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Assignment 2

Problem Set

Question 1

Derive the analytic form for the *causal* signal that has the following z-transform:

$$X(z) = \frac{2 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}}$$

$$\begin{aligned} X(z) &= \frac{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}} + \frac{1}{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}} \\ &= 1 + \frac{1}{1 - \frac{1}{2}z^{-1} + \frac{3}{5}z^{-2}} = 1 + \frac{z^2}{z^2 - 0.5z + 0.6} \\ \frac{X(z)}{z} &= \frac{1}{z} + \frac{z}{z^2 - 0.5z + 0.6} \end{aligned}$$

$$\alpha = 0.25 + j\frac{\sqrt{2.15}}{2} = 0.7746e^{j71.1708^\circ} \quad \alpha^* = 0.25 - j\frac{\sqrt{2.15}}{2} = 0.7746e^{-j71.1708^\circ}$$

$$\frac{z}{(z-\alpha)(z-\alpha^*)} = \frac{A}{z-\alpha} + \frac{A^*}{z-\alpha^*} \qquad z = A(z-\alpha^*) + A^*(z-\alpha)$$

$$A = \frac{\alpha}{\alpha - \alpha^*} = -j\left(\frac{0.25}{\sqrt{2.15}}\right) + 0.5 = 0.5283e^{j18.8293^\circ}$$

$$A^* = \frac{\alpha^*}{\alpha^* - \alpha} = j\left(\frac{0.25}{\sqrt{2.15}}\right) + 0.5 = 0.5283e^{-j18.8293^\circ}$$

$$\frac{X(z)}{z} = \frac{1}{z} + \frac{z}{z^2 - 0.5z + 0.6} = \frac{1}{z} + \frac{A}{z-\alpha} + \frac{A^*}{z-\alpha^*}$$

$$X(z) = 1 + \frac{A}{1-\alpha z^{-1}} + \frac{A^*}{1-\alpha^* z^{-1}}$$

$$\begin{aligned} x[n] &= \delta[n] + (A\alpha^n + A^*(\alpha^*)^n)u[n] \\ &= \delta[n] + (0.5283)0.7746^n(e^{j(n71.1708^\circ + 18.8293^\circ)} + e^{-j(n71.1708^\circ + 18.8293^\circ)})u[n] \\ &= \delta[n] + (1.057)0.7746^n(\cos{(n71.1708^\circ + 18.8293^\circ)})u[n] \end{aligned}$$

Question 2

A linear interpolator constructs a continuous signal by connecting successive samples. For example:

$$\hat{x}(t) = x(nT_s - T_s) + \frac{x(nT_s) - x(nT_s - T_s)}{T_s}(t - nT_s), nT_s \leq t \leq (n+1)T_s$$

This interpolator has a delay of Ts seconds. I.e. $\hat{x}(nT_s) = x(nT_s - T_s)$.

1. Derive the impulse response of the linear filter that yields the interpolation function described above when applied to an impulse train $\sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s)$.

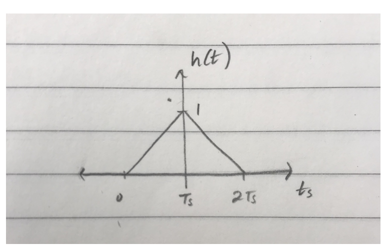
$$\begin{aligned}\hat{x}(t) &= x(nT_s - T_s) + \frac{x(nT_s) - x(nT_s - T_s)}{T_s}(t - nT_s) \\ &= x((n-1)T_s)\left[1 - \frac{t-nT_s}{T_s}\right] + x(nT_s)\left[\frac{t-nT_s}{T_s}\right]\end{aligned}$$

$$\begin{aligned}\hat{x}(t) &= h(t) * \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s) \\ &= \sum_{k=-\infty}^{\infty} x(kT_s)h(t-kT_s) \\ &= x(nT_s)h(t-nT_s) + x((n-1)T_s)h(t-(n-1)T_s)\end{aligned}$$

$$h(t-nT_s) = \frac{t-nT_s}{T_s}, \quad h(t-(n-1)T_s) = 1 - \frac{t-nT_s}{T_s}$$

$$\begin{aligned}t = nT_s \rightarrow \quad & h(nT_s - nT_s) = h(0) = 0 \\ & h(nT_s - (n-1)T_s) = h(T_s) = 1 - \frac{nT_s - nT_s}{T_s} = 1\end{aligned}$$

$$\begin{aligned}t = (n+1)T_s \rightarrow \quad & h((n+1)T_s - nT_s) = h(T_s) = \frac{T_s}{T_s} = 1 \\ & h((n+1)T_s - (n-1)T_s) = h(2T_s) = 1 - \frac{2T_s - T_s}{T_s} = 0\end{aligned}$$



2. Is this a causal filter?

   This is a causal filter because it only depends on previous values.

3. Derive the corresponding frequency response. How does this correspond to the ideal interpolator's frequency response?

Linear interpolation is a convolution of two rectangular pulses $h(t) = (p * p)(t)$, which outputs a triangular pulse.

$$p(t) = \begin{cases} 1, & \text{if } \frac{T_s}{2} \leq t \leq \frac{3T_s}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} p(t) \leftrightarrow P(w) &= \int_{T_s/2}^{3T_s/2} p(t) e^{-j\omega t} dt \\ &= \frac{-1}{j\omega} \left[e^{-j\frac{\omega T_s}{2}} - e^{-j\frac{\omega 3T_s}{2}} \right] \\ &= \frac{1}{j\omega} \left[e^{j\frac{\omega T_s}{2}} - e^{-j\frac{\omega T_s}{2}} \right] (e^{j\omega T_s}) \\ &= \frac{2}{\omega} \sin\left(\frac{\omega T_s}{2}\right)(e^{j\omega T_s}) \\ &= T_s \frac{\sin\left(\frac{\omega T_s}{2}\right)}{\frac{\omega T_s}{2}} (e^{j\omega T_s}) \\ &= T_s \operatorname{sinc}\left(\frac{\omega T_s}{2}\right)(e^{j\omega T_s}) \end{aligned}$$

$$h(t) = (p * p)(t) \leftrightarrow H(w) = P(w)P(w) = T_s^2 \operatorname{sinc}^2\left(\frac{\omega T_s}{2}\right)(e^{j2\omega T_s})$$

The frequency of an ideal interpolator is an ideal low pass filter.

$$H_{ideal}(w) = \begin{cases} 1, & \text{if } -\omega_c \leq \omega \leq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

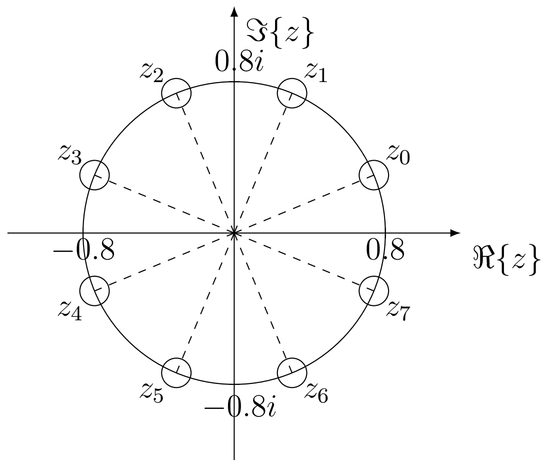
The ideal interpolator, unlike the interpolator above, is not causal. Additionally it does not have any amplification properties as the height of the low-pass filter is 1. The interpolator above is causal and can modify the amplitude due to the T_s^2 value.

Question 3

Consider a finite duration sequence $x(n)$, with non-zero values for $0 \leq n \leq 7$. Its z-transform is $X(z)$. Consider following points on the z-plane:

$$z_k = 0.8 e^{j[\frac{2\pi k}{8} + \frac{\pi}{8}]}, 0 \leq k \leq 7$$

1. Sketch these points in the complex z-plane.



The first zero starts at $\frac{\pi}{8}$ and each zero is $\frac{\pi}{4}$ shifted from the previous zero. The circle has a radius of 0.8.

2. Determine a sequence $s(n)$ such that its DFT will give the samples of $X(z)$ at these points. Note we are looking for an expression for $s(n)$ in terms of $x(n)$.

$$\begin{aligned} X(k) = X(z)|_{z=z_k} &= \sum_{n=0}^{7} x(n) z_k^{-n} \\ &= \sum_{n=0}^{7} x(n) \left[0.8e^{j\left(\frac{2\pi k}{8}+\frac{\pi}{8}\right)}\right]^{-n} \\ &= \sum_{n=0}^{7} x(n) \left(0.8e^{j\left(\frac{\pi}{8}\right)}\right)^{-n} \left(e^{-j\left(\frac{2\pi nk}{N}\right)}\right) \\ &= \sum_{n=0}^{7} s(n) e^{-j\frac{2\pi kn}{N}} = S(k) \\ s(n) &= x(n)\left(0.8e^{-j\frac{\pi n}{8}}\right) \end{aligned}$$

Question 4

Consider a band-limited analog signal $x_a(t)$, where $X_a(\omega) = 0$, for all $|\omega| > 2\pi B$. Derive the Nyquist rate for:

1. $\frac{dx_a(t)}{dt}$

$$x_a(t) \leftrightarrow X_a(\omega)$$
$$\frac{dx_a(t)}{dt} \leftrightarrow j\omega X_a(\omega)$$

Bandwidth doesn't change, so the Nyquist rate is still 2B.

2. $x_a(t)cos(6\pi Bt)$

$$\begin{aligned} x_a(t)cos(6\pi Bt) \leftrightarrow & \frac{1}{2\pi}X_a(\omega) * \pi\left[\delta(\omega - 6\pi B) + \delta(\omega + 6\pi B)\right] \\ &= \frac{1}{2}\left[X_a(\omega - 6\pi B) + X_a(\omega + 6\pi B)\right] \end{aligned}$$

Bandwidth doesn't change, so the Nyquist rate is still 2B.

3. $x_a(3t)$

$$x_a(3t) \leftrightarrow 3X_a(\omega/3)$$

Nyquist rate is 6B.

Question 5

If $X(k)$ is the N-point DFT of $x(n)$, what is the N-point DFT of:

$$x_c(n) = x(n)\cos(\frac{2\pi k_0 n}{N}), 0 \le n \le N-1$$

in terms of $X(k)$?

$$\cos(\frac{2\pi k_0 n}{N}) = \frac{1}{2}(e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}})$$

$$\begin{aligned} X_c(k) = \sum_{n=0}^{N-1} x_c(n)e^{-j\frac{2\pi kn}{N}} &= \sum_{n=0}^{N-1} \frac{1}{2}\left[x(n)(e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}})\right]e^{-j\frac{2\pi kn}{N}} \\ &= \frac{1}{2}\sum_{n=0}^{N-1} x(n)\left[(e^{j\frac{2\pi(k_0-k)n}{N}} + e^{-j\frac{2\pi(k_0+k)n}{N}})\right] \\ &= \frac{1}{2}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k-k_0)n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k_0-k)n}{N}} \\ &= \frac{1}{2}X(k-k_0)_{\text{mod N}} + \frac{1}{2}X(k+k_0)_{\text{mod N}} \end{aligned}$$

Question 6

Consider $x(n)$ such that $x(-2) = 1, x(-1) = 2, x(0) = 3, x(1) = 2, x(2) = 1, x(3) = 0$ and all else is 0. Determine its Fourier transform $X(\omega)$. Now, compute the 6-point DFT $Y(k)$ of $y(n) = [3, 2, 1, 0, 1, 2]$. How is $X(\omega)$ and $Y(k)$ related?

$$x(n) = \delta(n+2) + 2\delta(n+1) + 3\delta(n) + 2\delta(n-1) + \delta(n-2)$$

$$\begin{aligned} x(n) \leftrightarrow X(w) &= e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega} \\ &= 3 + 2(e^{j\omega} + e^{-j\omega}) + e^{j2\omega} + e^{-j2\omega} \\ &= 3 + 4\cos(\omega) + 2\cos(2\omega) \end{aligned}$$

$$Y(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{5} x(n) e^{-j\frac{\pi nk}{3}}, \quad k = 0, ..., 5$$

$$Y(k) = X(w)|_{w=\frac{\pi k}{3}}$$