ECE 4250
Assignment 1 Solutions

Problem Set

Question 1. 1. $y(n) = x(2-n)$

2. $y(n) = \text{sign}(x(n))$

3. $y(n) = x(7n)$

4. $y(n) = \sin(x(n))$

5. $y(n) = |x(n)|$

0.5 points for each

System	**Linear?**	**Time Invariant?**	**Causal?**	**Stable?**
(1)	Yes	No	No	Yes
(2)	No	Yes	Yes	Yes
(3)	Yes	No	No	Yes
(4)	No	Yes	Yes	Yes
(5)	No	Yes	Yes	Yes

Question 2. 10 points total, broken down into the following steps

The even part of the signal is: $x_e = \dfrac{x(n) + x(-n)}{2}$

The even part of the signal is: $x_o = \dfrac{x(n) - x(-n)}{2}$

Claim:

$$\sum_{-\infty}^{\infty}(x(n))^2 = \sum_{-\infty}^{\infty}(x_e(n))^2 + \sum_{-\infty}^{\infty}(x_o(n))^2$$

Give 6 points minimum for writing above equations. Take away 1 point for each mistake made in following steps.

Proof:

$$= \sum_{-\infty}^{\infty}\left(\frac{x(n) + x(-n)}{2}\right)^2 + \left(\frac{x(n) - x(-n)}{2}\right)^2$$

$$= \sum_{-\infty}^{\infty}\frac{(2x(n))^2 + 2x(n)x(-n) - 2x(n)x(-n) + x(-n))^2)}{4}$$

$$= \sum_{-\infty}^{\infty} \frac{(x(n))^2 + (x(-n))^2}{2}$$

$$= \sum_{-\infty}^{\infty} (x(n))^2$$

Question 3. 10 points total

If $x(n)$ is periodic, there is some $N \in \mathbb{Z}$ such that $x(n + k_1 N) = x(n), \forall n, k_1 \in \mathbb{Z}$.
I.e., $x_a(nT_s + k_1 NT_s) = x_a(nT_s), \forall n, k_1 \in \mathbb{Z}$.
Recall $x_a(t)$ is periodic: $x_a(t) = x_a(t + k_2 T_o), \forall k_2 \in \mathbb{Z}, t \in \mathbb{R}$
So: $x_a(nT_s) = x_a(nT_s + k_2 T_o) = x_a(nT_s + k_1 NT_s)$, where the second equality is from the periodicity of $x(n)$ and k_1 depends on the choice of k_2.
Thus: $N = \frac{T_o}{T_s}\frac{k_1}{k_2} \in \mathbb{Z}$.
This proves that if $x(n)$ is periodic, $\frac{T_o}{T_s}$ or $\frac{T_s}{T_o}$ needs to be a rational number.
A valid proof of this direction will earn you 5 points

Next let's prove the other direction of the statement. We shall assume $\frac{T_o}{T_s}$ is a rational number.
$x(n) = x_a(nT_s) = x_a(nT_s + kT_0), \ \forall k \in \mathbb{Z}$
$x(n) = x_a(nT_s + kT_0 \frac{T_s}{T_s} \frac{\mathrm{lcm}(T_0,T_s)}{\mathrm{lcm}(T_0,T_s)}), \forall k \in \mathbb{Z}$, where $\mathrm{lcm}(T_0, T_s)$ denotes the least common multiple of the two integers.
$x(n) = x_a(nT_s + \left(k \frac{T_0}{\mathrm{lcm}(T_0,T_s)} \frac{\mathrm{lcm}(T_0,T_s)}{T_s}\right) T_s), \ \forall k \in \mathbb{Z}$.

$$x(n) = x_a(\left(n + k \frac{T_0}{\mathrm{lcm}(T_0, T_s)} \frac{\mathrm{lcm}(T_0, T_s)}{T_s}\right) T_s)$$

Choose $k = \frac{\mathrm{lcm}(T_0,T_s)}{T_0}$ and define $N = \frac{\mathrm{lcm}(T_0,T_s)}{T_s}$, we have:

$$x(n) = x(n + N).$$

Thus the fundamental period is: $N = \frac{\mathrm{lcm}(T_0,T_s)}{T_s}$.
A valid proof of this direction will earn you 5 points

Question 4.

The impulse response from the unit-step response can be derived from the following relationship:
3 points for setup
$\delta(n) = u(n) - u(n-1)$
$u(n) \xrightarrow[LTISystem]{} s(n)$. Since the system is LTI, then:

3 points for expressing this h function as follows
$h(n) = s(n) - s(n-1)$.
4 points for full mathematical correct answer below
For a given input $x(n)$, the output is therefore:

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(k)(s(n-k) - s(n-k-1))$$

Question 5. Two discrete-time signals $x(n)$ and $y(n)$ are called *orthonormal* on an interval $[N_1, N_2]$, if

$$\sum_{n=N_1}^{N_2} x(n)y^*(n) = \begin{cases} 1, & \text{if } x(n) = y(n), \forall n \in [N_1, N_2] \\ 0, & \text{otherwise} \end{cases}$$

Show that harmonically related signals: $x_k(n) = \frac{1}{\sqrt{N}} e^{j2\pi kn/N}$ are orthonormal on an interval of length N. I.e,

$$\sum_{n=0}^{N-1} x_k(n) x_l^*(n) = \begin{cases} 1, \text{if } k = l \\ 0, \text{ otherwise} \end{cases}$$

10 points total

$$\sum_{n=0}^{N-1} x_k(n) x_l^*(n) = \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{j2\pi k \dfrac{n}{N}} \frac{1}{\sqrt{N}} e^{-j2\pi l \dfrac{n}{N}}$$

5 points for correct expression

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi(k-l) \dfrac{n}{N}}$$

5 points for correct explanation

The expression $\sum_{n=0}^{N-1} e^{j2\pi(k-l)\frac{n}{N}}$ will return N for $k = l$ and will return 0 otherwise because summing cosines and sines over a period always returns zero ($e^{j2\pi(k-l)\frac{n}{N}} = \cos(j2\pi(k-l)\frac{n}{N}) + j\sin(2\pi(k-l)\frac{n}{N})$).