Sheet 4

1)

$p(\gamma) = 0.1$ $\qquad$ $p(\text{background}) = 0.9$

$P(\text{target} \mid \gamma) = 0.95$ $\qquad$ $P(\text{target} \mid \text{background}) = 0.1$

$\boxed{p(\gamma \mid \text{target})}$

$$p(\gamma \mid \text{target}) = \frac{P(\text{target} \mid \gamma)\, P(\gamma)}{P(\text{target})}$$

Need $P(\text{target})$

$P(\text{target} \cap \text{background}) = 0.9 \times 0.1$

$P(\text{target} \cap \gamma) = 0.95 \times 0.1$

$P(\text{target}) = 0.09 + 0.095$

$= 0.185$

$P(\gamma \mid \text{target}) = 0.514$

$\Rightarrow$ ~ 51% chance γ came from obs. target

2)

$L(y,\hat{y})$	$\hat{y}=0$	$\hat{y}=1$
$y=0$	0	1
$y=1$	10	0

Risk $R(f) = E_{xy}[L(Y=y, \underbrace{f(X=x)}_{\text{a prediction}})]$

$= E_{y|x} E_x [\cdots] = E_x E_{y|x} [\cdots]$

$= \int E_{y|x}[\cdots] \underbrace{p(x)}_{\text{could be multivariate}} dx$

min. R w.r.t f

$E_{y|x}[L(Y=y, f(X=x))] = \sum_{y=Y} L(y, f(X=x)) p(y|x)$

Expected loss for $\hat{y}=0$:

$L(\hat{y}=0) = \underbrace{L(0,0)}_{0} P(y=0|x) + \underbrace{L(1,0)}_{10} P(y=1|x)$

$= 10\, P(y=1|x)$

" for $\hat{y}=1$

$L(\hat{y}=1) = L(0,1) P(y=0|x) + L(1,1) P(y=1|x)$

$= P(y=0|x)$

$\Rightarrow$

Choose $\hat{y}=0$ iff $10 P(y=1|x) < P(y=0|x)$

$\sum p(y|x) = 1$

$\Rightarrow 10 P(y=1|x) < 1 - P(y=1|x)$

$\Rightarrow P(y=1|x) < \frac{1}{11}$

Predict $\hat{y} = \begin{cases} 0 & P(y=1|x) < 1/11 \\ 1 & P(y=1|x) \geq 1/11 \end{cases}$

$\boxed{\text{more likely to predict } \hat{y}=1}$

↳ use when e.g. false -ves have much larger costs than false +ve

es. flagging illness or fraud

---

b) $y \in \{1, \dots k\}$ $\hat{y} = 0, 1, \dots, k$

$L(y,\hat{y}) = 1 - \delta_{y\hat{y}}$ $y, \hat{y} = 1, \dots, k$

$L(y,0) = \alpha$ $y = 1, \dots, k$

$\alpha \in (0,1)$

$n \neq 0$

$L(\hat{y}=n) = \sum_y (1-\delta_{y\hat{y}}) P(y \in \{1, \dots, k\})$

$= \boxed{1 - P(y=n)}$

$L(\hat{y}=0) = \sum \alpha P(y \in \{1, \dots, k\})$

$\boxed{= \alpha}$

choose $\hat{y}=0$ iff $\alpha < 1 - \max_{n=1,\dots,k} P(y=n|x)$

$\boxed{\max_{n=1,\dots,k} P(y=n|x) < 1-\alpha}$

↳ choose $\hat{y}=0$

$\boxed{\max_{n=1,\dots,k} P(y=n|x) \geq 1-\alpha}$

↳ choose $\boxed{\hat{y}=n}$

So $\hat{y}=0 \rightarrow$ special class

↳ depends on α

if $\alpha \to 1 \to$ never choose class $\hat{y}=0$

$\alpha \to 0 \to$ always put in $\hat{y}=0$

α acts as reject class

↳ don't want to choose any of real classes

↳ choose $\alpha \to 0$ if high consequences for making false prediction

4)

a) (300, 100) ; (100, 300)

Class 1: $p(y=1) = 3/4$ $p(y=2) = 1/4$

Class 2: $p(y=1) = 1/4$ $p(y=2) = 3/4$

Class 1: misclassification: $p(y=2|1) = \boxed{1/4}$

Gini impurity:

$$\mathcal{H} = 1 - \sum_{c=1}^{C} p(y=c)^2$$

$$= 1 - \left[(3/4)^2 + (1/4)^2 \right]$$

$$= 1 - 10/16 = 6/16 = \boxed{3/8}$$

Entropy:

$$\mathcal{H} = -\sum_{c=1}^{C} p(y=c) \log p(y=c)$$

$$= -\left[3/4 \ln 3/4 + 1/4 \ln 1/4 \right]$$

$$= \ln 4 - 3/4 \ln 3 = 0.562$$

Class 2: misclassification: $p(y=1) = \boxed{1/4}$

$\mathcal{H}_{gini} = 3/8$

$\mathcal{H}_{entropy} = 0.562$ → Symmetric

Can compute total weighted split

→ here again symmetric

⇒ $\mathcal{H}_{gini} = 3/8$ $\mathcal{H}_{entropy} = 0.562$

Case 2: (200, 0) (200, 400)

Node 1: misclassification = 0

$\mathcal{H}_{gini} = 1 - 1 = 0$

$\mathcal{H}_{entropy} = 0$

Node 2: misclassification = $1/3$

$$\mathcal{H}_{gini} = 1 - \left[(1/3)^2 + (2/3)^2 \right]$$

$$= 1 - 5/9 = 4/9$$

$$\mathcal{H}_{entropy} = -\left[1/3 \ln 1/3 + 2/3 \ln 2/3 \right]$$

$$= \ln 3 - 2/3 \ln 2$$

$$= 0.636$$

Weighted split

misclass: $\frac{200}{800} \times 0 + \frac{600}{800} \times 1/3 = \boxed{1/4}$

$\mathcal{H}_{gini} = 3/4 \times 4/9 = \boxed{3/9}$ $< 3/8$

$\mathcal{H}_{entropy} = 3/4 \times 0.636 = 0.477$ < 0.562

Same misclassification rate

↳ Split B has lower $\mathcal{H}_{gini}$ + lower $\mathcal{H}_{entropy}$

⇒ Split B preferred

Sheet 5

In [2]:
```
import numpy as np
from matplotlib import pyplot as plt
```

1 QDA

(a)

In [3]:
```
pts = np.load('data/data1d.npy')
labels = np.load('data/labels1d.npy')
```

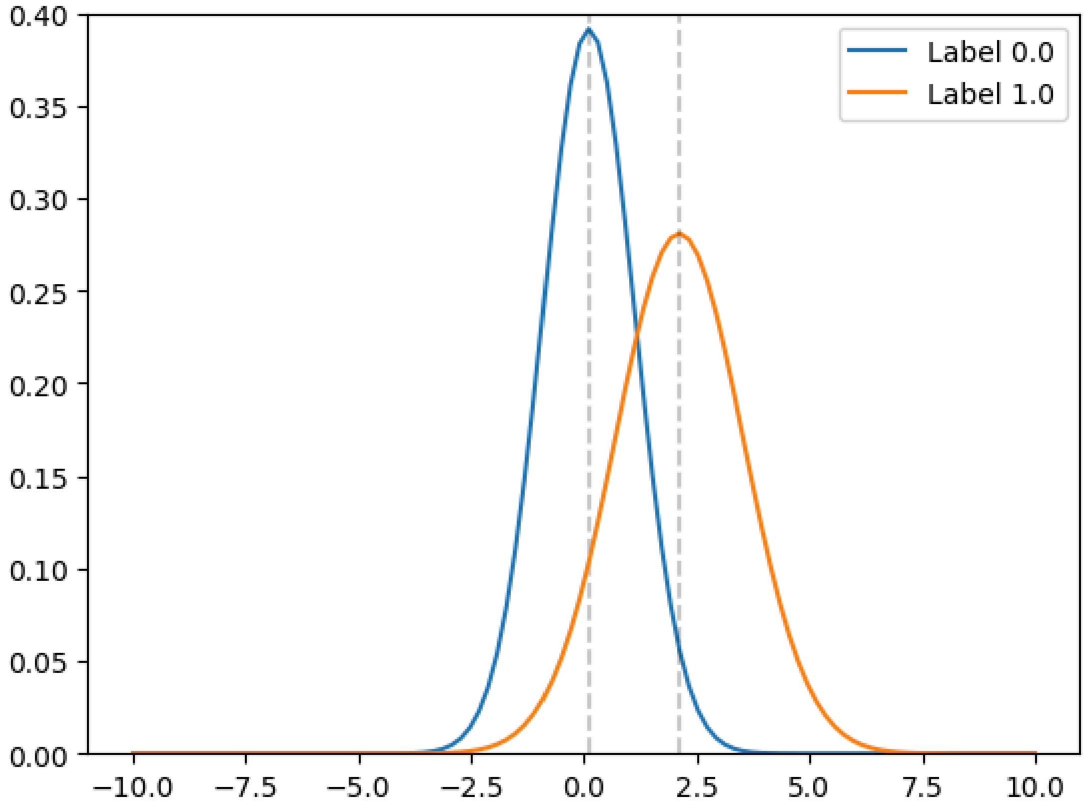
In [4]:
```
# Check label options
print(f"Unique labels:{np.unique(labels)}\n Number of unique labels:{len(np.uniq
# Split the data by the labels
data = {}
for label in np.unique(labels):
    data[label] = pts[labels == label]
```

```
Unique labels:[0. 1.]
 Number of unique labels:2
```

In [5]:
```
# Obtain mean and std for each label
mean = {}
std = {}
for label in data.keys():
    mean[label] = np.mean(data[label])
    std[label] = np.std(data[label])
print(f"Mean:{mean}\nStd:{std}")
```

```
Mean:{0.0: 0.10577655907233517, 1.0: 2.105667088262294}
Std:{0.0: 1.0185184414859962, 1.0: 1.4196573388063498}
```

In [6]:
```
#Plot gaussian distributions based on the mean and std
# Add dashed lines for the means
x = np.linspace(-10, 10, 100)
for label in data.keys():
    y = 1/(std[label]*np.sqrt(2*np.pi))*np.exp(-0.5*(x-mean[label])**2/std[label
    plt.plot(x, y, label=f"Label {label}")
    plt.axvline(mean[label], color='k', linestyle='--', alpha = 0.2,)
plt.ylim(0, 0.4)
plt.legend()
plt.show()
```



(b)

In [7]:

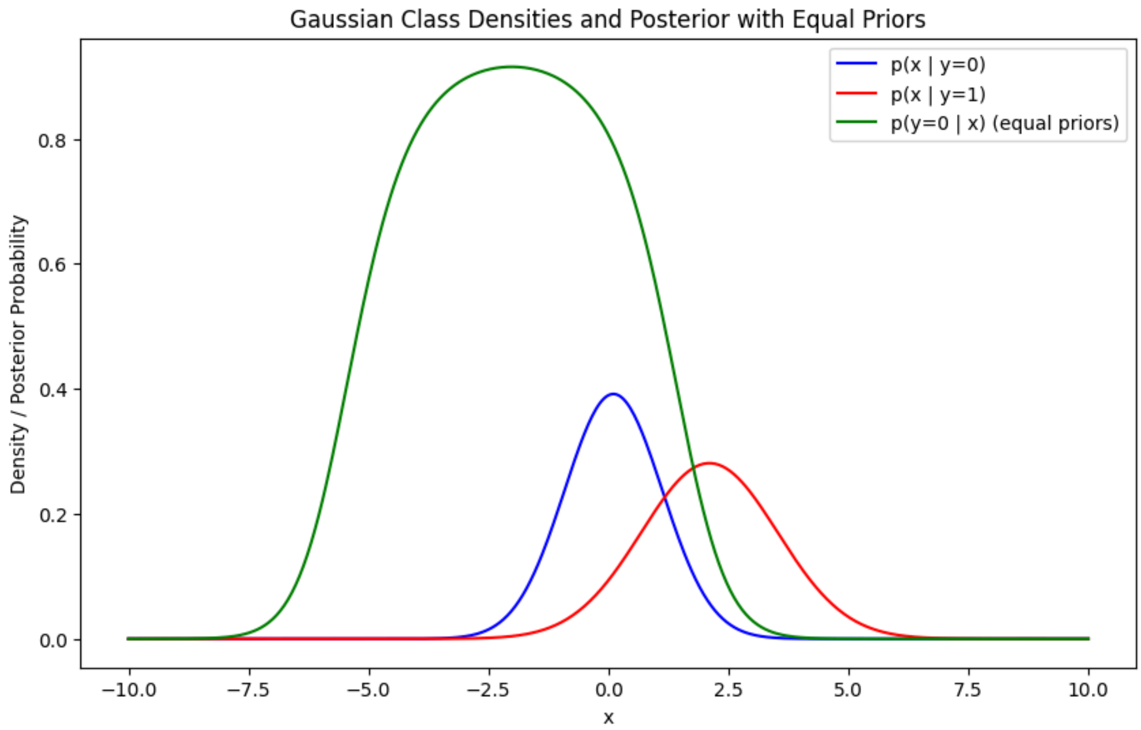
```
data = pts

# Separate data by class
data_class_0 = data[labels == 0]
data_class_1 = data[labels == 1]

# Compute mean and standard deviation for each class
mean_0, std_0 = np.mean(data_class_0), np.std(data_class_0)
mean_1, std_1 = np.mean(data_class_1), np.std(data_class_1)

print(f"Class 0: Mean = {mean_0}, Std Dev = {std_0}")
print(f"Class 1: Mean = {mean_1}, Std Dev = {std_1}")
```

```
Class 0: Mean = 0.10577655907233517, Std Dev = 1.0185184414859962
Class 1: Mean = 2.105667088262294, Std Dev = 1.4196573388063498
```

In [8]:

```
from scipy.stats import norm

# Define the range for plotting
x_values = np.linspace(-10, 10, 500)

# Calculate Gaussian class densities
density_0 = norm.pdf(x_values, mean_0, std_0)
density_1 = norm.pdf(x_values, mean_1, std_1)

# Define prior probabilities
p_y_0 = 0.5  # Initially equal priors
p_y_1 = 0.5

# Compute posterior for p(y=0|x) with equal priors
posterior_y0_equal_prior = (density_0 * p_y_0) / (density_0 * p_y_0 + density_1
posterior_y1_equal_prior = (density_1 * p_y_1) / (density_0 * p_y_0 + density_1
```

```
# Plot the class densities and posterior with equal priors
plt.figure(figsize=(10, 6))
plt.plot(x_values, density_0, label="p(x | y=0)", color="blue")
plt.plot(x_values, density_1, label="p(x | y=1)", color="red")
plt.plot(x_values, posterior_y0_equal_prior, label="p(y=0 | x) (equal priors)",
plt.title("Gaussian Class Densities and Posterior with Equal Priors")
plt.xlabel("x")
plt.ylabel("Density / Posterior Probability")
plt.legend()
plt.show()
```



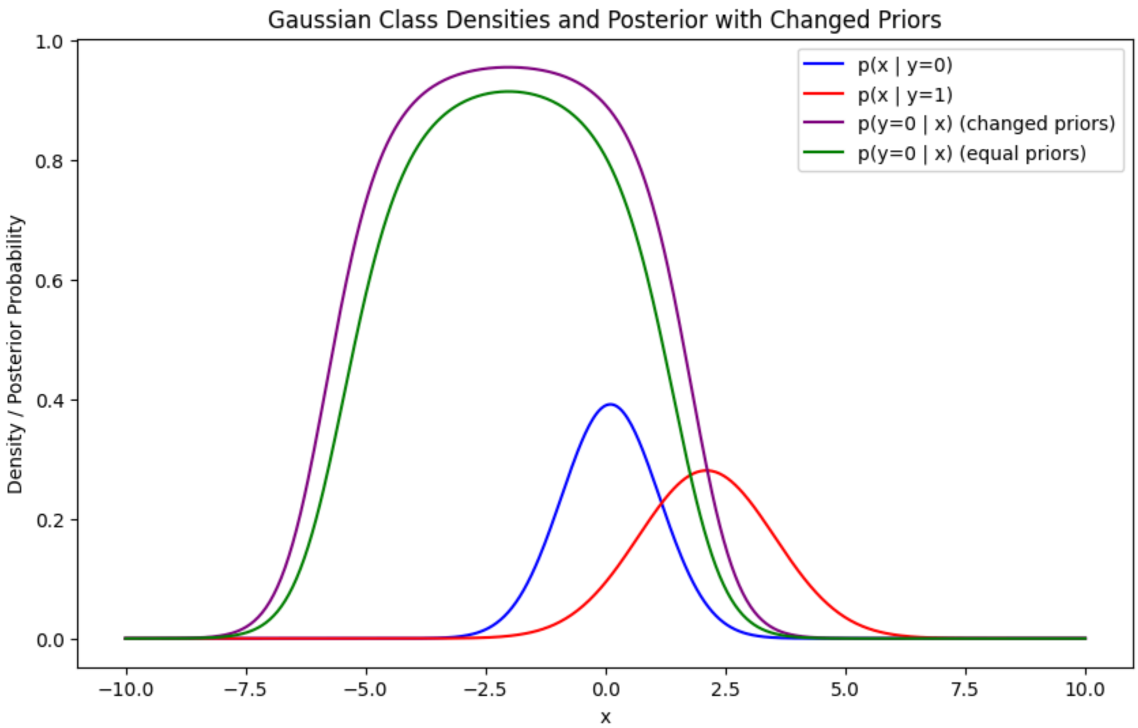
See something reasonable around the regions of the means, with the 0 class initially dominating, and then the 1 class taking over as we move to larger x. There are issue at the tails however:

- Below -5 the posterior significantly drops off again, suggesting class 1 dominance, which is not the case.
- Class 1 also dominates for large x

Clearly we have some issue with modelling outside the ranges of the standard deviations.

In [9]:
```
# Change the prior probabilities
p_y_0 = 2 / 3  # Now p(y=0) = 2 * p(y=1)
p_y_1 = 1 / 3

# Compute posterior for p(y=0|x) with changed priors
posterior_y0_changed_prior = (density_0 * p_y_0) / (density_0 * p_y_0 + density_
posterior_y1_changed_prior = (density_1 * p_y_1) / (density_0 * p_y_0 + density_

# Plot the class densities and posterior with changed priors
plt.figure(figsize=(10, 6))
plt.plot(x_values, density_0, label="p(x | y=0)", color="blue")
plt.plot(x_values, density_1, label="p(x | y=1)", color="red")
plt.plot(x_values, posterior_y0_changed_prior, label="p(y=0 | x) (changed priors
```

```
plt.plot(x_values, posterior_y0_equal_prior, label="p(y=0 | x) (equal priors)",
#plt.plot(x_values, posterior_y1_changed_prior, label="p(y=0 | x) (equal priors)
plt.title("Gaussian Class Densities and Posterior with Changed Priors")
plt.xlabel("x")
plt.ylabel("Density / Posterior Probability")
plt.legend()
plt.show()
```



Posterior gets broader and taller - signifying increased dominance of 0 class over 1 class. Same issues beyond reasonable ranges of standard deviations.

3 Trees and Random Forests

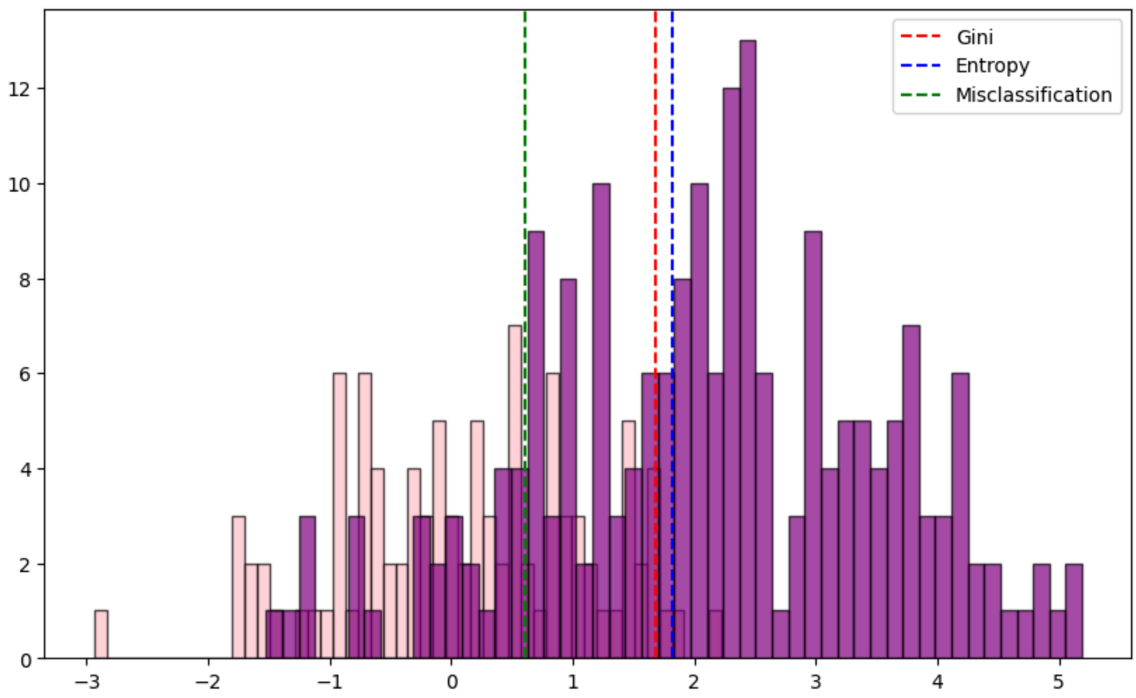
(b)

In [16]:

```
# Load the data
pts = np.load('data/data1d.npy')
labels = np.load('data/labels1d.npy')

# TODO: Sort the points to easily split them
sorted_indices = np.argsort(pts)
pts = pts[sorted_indices]
labels = labels[sorted_indices]
# Examine the labels
print(f"Unique labels:{np.unique(labels)}\n Number of unique labels:{len(np.uniq

# TODO: Implement or find implementation for Gini impurity, entropy and misclass

def probabilities(partition):
    # divide counts by size of dataset to get cluster probabilites
    return np.unique(partition, return_counts=True)[1] / len(partition)

def compute_split_measure(l, l0, l1, method):
    p0 = probabilities(l0)
```

```
    p1 = probabilities(l1)
    p = probabilities(l)
    return method(p) - (len(l0) * method(p0) + len(l1) * method(p1)) / (len(l))

def gini(p):
    return 1 - np.sum(p**2)

def entropy(p):
    return -np.sum(p * np.log(p))

# TODO: Iterate over the possible splits, evaulating and saving the three criter
#       (e.g. in three lists)
gini_splits = []
entropy_splits = []
misclassification_splits = []
for i in range(1, len(pts)):
    l0 = labels[:i]
    l1 = labels[i:]
    gini_splits.append(compute_split_measure(labels, l0, l1, gini))
    entropy_splits.append(compute_split_measure(labels, l0, l1, entropy))
    misclassification_splits.append(compute_split_measure(labels, l0, l1, lambda

# TODO: Then, Compute the split that each criterion favours and visualize them
#       (e.g. with a histogram for each class and vertical lines to show the spl
gini_best = np.argmax(gini_splits)
entropy_best = np.argmax(entropy_splits)
misclassification_best = np.argmax(misclassification_splits)

plt.figure(figsize=(10, 6))
plt.hist(pts[labels == 0], bins=50, color='pink', edgecolor='black', alpha=0.7)
plt.hist(pts[labels == 1], bins=50, color='purple', edgecolor='black', alpha=0.7
plt.axvline(pts[gini_best], color='red', linestyle='--', label='Gini')
plt.axvline(pts[entropy_best], color='blue', linestyle='--', label='Entropy')
plt.axvline(pts[misclassification_best], color='green', linestyle='--', label='M
plt.legend()
plt.show()
```

```
Unique labels:[0. 1.]
 Number of unique labels:2
```



(b)

In [17]:
```
# Load the dijet data
features = np.load('data/dijet_features_normalized.npy')
labels = np.load('data/dijet_labels.npy')

# TODO: define train, val and test splits as specified (make sure to shuffle the
```

In [32]:
```
# Check labels
print(np.unique(labels, return_counts=True))
features.shape, labels.shape
```

```
(array([0., 1., 2.]), array([999, 864, 370], dtype=int64))
```

Out[32]:
```
((116, 2233), (2233,))
```

In [25]:
```
from sklearn.ensemble import RandomForestClassifier
from sklearn.model_selection import train_test_split
from sklearn.metrics import accuracy_score, classification_report


# TODO: train a random forest classifier for each combination of specified hyper
#       and evaluate the performances on the validation set.

# Split into train, validation and test sets
X_train, X_temp, y_train, y_temp = train_test_split(features.T, labels, test_siz
X_val, X_test, y_val, y_test = train_test_split(X_temp, y_temp, test_size=200, r

# Check lengths
#len(X_train), len(X_val), len(X_test)

results = []
# Define hyperparameters
hyperparameters = {
    'n_estimators': [5, 10, 20, 100],
    'criterion': ['gini', 'entropy'],
    'max_depth': [2, 5, 10, None],
}

# Consider each combination of hyperparameters
for n_trees in hyperparameters['n_estimators']:
    for criterion in hyperparameters['criterion']:
        for max_depth in hyperparameters['max_depth']:
            # Train the model
            model = RandomForestClassifier(n_estimators=n_trees, criterion=crite
            model.fit(X_train, y_train)
            # Evaluate the model
            y_val_pred = model.predict(X_val)
            accuracy = accuracy_score(y_val, y_val_pred)

            # Store results
            results.append({
                "n_estimators": n_trees,
                "criterion": criterion,
                "max_depth": max_depth,
                "accuracy": accuracy
            })

# Identify the best hyperparameters
```

```
best_model = max(results, key=lambda x: x["accuracy"])
print("Best Hyperparameters:", best_model)
```

```
Best Hyperparameters: {'n_estimators': 100, 'criterion': 'gini', 'max_depth': 1
0, 'accuracy': 0.805}
```

In [30]:

```
# TODO: for your preferred configuration, evaluate the performance of the best c
# Train the best model
model = RandomForestClassifier(
    n_estimators=best_model['n_estimators'],
    criterion=best_model['criterion'],
    max_depth=best_model['max_depth'],
    random_state=42
)
model.fit(X_train, y_train)
#model.fit(np.vstack((X_train, X_val)), np.hstack((y_train, y_val)))
y_test_pred = model.predict(X_test)
accuracy = accuracy_score(y_test, y_test_pred)
print("Test Accuracy:", accuracy)
print("Classification Report:\n", classification_report(y_test, y_test_pred))
```

```
Test Accuracy: 0.78
Classification Report:
               precision    recall  f1-score   support

         0.0       0.76      0.78      0.77        94
         1.0       0.67      0.65      0.66        65
         2.0       1.00      1.00      1.00        41

    accuracy                           0.78       200
   macro avg       0.81      0.81      0.81       200
weighted avg       0.78      0.78      0.78       200
```

4. (a) ~~misclaf~~ miss rate: $A = B$

A: ~~Left~~ $H_{Left} = 1 - \frac{300}{400} = \frac{1}{4} = H_{Right}$, $H_{total} = \frac{1}{4}$

B: $H_{Left} = 1 - \frac{200}{200} = 0$, $H_{Right} = 1 - \frac{400}{600} = \frac{1}{3}$, $H_{total} = \frac{1}{3} \times \frac{3}{4} + 0 \times \frac{1}{4} = \frac{1}{4}$

Entropy: prefer B.

A: $H_{Left} = -\frac{3}{4}\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = -\frac{1}{4}\log_2\frac{27}{256} = 2 - \frac{1}{4}\log_2 27 = H_{right} = H_{tot}$

B: $H_L = 0$, $H_R = -\frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3} = \frac{1}{3}\log_2\frac{27}{2^3} = \frac{1}{3}\log_2 27 - 1$, 0.811

$H_{tot} = \frac{1}{4}H_L + \frac{3}{4}H_R = \frac{1}{4}\log_2 27 - \frac{3}{4} \approx 0.439$

Gini: prefer B

A: $H_L = 1 - (\frac{3}{4})^2 - (\frac{1}{4})^2 = \frac{3}{8} = H_R = H_{tot} = 0.375$

B: $H_L = 0$, $H_R = 1 - (\frac{1}{3})^2 - (\frac{2}{3})^2 = \frac{4}{9}$, $H_{tot} = \frac{1}{4}H_L + \frac{3}{4}H_R = \frac{1}{3} = 0.333$

5. (a) $p(\vec{x}) = \frac{\sqrt{|\Lambda|}}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2\end{pmatrix}^T \begin{pmatrix}\Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22}\end{pmatrix}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2\end{pmatrix}\right)$

$= \frac{\sqrt{|\Lambda|}}{2\pi} \exp\left[-\frac{1}{2}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2\end{pmatrix}^T \begin{pmatrix}\Lambda_{11}(x_1-\mu_1) + \Lambda_{12}(x_2-\mu_2) \\ \Lambda_{21}(x_1-\mu_1) + \Lambda_{22}(x_2-\mu_2)\end{pmatrix}\right]$

$= \frac{\sqrt{|\Lambda|}}{2\pi} \exp\left[-\frac{1}{2}\left[(x_1-\mu_1)^2\Lambda_{11} + (x_1-\mu_1)(x_2-\mu_2)(\Lambda_{12}+\Lambda_{21}) + (x_2-\mu_2)^2\Lambda_{22}\right]\right]$

$= \frac{\sqrt{|\Lambda|}}{2\pi} \exp\left[-\frac{1}{2}\Lambda_{11}\left[x_1 - \mu_{1|2}\right]^2 + C_1\right]$

$\mu_{1|2} = \mu_1 + \frac{1}{2}\left(\frac{\Lambda_{12}+\Lambda_{21}}{\Lambda_{11}}\right)(\mu_2 - x_2) = \mu_1 + \frac{\Lambda_{12}}{\Lambda_{11}}(\mu_2 - x_2)$

$C_1 = -\frac{1}{2}(x_2-\mu_2)^2\left[\Lambda_{22} - \frac{1}{4}\frac{(\Lambda_{12}+\Lambda_{21})^2}{\Lambda_{11}}\right] = -\frac{1}{2}(x_2-\mu_2)^2\left[\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}}\right]$

$\Sigma_{1|2} = \cancel{\sqrt{\frac{|\Lambda|}{\Lambda_{11}}}}\ \Lambda_{11}$ $\quad(\Lambda_{12} = \Lambda_{21})$

$\begin{cases} \Lambda_{11} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\ \Lambda_{12} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1} = \Lambda_{11}\Sigma_{12}\Sigma_{22}^{-1} \end{cases}$

$\Rightarrow \begin{cases} \mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mu_2 - x_2), \\ \Sigma_{1|2} = \cancel{\sqrt{}}\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, \\ C_1 = -\frac{1}{2}\cancel{(x_2-\mu_2)^2} \end{cases}$

5. (b) $p(x_1, x_2) = \frac{\sqrt{|\Lambda|}}{2\pi} \exp\left[\cancel{-\frac{1}{2}(x_2} - \frac{1}{2}\Lambda_{22}(x_2 - \mu_{2|1})^2 + C_2 \right]$

$$\mu_{2|1} = \mu_2 + \frac{\Lambda_{12}}{\Lambda_{22}}(\mu_1 - x_1)$$

$$C_2 = -\frac{1}{2}(x_1 - \mu_1)^2 \left(\Lambda_{11} - \frac{\Lambda_{12}^2}{\Lambda_{22}} \right)$$

$$p(x_1) = \int_{-\infty}^{+\infty} p(x_1, x_2)\, dx_2$$

$$= \frac{\sqrt{|\Lambda|}}{2\pi} e^{C_2} \int_{-\infty}^{+\infty} \exp\left[-\frac{\Lambda_{22}}{2}(x_2 - \mu_{2|1})^2 \right] dx_2$$

$$= \frac{\sqrt{|\Lambda|}}{2\pi} e^{C_2} \cdot \frac{\sqrt{2\pi}}{\sqrt{\Lambda_{22}}} = \sqrt{\frac{|\Lambda|}{2\pi}} \cdot \frac{1}{\sqrt{\Lambda_{22}}} \exp\left[-\frac{1}{2}\left(\Lambda_{11} - \frac{\Lambda_{12}^2}{\Lambda_{22}} \right)(x_1 - \mu_1)^2 \right]$$

$$\tilde{\mu}_1 = \mu_1$$

$$\tilde{\Sigma}_1 = \frac{1}{\Lambda_{11} - \frac{\Lambda_{12}^2}{\Lambda_{22}}} = \frac{\Lambda_{22}}{|\Lambda|} = \frac{1}{\Sigma_{11}}$$