) + (N-1)^2 - \frac{2n-(N-1)}{2}(4n-N)$$
$$= 10n^2 - (6N-5)n + (N-1)^2 - \frac{1}{2}(8n^2 - (6N-4)n + N(N-1))$$
$$= 10n^2 - (6N-5)n + (N-1)^2 - 4n^2 + (3N-2)n - \frac{1}{2}N(N-1)$$
$$= 6n^2 - 3(N-1)n + (N-1)\left(\frac{N}{2}-1\right)$$

$$\langle C\rangle = C_i = \frac{2L_i}{k_i(k_i-1)} = \frac{6n^2-3(N-1)n+(N-1)\left(\frac{N}{2}-1\right)}{n(2n-1)}$$

1.(e) $N=1000$ $p=0.03$ $\langle k\rangle=10$ $\rightarrow$ Set python code

Using python code: $C_{avg}[p_{set}.index(0.03)] = 0.6147$

$\rightarrow$ Average clustering coefficient 0.6147

$d_{avg}[p_{set}.index(0.03)] = 6.13$

$\rightarrow$ Average distance 6.13

2.(a) Degree distribution

$N = 1+3+3\times2+3\times2\times2+3\times2\times2\times2+3\times2\times2\times2\times2$

$= 1+3\times\sum_{k=0}^{4}2^k = 1+3\times\frac{2^5-1}{2-1} = 94$

$N_{k=3} = 1+3\times\sum_{k=0}^{3}2^k = 46$

$N_{k=1} = 3\times2^4 = 48$

$P_{k=3} = \frac{46}{94} = \frac{23}{47}$ $\quad P_{k=1} = \frac{48}{94} = \frac{24}{47}$

(b) After k steps, $N_k = 1+3\times\sum_{i=0}^{k-1}2^i = 1+3\times\frac{1\cdot(2^k-1)}{2-1} = 1+3\times(2^k-1)$

$\hookrightarrow k = \log_2\left(\frac{N-1}{3}+1\right)$

$d_{max} = 2k = 2\log_2\left(\frac{N-1}{3}+1\right) =$

from one end point to other end point passing 0.

Since $\langle d\rangle \le d_{max} = 2\log_2\left(\frac{N+2}{3}\right) = 2(\log_2(N+2)-\log 3)$

$\sim 2\log_2 N$.

$\therefore \langle d\rangle$ increases slower than d_{max}, that is $\log_2 N$.

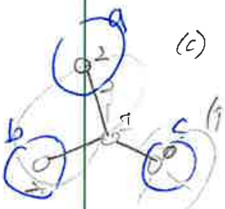
Therefore, it has small world property.

(c) betweenness centrality, $\sum_{u\neq i\neq v}\frac{\sigma_{uv}^i}{\sigma_{uv}} = x_i$

– Shortest path is unique in Cayley Tree.

$\frac{\sigma_{uv}^i}{\sigma_{uv}} = \begin{cases}1 & u,v \text{ pass } i\\ 0 & u,v \text{ doesn't pass } i\end{cases}$

$\Rightarrow$ problem for how many pair of nodes whose shortest path pass i.

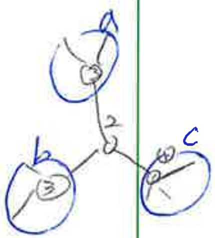


(i) red, # of nodes $a_r = 1+2+2^2+2^3$
$= 15$

# of nodes $b_r = 15$

# of nodes $c_r = 1+2+2^2+2^3+2^4+2^5$
$= 1\times\frac{2^6-1}{2-1} = 63$

$bc_r = 15\times15+15\times63+15\times63 = 2115$



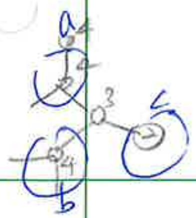
(ii) orange # of nodes $a_o = 1+2+2^2 = 7$

$b_o = 7$

$c_o = 1+\sum_{k=0}^{3}2^k+\sum_{k=0}^{5}2^k = 1+\frac{2^4-1}{2-1}+\frac{2^6-1}{2-1}$

$= 1+15+63 = 79$

$bc_o = 7\times7+7\times79+7\times79 = 1155$



(iii) yellow # of nodes $a_y = 3$

$b_y = 3$

$c_y = N-1-a_y-b_y = 94-1-3-3 = 87$

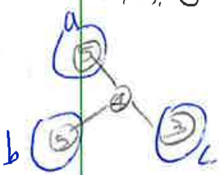
$bc_y = 3\times3+3\times87+3\times87 = 531$

182 $\quad 1+2+4+$

2, (c). green. $\quad$ # of nodes $\quad a_g = 1$

$b_g = 1$

$c_g = 94 - 1 - 1 = 91$



$bc_g = |x| + |y_4| + |x_{91}| = 183$

~~blue~~ $bc_b = 0 \quad (a_g = 0)$.

red has the biggest centrality

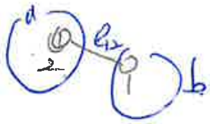
In my intuition, as node is near the center node ($i=0$), it has bigger centrality. Therefore, I agree with my calculation result.

(d). (i) e_{01} $\quad$ # of nodes $\quad a = \sum_{k=0}^{4} 2^k = \frac{2^5-1}{2-1} = 31$

$b = 94 - 31 = 63$

$bc = 31 \times 63 = 1953$

(ii) e_{12}

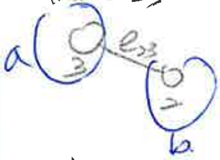


$a = \sum_{k=0}^{3} 2^k = \frac{2^4-1}{2-1} = 15$

$b = 94 - 15 = 79$

$bc = 15 \times 79 = 1185$

(iii) e_{23}

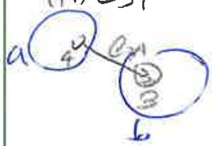


$a = \sum_{k=0}^{2} 2^k = \frac{2^3-1}{2-1} = 7$

$b = 94 - 7 = 87$

$bc = 87 \times 7 = 609$

(iv) e_{34}



$a = \sum_{k=0}^{1} 2^k = \frac{2^2-1}{2-1} = 3$

$b = 94 - 3 = 91$

$bc = 91 \times 3 = 273$

(v) e_{45}

$a = 1$

$b = 94 - 1 = 93$

$bc = 1 \times 93 = 93$

* e_{01} has the biggest centrality

Cayley tree extends from 0 (origin) node. Therefore, "0" node should be passed for going from one branch to other branch. ∴ Link closed to 0 node is important in my intuition.