Задание №3

$$p(x) = \begin{cases} \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \theta > 0, \ n = 3$$

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\tilde{\theta}_1 = \bar{x}, \quad \tilde{\theta}_2 = x_{(2)}$$

$$M[\xi] = \int_{-\infty}^{\infty} x p(x) dx = \int_0^{+\infty} x \cdot e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta} dx =$$

$$= \left\{ \frac{x}{\theta} = t \right\} = \frac{1}{\theta} \cdot \theta^2 \int_0^{+\infty} t \cdot e^{-t} dt = \theta$$

$$M[\xi^2] = \int_{-\infty}^{\infty} x^2 p(x) dx = \int_0^{+\infty} x^2 \cdot e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta} dx =$$

$$= \left\{ \frac{x}{\theta} = t \right\} = \theta^2 \int_0^{+\infty} t^2 e^{-t} dt = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = 2\theta^2 - \theta^2 = \theta^2$$

а) $? \ M[\tilde{\theta}_1] = \theta$

$$M[\tilde{\theta}_1] = M[\bar{x}] = M\left[\frac{1}{n} \sum_i^n x_i\right] =$$

$$= \frac{1}{n} \sum^n M[x_i] = M[\xi] = \theta$$

• несмещенная

$$? \; M[\tilde{\Theta}_2] = \Theta$$

$$f(y) = n \cdot \frac{e^{-\frac{y}{\Theta}}}{\Theta} \cdot C_{n-1}^{1} \left(1 - (1 - e^{-\frac{y}{\Theta}})\right)^{n-2} (1 - e^{-\frac{y}{\Theta}}) =$$

$$= \frac{n(n-1)}{\Theta} \cdot \left(e^{-\frac{y}{\Theta}(n-1)} - e^{-\frac{y}{\Theta}n} \right)$$

$$M[\tilde{\Theta}_2] = \int_0^{+\infty} \frac{n(n-1)y}{\Theta} \left(e^{-\frac{y}{\Theta}(n-1)} - e^{-\frac{y}{\Theta}n} \right) dy =$$

$$= \int_0^{+\infty} \frac{n(n-1)y}{\Theta} \cdot e^{-\frac{y}{\Theta}(n-1)} dy -$$

$$- \int_0^{+\infty} \frac{n(n-1)y}{\Theta} e^{-\frac{y}{\Theta}n} dy = n \frac{\Theta}{n-1} \int_0^{+\infty} t e^{-t} dt -$$

$$- (n-1) \frac{\Theta}{n} \int_0^{+\infty} t e^{-t} dt = \frac{n^2 - (n-1)^2}{n(n-1)} \Theta =$$

$$= \frac{2n-1}{n(n-1)} \Theta \quad \text{- смещенная}$$

$$\tilde{\Theta}_2' = \frac{n(n-1)}{2n-1} x_{(2)} \qquad M[\tilde{\Theta}_2'] = \Theta$$

в) $$M[\tilde{\Theta}_2^2] = \int_0^{+\infty} \frac{n(n-1)y^2}{\Theta} \left[e^{-\frac{y}{\Theta}(n-1)} - e^{-\frac{y}{\Theta}n} \right] dy =$$

$$= \int_0^{+\infty} \frac{n(n-1)y^2}{\Theta} e^{-\frac{y}{\Theta}(n-1)} dy - \int_0^{+\infty} \frac{n(n-1)y^2}{\Theta} e^{-\frac{y}{\Theta}n} dy =$$

$$= n \frac{\Theta^2}{(n-1)^2} \int_0^{+\infty} t^2 e^{-t} dt - (n-1) \frac{\Theta^2}{n^2} \int_0^{\infty} t^2 e^{-t} dt =$$

$$= 2\Theta^2 \left(\frac{3n^2 - 3n + 1}{n^2 (n-1)^2} \right)$$

$$\searrow D$$

$$D[\tilde{\theta}_2] = M[\tilde{\theta}_2^2] - M^2[\tilde{\theta}_2] = \theta^2\left(\frac{2n^2-2n+1}{n^2(n-1)^2}\right)$$

$$\tilde{\theta}_2' = \frac{n(n-1)}{2n-1}\hat{\theta}_2$$

$$D[\tilde{\theta}_2'] = \frac{n^2(n-1)^2}{(2n-1)^2}\cdot D[\hat{\theta}_2] = \theta^2\left(\frac{2n^2-2n+1}{4n^2-4n+1}\right)$$

$$D[\tilde{\theta}_1] = D\left[\frac{1}{n}\sum_i^n x_i\right] = \frac{1}{n^2}\cdot nD[\xi] = \frac{\theta^2}{n}$$

$$\forall n \in \mathbb{N} \hookrightarrow D[\tilde{\theta}_1] < D[\tilde{\theta}_2']$$

c) I) $$p(x) = \begin{cases} \frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Расм-м ~~от~~ интервал $(0, +\infty)$, на котором непр. диф-ма $p(x)$

II) $$\int_0^{+\infty}\left(\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}\cdot\frac{x}{\theta^2} - \frac{1}{\theta^2}e^{-\frac{x}{\theta}}\right)dx =$$

$$= \frac{1}{\theta} - \frac{1}{\theta} = 0$$

$$\frac{\partial}{\partial\theta}\left(\int_0^{+\infty}\frac{1}{\theta}e^{-\frac{x}{\theta}}dx\right) = 0$$

III)
$$\ln\left(\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}\right) = -\frac{x}{\theta} - \ln\theta$$

$$\left(\frac{\partial \ln p}{\partial\theta}\right)^2 = \left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] =$$

$$= \int_0^{+\infty}\left(\frac{x^2}{\theta^4} - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right)\frac{1}{\theta}e^{-\frac{x}{\theta}}dx =$$

$$= \frac{1}{\theta^2}\left(\int_0^{+\infty} t^2 e^{-t} dt - 2\int_0^{+\infty} t e^{-t} dt + \int_0^{+\infty} e^{-t} dt\right) =$$

$$= \frac{1}{\theta^2}(2 - 2 + 1) = \frac{1}{\theta^2}$$

$I(\theta)$ непр. на $\Theta = (0, +\infty)$ и > 0

$\Downarrow$

модель регулярна

$\left.\begin{array}{l}\tilde{\theta}_1 \text{ – не смещенная оценка } \theta \\ D[\tilde{\theta}_1] = \dfrac{\theta^2}{n} \text{ – огранич.}\end{array}\right\} \Rightarrow$

$\Rightarrow \tilde{\theta}_1$ регулярна

$\left.\begin{array}{l}\tilde{\theta}_2' \text{ – не смещенная оценка } \theta \\ D[\tilde{\theta}_2'] = \theta^2\left[\dfrac{2n^2 - 2n + 1}{4n^2 - 4n + 1}\right] \text{ – огранич.}\end{array}\right\} \Rightarrow$

$\Rightarrow \tilde{\theta}_2'$ регулярна

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а) $\tilde{\theta}_1$, все условия (Т) выполнены

$\left.\begin{array}{l}D[\tilde{\theta}_1] \geqslant \dfrac{g'^2(\theta)}{nI(\theta)} = \dfrac{1}{3}\cdot\theta^2 = \dfrac{\theta^2}{3} \\ n = 3 \Rightarrow D[\tilde{\theta}_1] = \dfrac{\theta^2}{3}\end{array}\right\} \Rightarrow$

$\Rightarrow \tilde{\theta}_1$ – эффективная б) $\tilde{\Theta}_2$, все условия (Т) выполнены

$$D[\tilde{\Theta}_2] \geq \frac{g'^2(\Theta)}{nI(\Theta)} = \frac{\Theta^2}{3}$$

$$n = 3 \Rightarrow D[\tilde{\Theta}_2] = \Theta^2\left(\frac{18-6+1}{36-12+1}\right) = \Theta^2\left(\frac{13}{25}\right)$$

$$D[\tilde{\Theta}_2] \neq \frac{\Theta^2}{3} \Rightarrow \tilde{\Theta}_2 \text{ - не эффективная}$$