Matrix Calculus

Chain rule

Vector chain rule

Vector chain rule

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} ln(x^2) \\ sin(3x) \end{bmatrix}$$

$$\begin{bmatrix} f_1(\mathbf{g}) \\ f_2(\mathbf{g}) \end{bmatrix} = \begin{bmatrix} ln(g_1) \\ sin(g_2) \end{bmatrix}$$

x^2

$3x$

$$\frac{\partial \mathbf{y}}{\partial x}$$

Vector chain rule

$$\frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial f_1(\mathbf{g})}{\partial x} \\ \frac{\partial f_2(\mathbf{g})}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial g_1}\frac{\partial g_1}{\partial x} + \frac{\partial f_1}{\partial g_2}\frac{\partial g_2}{\partial x} \\ \frac{\partial f_2}{\partial g_1}\frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial g_2}\frac{\partial g_2}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{1}{g_1}2x + 0 \\ 0 + cos(g_2)3 \end{bmatrix} = \begin{bmatrix} \frac{2}{x} \\ 3cos(3x) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} \\ \frac{\partial f_2}{\partial g_1} & \frac{\partial f_2}{\partial g_2} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_2}{\partial x} \end{bmatrix} = \frac{\partial \mathbf{f}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial x}$$

Vector chain rule

$$\begin{bmatrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} \\ \frac{\partial f_2}{\partial g_1} & \frac{\partial f_2}{\partial g_2} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_2}{\partial x} \end{bmatrix}$$

Jacobian matrix

$$\begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\mathbf{x}) & \frac{\partial}{\partial x_2} f_1(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n} f_1(\mathbf{x}) \\ \frac{\partial}{\partial x_1} f_2(\mathbf{x}) & \frac{\partial}{\partial x_2} f_2(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n} f_2(\mathbf{x}) \\ & & \dots & \\ \frac{\partial}{\partial x_1} f_m(\mathbf{x}) & \frac{\partial}{\partial x_2} f_m(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n} f_m(\mathbf{x}) \end{bmatrix}$$

Vector chain rule

$$\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{g}(\mathbf{x})) = \begin{bmatrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} & \cdots & \frac{\partial f_1}{\partial g_k} \\ \frac{\partial f_2}{\partial g_1} & \frac{\partial f_2}{\partial g_2} & \cdots & \frac{\partial f_2}{\partial g_k} \\ & & \cdots & \\ \frac{\partial f_m}{\partial g_1} & \frac{\partial f_m}{\partial g_2} & \cdots & \frac{\partial f_m}{\partial g_k} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ & & \cdots & \\ \frac{\partial g_k}{\partial x_1} & \frac{\partial g_k}{\partial x_2} & \cdots & \frac{\partial g_k}{\partial x_n} \end{bmatrix}$$

$$= \; diag(\frac{\partial f_i}{\partial g_i})diag(\frac{\partial g_i}{\partial x_i}) = diag(\frac{\partial f_i}{\partial g_i}\frac{\partial g_i}{\partial x_i})$$

Vector chain rule