# Linear Algebra Problems and Working Solutions

# Chapter 1

# Vector Spaces

Topics:

1. Vector operations.
2. Vector spaces.
3. Basis and linear independence.

Problems:
**(1.1 Treil)** Let $\mathbf{x} = (1, 2, 3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$, $\mathbf{z} = (4, 2, 1)^T$.
Compute $2\mathbf{x}, 3\mathbf{y}, \mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.
*Answer:*

$$2\mathbf{x} = 2\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix},$$

$$2\mathbf{y} = 2\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}3 \cdot y_1\\3 \cdot y_2\\3 \cdot y_3\end{bmatrix},$$

$$\mathbf{x} + 2\mathbf{y} - 3\mathbf{z} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}2 \cdot y_1\\2 \cdot y_2\\2 \cdot y_3\end{bmatrix} - \begin{bmatrix}12\\6\\3\end{bmatrix} = \begin{bmatrix}2y_1 - 11\\2y_2 - 4\\2y_3\end{bmatrix}$$

$\square$

**(1.8 Treil)** Prove that for any vector $\mathbf{v}$ its additive inverse is $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.
*Answer:*

The additive inverse axiom for vector spaces says that, given vector space $V, \forall \mathbf{v} \in V, \exists \mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. If $\mathbf{v}$ is in field $\mathbb{R}^n$,

$\mathbf{v} = (v_1, v_2, ..., v_n)$.

$(-1)\mathbf{v} = (-1)(v_1, v_2, ..., v_n) = (-v_1, -v_2, ..., -v_n)$ $\mathbf{v} + (-1)\mathbf{v} = (0, 0, ...0) = \mathbf{0}$. Thus,$(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$.
□

**(2.2.18 Shields)**

Suppose that $\mathbf{u}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ and that $\mathbf{v_1}$ and $\mathbf{v_2}$ are each linear combinations of $\mathbf{w_1}$ and $\mathbf{w_2}$. Is $\mathbf{u}$ a linear combination of $\mathbf{w_1}$ and $\mathbf{w_2}$? Why?

*Answer:*

Since $\mathbf{u}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$. By the same property, $\mathbf{v_1} = c_3\mathbf{w_1} + c_4\mathbf{w_2}$ and $\mathbf{v_2} = c_5\mathbf{w_1} + c_6\mathbf{w_2}$, where $c_i$ are constants. So, $\mathbf{u} = c_1(c_3\mathbf{w_1} + c_4\mathbf{w_2}) + c_2(c_5\mathbf{w_1} + c_6\mathbf{w_2}) = a\mathbf{w_1} + b\mathbf{w_2}$, where $a$ and $b$ are constants. Thus, $\mathbf{u}$ is a linear combination of $\mathbf{w_1}$ and $\mathbf{w_2}$.
□