№ 5 $\xi \sim R[\theta, 2\theta]$

$\vec{x}_n$ - выборка

$p(x,\theta) = \frac{1}{\theta}\{(\theta, 2\theta)\}$

• Метод моментов (Пирсон)

а,в) $\alpha_1 = \int_\theta^{2\theta} x \cdot \frac{1}{\theta} dx = \frac{x^2}{2} \cdot \frac{1}{\theta}\Big|_\theta^{2\theta} = 2\theta - \frac{\theta}{2} = \frac{3\theta}{2}$

$\alpha_2 = \int_\theta^{2\theta} \frac{x^2}{\theta} dx = \frac{x^3}{3} \frac{1}{\theta}\Big|_\theta^{2\theta} = \frac{8\theta^2}{3} - \frac{\theta^2}{3} = \frac{7\theta^2}{3}$

$\mu_2 = \alpha_2 - \alpha_1^2 = \frac{7}{3}\theta^2 - \left(\frac{3\theta}{2}\right)^2 =$

$= \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{1}{3}\theta^2 - \frac{1}{4}\theta^2 = \frac{1}{12}\theta^2 = D[\xi]$

$\alpha_1 = \bar{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

$\frac{3}{2}\theta = \bar{x} \Rightarrow \underline{\underline{\tilde{\theta}_1 = \frac{2}{3}\bar{x}}}$

$M[\tilde{\theta}_1] = M[\frac{2}{3}\bar{x}] = \frac{2}{3} M[\bar{x}] =$

$= \frac{2}{3} M[\xi] = \theta$ • несмещенная

$D[\tilde{\theta}_1] = D[\frac{2}{3}\bar{x}] = \frac{4}{9} D[\bar{x}] =$

$= \frac{4}{9n} D[\xi] = \frac{1}{27n}\theta^2 \to 0,\ n \to \infty$ • состоят.

• Метод максимального правдоподобия (Фишер)

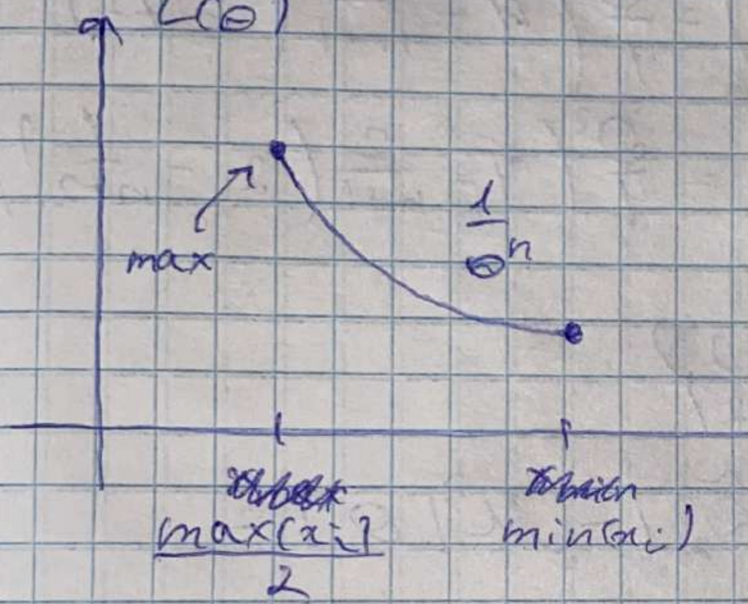
↘

$$L(\Theta) = \prod_{i=1}^{n} p(x_i, \Theta) = \frac{1}{\Theta^n}\{\forall i,\ \Theta < x_i < 2\Theta\} =$$

$$= \frac{1}{\Theta^n}\{\max(x_i) < 2\Theta,\ \min(x_i) > \Theta\} =$$

$$= \frac{1}{\Theta^n}\left\{\frac{\max(x_i)}{2} < \Theta,\ \Theta < \min(x_i)\right\} =$$

$$= \frac{1}{\Theta^n}\left\{\frac{\max(x_i)}{2} < \Theta < \min(x_i)\right\}$$



$L(\Theta) \to \max$

$$\hat{\Theta} = \frac{1}{2}\max(x_i) = \frac{x_{max}}{2}$$

$$\tilde{\Theta}_2 = \frac{x_{max}}{2}$$

$$M[\tilde{\Theta}_2] = \frac{1}{2}M[x_{max}] = \frac{1}{2}\int_{\Theta}^{2\Theta} \frac{nz}{\Theta}\left(\frac{z}{\Theta} - 1\right)^{n-1} dz =$$

$$= \left\{\frac{z}{\Theta} = t\right\} = \frac{\Theta}{2}\int_1^2 n\, t(t-1)^{n-1}\, dt =$$

$$= \frac{\Theta}{2}\left[t(t-1)^n \Big|_1^2 - \int_1^2 (t-1)^n\, dt \right] =$$

$$= \frac{\Theta}{2}\left(t(t-1)\Big|_1^2 - \frac{(t-1)^{n+1}}{n+1}\Big|_1^2 \right) = \frac{\Theta}{2}\left(2 - \frac{1}{n+1}\right) =$$

$$= \Theta - \frac{\Theta}{2(n+1)} = \frac{\Theta(2n+1)}{2(n+1)}$$ • смещен.

$$\tilde{\Theta}_2' = \frac{2(n+1)}{2n+1} x_{max} \cdot \frac{1}{2} = x_{max} \frac{n+1}{2n+1}$$ • несмещ.

$$M[\tilde{\Theta}_2^2] = \frac{1}{4} \int_{\Theta}^{2\Theta} \frac{nz^2}{\Theta} \left(\frac{z}{\Theta} - 1\right)^{n-1} dz = \left\{ \frac{z}{\Theta} = t \right\} =$$

$$= \frac{\Theta^2}{4} \int_1^2 n t^2 (t-1)^{n-1} dt = \frac{\Theta^2}{4} \left(t^2 (t-1)^n \Big|_1^2 - \right.$$

$$\left. - 2 \int_1^2 t(t-1)^n dt \right) = \frac{\Theta^2}{4} \left(4 - \frac{2}{n+1} \left(t(t-1)^{n+1} \Big|_1^2 - \right.\right.$$

$$\left.\left. - \frac{(t-1)^{n+2}}{n+2} \Big|_1^2 \right) \right) = \frac{\Theta^2}{4} \left(4 - \frac{2}{n+1} \left(2 - \frac{1}{n+2} \right) \right) =$$

$$= \frac{\Theta^2 (4n^2 + 8n + 2)}{4(n+1)(n+2)}$$

$$D[\tilde{\Theta}_2] = M[\tilde{\Theta}_2^2] - M^2[\tilde{\Theta}_2] =$$

$$= \frac{\Theta^2 (4n^2 + 8 + 2)}{4(n+1)(n+2)} - \left(\frac{\Theta(2n+1)}{2(n+1)} \right)^2 =$$

$$= \frac{\Theta^2}{4} \left(\frac{\cancel{4n^3} + \cancel{8n^2} + 2n + \cancel{4n^2} + \cancel{8n} + \cancel{2} - \cancel{4n^3} - \cancel{4n^2} - n -}{(n+1)^2 (n+2)} \right.$$

$$\left. \frac{- \cancel{8n^2} - \cancel{8n} - \cancel{2}}{} \right) = \frac{n \Theta^2}{4(n+1)^2 (n+2)} \to 0, \ n \to \infty$$

• состоят.

$$D[\hat{\Theta}_2] = \left(\frac{2(n+1)}{2n+1} \right)^2 D[\tilde{\Theta}_2] =$$

$$= \frac{\cancel{4(n+1)^2}}{(2n+1)^2} \cdot \frac{n \Theta^2}{\cancel{4(n+1)^2}(n+2)} = \frac{n \Theta^2}{(2n+1)^2 (n+2)} \to 0, \ n \to \infty$$

• состоят.

c) Сравниваем эффективность

$$D[\tilde{\Theta}_1] = \frac{\Theta^2}{2+n} \geq \frac{n\Theta^2}{(n+2)(2n+1)^2} = D[\tilde{\Theta}_2'] \text{ при } n \geq 4$$

$\tilde{\Theta}_2'$ эффективнее чем $\tilde{\Theta}_1$

d) Точный доверительный интервал

$$f(\vec{x}_n, \Theta) = \frac{x_{max}}{\Theta} - 1$$

$$P(f < t) = P(x_{max} < \Theta t + \Theta) = F^n(\Theta t + \Theta)$$

$$F(x) = \begin{cases} 0, & x \leq \Theta \\ \frac{x}{\Theta} - 1, & \Theta < x \leq 2\Theta \\ 1, & x > 2\Theta \end{cases} \Rightarrow$$

$$\Rightarrow F^n(\Theta t + \Theta) = \begin{cases} 0, & t \leq 0 \\ t^n, & 0 < t \leq 1 \\ 1, & t > 1 \end{cases}$$

$$t_1 = q_{\alpha/2} = \sqrt[n]{\frac{\alpha}{2}} = \sqrt[n]{\frac{1-\beta}{2}}$$

$$t_2 = q_{1-\alpha/2} = \sqrt[n]{1 - \frac{\alpha}{2}} = \sqrt[n]{\frac{1+\beta}{2}}$$

$$P\left(t_1 < \frac{x_{max}}{\Theta} - 1 < t_2\right) = \beta$$

$$t_1 + 1 < \frac{x_{max}}{\Theta} < t_2 + 1$$

↘

$$\frac{1}{1+t_2} < \frac{\Theta}{x_{max}} < \frac{1}{1+t_1}$$

$$P\left(\frac{x_{max}}{1+\sqrt[n]{\frac{1+\beta}{2}}} < \Theta < \frac{x_{max}}{1+\sqrt[n]{\frac{1-\beta}{2}}}\right) = \beta$$

2) Асимптотический доверительный интервал

Делаем по ОММ

$$\tilde{\Theta}_1 = \frac{2}{3}\overline{x} = \frac{2}{3}\tilde{\alpha}_1 = g(\tilde{\alpha}_1)$$

$$g(\alpha_1) = \frac{2}{3}\alpha_1 = \Theta$$

$$\nabla g = \frac{2}{3}$$

$$K_1 = \alpha_2 - \alpha_1^2$$

$$\tilde{\mu}_2 = \tilde{\alpha}_2 - \tilde{\alpha}_1^2 = \frac{S^2(n-1)}{n}$$

$$\frac{\sqrt{n}(\tilde{\Theta}_1 - \Theta)}{\frac{2}{3}S\sqrt{\frac{n-1}{n}}} = \frac{3n(\tilde{\Theta}_1 - \Theta)}{2S\sqrt{n-1}} \rightarrow N(0,1)$$

$$P\left(t_1 < \frac{3n(\tilde{\Theta}_1 - \Theta)}{2S\sqrt{n-1}} < t_2\right) = \beta$$

$$P\left(\tilde{\Theta}_1 - \frac{2St_2\sqrt{n-1}}{3n} < \Theta < \tilde{\Theta}_1 - \frac{2St_1\sqrt{n-1}}{3n}\right) = \beta$$