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Assignment 1

ECE 4250
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Problem Set

Question 1

For each of the following systems, classify whether they are (i) linear or not, (ii) time invariant or not, (iii) causal or not, and (iv) stable or not:

- $y(n) = x(2 - n)$
  (i) linear
  (ii) not time-invariant
  (iii) not causal
  (iv) stable
- $y(n) = sign(x(n))$
  (i) not linear
  (ii) time-invariant
  (iii) causal
  (iv) stable
- $y(n) = x(7n)$
  (i) linear
  (ii) not time-invariant
  (iii) not causal
  (iv) stable
- $y(n) = sin(x(n))$
  (i) not linear
  (ii) time-invariant
  (iii) causal
  (iv) stable
- $y(n) = |x(n)|$
  (i) not linear
  (ii) time-invariant
  (iii) causal
  (iv) stable

Question 2

Show that the energy of a real-valued signal can be decomposed into the sum of the energies of the even and odd parts of the signal.

Consider a real-valued signal $x(n)$:

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \quad x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$\sum_{n=-\infty}^{\infty} x_e(n)x_o(n) = \sum_{n=-\infty}^{\infty} \frac{1}{4}[x^2(n) + x(-n)x(n) - x(-n)x(n) - x^2(n)] = \frac{1}{4}\sum_{n=-\infty}^{\infty} 0 = 0$$

$$\sum_{n=-\infty}^{\infty} x^2(n) = \sum_{m=-\infty}^{\infty} [x_e(n) + x_o(n)]^2$$

$$= \sum_{m=-\infty}^{\infty} x_e^2(n) + 2\sum_{m=-\infty}^{\infty} x_e(n)x_o(n) + \sum_{m=-\infty}^{\infty} x_o^2(n) = E_e + 0 + E_o = E_e + E_o$$

Question 3

Consider a continuous-time sinusoid $x_a(t)$ with a period $T_0 = 1/F_0$. Assume this signal is sampled at a rate of $F_s = 1/Ts$ to produce a discrete-time signal $x(n) = x_a(nT_s)$.

- What is the condition on T_s and T_0 for $x(n)$ to be periodic?

$$x(n) = x(n+N), \quad \omega_0 = 2\pi T_0$$

$$x(nT_s) = x((n+N)T_s)$$

$$e^{j\frac{2\pi}{T_0}(n)T_s} = e^{j\frac{2\pi}{T_0}(n+N)T_s}$$

$$e^{j\frac{2\pi}{T_0}(n)T_s} = (e^{j\frac{2\pi}{T_0}(n)T_s})(e^{j\frac{2\pi}{T_0}(N)T_s})$$

$$e^{j\frac{2\pi}{T_0}(N)T_s} = 1$$

$$\frac{2\pi}{T_0}(N)T_s = 2\pi k$$

$$\frac{T_0}{T_s} = \frac{N}{k}$$

  for positive integers N and k for which $\frac{N}{k}$ are rational.

- If $x(n)$ is periodic, what is its fundamental (baseline) period?

  The fundamental period is the smallest value of N such such that $k\frac{T_0}{T_s} = 1$.

Question 4

Assume you are given an LTI system with a step response $s(n)$, which is defined as the output of the system when excited with (input) a step function $u(n)$. Can you derive the expression for the output $y(n)$ in terms of $s(n)$ and an arbitrary input $x(n)$?

$$s(n) = u(n) - u(n-1)$$
$$h(n) = s(n) - s(n-1)$$
$$y(n) = x(n) * h(n) = x(n) * (s(n) - s(n-1))$$

Question 5

Two discrete-time signals $x(n)$ and $y(n)$ are called orthonormal on an interval $[N1, N2]$, if

$$\sum_{n=N_1}^{N_2} x(n)y^*(n) = \begin{cases} 1, & \text{if } x(n) = y(n)\ \forall n \in [N1, N2] \\ 0, & \text{otherwise} \end{cases}$$

Show that harmonically related signals: $x_k(n) = \frac{1}{\sqrt{N}} e^{j2kn/N}$ are orthonormal on an interval of length N.

$$x_k(n) = \frac{1}{\sqrt{N}} e^{j2kn/N} = \frac{1}{\sqrt{N}}(cos(2\pi kn/N) + jsin(2\pi kn/N))$$
$$x_l^*(n) = \frac{1}{\sqrt{N}}(cos(2\pi ln/N) - jsin(2\pi ln/N))$$

$$\sum_{n=0}^{N-1} x_k(n)x_l^*(n) = \sum_{n=0}^{N-1} \frac{1}{N}(cos(2\pi ln/N)cos(2\pi kn/N) - jsin(2\pi ln/N)cos(2\pi kn/N)$$
$$+jsin(2\pi kn/N)cos(2\pi ln/N) + sin(2\pi kn/N)sin(2\pi ln/N))$$

$$= \sum_{n=0}^{N-1} \frac{1}{N}(\frac{cos(2\pi(k-l)n/N) + cos(2\pi(k+l)n/N)}{2} - j\frac{sin(2\pi(k+l)n/N) + sin(2\pi(l-k)n/N)}{2}$$
$$+j\frac{sin(2\pi(k+l)n/N) + sin(2\pi(k-l)n/N)}{2} + \frac{cos(2\pi(k-l)n/N) - cos(2\pi(k+l)n/N)}{2})$$
$$= \sum_{n=0}^{N-1} \frac{1}{N}(cos(2\pi(k-l)n/N) + jsin(2\pi(k-l)n/N = \sum_{n=0}^{N-1} \frac{1}{N} e^{\frac{j2\pi(k-l)n}{N}}$$

if $k = l$:

$$= \sum_{n=0}^{N-1} \frac{1}{N} e^{\frac{j2\pi(0)n}{N}} = \sum_{n=0}^{N-1} \frac{1}{N} = \frac{1}{N}(N) = 1$$

if $k \neq l$:

$$= \sum_{n=0}^{N-1} \frac{1}{N} e^{\frac{j2\pi(k-l)n}{N}} = \frac{1}{N} \left[1 + e^{\frac{j2\pi(k-l)n}{N}} + (e^{\frac{j2\pi(k-l)n}{N}})^2 + \ldots + (e^{\frac{j2\pi(k-l)n}{N}})^{(N-1)} \right]$$

$$= \frac{1}{N} \left[\frac{1 - e^{j2\pi(k-l)}}{1 - e^{j2\pi(k-l)/N}} \right] = \frac{1}{N} \left[\frac{1-1}{1-1^{\frac{1}{N}}} \right] = 0$$

$$\sum_{n=0}^{N-1} x_k(n) x_l^*(n) = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{otherwise} \end{cases}$$

Programming Questions

Question 1

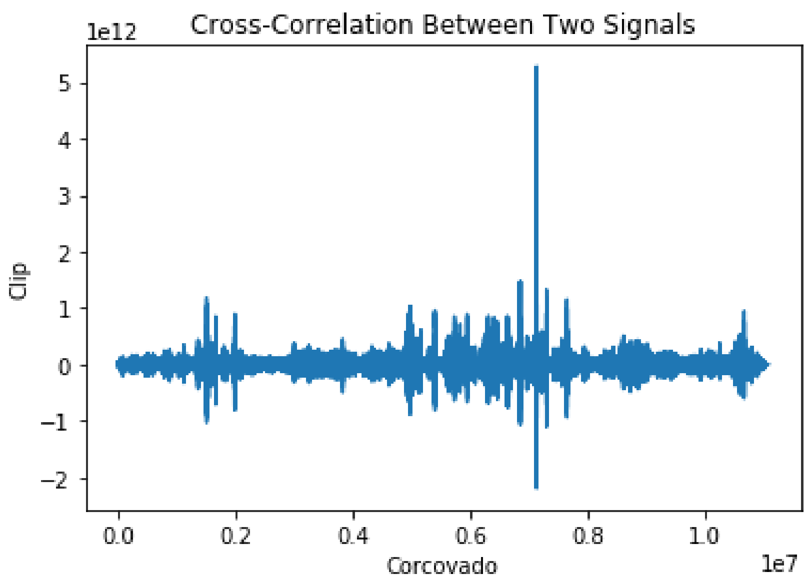
a) See code.

b) This convolution shows how the shape of signal x is affected by a single square pulse h.

c) Because the value of h at every index is the same, we can multiply the sum of the values at every index of x with the constant value of h. By doing so, we can evaluate the convolution with a single pass.

d) With the regular technique, the convolution took around 0.229 seconds, while with the efficient technique, the convolution only took 0.00139 seconds.

Question 2



The peak was found at around 161 seconds, which means that the clip begins at roughly 2 minutes and 41 seconds into the song. The sampling rate for both the corcovado and the clip wav files were 44100 Hz.

Question 3

a) See code.

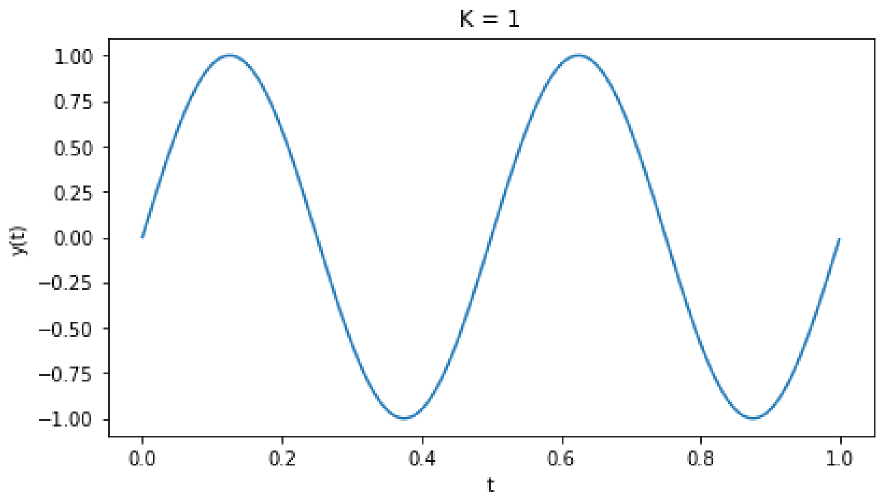
b) See code.

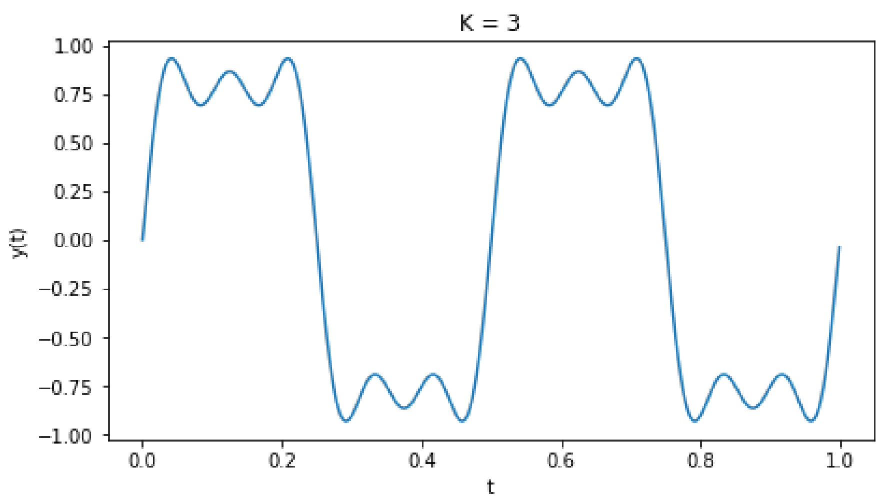
c) See code.

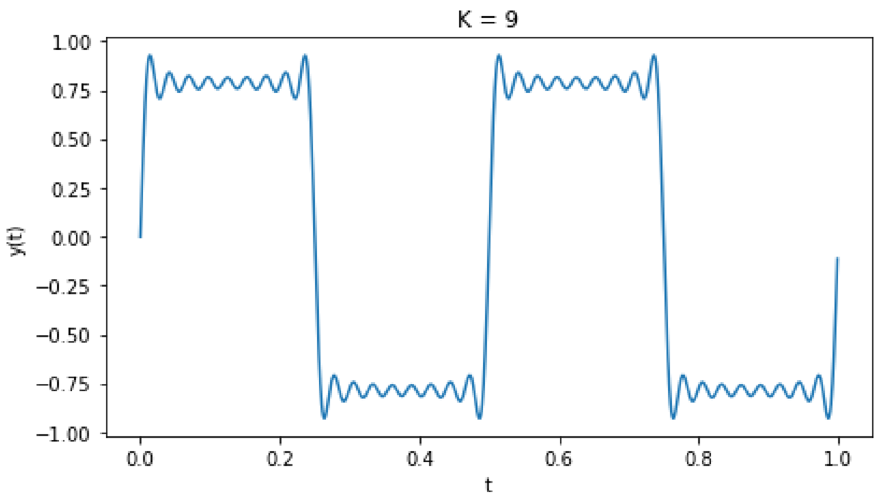
d) K = 3 plot doesn't look exactly like a square wave.

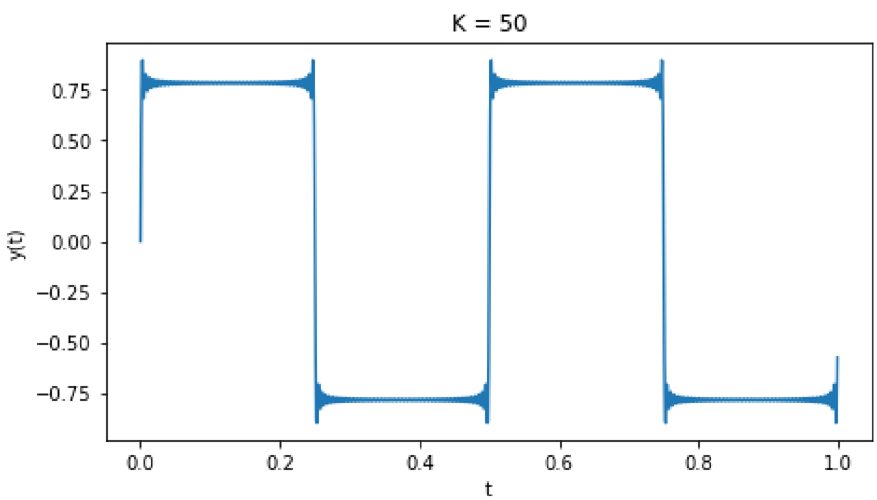
e) We can observe that as K increases, the shape of the plot becomes closer to a square wave.

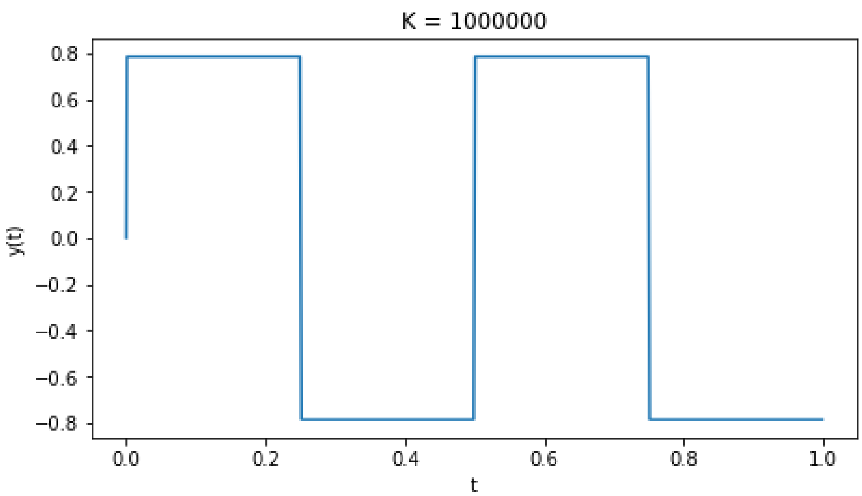
f) At some K values, we can observe Gibbs phenomenon, which is an overshoot of the Fourier series near a discontinuity.



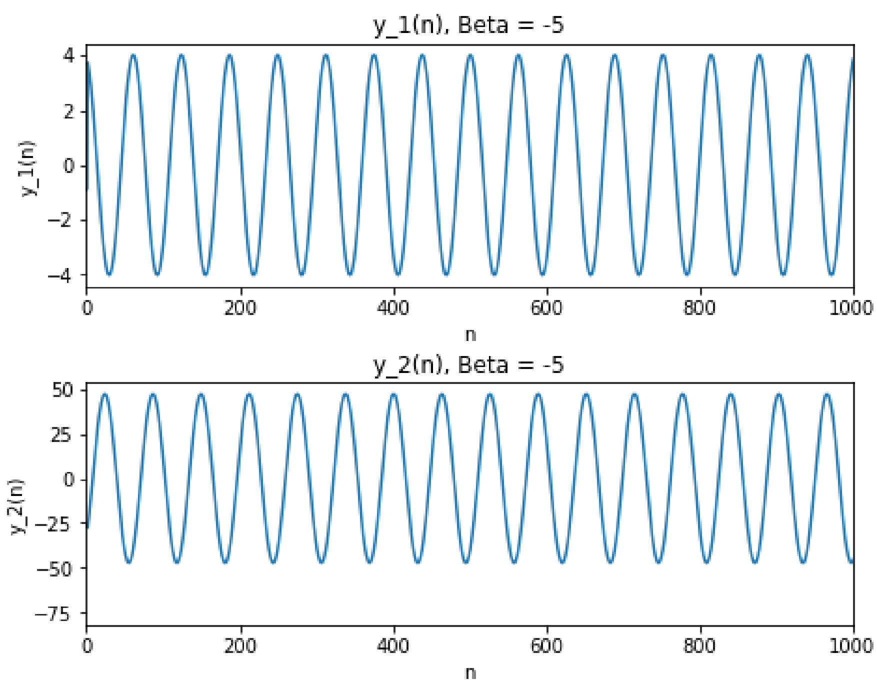


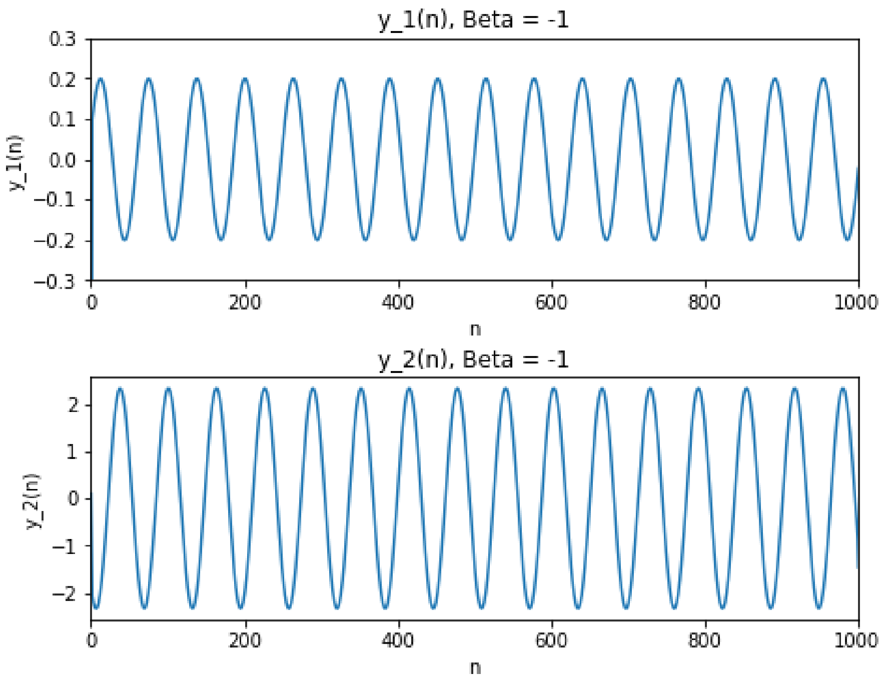

K = 9
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y(t)
0.0
0.2
0.4
0.6
0.8
1.0
t

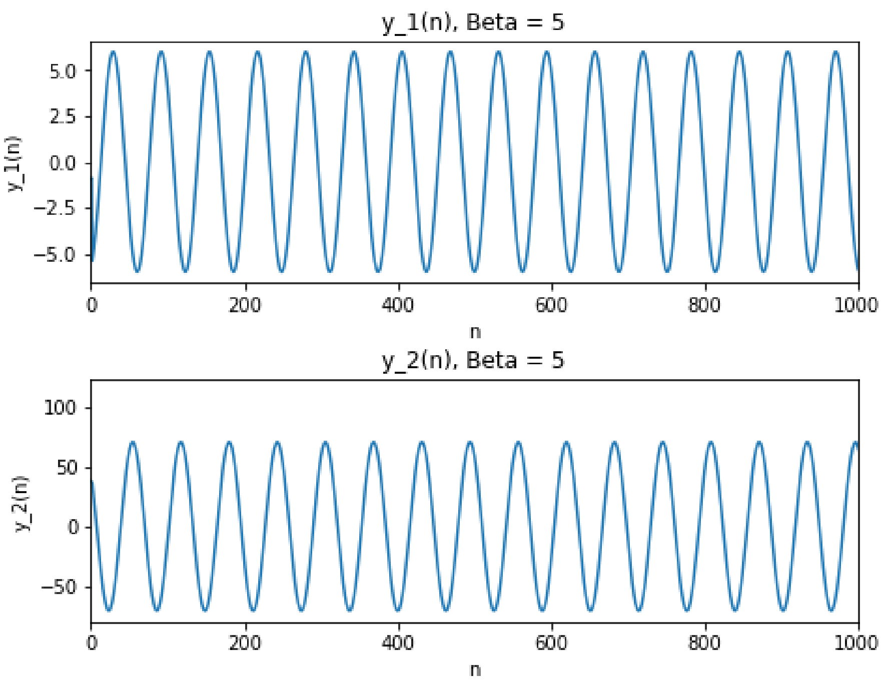

K = 50
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
y(t)
0.0
0.2
0.4
0.6
0.8
1.0
t


K = 1000000
y(t)
t
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
0.0
0.2
0.4
0.6
0.8
1.0

Question 4


y_1(n), Beta = -5
y_1(n)
n
4
2
0
−2
−4
0
200
400
600
800
1000
y_2(n), Beta = -5
y_2(n)
n
50
25
0
−25
−50
−75
0
200
400
600
800
1000





Because $y_1(n)$ is the output of the convolution of $x(n)$ and $h_1(n)$, it is the sum of a sine function and a shifted sine function. The magnitude of β changes the amplitude of the sinusoid. The sign of β determines whether the input sine function is negative or not. $y_1(n)$ has the same frequency as $x(n)$.

$h_2(n)$ is a unit step function shifted to the right by 200 subtracted by a unit step function. $y_1(n)$, $y_2(n)$, and x all have the same frequency. The magnitude of β changes the amplitude of $y_2(n)$.