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Assignment 3

ECE 4250
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Problem Set

Question 1

Consider the signal $x(n) = a^n u(n)$, with $|a| < 1$.

1. Determine the spectrum $X(\omega)$.

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \left[ae^{-j\omega}\right]^n \\
&= \frac{1}{1 - ae^{-j\omega}} \\
X(\omega) &= \frac{1}{1 - ae^{-j\omega}}
\end{aligned}$$

2. The signal $x(n)$ is applied to a decimator that reduces the rate by a factor of 2. Determine the output spectrum by working out the effect of decimation in the Fourier domain.

$$\begin{aligned}
Y(\omega_y) &= \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega_y - 2\pi k}{D}\right), \quad D = 2 \\
&= \frac{1}{2}\left[X\left(\frac{\omega_y}{2}\right) + X\left(\frac{\omega_y - 2\pi}{2}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{1 - ae^{-j\frac{\omega_y}{2}}} + \frac{1}{1 - ae^{-j\frac{\omega_y - 2\pi}{2}}}\right] \\
&= \frac{1}{2}\left[\frac{1 - ae^{-j(\frac{\omega_y}{2} - \pi)} + 1 - ae^{-j\frac{\omega_y}{2}}}{1 - ae^{-j\frac{\omega_y}{2}} - ae^{-j(\frac{\omega_y}{2} - \pi)} + a^2 e^{-j(\omega - \pi)}}\right] \\
&= \frac{1}{2}\left[\frac{2}{1 - a^2 e^{-j\omega}}\right] \\
&= \frac{1}{1 - a^2 e^{-j\omega}}
\end{aligned}$$

3. Show that the spectrum in part (b) is the same as the Fourier transform of $x(2n)$.

$$\omega_y = 2\omega_x$$

$$
\begin{aligned}
X(\omega_x) &= \sum_{n=-\infty}^{\infty} x[2n]e^{-j2\omega_x n}, \quad m = 2n \\
&= \sum_{m=-\infty}^{\infty} x[2n]e^{-j\omega_y n} \\
&= \frac{1}{1-a^2e^{-j\omega}}
\end{aligned}
$$

Question 2

Consider the two different ways of cascading a decimator and and interpolator, as seen in Figure below. Note that we are omitting the low-pass filter(s) that would be typically used.

1. If $D = I \neq 1$, show that the two outputs are not the same.

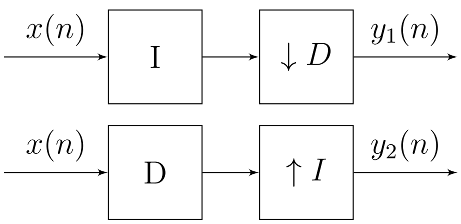
   Let $D = I = 2$, and $x(n) = \{x_0, x_1, x_2, ..., x_n\}$.

   Decimation then Interpolation: $y_1(n) = \{x_0, x_2, x_4, ...\}$, $\quad z_1(n) = \{x_0, 0, x_2, 0, x_4, 0, ...\}$

   Interpolation then Decimation: $y_2(n) = \{x_0, 0, x_1, 0, x_2, 0, ...\}$, $\quad z_2(n) = \{x_0, x_1, x_2, ...\}$

   $z_1(n) \neq z_2(n)$

2 What is the condition, under which the two outputs would be identical?



Decimation then Interpolation:

$$
x_1(n) = \begin{cases} x\left(\frac{n}{I}\right), & \text{if } n = \text{multiple of I} \\ 0, & \text{otherwise} \end{cases}
$$

$$
y_1(n) = x_1(Dn) = \begin{cases} x\left(\frac{Dn}{I}\right), & \text{if } Dn = \text{multiple of I} \\ 0, & \text{otherwise} \end{cases}
$$

Interpolation then Decimation:

$$
\begin{aligned}
x_2(n) &= x(Dn) \\
y_1(n) = x_1(Dn) &= \begin{cases} x_2\left(\frac{n}{I}\right), & \text{if } n = \text{multiple of I} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} x\left(\frac{Dn}{I}\right), & \text{if } n = \text{multiple of I} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$\frac{lcm(D,I)}{D} = I \rightarrow \text{D and I are relatively prime.}$$

Order of up-sampling (interpolation) and down-sampling (decimation) can be swapped if D and I are relatively prime.