ECE 4250
Assignment 2 Solutions

Question 1. 10 points total

$$X(z) = \frac{2 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}}$$

You can re-write $X(z)$ ias:

$$X(z) = 1 + \frac{1}{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}} = X_1(z) + X_2(z)$$

The first term, $X_1(z) = 1$, corresponds to a delta function $x_1(n) = \delta(n)$
Now, let's focus on $X_2(z)$.

We can re-write it as:

$$\frac{X_2(z)}{z} = \frac{z}{z^2 - \frac{1}{2}z + \frac{3}{5}} = \frac{z}{(z - r_1)(z - r_2)} = \frac{A}{z - r_1} + \frac{B}{z - r_2},$$

where $r_1 \approx 0.25 + j0.733$ and $r_2 \approx 0.25 - j0.733$ are a conjugate pair.

We can solve for A and B by multiplying with $z - r_1$ or $z - r_2$, and evaluating at $z = r_1$ or $z = r_2$, respectively. You get $A \approx 0.5 - j0.17$ and $B =\approx 0.5 + j0.17$.

So:

$$\begin{aligned} x_2(n) &= (0.5 - j0.17)(0.25 + j0.733)^n u(n) + (0.5 + j0.17)(0.25 - j0.733)^n u(n) \\ &\approx 1.06(0.77)^n \cos(1.24n - 0.33)u(n) \end{aligned}$$

Since $x(n) = x_1(n) + x_2(n)$, we have:

$$x(n) \approx \delta(n) + 1.06(0.77)^n \cos(1.24n - 0.33)u(n)$$

It is possible to get expressions that look different. One important check is that the answer $x(n)$ needs to be a real signal (the poles are conjugate pairs). Also for $n < 0$ $x(n)$ should be zero (causality). For $n > 0$, we should have a discrete cosine. You can also check if your result is correct by evaluating the answers for $n = 0, n = 1, n = 2$, etc.

If the answer is not expressed as a real function, we will deduct 2 points. If the answer is expressed as a real function, but the expression is wrong, we will deduct 1 point. If approach is correct but numerical values for the complex expression are wrong, you will get 6 points. If the complex expression is correct, you will get 8 points.

Question 2. 10 points total: 4pts (a) + 2pts (b) + 4pts (c)

a)

The impulse response function is:

$$h(t) = \begin{cases} t/T_s, 0 \leq t < T_s \\ 2 - t/T_s, T_s \leq t < 2T_s \\ 0, \text{otherwise} \end{cases}$$

b)

This is causal since $h(t) = 0, \forall t < 0$

c)

The impulse response function, which has a triangular form, can be obtained by convolving two box functions:

$$h(t) = b \star b(t),$$

where

$$b(t) = \begin{cases} 1, T_s/2 \leq t < T_s \\ 0, \text{otherwise} \end{cases}$$

$b(t)$ has a Fourier transform: $B(\omega) = \text{sinc}(\omega T_s/2)e^{-j\omega 3T_s/4}$. Hence, the FT of h is $H(\omega) = B^2(\omega) = \text{sinc}^2(\omega T_s/2)e^{-j\omega 3T_s/2}$.

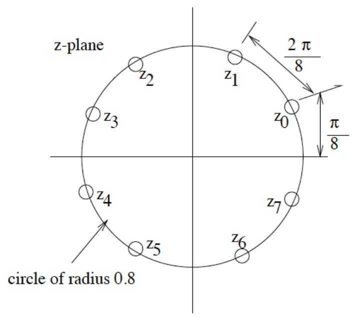
The ideal interpolator has a finite support FT, which is an ideal low-pass filter:

$$H_{ideal}(\omega) = \begin{cases} 1, |\omega| \leq 2\pi/T_s \\ 0, \text{otherwise} \end{cases}$$

Ideal interpolator is not causal, has no frequency attenuation, amplification. The linear interpolator above is causal and modifies the frequency spectrum.

Question 3. 10 points total: 5pts (a) + 5pts (b)

a)



b)

$$
\begin{aligned}
X(k) &= X(z)|_{z=z_k} \\
&= \sum_{n=0}^{7} x(n)[0.8e^{j(2\pi k/8+\pi/8)}]^{-n} \\
s(n) &= x(n)0.8^{-n}e^{-j\pi/8n}
\end{aligned}
$$

Question 4. 10 points total

Note: ω is in radians, however Nyquist sampling rate is in Hertz. Let $X_a(\omega)$ denote the Fourier transform of $x_a(t)$. The Nyquist rate of $x_a(t)$ is $2B$.

Take away up to 3 points for getting any of the following sub-parts wrong

a)

Fourier transform of $\frac{dx_a(t)}{dt}$ is $j\omega X_a(\omega)$, which has same bandwidth as $X_a(\omega)$. Thus Nyquist rate is $2B$.

b)

Fourier transform of $x_a(t)\cos 6\pi Bt$ is $\frac{1}{2}X_a(\omega + 6\pi B) + \frac{1}{2}X_a(\omega - 6\pi B)$. Note that this is a band-limited signal, with $2B$ bandwidth. Although a naive application of sampling theorem would suggest we need to sample at a rate $\geq 8B$), we can still recover the signal if we sample at $2B$. Hence, the Nyquist rate for this signal is $2B$. However, we will accept your answer if you've stated it to be $8B$.

c)

The Fourier transform of $x_a(3t)$ is $3X_a(\omega/3)$. The Nyquist rate is $6B$.

Question 5. 10 points total

$$
\begin{aligned}
X_c(k) &= \sum_{n=0}^{N-1} \frac{1}{2} x(n)(e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}})e^{-\frac{2\pi k n}{N}} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k-k_0)n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+k_0)n}{N}} \\
&= \frac{1}{2}X(k-k_0)_{\text{mod } N} + \frac{1}{2}X(k+k_0)_{\text{mod } N}
\end{aligned}
$$

Question 6. 10 points total

$$\begin{aligned} X(\omega) &= e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega} \\ &= 3 + 2\cos(2\omega) + 4\cos(\omega) \end{aligned}$$

Using DFT definition, one can show:

$$Y(k) = 3 + 4\cos(\pi/3k) + 2\cos(2\pi/3k)$$

Hence: $Y(k) = X(\omega)|_{\omega=\frac{\pi k}{3}}$.

This should also be apparent from the fact that y is one period of a periodic signal obtained by repeating x.