Ordinary Differential Equations of First Order 4

4.1 INTRODUCTION

Typically, phenomena in the natural sciences such as physical, biological and chemical sciences can be described or modeled by equations involving of one or more unknown functions. Such equations are called differential equations. A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. Differential equations are essential for a mathematical description of nature, they lie at the core of many physical theories.

In mathematics, history of differential equations traces the development of "differential equations" from calculus, itself independently invented by English physicist Isaac Newton and German mathematician Gottfried Leibniz. The exact chronological origin and history to the subject of differential equations is a bit of a murky subject; for what seems to be a number of reasons: one being secretiveness, two being private publication issues (private works published only decades latter), and three being the nature of the battle of mathematical and scientific discovery, which is a type of intellectual "war".

Ordinary differential equations of first order have wide range of applications in science and Engineering. One of the important applications is growth problem which occurs in various fields like economic growth, growth of bacteria, decay of radioactive materials in chemistry etc. The flow problem, voltage-current problems of electric circuits and study of spread of epidemics are some of the other applications.

Definition: An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.

Examples:

(1) $\dfrac{dy}{dx}+4xy=x^3$

(2) $\dfrac{dy}{dx}=\dfrac{3x^2-2xy-5}{x^2+y^2-2y}$

(3) $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=0$

(4) $y''-4y'+4y=e^{2x}$

(5) $\left(D^3+2D^2+D\right)y=\sin x$

Types of differential equations

There are two classes of differential equations. They are as follows:

(1) Ordinary differential equations

(2) Partial differential equations

(1) **Ordinary differential equations**: Equations involving derivatives with respect to a single independent variable are called Ordinary differential equations.

Examples:

(i) $xy'' + y' + xy = 0$

(ii) $(D^3 + 2D^2 + D + 1)y = e^{-x}$

(iii) $\dfrac{dy}{dx} = \sin x + \cos x$

(iv) $(4x + 5y)\dfrac{dy}{dx} + (3x + 2y + 5) = 0$

(2) **Partial differential equations**: Equations involving partial derivatives with respect to two or more independent variables are called Partial differential equations.

Examples:

(i) $\dfrac{\partial^2 z}{\partial t^2} = \dfrac{1}{c^2}\dfrac{\partial^2 z}{\partial x^2}$

(ii) $(D^3 + 2D^2D' + DD'^2 + 1)z = e^{-x+2y}$

(iii) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial^2 z}{\partial t^2} = 0$

(iv) $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial x} = kz$

Order and Degree of Differential equations: The order of the differential equation is the order of the highest derivative involved in the differential equation.

The Degree of a differential equation is the power of highest order derivative involved in the differential equation, when the differential equation is free from radicals and fractions.

Examples:

(i) The order and degree of the differential equations $\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} + \sin x = 0$ are 1 and 2 respectively.

(ii) The order and degree of the differential equations $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ are 2 and 1 respectively.

(iii) The order and degree of the differential equations $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \left(c\dfrac{d^2y}{dx^2}\right)^{1/3}$ are 2 and 2 respectively.

(iv) The order and degree of the differential equations $\dfrac{d^2y}{dx^2} = \left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{3/2}$ are 2 and 2 respectively.

Linear and Non-linear Differential equations: A differential equation in which the dependent variable and all its derivatives occur in the first degree only and no products of dependent variables and/or derivatives occur is known as linear differential equation. A differential equation which is not linear is known as non-linear differential equation.

Examples:

(i) $\dfrac{dy}{dx} = \sin x + \cos x$ is a linear differential equation

(ii) $\dfrac{d^2y}{dx^2} = -kx$ is a linear differential equation

(iii) $\dfrac{d^2y}{dx^2} = \left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{3/2}$ is a non-linear differential equation

(iv) $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \left(c\dfrac{d^2y}{dx^2}\right)^{1/3}$ is a non-linear differential equation

Solution of a differential equation: A Solution of a differential equation is a relation between the dependent and independent variables, not involving the derivatives such that this relation and derivatives obtained from it satisfies the given differential equation.

Examples:

(i) $y = ce^{2x}$ is the solution of the differential equation $\dfrac{dy}{dx} - 2y = 0.$

(ii) $y = Ae^{-5x} + Be^{-x}$ is the solution of the differential equation $\left(D^2+6D+5\right)y = 0$

4.2 SOLUTION OF EQUATIONS IN VARIABLES SEPARABLE FORM

The equation of the form $F(x)G(y)dx + f(x)g(y)dy = 0$ (1)
is called an equation with Variables separable (or) simply separable equation, because the variables x and y can be separated.

Rewriting (1) we get $F(x)G(y)dx = -f(x)g(y)dy$

$$\frac{F(x)}{f(x)}dx = -\frac{g(y)}{G(y)}dy$$

$$M(x)dx = -N(y)dy \qquad (2), \text{ where}$$

$M(x) = \dfrac{F(x)}{f(x)}$, a function of x only and $N(y) = \dfrac{g(y)}{G(y)}$, a function of y only

Integrating (2) we get

$$\int M(x)dx = -\int N(y)dy + c \qquad (3), \text{ where}$$

c is integrating constant

Equation (3) is the Solution of (1).

Problems

1. **Solve $3e^x \tan y\, dx + \left(1-e^x\right)\sec^2 y\, dy = 0.$**

Solution: The given equation is $3e^x \tan y\, dx + \left(1-e^x\right)\sec^2 y\, dy = 0$ (1)

$$3e^x \tan y\,dx = -\left(1-e^x\right)\sec^2 y\,dy$$

$$\frac{3e^x}{1-e^x}dx = -\frac{\sec^2 y}{\tan y}dy$$

Integrating on both sides $-3\int \frac{e^x}{1-e^x}dx = \int \frac{\sec^2 y}{\tan y}dy$

$$3\log\left(1-e^x\right) = \log\left(\tan y\right) + \log c$$

$$\log\left(1-e^x\right)^3 = \log\left(\tan y . c\right)$$

$$\left(1-e^x\right)^3 = c\tan y$$

$\left(1-e^x\right)^3 = c\tan y$ is the required Solution of equation (1). ■

2. Solve $(x-y^2x)dx=(y-x^2y)dy$.

Solution: The given equation is $\left(x-y^2x\right)dx=\left(y-x^2y\right)dy$ (1)

$$x\left(1-y^2\right)dx = y\left(1-x^2\right)dy$$

$$\frac{x}{1-x^2}dx = \frac{y}{1-y^2}dy$$

Integrating on both sides $\int \frac{x}{1-x^2}dx = \int \frac{y}{1-y^2}dy$

$$-\frac{1}{2}\log\left(1-x^2\right) = -\frac{1}{2}\log\left(1-y^2\right) + \log c$$

$$\log\left(1-x^2\right)^{-1/2} = \log\left(1-y^2\right)^{-1/2} + \log c$$

$$\log\left(\frac{1}{\sqrt{1-x^2}}\right) = \log\left(\frac{1}{\sqrt{1-y^2}}\right) + \log c$$

$$\log\left(\frac{1}{\sqrt{1-x^2}}\right) = \log\left(\frac{1}{\sqrt{1-y^2}}\times c\right)$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{c}{\sqrt{1-y^2}}$$

$$c\sqrt{1-x^2} = \sqrt{1-y^2}$$

$c\sqrt{1-x^2} = \sqrt{1-y^2}$ is the required solution of equation (1). ■

3. Solve $\frac{dy}{dx} = e^{2x-y} + x^3e^{-y}$.

Solution: The given equation is $\frac{dy}{dx} = e^{2x-y} + x^3e^{-y}$ (1)

$$\frac{dy}{dx} = e^{2x}e^{-y} + x^3e^{-y}$$

$$\frac{dy}{dx} = e^{-y}\left(e^{2x} + x^3\right)$$

$$\frac{dy}{e^{-y}} = \left(e^{2x} + x^3\right)dx$$

$$e^y dy = \left(e^{2x} + x^3\right)dx$$

Integrating on both sides $\int e^y dy = \int \left(e^{2x} + x^3\right)dx$

$$e^y = \frac{e^{2x}}{2} + \frac{x^4}{4} + c$$

$$e^y - \frac{e^{2x}}{2} - \frac{x^4}{4} = c$$ is the required solution of equation (1). ■

4. Solve $\tan x \sin^2 y\, dx + \cos^2 x \cot y\, dy = 0$.

Solution: The given equation is $\tan x \sin^2 y\, dx + \cos^2 x \cot y\, dy = 0$. (1)

$$\tan x \sin^2 y\, dx = -\cos^2 x \cot y\, dy$$

$$\frac{\tan x}{\cos^2 x} dx = -\frac{\cot y}{\sin^2 y} dy$$

$$\sec^2 x \tan x\, dx = -\operatorname{cosec}^2 y \cot y\, dy$$

Integrating on both sides $\int \sec^2 x \tan x\, dx = -\int \operatorname{cosec}^2 y \cot y\, dy$

$\int u du = \int t\, dt \quad \left[u = \tan x \Rightarrow du = \sec^2 x dx \text{ and } t = \cot y \Rightarrow dt = -\operatorname{cosec}^2 y dy\right]$

$$\frac{u^2}{2} = \frac{t^2}{2} + c$$

$$\frac{\tan^2 x}{2} = \frac{\cot^2 y}{2} + c$$

$$\frac{\tan^2 x}{2} - \frac{\cot^2 y}{2} + c$$ is the required solution of equation (1) ■

5. Solve $e^{-4y} \log x\, dx + x \cos 3y\, dy = 0$.

Solution: The given equation is $e^{-4y} \log x\, dx + x \cos 3y\, dy = 0$

$$e^{-4y} \log x\, dx = -x \cos 3y\, dy$$

$$\frac{\log x}{x} dx = -\frac{\cos 3y}{e^{-4y}} dy$$

Integrating on both sides $\frac{\log x}{x} dx = -e^{4y} \cos 3y\, dy$

$\int \frac{\log x}{x} dx = -\int e^{4y} \cos 3y\, dy \qquad u = \log x, dv = \frac{1}{x} dx \Rightarrow du = \frac{1}{x} dx, v = \log x$

$\frac{(\log x)^2}{2} = -\frac{e^{4y}}{4^2 + 3^2}\left[4\cos 3y + 3\sin 3y\right] + c \qquad \int u dv = uv - \int v du \Rightarrow \int \frac{\log x}{x} dx = (\log x)^2 - \int \frac{\log x}{x} dx$

$\frac{(\log x)^2}{2} = -\frac{e^{4y}}{25}\left[4\cos 3y + 3\sin 3y\right] + c \qquad \int \frac{\log x}{x} dx + \int \frac{\log x}{x} dx = (\log x)^2 \Rightarrow \int \frac{\log x}{x} dx = \frac{(\log x)^2}{2}$

$$\frac{(\log x)^2}{2}+\frac{e^{4y}}{25}\left[4\cos 3y+3\sin 3y\right]=c \text{ is the required solution of equation (1).}$$

6. Solve $xdy+ydx+4\sqrt{1-x^2y^2}\,dx=0$.

Solution: The given equation is $xdy+ydx+4\sqrt{1-x^2y^2}\,dx=0$ (1)

$$d(xy)+4\sqrt{1-x^2y^2}\,dx=0$$

$$d(xy)=-4\sqrt{1-x^2y^2}\,dx$$

$$\frac{d(xy)}{\sqrt{1-x^2y^2}}=-4dx$$

Integrating on both sides $\displaystyle\int\frac{d(xy)}{\sqrt{1-x^2y^2}}=-4\int dx$

$$\sin^{-1}(xy)=-4x+c$$

$\sin^{-1}(xy)=-4x+c$ is the required solution of equation (1). ■

7. Solve $\frac{dy}{dx}=(x+y+2)^2$.

Solution: The given equation is $\frac{dy}{dx}=(x+y+2)^2$ (1)

Let $u=x+y+2$

$$\frac{du}{dx}=1+\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{du}{dx}-1 \quad (2)$$

Substituting (2) in (1) we get $\frac{du}{dx}-1=u^2$

$$\frac{du}{dx}=u^2+1$$

$$\frac{du}{u^2+1}=dx$$

Integrating on both sides $\displaystyle\int\frac{du}{u^2+1}=\int dx$

$$\tan^{-1}u=x+c$$

$\tan^{-1}(x+y+2)-x=c$ is the required solution of equation (1). ■

8. Solve $y'=\sin(x-y+1)$.

Solution: The given equation is $y'=\sin(x-y+1)$

$$\frac{dy}{dx}=\sin(x-y+1) \quad (1)$$

$$\text{Let } u = x - y + 1$$

$$\frac{du}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{du}{dx} \qquad (2)$$

Substituting (2) in (1)

$$1 - \frac{du}{dx} = \sin u$$

$$\frac{du}{dx} = 1 - \sin u$$

$$\frac{du}{1-\sin u} = dx$$

Integrating on both sides

$$\int \frac{du}{1-\sin u} dx = \int dx$$

$$\int \frac{1+\sin u}{(1-\sin u)(1+\sin u)} du = \int dx$$

$$\int \frac{1+\sin u}{1-\sin^2 u} du = \int dx$$

$$\int \frac{1+\sin u}{\cos^2 u} du = \int dx$$

$$\int \left(\frac{1}{\cos^2 u} + \frac{\sin u}{\cos^2 u} \right) du = \int dx$$

$$\int \left(\sec^2 u + \tan u \sec u \right) du = \int dx$$

$$\tan u + \sec u = x + c$$

$\tan(x+y-1) + \sec(x+y-1) = x + c$ is the required solution of equation (1). ■

9. Solve $\frac{dy}{dx} - x\tan(y-x) = 1$.

Solution: The given equation is $\frac{dy}{dx} - x\tan(y-x) = 1 \qquad (1)$

Let $u = y - x$

$$\frac{dy}{dx} - x\tan u = 1$$

$$\frac{dy}{dx} = 1 + x\tan u \qquad (2)$$

Here $u = y - x$

$$\frac{du}{dx} = \frac{dy}{dx} - 1$$

$$\frac{dy}{dx} = \frac{du}{dx} + 1 \qquad (3)$$

Substituting (3) in (2)
$$\frac{du}{dx}+1=1+x\tan u$$
$$\frac{du}{dx}=x\tan u$$
$$\frac{du}{\tan u}=x\,dx$$

Integrating on both sides
$$\int\frac{du}{\tan u}=\int x dx$$
$$\int\cot u\,du=\int x dx$$
$$\log(\sin u)=\frac{x^2}{2}+c$$
$$\log(\sin u)-\frac{x^2}{2}=c$$

$\therefore\ \log\sin(y-x)-\frac{x^2}{2}=c$ is the required solution of equation (1). ■

10. Solve $(x-2\sin y+1)=2(x-2\sin y)\cos y\frac{dy}{dx}$.

Solution: The given equation is $(x-2\sin y+1)=2(x-2\sin y)\cos y\frac{dy}{dx}$ (1)

$$\text{Let } u=x-2\sin y$$
$$\frac{du}{dx}=1-2\cos y\frac{dy}{dx}$$
$$2\cos y\frac{dy}{dx}=1-\frac{du}{dx} \qquad (2)$$

Substituting (2) in (1)
$$u+1=u\left(1-\frac{du}{dx}\right)$$
$$u+1=u-u\frac{du}{dx}$$
$$u\frac{du}{dx}=-1$$
$$u\,du=-dx$$

Integrating on both sides
$$\int udu=-\int dx$$
$$\frac{u^2}{2}=-x+c$$
$$\frac{u^2}{2}+x=c$$

$\therefore\ \frac{(x-2\sin y)^2}{2}+x=c$ is the required solution of equation (1). ■

11.

11. Obtain the particular solution of $\frac{dy}{dx}=(4x+y+1)^2, y(0)=1.$

Solution: The given equation is $\frac{dy}{dx}=(4x+y+1)^2$ (1)

Let $u=4x+y+1$

$$\frac{du}{dx}=4+\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{du}{dx}-4 \qquad (2)$$

Substituting (2) in (1)

$$\frac{du}{dx}-4=u^2$$

$$\frac{du}{dx}=u^2+4$$

$$\frac{du}{u^2+4}=dx$$

Integrating on both sides

$$\int\frac{du}{u^2+4}=\int dx$$

$$\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)=x+c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)=x+c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right)=x+c \qquad (2)$$

Applying the initial condition $y(0)=1$ in (2)

$$\frac{1}{2}\tan^{-1}\left(\frac{4(0)+1+1}{2}\right)-0=c$$

$$\frac{1}{2}\tan^{-1}(1)=c$$

(i.e.) $c=\frac{\pi}{8}$

Substituting the value of c in (2) we get

$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right)-x=\frac{\pi}{8}$ is the required solution of equation (1). ■

EXERCISE 4.1

1. Solve $(1+x^2)dy=(1+y^2)dx.$
2. Solve $x\sqrt{1+y^2}\,dx+y\sqrt{1+x^2}\,dy=0.$
3. Solve $(x^2-x^2y)dy+(y^2+xy^2)dx=0.$
4. Solve $(1+e^x)ydy=(1+y)e^x dx.$

5. Solve $\tan x.\sin^2 y\, dx + \cos^2 x.\cot y\, dy = 0$.

6. Solve $\frac{dy}{dx} + x^2 = x^2 e^{3y}$.

7. Solve $\frac{dy}{dx} = e^{x-y} + e^{2\log x - y}$.

8. Solve $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$.

9. Solve $\left(y - x\frac{dy}{dx}\right)x = y$.

10. Solve $(x - y^2 x)dx - (y - x^2 y)dy = 0$.

11. Solve $x(1+y^2)dx + y(1+x^2)dy = 0$.

12. Solve $\frac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}$.

13. Solve $x\cos^2 y dx = y\cos^2 x dy$.

14. Solve $y - x\frac{dy}{dx} = 3\left(1 + x^2\frac{dy}{dx}\right)$.

15. Solve $(1+x^2)dy + x\sqrt{1-y^2}\,dx = 0$.

ANSWERS TO EXERCISE 4.1

1. $(y - x) = c(1 + xy)$
2. $\sqrt{1+x^2} + \sqrt{1+y^2} = c$
3. $\log\left(\frac{x}{y}\right) - \frac{y+x}{xy} = c$
4. $e^y = c(1+y)(1+e^x)$
5. $\tan^2 x - cot^2 y = c$
6. $(e^{3y} - 1) = ce^{(3y+x^2)}$
7. $e^y = e^x + \frac{x^3}{3} + c$
8. $y\sin y = x^2\log x + c$
9. $\log y = \log x + \frac{1}{x} + c$
10. $1 - y^2 = c(1 - x^2)$
11. $(1+y^2)(1+x^2) = c$
12. $2e^{3y} = 3e^{2x} + 8x^3 + c$
13. $y\tan y + \log\cos y = x\tan x + \log\cos x + c$
14. $(y-3)(3x+1) = cx$
15. $\sin^{-1} y + \frac{\log(1+x^2)}{2} = c.$

4.3 EXACT DIFFERENTIAL EQUATION

Definition: A differential Equation which is obtained from its Primitive by mere differentiation without any further operation is called Exact differential Equation.

Examples:

1) $d(xy) = 0$

2) $xdy + ydx = 0$

3) $\frac{xdy - ydx}{x^2}$

Necessary and Sufficient Condition for Exactness:

The necessary and sufficient condition for a differential equation of the form $M\,dx + N\,dy = 0$ is to be exact in $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution of Exact differential Equation:

Consider, the equation $M(x, y)dx + N(x, y)dy = 0$ (1)

The Solution of equation (1) is given by

$\int M dx + \int (\text{term of } N \text{ not containing } x)dy = C$ where 'C' is integrating constant

Problems:

1. **Solve $e^y dx + (xe^y + 2y)\,dy = 0$**

Solution: The given equation is $e^y dx + (xe^y + 2y)\,dy = 0$ (1)

Here $M = e^y$ and $N = xe^y + 2y$

$$\text{Now } \frac{\partial M}{\partial y} = e^y, \ \frac{\partial N}{\partial x} = e^y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The Equation (1) is Exact.

Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x)dy = C$

$$\int e^y dx + \int 2y dy = C$$

$$xe^y + 2\frac{y^2}{2} = C$$

$\therefore xe^y + y^2 = C$ is the required solution ■

2. **Solve $(y^2 - 2xy)dx = (x^2 - 2xy)dy$**

Solution: The given equation is $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

$$(y^2 - 2xy)dx - (x^2 - 2xy)dy = 0 \quad (1)$$

Here $M = y^2 - 2xy$ and $N = -(x^2 - 2xy)$

$$\text{Now } \frac{\partial M}{\partial y} = 2y - 2x \ \ \frac{\partial N}{\partial x} - (2x - 2y) = 2y - 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of 0 is The equation (1) is exact

$$\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$$

$$\int (y^2 - 2xy) dx = C$$

$$y^2 x - \frac{2x^2}{2} y = C$$

$\therefore y^2x - x^2y = C$ is the required solution ■

3. Solve $(e^y + 1)\cos x dx + e^y \sin x dy = 0$

Solution: The given equation is $(e^y + 1)\cos x dx + e^y \sin x dy = 0$ (1)

Here $M = (e^y + 1)\cos x$ and $N = e^y \sin x$

$$\text{Now } \frac{\partial M}{\partial y} = e^y \cos x,$$

$$\frac{\partial N}{\partial x} = e^y \cos x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The equation (1) is Exact.

Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

$$\int (e^y + 1)\cos x dx = C$$

$\therefore (e^y + 1) \sin x = C$ is the required solution

4. Solve $\dfrac{dy}{dx} + \dfrac{3x - 2y + 5}{-2x + 4y + 1} = 0$

Solution: The given equation is $\dfrac{dy}{dx} + \dfrac{3x - 2y + 5}{-2x + 4y + 1} = 0$

$$\frac{dy}{dx} = \frac{-(3x - 2y + 5)}{-2x + 4y + 1}$$

$$(-2x + 4y + 1)dy = -(3x - 2y + 5)dx$$

$$(3x - 2y + 5)dx + (-2x + 4y + 1)dy = 0$$

Here $M = 3x - 2y + 5$ and $N = -2x + 4y + 1$

$$\text{Now } \frac{\partial M}{\partial y} = -2, \ \frac{\partial N}{\partial x} = -2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The equation (1) is Exact

Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

$$\int (3x - 2y + 5) dx + \int (4y + 1) dy = C$$

$$3\frac{x^2}{2} + 5x - 2yx + \frac{4y^2}{2} + y = C$$

$$\therefore 3\frac{x^2}{2} - 2xy + 5x + 2y^2 + y = C$$ ■

5. Solve $\frac{dy}{dx} = \frac{3x^2 - 2xy - 5}{x^2 + y^2 - 2y}$

Solution: The given equation is $\frac{dy}{dx} = \frac{3x^2 - 2xy - 5}{x^2 + y^2 - 2y}$

$$(x^2 + y^2 - 2y) dy = (3x^2 - 2xy - 5) dx$$

$$(3x^2 - 2xy - 5) dx - (x^2 + y^2 - 2y) dy = 0 \quad (1)$$

Here $M = 3x^2 - 2xy - 5$ and $N = -(x^2 + y^2 - 2y)$

$$\text{Now } \frac{\partial M}{\partial y} = -2x, \ \frac{\partial N}{\partial x} = -2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The equation (1) is Exact

The Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

$$\int (3x^2 - 2xy - 5) dx + \int (-y^2 + 2y) dy = C$$

$$\frac{3x^3}{3} - \frac{-2x^2}{2} y - 5x - \frac{y^3}{3} + \frac{2y^2}{2} = C$$

$\therefore x^3 - x^2 y - 5x - \frac{y^3}{3} + y^2 = C$ is the required solution ■

6. **Solve $(1 + e^{x/y})dx + \left(1 - \frac{x}{y}\right) e^{x/y}\, dy = 0$**

Solution: The given equation is $(1 + e^{x/y})dx + (1 - \frac{x}{y})e^{x/y}\, dy = 0$ (1)

$$\text{Here } M = (1 + e^{x/y}) \text{ and } N = \left(1 - \frac{x}{y}\right) e^{x/y} = e^{x/y} - e^{x/y}\frac{x}{y}$$

$$\text{Now } \frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right),\ \frac{\partial N}{\partial x} = e^{x/y}\left(\frac{1}{y}\right) - \left[\frac{x}{y}e^{x/y}\left(\frac{1}{y}\right) - e^{x/y}\left(\frac{1}{y}\right)\right]$$

$$= \frac{e^{x/y}}{y} - \frac{x}{y^2}e^{x/y} - \frac{e^{x/y}}{y}$$

$$= e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The Equation is Exact

The Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x)dy = C$

$$\int (1 + e^{x/y})dx + \int 0 dy = C \quad \left(\because \int e^{x/y} dx = \frac{e^{x/y}}{\frac{1}{y}}\right) = ye^{x/y}$$

$\therefore x + e^{x/y}\, y = C$ is the required solution ■

7. **Solve $e^x(\sin x + \cos x) \sec y dy + e^x \sin x \sec y \tan y dy = 0$**

Solution: The given equation is $e^x(\sin x + \cos x) \sec y dy + e^x \sin x \sec y \tan y$ (1)
$dy = 0$

Here $M = e^x (\sin x + \cos x) \sec y$ and $N = e^x (\sin x \sec y \tan y)$

$$\text{Now } \frac{\partial M}{\partial y} = e^x(\sin x + \cos x)(\sec y \tan y), \frac{\partial N}{\partial x} = \left(e^x \cos x + \sin x e^x\right)\left(\sec y \tan y\right)$$

$$\frac{\partial N}{\partial x} = e^x (\sin x + \cos x)(\sec y \tan y)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The equation (1) is Exact.

The Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

$$\int e^x (\sin x + \cos x) \sec y dx = C \quad \left| e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) \right|$$

$$\int e^x \sin x \sec y dx + \int e^x \cos x \sec y dx = C$$

$$\frac{e^x}{2}(\sin x - \cos x)\sec y + \frac{e^x}{2}(\cos x + \sin x)\sec y = C$$

$$\frac{e^x \sin x \sec y}{2} + \frac{e^x \sin x \sec y}{2} = C$$

$$2\frac{e^x \sin x \sec y}{2} = C$$

$\therefore e^x \sin x \sec y = C$ is the required solution

8. Solve (sin x sin y – xe^y)dy = (e^y + cos x cos y)dx

Solution: The given equation is $(\sin x \sin y - xe^y)dy = (e^y + \cos x \cos y)dx$ (1

$$(e^y + \cos x \cos y)dx - (\sin x \sin y - xe^y)dy = 0$$

Here $M = e^y + \cos x \cos y$ *and*

$$N = -(\sin x \sin y - xe_y)$$

$$\frac{\partial M}{\partial y} = e^y + (-\cos x \sin y)$$

$$\frac{\partial N}{\partial y} = -(\cos x \sin y - e^y)$$

$$= e^y - (\cos x \sin y)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is Exact.

Solution is of equation (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

integration of e^x f(x)+fdash(X) dx = e^f(x)+c

$$\int (e^y + \cos x \cos y) dx = C$$

$$xe^y + \sin x \cos y = C$$

$\therefore xe^y + \sin x \cos y = C$ is the required solution ■

9. Determine whether the equation $(\cos x - x\cos y)\,dy - (\sin y + y\sin x)dx = 0$ is exact and if so solve it.

Solution: The given equation is $(\cos x - x\cos y)dy - (\sin y + y\sin x)dx = 0$

$$-(\sin y + y\sin x)dx + (\cos x - x\cos y)dy = 0 \quad (1)$$

Here $M = -(\sin y + y\sin x)$ and $N = \cos x - x\cos y$

$$\text{Now } \frac{\partial M}{\partial y} = -(\cos y + \sin x), \quad \frac{\partial N}{\partial x} = -\sin x - \cos y$$

$$= -(\sin x + \cos y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is exact.

Solution (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

$$\int -(\sin y + y\sin x) dx = C$$

$$-(x\sin y + y(-\cos x)) = C$$

$$y\cos x - x\sin y = C$$

$\therefore y\cos x - \sin y = C$ is the required solution ■

). Solve $y\sin 2x\, dx = (y^2 + \cos^2 x)dy$

Solution: The given equation is $y\sin 2x dx = (y^2 + \cos^2 x)dy$

$$y\sin 2x dx - (y^2 + \cos^2 x)dy = 0 \quad (1)$$

Here $M = y\sin 2x$ and $N = -(y^2 + \cos^2 x)$

$$N = -\left(y^2 + \frac{1+\cos 2x}{2}\right)$$

$$N = -y^2 - \frac{1}{2} - \frac{\cos 2x}{2}$$

$$\frac{\partial M}{\partial y} = \sin 2x, \quad \frac{\partial N}{\partial x} = \sin 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is Exact

Solution of (1) is $\int M dx + \int (\text{term of } N \text{ not containing } x) dy = C$

$$\int y \sin 2x dx + \int \left(-y^2 - \frac{1}{2}\right) dy = C$$

$$-\frac{y \cos 2x}{2} - \frac{y^3}{3} - \frac{y}{2} = C$$

$\therefore \frac{y \cos 2x}{2} + \frac{y^3}{3} + \frac{y}{2} = C$ is the required solution

EXERCISE 4.2

1. Solve $\left(2x^3 - xy^2 - 2y + 3\right) dx - \left(x^2 y + 2x\right) dy = 0$.
2. Solve $\left(y - x^3\right) dx + \left(x + y^3\right) dy = 0$.
3. Solve $\left(3x^2 y + \frac{y}{x}\right) dx + \left(x^3 + \log x\right) dy = 0$.
4. Solve $\left(1 + 4xy + 2y^2\right) dx + \left(1 + 4xy + 2x^2\right) dy = 0$.
5. Solve $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0$.
6. Solve $\left(3x^2 + 6xy^2\right) dx + \left(6x^2 y + 4y^3\right) = 0$.
7. Solve $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0$.
8. Solve $(2xy + y - \tan y) dx + \left(x^2 - x \tan^2 y + \sec^2 y\right) dy = 0$.
9. Solve $(\cos x \tan y + \cos(x + y)) dx + \left(\sin x \sec^2 y + \cos(x + y)\right) dy = 0$.
10. Solve $\left(x^4 - 2xy^2 + y^4\right) dx - \left(2x^2 y - 4xy^3 + \sin y\right) dy = 0$.
11. Solve $x dx + y dy + \frac{x dy - y dx}{x^2 + y^2} = 0$.
12. Solve $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + \left(x + \log x - x \sin y\right) dy = 0$.
13. Solve $\left(3x^2 y + \frac{y}{x}\right) dx + \left(x^3 + \log x\right) dy = 0$.
14. Solve $(\cos x . \cos y - \cot x) dx - (\sin x . \sin y) dy = 0$.

ANSWERS TO EXERCISE 4.2

1. $x^4 - x^2y^2 - 4xy + 6x = c$
2. $4xy - x^4 + y^4 = c$
3. $x^3y + y\log x = c$
4. $x + 2x^2y + 2xy^2 + y = c$
5. $x^4 - y^3 + e^{xy^2} = c$
6. $x^3 + 3x^2y^2 + y^4 = c$
7. $e^{xy^2} + x^4 - y^3 = c$
8. $x^2y + xy - x\tan y + \tan y = c$
9. $\tan y \sin x + \sin(x+y) = c$
10. $\frac{y^5}{5} - x^2y^2 + xy^4 + \cos y = c$
11. $x^2 - 2\tan^{-1}\left(\frac{x}{y}\right) + y^2 = c$
12. $y(x + \log x) + x\cos y = c$
13. $x^3y + y\log x = c$
14. $\sin x.\cos y = \log(c\sin x)$

4.4 INTEGRATING FACTORS

A non-exact differential equation can always be made exact by multiplying it by some functions of x and y. Such a function is called Integrating factor. There is no general method for finding such integrating factors. In this section we shall see some of the methods for determining integrating factors.

Definition: A factor, which when multiplied to a non-exact differential equation makes it exact, is known as an integrating factor.

Examples:

(1) $\frac{1}{y^2}$ is the integrating factor of the non-exact differential equation $ydx - xdy = 0$.

(2) $\frac{1}{xy}$ is the integrating factor of the non-exact differential equation $a(xdy + 2ydx) = xydy$.

(3) $\frac{1}{x^2}$ is the integrating factor of the non-exact differential equation $ydx - xdy + \log x\, dx = 0$.

(4) e^x is the integrating factor of the non-exact differential equation $(x - y)dx - dy = 0$.

Method 1:

In some cases the integrating factors is found by inspection after proper grouping of terms. The following table gives the list of integrating factors.

S.NO	Group of Terms	Integrating Factor	Exact Differential
1	$xdy + ydx$	1	$d(xy)$
2	$xdy + ydx$	$\frac{1}{xy}$	$d(\log xy)$

3	$xdy + ydx$	$\frac{1}{(xy)^a}$	$d\left(\frac{xy^{1-a}}{1-a}\right), a \neq 1$
4	$xdx + ydy$	$\frac{2}{x^2 + y^2}$	$d\left(\log\left(x^2 + y^2\right)\right)$
5	$xdx + ydy$	$\frac{2}{\left(x^2 + y^2\right)^a}$	$d\left(\frac{\left(x^2 + y^2\right)^{1-a}}{1-a}\right), a \neq 1$
6	$xdx + ydy$	2	$d\left(x^2 + y^2\right)$
7	$xdy - ydx$	$\frac{1}{x^2}$	$d\left(\frac{y}{x}\right)$
8	$xdy - ydx$	$-\frac{1}{y^2}$	$d\left(\frac{x}{y}\right)$
9	$xdy - ydx$	$\frac{1}{xy}$	$d\left(\log\frac{y}{x}\right)$
10	$xdy - ydx$	$\frac{1}{x^2 + y^2}$	$d\left(tan^{-1}\frac{y}{x}\right)$
11	$xdy - ydx$	$-\frac{1}{x^2 + y^2}$	$d\left(tan^{-1}\frac{x}{y}\right)$
12	$ydx - xdy$	$\frac{1}{y^2}$	$d\left(\frac{x}{y}\right)$
13	$ydx - xdy$	$\frac{1}{xy}$	$d\left(\log\frac{x}{y}\right)$
14	$ydx - xdy$	$\frac{1}{x^2 + y^2}$	$d\left(tan^{-1}\frac{x}{y}\right)$
15	$2yxdx - x^2dy$	$\frac{1}{y^2}$	$d\left(\frac{x^2}{y}\right)$
16	$2xydx - y^2dx$	$\frac{1}{x^2}$	$d\left(\frac{y^2}{x}\right)$

17	$ye^x dx - e^x dy$	$\frac{1}{y^2}$	$d\left(\frac{e^x}{y}\right)$
18	$xe^y dy - e^y dx$	$\frac{1}{x^2}$	$d\left(\frac{e^y}{x}\right)$

Method 2: If the differential equation $Mdx + Ndy$ is homogeneous and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is the integrating factor.

Method 3: If the differential equation $Mdx + Ndy$ is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is the integrating factor.

Method 4: The differential equation $Mdx + Ndy = 0$ has $e^{\int f(x)dx}$ as the integrating factor if $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ is a function of $x\left[\text{say } f(x)\right]$.

Method 5: The differential equation $Mdx + Ndy = 0$ has $e^{\int f(y)dy}$ as the integrating factor if $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$ is a function of $y\left[\text{say } f(y)\right]$.

Method 6: If $Mdx + Ndy = 0$ is of the form
$x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$, where a, b, c, d, m, n, p and q are all constants such that $mq - np \neq 0$, then $x^h y^k$ is an integrating factor of the equation for some suitable constants h, k to be determined form $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ and $\frac{c+h+1}{p} = \frac{d+k+1}{q}$.

Problems:

1. **Solve $ydx - xdy = xy^3 dy$.**

Solution: The given equation is $ydx - xdy = xy^3 dy$ (1)

The given equation is not exact.

Dividing equation (1) by $\frac{1}{xy}$

$$\frac{1}{xy}[ydx - xdy] = \frac{xy^3 dy}{xy}$$

$$\frac{ydx - xdy}{xy} = y^2 dy$$

$$d\left(\log\frac{x}{y}\right) = y^2 dy$$

Integrating on both sides we get

$$\log\left(\frac{x}{y}\right) = \frac{y^3}{3} + c \text{ is the required Solution of (1)}$$

2. Solve $x\,dx + y\,dy + \left(x^2 + y^2\right)\tan x\,dx = 0$.

Solution: The given equation is $xdx + ydy + \left(x^2 + y^2\right)\tan xdx = 0$ (1)

The given equation is not exact.

Dividing equation (1) by $\dfrac{2}{x^2 + y^2}$ we get

$$\frac{2(xdx + ydy)}{x^2 + y^2} + \frac{2\left(x^2 + y^2\right)\tan xdx}{x^2 + y^2} = 0$$

$$d\left(\log\left(x^2 + y^2\right)\right) + 2\tan x\,dx = 0$$

Integrating on both sides we get

$$\log\left(x^2 + y^2\right) + 2\log(\sec x) = 0$$

$$\log\left(x^2 + y^2\right) + \log(\sec x)^2 = 0$$

$\left(x^2 + y^2\right)\sec^2 x = c$ is required Solution of equation (1)

3. Solve $(y + x)dy = (y - x)dx$.

Solution: The given equation is $(y + x)dy = (y - x)dx$

$$ydy + xdy - ydx + xdx = 0$$

$$(ydy + xdx) + (xdy - ydx) = 0 \qquad (1)$$

The given equation is not exact.

Dividing equation (1) by $\dfrac{2}{x^2 + y^2}$

$$\frac{2}{x^2 + y^2}[ydy + xdx] = \frac{2}{x^2 + y^2}[xdy - ydx]$$

$$d\left(\log\left(x^2 + y^2\right)\right) = 2\left(d\left(tan^{-1}\frac{y}{x}\right)\right)$$

Integrating on both sides we get

$\left(x^2 + y^2\right) + 2\tan^{-1}\left(\dfrac{y}{x}\right) = c$ is required Solution of equation (1)

4. Solve $y\,dx - x\,dy + \log x = 0$.

Solution: The given equation is $ydx - xdy + \log x = 0$ (1)

The given equation is not exact.

Dividing equation (1) by x^2

$$\frac{1}{x^2}[ydx - xdy + \log x] = 0$$

$$\frac{ydx - xdy}{x^2} + \frac{\log x}{x^2} = 0$$

$$-d\left(\frac{y}{x}\right)+d\left(\int\frac{1}{x^2}\log x\,dx\right)=0$$

$$d\left(\int\frac{1}{x^2}\log x\,dx-\frac{y}{x}\right)=0$$

Integrating on both sides

$$\int\frac{1}{x^2}\log x\,dx-\frac{y}{x}=c$$

$$-\frac{\log x}{x}-\int\left(-\frac{1}{x}\right)\frac{1}{x}dx-\frac{y}{x}=c$$

$$-\frac{\log x}{x}-\frac{1}{x}-\frac{y}{x}=c$$

$$-\log x-1-y=cx$$

$\therefore y+\log x+1+cx=0$ is required solution of equation (1)

5. Solve $x^2ydx-\left(x^3+y^3\right)dy=0.$

Solution: The given equation is $x^2ydx-\left(x^3+y^3\right)dy=0$ (1)

Here $M=x^2y,\ N=-\left(x^3+y^3\right)$

Now $M.x+N.y=x^2y.x-\left(x^3+y^3\right).y$

$=x^3y-x^3y-y^4=-y^4\neq 0$

Hence the integrating factor $=\dfrac{1}{M.X+N.y}=-\dfrac{1}{y^4}$

Multiplying equation (1) by $-\dfrac{1}{y^4}$

$$\frac{-1}{y^4}x^2ydx+\frac{\left(x^3+y^3\right)dy}{y^4}=0$$

$$\Rightarrow\frac{-x^2dx}{y^3}+\frac{\left(x^3+y^3\right)dy}{y^4}=0 \qquad (2)$$

Now $M=\dfrac{-x^2}{y^3},\ N=\dfrac{\left(x^3+y^3\right)}{y^4}$

Then $\dfrac{\partial M}{\partial y}=\dfrac{3x^2}{y^4},\ \dfrac{\partial N}{\partial x}=\dfrac{3x^2}{y^4}$, which implies the equation (2) is exact.

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int\frac{-x^2}{y^3}dx+\int\frac{1}{y}dy=c$$

$-\dfrac{x^3}{3y^3}+\log y=c$ is the required solution of equation (1)

6. **Solve $\left(x^2y-2xy^2\right)dx-\left(x^3-3x^2y\right)dy=0$.**

Solution: The given equation is $\left(x^2y-2xy^2\right)dx-\left(x^3-3x^2y\right)dy=0$ (1)

Here $M=x^2y-2xy^2$, $N=x^3-3x^2y$

Now $M.x+N.y=\left(x^2y-2xy^2\right).x-\left(x^3-3x^2y\right).y$

$$=x^3y-2x^2y^2-x^3y+3x^2y^2=x^2y^2\neq 0$$

Hence the integrating factor $=\dfrac{1}{M.X+N.y}=\dfrac{1}{x^2y^2}$

Multiplying equation (1) by $\dfrac{1}{x^2y^2}$

$$\frac{\left(x^2y-2xy^2\right)dx}{x^2y^2}-\frac{\left(x^3-3x^2y\right)dy}{x^2y^2}=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{3}{y}-\frac{x}{y^2}\right)dy=0$$

Now $M=\dfrac{1}{y}-\dfrac{2}{x}$, $N=\dfrac{3}{y}-\dfrac{x}{y^2}$

Then $\dfrac{\partial M}{\partial y}=\dfrac{-1}{y^2}$, $\dfrac{\partial N}{\partial x}=\dfrac{-1}{y^2}$, which implies the equation (2) is exact.

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$-\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$-\frac{x}{y}+2\log x+3\log y=c$$

$2\log x+3\log y-\dfrac{x}{y}=c$ is the required Solution of equation (1)

7. **Solve $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$.**

Solution: The given equation is $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$ (1)

The given equation is not exact.

Equation (1) is of the form $f_1(xy)\,ydx+f_2(xy)\,xdy=0$

Here $M=y\left(xy+2x^2y^2\right)$, $N=x\left(xy-x^2y^2\right)$

Now $M.x-N.y=y\left(xy+2x^2y^2\right).x-x\left(xy-x^2y^2\right).y$

$$=x^2y^2+2x^3y^3-x^2y^2+x^3y^3=3x^3y^3\neq 0$$

Hence the integrating factor $=\dfrac{1}{M.\text{x}-N.y}=\dfrac{1}{3x^3y^3}$

Multiplying equation (1) by $\dfrac{1}{3x^3y^3}$

$$\frac{y\left(xy+2x^2y^2\right)dx}{3x^3y^3}+\frac{x\left(xy-x^2y^2\right)dy}{3x^3y^3}=0$$

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0$$

$$\text{Here } M=\left(\frac{1}{3x^2y}+\frac{2}{3x}\right), N=\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)$$

$\dfrac{\partial M}{\partial y}=-\dfrac{1}{3x^2y^2},\dfrac{\partial N}{\partial x}=-\dfrac{1}{3x^2y^2}$, which implies the equation (2) is exact.

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx-\int\frac{1}{3y}\,dy=c$$

$$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c$$

$\dfrac{2}{3}\log x-\dfrac{1}{3xy}-\dfrac{1}{3}\log y=c$ is the required Solution of equation (1)

8. Solve $y(1+xy)dx+x(1-xy)dy=0$.

Solution: The given equation is $y(1+xy)dx+x(1-xy)dy=0$ (1)

The given equation is not exact.

Equation (1) is of the form $f_1(xy)\,ydx+f_2(xy)\,xdy=0$

Here $M=y(1+xy)$, $N=x(1-xy)$

Now $M.x-N.y=y(1+xy).x-x(1-xy).y$

$=xy+x^2y^2-xy+x^2y^2=2x^2y^2\neq 0$

Hence the integrating factor $=\dfrac{1}{M.\text{x}-N.y}=\dfrac{1}{2x^2y^2}$

Multiplying equation (1) by $\dfrac{1}{2x^2y^2}$

$$\frac{y(1+xy)dx}{2x^2y^2}+\frac{x(1-xy)dy}{2x^2y^2}=0$$

$$\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0 \tag{2}$$

Here $M=\dfrac{1}{2x^2y}+\dfrac{1}{2x},N=\dfrac{1}{2xy^2}-\dfrac{1}{2y}$

$$\frac{\partial M}{\partial y}=-\frac{1}{2x^2y^2},\frac{\partial N}{\partial x}=-\frac{1}{2x^2y^2}, \text{ which implies the equation (2) is exact.}$$

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx-\int\frac{1}{2y}\,dy=c$$

$$-\frac{1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$\frac{1}{2}\log x-\frac{1}{2xy}-\frac{1}{2}\log y=c$ is the required solution of equation (1)

9. Solve $2ydx+(2x\log x-xy)dy=0.$

Solution: The given equation is $2ydx+(2x\log x-xy)dy=0$ (1)

The given equation is not exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2x\log x-xy}(2-2-2\log x+y)$

$$=\frac{(y-2\log x)}{2x\log x-xy}=\frac{(y-2\log x)}{-x(y-x\log x)}$$

$$=-\frac{1}{x}=f(x)$$

Since $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]$ is a function of x, the integrating factor is given by $e^{\int f(x)dx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Multiplying equation (1) by $\frac{1}{x}$ we get

$$\frac{2ydx}{x}+\frac{(2x\log x-xy)dy}{x}=0$$

$$\frac{2y}{x}dx+(2\log x-y)dy=0 \tag{2}$$

Here $M=\frac{2y}{x}, N=2\log x-y$

$$\frac{\partial M}{\partial y}=\frac{2}{x},\frac{\partial N}{\partial x}=\frac{2}{x}, \text{ which implies the equation (2) is exact.}$$

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int\frac{2y}{x}dx-\int y\,dy=c$$

$$2y\log x-\frac{y^2}{2}=c$$

$$2y\log x-\frac{y^2}{2}=c \text{ is the required Solution of equation (1)}$$

10. Solve $\left(x^2+y^2+2x\right)dx+2ydy=0.$

Solution: The given equation is $\left(x^2+y^2+2x\right)dx+2ydy=0$ (1)

The given equation is not exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2y}(2y-0)=1$

Since $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]$ is a constant, the integrating factor is given by $e^{\int f(x)dx}=e^{\int 1dx}=e^x$

Multiplying equation (1) by e^x we get

$$e^x\left(x^2+y^2+2x\right)dx+2ye^xdy=0 \quad (2)$$

Here $M=e^x\left(x^2+y^2+2x\right), N=2ye^x$

$$\frac{\partial M}{\partial y}=2ye^x, \frac{\partial N}{\partial x}=2ye^x, \text{ which implies the equation (2) is exact.}$$

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int e^x\left(x^2+y^2+2x\right)dx=c$$

$$\int\left(x^2+2x\right)e^xdx+\int y^2e^xdx=c$$

$$\left(x^2+2x\right)e^x-(2x+2)e^x+2e^x+y^2e^x=c$$

$$x^2e^x+2xe^x-2xe^x-2e^x+2e^x+y^2e^x=c$$

$$\left(x^2+y^2\right)e^x=c \text{ is the required Solution of equation (1)}$$

11. Solve $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0.$

Solution: The given equation is $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$ (1)

The given equation is not exact.

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{3x^2y^4+2xy}\left(6x^2y^3-2x-12x^2y^3-2x\right)$

$$=\frac{-\left(6x^2y^3+4x\right)}{3x^2y^4+2xy}=\frac{-2\left(3x^2y^3+2x\right)}{y\left(3x^2y^3+2x\right)}$$

$$=-\frac{2}{y}=f(y)$$

Since $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y, the integrating factor is given by $e^{\int f(y)dy} = e^{-2\int \frac{1}{y}dx} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$

Multiplying equation (1) by $\frac{1}{y^2}$ we get

$$\frac{\left(3x^2y^4+2xy\right)dx}{y^2}+\frac{\left(2x^3y^3-x^2\right)dy}{y^2}=0 \qquad (2)$$

Here $M=\frac{\left(3x^2y^4+2xy\right)}{y^2}, N=\frac{\left(2x^3y^3-x^2\right)}{y^2}$

$\frac{\partial M}{\partial y}=6x^2\,\mathrm{y}-\frac{2x}{y^2}, \frac{\partial N}{\partial x}=6x^2\,\mathrm{y}-\frac{2x}{y^2}$, which implies the equation (2) is exact.

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int\frac{\left(3x^2y^4+2xy\right)}{y^2}dx=c$$

$$\int\left(3x^2y^2+2\frac{x}{y}\right)dx=c$$

$$\frac{3x^3y^2}{3}+\frac{2x^2}{2y}=c$$

$x^3y^2+\frac{x^2}{y}=c$ is the required solution of equation (1)

12. Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0.$

Solution: The given equation is $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0 \qquad (1)$

The given equation is not exact.

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y^4+2y}\left(y^3-4-4y^3-2\right)$

$$=\frac{-3y^3-6}{y^4+2y}=\frac{-3\left(y^3+2\right)}{y\left(y^3+2\right)}$$

$$=-\frac{3}{y}=f(y)$$

Since $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y, the integrating factor is given by

$e^{\int f(y)dy} = e^{-3\int \frac{1}{y}dx} = e^{-3\log y} = e^{\log y^{-3}} = \frac{1}{y^3}$

Multiplying equation (1) by $\frac{1}{y^3}$ we get

$$\frac{(y^4+2y)dx}{y^3}+\frac{(xy^3+2y^4-4x)dy}{y^3}=0 \tag{2}$$

Here $M=y+\frac{2}{y^2}, N=x+2y-\frac{4x}{y^3}$

$$\frac{\partial M}{\partial y}=1-\frac{4}{y^3}, \frac{\partial N}{\partial x}=1-\frac{4}{y^3},$$ which implies the equation (2) is exact.

The Solution is $\int M\,dx+\int(\text{Terms of } N \text{ not containg } x)\,dy=c$

$$\int y+\frac{2}{y^2}\,dx+\int 2y\,dy=c$$

$$xy+\frac{2x}{y^2}+\frac{2y^2}{2}=c$$

$$xy+\frac{2x}{y^2}+y^2=c$$

$x^3y^2+\frac{x^2}{y}=c$ is the required solution of equation (1)

13. Solve $x(4y\,dx+2x\,dy)+y^3(3y\,dx+5x\,dy)=0$.

Solution: The given equation is $x(4ydx+2xdy)+y^3(3ydx+5xdy)=0$ (1)

Here $a=1,b=0,c=0,d=3,m=4,n=2,p=3$ and $q=5$

Let x^hy^k be the integrating factor.

Multiplying equation (1) by x^hy^k we get

$$x(4ydx+2xdy).x^hy^k+y^3(3ydx+5xdy).x^hy^k=0$$

$$(4x^{h+1}y^{k+1}dx+2x^{h+2}y^kdy)+(3x^hy^{k+4}dx+5x^{h+1}y^{k+3}dy)=0$$

$$(4x^{h+1}y^{k+1}+3x^hy^{k+4})dx+(2x^{h+2}y^k+5x^{h+1}y^{k+3})dy=0 \tag{2}$$

Here $M=4x^{h+1}y^{k+1}+3x^hy^{k+4}, N=2x^{h+2}y^k+5x^{h+1}y^{k+3}$

If the equation (2) is to be exact $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$\Rightarrow 4(k+1)x^{h+1}y^k+3(k+4)x^hy^{k+3}=2(h+2)x^{h+1}y^k+5(h+1)x^hy^{k+3}$$

Equating the coefficients of $x^{h+1}y^k$ and x^hy^{k+3} we get

$$4(k+1)=2(h+2)\Rightarrow 2k+2=h+2\Rightarrow 2k-h=0 \tag{3}$$

$$3(k+4)=5(h+1)\Rightarrow 3k+12=5h+5\Rightarrow 3k-5h=-7 \tag{4}$$

Solving equations (3) and (4) we get $k=1,h=2$

Hence the integrating factor is x^2y

Multiplying given equation by x^2y

$$x(4ydx+2xdy).x^2y+y^3(3ydx+5xdy).x^2y=0$$
$$4x^3y^2dx+2x^4ydy+3x^2y^5dx+5x^3y^4dy=0$$
$$d\left(x^4y^2+x^3y^5\right)=0$$

Integrating we get $x^4y^2+x^3y^5=c$ is the required Solution of equation (1)

14. Solve $xy(y\,dx+x\,dy)+x^2y^2(2\,y\,dx-x\,dy)=0$.

Solution: The given equation is $xy(ydx+xdy)+x^2y^2(2ydx-xdy)=0$ (1)

Here $a=1,b=1,c=2,d=2,m=1,n=1,p=2$ and $q=-1$

Let x^hy^k be the integrating factor.

Multiplying equation (1) by x^hy^k we get

$$xy(ydx+xdy).x^hy^k+x^2y^2(2ydx-xdy).x^hy^k=0$$
$$\left(x^{h+1}y^{k+2}dx+x^{h+2}y^{k+1}dy\right)+\left(2x^{h+2}y^{k+3}dx-x^{h+3}y^{k+2}dy\right)=0$$
$$\left(x^{h+1}y^{k+2}+2x^{h+2}y^{k+3}\right)dx+\left(x^{h+2}y^{k+1}-x^{h+3}y^{k+2}\right)dy=0$$

Here $M=x^{h+1}y^{k+2}+2x^{h+2}y^{k+3}, N=x^{h+2}y^{k+1}-x^{h+3}y^{k+2}$

If the equation (2) is to be exact $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

$$\Rightarrow (k+2)x^{h+1}y^{k+1}+2(k+3)x^{h+2}y^{k+2}=(h+2)x^{h+1}y^{k+1}-(h+3)x^{h+2}y^{k+2}$$

Equating the coefficients of $x^{h+1}y^{k+1}$ and $x^{h+2}y^{k+2}$ we get

$$(k+2)=(h+2)\Rightarrow k=h \tag{3}$$

$$3(k+3)=-(h+3)\Rightarrow 3k+9=-h-3\Rightarrow 3k+h=-12 \tag{4}$$

Solving equations (3) and (4) we get $k=-3,h=-3$

Hence the integrating factor is $x^{-3}y^{-3}$

Multiplying given equation by $\dfrac{1}{x^3y^3}$

$$xy(ydx+xdy)\frac{1}{x^3y^3}+x^2y^2(2ydx-xdy)\frac{1}{x^3y^3}=0$$
$$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$$
$$\frac{dx}{x^2y}+\frac{dy}{xy^2}+\frac{2}{x}dx-\frac{1}{y}dy=0$$
$$-d\left(\frac{1}{xy}\right)+\frac{2}{x}dx-\frac{1}{y}dy=0$$

Integrating we get $-\dfrac{1}{xy}+2\log x-\log y=c$ is the required Solution of equation (1)

EXERCISE 4.3

1. Solve $ydx - xdy + 3x^2y^2e^{x^2}dx = 0$.

2. Solve $(1+xy)\,ydx + (1-xy)\,xdy = 0$.

3. Solve $y(2xy + e^x)dx = e^x dy$.

4. Solve $x^2\dfrac{dy}{dx} + xy = \sqrt{1-x^2y^2}$.

5. Solve $\dfrac{dy}{dx} = \dfrac{x^3+y^3}{xy^2}$.

6. Solve $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$.

7. Solve $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,y\,dy$.

8. Solve $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$.

9. Solve $\left(xy^2 - e^{\frac{1}{x^2}}\right)dx - x^2ydy = 0$.

10. Solve $y(2x^2 - xy + 1)dx + (x - y)dy = 0$.

11. Solve $x\sin x\dfrac{dy}{dx} + y(x\cos x - \sin x) = 2$.

12. Solve $\left(y + \dfrac{1}{3}y^3 + \dfrac{1}{2}x^2\right)dx + \dfrac{1}{4}(1+y^2)\,x\,dy = 0$.

13. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$.

14. Solve $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$.

ANSWERS TO EXERCISE 4.3

1. Integrating factor is $\dfrac{1}{y^2}$ and solution is $\dfrac{x}{y} + e^{x^2} = c$

2. Integrating factor is $\dfrac{1}{x^2y^2}$ and the solution is $-\dfrac{1}{xy} + \log\left(\dfrac{x}{y}\right) = c$

3. Integrating factor is $\dfrac{1}{y^2}$ and the solution is $x^2 + \dfrac{e^x}{y} = c$

4. Integrating factor is $\dfrac{1}{x}$ and the solution is $\sin^{-1}(xy) - \log x = c$

5. Integrating factor is $\dfrac{1}{x^4}$ and the solution is $\log x - \dfrac{y^3}{3x^3} = c$

6. Integrating factor is $\frac{1}{x^2y^2}$ and the solution is $3\log x+\frac{y}{x}-2\log y=c$

7. Integrating factor is $\frac{1}{2xy\cos xy}$ and the solution is $\log\left(\frac{x}{y}\sec xy\right)=2c$

8. Integrating factor is $\frac{1}{2x^2y^2}$ and the solution is $\log\left(\frac{x}{y}\sec xy\right)=2c$

9. Integrating factor is $\frac{1}{x^4}$ and the solution is $-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c$

10. Integrating factor is e^{x^2} and the solution is $e^{x^2}\left(2xy-y^2\right)=c$

11. Integrating factor is $\frac{1}{x^2}$ and the solution is $\frac{2}{x}+\frac{y\sin x}{x}=c$

12. Integrating factor is x^3 and the solution is $x^3\left(3xy+xy^3+x^3\right)=12c$

13. The solution is $4\sqrt{xy}-\frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}}=c$

14. The solution is $5x^{\frac{-36}{13}}y^{\frac{24}{13}}-12x^{\frac{-10}{13}}y^{\frac{-15}{13}}=c$

4.5 LINEAR DIFFERENTIAL EQUATION

Definition: A differential equation of any order is said to linear when it is in the first degree in dependent value or variable and all its derivatives.

General form of first Linear differential Equation:

The equation of the form $\frac{dy}{dx}+Py=Q$ where P and Q are functions of x only is the general form of first order linear equation.

Integrating Factor:

$e^{\int Pdx}$ is called integrating factor of the linear differential equation. $\frac{dy}{dx}+Py=Q$

Solution of First Order Linear Differential Equation:

The general solution of first order linear differential equation $\frac{dy}{dx}+Py=Q$ is $ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C$ where C is the constant of integration.

Problems:

1. **Solve** $\frac{dy}{dx}+\frac{4xy}{x^2+1}=\frac{1}{(x^2+1)^3}$

Solution: The given equation is $\frac{dy}{dx} + \frac{4xy}{x^2+1} = \frac{1}{(x^2+1)^3}$ (1)

This is the form of $\frac{dy}{dx} + Py = Q$

Here $P = \frac{4x}{x^2+1}$ and $Q = \frac{1}{(x^2+1)^3}$

$$\text{Now } \int P dx = \int \frac{4x}{x^2+1} dx$$

$$= 2 \log (x^2 + 1)$$

$$\int P dx = \log(x^2+1)^2$$

$$\text{Integrating factor} = e^{\int P dx}$$

$$= e^{\log(x^2+1)^2}$$

$$= (x^2 + 1)^2$$

Solution is $ye^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$y(x^2+1)^2 = \int \frac{1}{(x^2+1)^3}(x^2+1)^2 dx + C$$

$$y(x^2+1)^2 = \int \frac{dx}{x^2+1}$$

$$\therefore y(x^2+1)^2 = \tan^{-1}(x) + C \qquad \left(\because \int \frac{1}{x^2+1} = \tan^{-1}(x)\right)$$

2. **Solve** $\frac{dy}{dx} + \frac{y}{x} = e^x + \sin x$

Solution: The given equation is $\frac{dy}{dx} + \frac{y}{x} = e^x + \sin x$ (1)

This is of the form $\frac{dy}{dx} + Py = Q$

$$\text{Here } P = \frac{1}{x} \text{ and } Q = e^x + \sin x$$

$$\text{Now } \int P dx = \int \frac{1}{x} dx$$

$$\int P dx = \log x$$

$$\text{Integrating factor} = e^{\int P dx} = e^{\log x}$$

$$e^{\int P dx} = x$$

Solution is $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$

$$\int udv = uv - u'v_1 + u''v_2$$

$$u = x \qquad dv = e^x$$

$$u' = 1 \qquad v = e^x$$

$$v_1 = e^x$$

$$yx = \int (e^x + \sin x)xdx$$

$$xy = \int xe^x dx + \int x\sin xdx$$

$$xy = xe^x - e^x + x(-\cos x) + \sin x + C$$

$$\therefore xy = xe^x - e^x - x\cos x + \sin x + C$$

$$\int udv = uv - u'v_1 + \dots$$

$$u = x \quad dv = \sin x$$

$$u' = 1 \quad v = -\cos x$$

$$v_1 = -\sin x$$

3. Solve $(1 + x^2)\dfrac{dy}{dx} + 2xy = \cot x$

Solution: The given equation is $(1 + x^2)\dfrac{dy}{dx} + 2xy = \cot x$

$$\frac{dy}{dx} + \frac{2xy}{(1+x^2)} = \frac{\cot x}{(1+x^2)} \tag{1}$$

Here $P = \dfrac{2x}{1+x^2}$ and $Q = \dfrac{\cot x}{(1+x^2)}$

Now $\int Pdx = \int \dfrac{2x}{(1+x^2)}dx$

$$\int Pdx = \log(1+x^2)$$

Integrating factor $e^{\int Pdx} = e^{\log(1+x^2)}$

$$\therefore e^{\int Pdx} = (1+x^2)$$

Solution is $y\,e^{\int Pdx} = \int Q e^{\int Pdx\,dx + C}$

$$y(1+x^2) = \int \frac{\cot x}{(1+x^2)}(1+x^2)dx$$

$$y(1+x^2) = \int \cot xdx \ \left(\because \int \cot xdx = \log(\sin x)\right)$$

$$\because y(1+x^2) = \log(\sin x) + C$$

4. Solve $xy' + y + 4 = 0$

Solution: The given equation is $xy' + y + 4 = 0$

$$x\frac{dy}{dx} + y = -4$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{-4}{x} \quad (1)$$

This is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \dfrac{1}{x}$ and $Q = \dfrac{-4}{x}$

$$\text{Now } \int P dx = \int \frac{1}{x} dx$$

$$= \log x$$

Integrating factor $e^{\int P dx} = e^{\log x}$

Solution of (1) is $ye^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$yx = \int \frac{-4}{x} x dx$$

$$= \int -4 dx$$

$$yx = -4x + C$$

$$\therefore yx + 4x = C$$

5. Solve $(x^2 - 1)\dfrac{dy}{dx} + 2xy = 1$

Solution: The given equation is $(x^2 - 1)\dfrac{dy}{dx} + 2xy = 1$

$$\frac{dy}{dx} + \frac{2xy}{(x^2 - 1)} = \frac{1}{(x^2 - 1)} \quad (1)$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \dfrac{2x}{(x^2 - 1)}$ and $Q = \dfrac{1}{(x^2 - 1)}$

$$\text{Now } \int P dx = \int \frac{2x}{(x^2 - 1)} dx$$

$$\int P dx = \log(x^2-1)$$

$$\text{Integrating factor } e^{\int P dx} = e^{\log(x^2-1)}$$

$$\therefore e^{\int P dx} = (x^2-1)$$

$$\text{Solution is } ye^{\int P dx} = \int Qe^{\int P dx} dx + C$$

$$y(x^2-1) = \int \frac{1}{(x^2-1)}(x^2-1)dx$$

$$= \int dx$$

$$= x + C$$

$$\therefore y(x^2-1) = x + C$$

6. Solve $y' + y\cot x = 2x\operatorname{cosec} x$

Solution: The given equation is $\dfrac{dy}{dx} + y\cot x = 2x\operatorname{cosec} x$ (1)

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \cot x$ and $Q = 2x \operatorname{cosec} x$

$$\text{Now } \int P dx = \int \cot x dx$$

$$= \log \sin x$$

$$\text{Integrating factor } e^{\int P dx} = e^{\log \sin x}$$

$$\therefore e^{\int P dx} = \sin x$$

$$\text{Solution is } ye^{\int P dx} = \int Qe^{\int P dx} dx + C$$

$$y \sin x = \int (2x\operatorname{cosec} x)\sin x dx$$

$$= \int 2x \frac{1}{\sin x} \sin x dx$$

$$= \int 2x dx$$

$$= \frac{2x^2}{2} + C$$

$$\therefore y \sin x = x^2 + C$$

7. **Solve** $(1+y^2)\dfrac{dx}{dy}x = e^{\tan^{-1}y}$

Solution: The given equation is $(1 + y^2)\dfrac{dx}{dy} + x = e^{\tan^{-1}y}$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}$$

It is of the form $\dfrac{dx}{dy}+Px=Q$

$$\text{Here } P=\frac{1}{1+y^2} \text{ and } Q=\frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\text{Now } \int Pdy=\int\frac{1}{1+y^2}dy$$

$$=\tan^{-1}y$$

$$\text{Integrating factor is } e^{\int Pdy}=e^{\tan^{-1}y}$$

$$\text{Solution is } xe^{\int Pdy}=\int Qe^{\int Pdy}dy+C$$

$$xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy \qquad \left[\text{Let } t=\tan^{-1}y\right.$$

$$=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy \qquad \left.dt=\frac{1}{1+y^2}dy\right]$$

$$=\int e^{2t}dt$$

$$=\frac{e^{2t}}{2}+C$$

$$\therefore xe^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C$$

8. **Solve** $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

Solution: The given equation is $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

$$\therefore \frac{dy}{dx}+\frac{y}{\cos^2 x}=\frac{\tan x}{\cos^2 x}$$

It is of the form $\dfrac{dy}{dx}+Py=Q$

$$\text{Here } P = \frac{1}{\cos^2 x} \text{ and } Q = \frac{\tan x}{\cos^2 x}$$

$$\text{Now } \int P dx = \int \frac{1}{\cos^2 x} dx$$

$$= \int \sec^2 x dx$$

$$= \tan x$$

Integrating factor is $e^{\int P dx} = e^{\tan x}$

Solution is $ye^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$ye^{\tan x} = \int \frac{\tan x}{\cos^2 x} e^{\tan x} dx \qquad \text{[Let } t = \tan x$$

$$= \int \tan x \sec^2 x e^{\tan x} dx \qquad dt = \sec^2 x dx]$$

$$ye^{\tan x} = \int te^t dt$$

$$= te^t - e^t + C$$

$$\therefore ye^{\tan x} = \tan x e^{\tan x} - e^{\tan x} + C$$

9. Solve $x\frac{dy}{dx} + y = \log x$

Solution: The given equation $\frac{x\,dy}{dx} + y = \log x$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x} \tag{1}$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1}{x}$ and $Q = \frac{\log x}{x}$

$$\text{Now } \int P dx = \int \frac{1}{x} dx$$

$$= \log x$$

Integrating factor is $e^{\int pdx} = e^{\log x}$ | Integration by parts

$e^{\int pdx} = x$ $\qquad \int u dv = uv - \int v du$

Solution is $ye^{\int pdx} = \int Qe^{\int pdx}dx + C$

$$yx = \int \frac{\log x}{x} x dx + C$$

$$= \int \log x dx + C$$

$$\therefore xy = x\log x - x + C$$

$u = \log x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$$\int \log x dx = x\log x - \int x\frac{1}{x}dx$$

$$= x\log x - \int dx$$

$$= x\log x - x$$

10. Solve $x\dfrac{dy}{dx} + 2y = x^2 \log x$

Solution: The given equation is $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{x^2 \log x}{x}$ (1)

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \dfrac{2}{x}$ and $Q = x\log x$

$$\text{Now } \int Pdx = \int \frac{2}{x}dx$$

$$= 2\int \frac{1}{x}dx$$

$$\int Pdx = 2\log x$$

Integrating factor is $e^{\int Pdx} = e^{2\log x} = e^{\log x^2}$

$$e^{\int Pdx} = x^2$$

Solution is $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$

$$yx^2 = \int x\log x \, x^2 dx$$

$$= \int x^3 \log x dx$$

Integration by parts

$$\int udv = uv - \int vdu$$

$u = \log x \quad dv = x^3 dx$

$du = \frac{1}{x}dx \quad v = \frac{x^4}{4}$

$$yx^2 = \frac{x^4}{4}\log x - \frac{x^4}{4}\left(\frac{1}{4}\right) + C \qquad \int x^3 \log x dx = \frac{x^4}{4}\log x - \int \frac{x^4}{4}\frac{1}{x}dx$$

$$\therefore yx^2 = \frac{x^4}{4}\log x - \frac{x^4}{16} + C \qquad = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\frac{x^4}{4}\bigg]$$

11. Solve $\dfrac{dy}{dx} + y \tan x = \cos^3 x$

Solution: The given equation is $\dfrac{dy}{dx} + y\tan x = \cos^3 x$ (1)

It is of the form $\dfrac{dy}{dx} + Py = Q$

$$\text{Here } P = \tan x \text{ and } Q = \cos^3 x$$

$$\int Pdx = \int \tan x dx$$

$$\int Pdx = \log(\sec x)$$

Integrating factor is $e^{\int Pdx} = e^{\log(\sec x)}$

$$e^{\int Pdx} = \sec x$$

Solution is $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$

$$y\sec x = \int \cos^3 x \sec x dx$$

$$\int cos^2 x\, dx$$

$$= \int\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \int \frac{1}{2}dx + \int \frac{\cos 2x}{2}dx$$

$$= \frac{1}{2}x + \frac{1}{2}\int \cos 2x dx$$

$$= \frac{x}{2} + \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + C$$

$$y\sec x = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

12. Solve $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x, y\left(\frac{\pi}{2}\right)=0$

Solution: The given equation is $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x$ (1)

It is of the form $\frac{dy}{dx}+Py=Q$

Here $P=\cot x$ and $Q=4x\operatorname{cosec} x$

$$\text{Now } \int Pdx=\int \cot x dx$$

$$=\log(\sin x)$$

Integrating factor is $e^{\int Pdx}=e^{\log(\sin x)}$

$$e^{\int Pdx}=(\sin x)$$

Solution is $ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C$

$$y\sin x=\int 4x\operatorname{cosec} x\sin xdx$$

$$=4\int x\frac{1}{\sin x}\sin dx$$

$$=4\int xdx$$

$$=\frac{4x^2}{2}$$

$$y\sin x=2x^2+C \quad (1)$$

Here $x=\frac{\pi}{2}$ and $y=0$

$$0\left(\sin\frac{\pi}{2}\right)=2\left(\frac{\pi}{2}\right)^2+C$$

$$0=2\left(\frac{\pi^2}{4}\right)+C$$

$$C=-\frac{\pi^2}{4} \quad (2)$$

Substitute (2) in (1)

$$\therefore y\sin x=2x^2-\frac{\pi^2}{2}$$

EXERCISE 4.4

1. Solve $(x^2-1)\dfrac{dy}{dx}+2xy=1$.
2. Solve $(1+x^2)\dfrac{dy}{dx}+2xy=\cos x$.
3. Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=\sin x^2$.
4. Solve $y^2dx+(3xy-1)dy=0$.
5. Solve $\cos^2 x\dfrac{dy}{dx}+y=\tan x$.
6. Solve $\dfrac{dy}{dx}=y\tan x-2\sin x$.
7. Solve $x\log x\dfrac{dy}{dx}+y=2\log x$.
8. Solve $\dfrac{dy}{dx}=\dfrac{x+1+y}{x+1}$.
9. Solve $\dfrac{dy}{dx}+\dfrac{4x}{x^2+1}y=\dfrac{1}{(x^2+1)^3}$.
10. Solve $x\dfrac{dy}{dx}+2y-x^2\log x=0$.
11. Solve $\sin x\dfrac{dy}{dx}+y\cos x=2\sin^2 x\cos x$.
12. Solve $(x^2-1)\dfrac{dy}{dx}+2(x+2)y=2(x+1)$.
13. Solve $\cos x\dfrac{dy}{dx}+y\sin x=1$.
14. Solve $\dfrac{dy}{dx}+\dfrac{2}{x}y=\sin x$.
15. Solve $\dfrac{dy}{dx}+2y\cot x=3x^2\operatorname{cosec}^2 x$.
16. Solve $\sin x\dfrac{dy}{dx}+y=\tan x$.
17. Solve $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\dfrac{dy}{dx}=0$.
18. Solve $(x+\log y)dy+ydx=0$.
19. Solve $ydx+(x-y^3)dy=0$.
20. Solve $\sqrt{1-y^2}\dfrac{dx}{dy}=\sin^{-1}y-x$.

ANSWERS TO EXERCISE 4.4

1. $y(x^2-1)=x+c$
2. $y(1+x^2)=\sin x+c$
3. $xy=-\dfrac{\cos^2 x}{2}+c$
4. $xy^3=\dfrac{y^2}{2}+c$
5. $y=(\tan x-1)+ce^{-\tan x}$
6. $y\cos x=\dfrac{\cos 2x}{2}+c$
7. $y\log x=(\log x)^2+c$
8. $\dfrac{y}{x+1}=\log(x+1)+c$
9. $y(x^2+1)^2=\tan^{-1}x+c$

10. $x^2y = \frac{x^4}{4}\log x - \frac{x^4}{16} + c$

11. $y\sin x = \frac{2\sin^3 x}{3} + c$

12. $y(x-1)^3 = (x+1)\left(x^2 - 6x + 8\log(x+1) + c\right)$

13. $y\sec x = \tan x + c$

14. $x^2y = -x^2\cos x + 2x\sin x + 2\cos x + c$

15. $y\sin^2 x = x^3 + c$

16. $y(\operatorname{cosec}x - \cot x) = \log(\sec x) - \log(\operatorname{cosec}x - \cot x)$

17. $xe^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + c$

18. $xy = -y\log y + y + c$

19. $xy = \frac{y^4}{4} + c$

20. $x = \sin^{-1}y - 1 + ce^{-\sin^{-1}y}$

4.6 BERNOULLI'S EQUATION

Definition: A first order first degree differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where n is any constant other than 0 and 1 is called Bernoulli's equation.

Method for finding Solution of bernoulli's Equation:

Step 1: Rewrite the Equation is standard bernoulli's Equation. Identify n, P and Q

Step 2: Introduce a new variable, $z = y^{1-n}$ and obtain the resultant first Order linear Equation in z.

Step 3: Solve the linear Equation in z by the method applied for solving linear differential equation.

1. Solve $\frac{dy}{dx} - y\tan x = -y^2\sec x$

Solution: The given Equation is $\frac{dy}{dx} - y\tan x = -y^2\sec x$ (1)

This is Bernaulli's Equation

Here $n = 2$

Multiplying Equation (1) by $\frac{1}{y^2}$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x \tag{2}$$

Put $z = y^{1-n} = y^{1-2}$

$$z = \frac{1}{y}$$

$$\therefore \frac{dz}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

Sub in (2)

$$\frac{-dz}{dx} - z\tan x = -\sec x$$

$$\frac{dz}{dx} + z\tan x = \sec x \tag{3}$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \tan x$ and $Q = \sec x$

Now $\int P\,dx = \int \tan x\,dx$

$$= \log(\sec x)$$

$$e^{\int P dx} = e^{\log(\sec x)}$$

$$e^{\int P dx} = \sec x$$

Solution of (3) is $ze^{\int P dx} = \int Qe^{\int P dx}dx + C$

$$z\sec x = \int \sec x \sec x\,dx$$

$$= \int \sec^2 x\,dx$$

$$z\sec x = \tan x + C$$

$\therefore \frac{1}{y}\sec x = \tan x + C$ is the required solution

2. Solve $y(2xy + e^x)\,dx = e^x\,dy$

Solution: The given equation is $y(2xy + e^x)dx = e^x dy$

$$\frac{y(2xy + e^x)}{e^x} = \frac{dy}{dx}$$

$$\frac{2xy^2 + ye^x}{e^x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2xy^2}{e^x} + y$$

$$\frac{dy}{dx} - y = \frac{2xy^2}{e^x} \qquad (1)$$

It is Bernoulli's Equation
Here $n = 2$
Multiplying Equation (1) by $\frac{1}{y^2}$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x} \qquad (2)$$

$$\text{Put } z = y^{1-n}$$

$$z = \frac{1}{y}$$

$$\frac{dz}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

Sub in equation (2)

$$\frac{-dz}{dx} - z = 2xe^{-x}$$

It is of the form $dy/dx + Py = Q$

Here $P = 1$ and $Q = -2xe^{-x}$

$$\text{Now } \int P\,dx = \int dx = x$$

$$e^{\int P\,dx} = e^x$$

Solution of (3) is $z\,e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + C$

$$z\,e^x = \int -2xe^{-x}e^x dx$$

$$= -2\int x\,dx$$

$$ze^x = -2\frac{x^2}{2} + C$$

$$ze^x = -x^2 + C$$

$$ze^x + x^2 = C$$

$$\therefore \frac{e^x}{y} + x^2 = C$$

3. **Solve** $xy\,(1+xy^2)\dfrac{dy}{dx}=1$

Solution: The given equation is $xy(1+xy^2)\dfrac{dy}{dx}=1$

$$xy(1+xy^2)=\frac{dx}{dy}$$

$$\frac{dx}{dy}=xy+x^2y^3$$

$$\frac{dx}{dy}-xy=x^2y^3 \qquad (1)$$

It is Bernoulli's Equation

Here $n=2$

Put $z=x^{1-n}$

$$z=x^{1-2}$$

$$z=\frac{1}{x}$$

$$\frac{dz}{dy}=\frac{-1}{x^2}\frac{dx}{dy}$$

Sub in (1)

$$-x^2\frac{dz}{dy}-xy=x^2y^3$$

$$\frac{dz}{dy}+\frac{y}{x}=-y^3$$

$$\frac{dz}{dy}+zy=-y^3$$

It is of the form $\dfrac{dx}{dy}+Py=Q$

Here $P=y \quad Q=-y^3$

Now $\int P\,dy=\int y\,dy$

$$=\frac{y^2}{2}$$

$$e^{\int p\,dy}=e^{\frac{y^2}{2}}$$

Solution is $ze^{\int P\,dy}=\int Q\,e^{\int P\,dy}\,dy+C$

$$ze^{\frac{y^2}{2}}=-\int y^3e^{\frac{y^2}{2}}\,dy$$

$$= -\int e^t y^2 (y\,dy)$$
$$= -\int e^t 2t\,dt$$
$$= -2\int te^t dt$$

$$ze^{\frac{y^2}{2}} = -2\left(te^t - e^t\right) + C \qquad \left[t = \frac{y^2}{2}\right.$$

$$= -2\left(\frac{y^2}{2}e^{\frac{y^2}{2}} - e^{\frac{y^2}{2}}\right) + C \qquad \left.dt = \frac{2y}{2}dy = dt = ydy\right]$$

$$= -y^2e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + C$$

$$ze^{\frac{y^2}{2}} = -y^2e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + C$$

$$\therefore \frac{1}{x}e^{\frac{y^2}{2}} = -y^2e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + C$$

4. **Solve $dy/dx = x^3 y^3 - yx$**

Solution: The given equation is $\dfrac{dy}{dx} = x^3y^3 - yx$

$$\frac{dy}{dx} + xy = x^3y^3 \qquad (1)$$

It is Bernoulli's Equation

Here $n = 3$

Put $z = y^{1-n}$

$$z = y^{1-3}$$
$$z = y^{-2}$$
$$z = \frac{1}{y^2}$$
$$\frac{dz}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$$

Sub in (1)

$$\frac{-y^3}{2}\frac{dz}{dx} + xy = x^3y^3$$

($\div$ by y^3)

$$\frac{-1}{2}\frac{dz}{dx} + \frac{x}{y^2} = x^3$$

$$\frac{dz}{dx} - \frac{2x}{y^2} = -2x^3$$

$$\frac{dz}{dx}-2zx=-2x^3 \qquad (2)$$

It is of the form $\frac{dy}{dx}+Py=Q$

Here $P=-2x$ and $Q=-2x^3$

$$\text{Now } \int P\,dx=-2\int x\,dx$$

$$=\frac{-2x^2}{2}=-x^2$$

$$e^{\int P\,dx}=e^{-x^2}$$

Solution is $ze^{\int P\,dx}=\int Qe^{\int P\,dx}dx+C$

$$ze^{-x^2}=-2\int x^3e^{-x^2}dx$$

$$=-\int te^{-t}dt$$

$$=-\left(-te^{-t}-e^{-t}\right)+C \qquad [u=t \quad dv=e^{-t}$$

$$ze^{-x^2}=te^{-t}+e^{-t}+C \qquad u'=1 \quad v=-e^{-t}$$

$$ze^{-x^2}=x^2e^{-x^2}+e^{-x^2}+C \qquad v_1=e^{-t}]$$

$$\therefore \frac{e^{-x^2}}{y^2}=x^2e^{-x^2}+e^{-x^2}+C$$

5. Solve $3y'+xy=xy^{-2}$

Solution: The given equation $3y'+xy=xy^{-2}$

$$3\frac{dy}{dx}+xy=xy^{-2}$$

$$\frac{dy}{dx}+\frac{xy}{3}=\frac{xy^{-2}}{3} \qquad (1)$$

It is a Bernoulli's Equation

Here $n=-2$

Put $z=y^{1-n}=y^{1+2}$

$$z = y^3$$

$$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$$

Sub in (1)

$$\frac{1}{3y^2}\frac{dz}{dx} + \frac{xy}{3} = \frac{xy^{-2}}{3}$$

(multiply y^2)

$$\frac{1}{3}\frac{dz}{dx} + \frac{xy^3}{3} = \frac{x}{3}$$

$$\frac{dz}{dx} + xy^3 = x$$

$$\frac{dz}{dx} + xz = x$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = x$ and $Q = x$

$$\text{Now } \int P\,dx = \int x\,dx$$

$$= \frac{x^2}{2}$$

$$e^{\int P\,dx} = e^{\frac{x^2}{2}}$$

Solution is $ze^{\int P\,dx} = \int Qe^{\int P\,dx}dx + C$ $\left[\text{Put } t = \frac{x^2}{2}\right.$

$$ze^{\frac{x^2}{2}} = \int xe^{\frac{x^2}{2}}dx$$

$$= \int e^t\,dt$$ $\left.dt = xdx\right]$

$$ze^{\frac{x^2}{2}} = e^t + C$$

$$ze^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} + C$$

$$\therefore\ y^3e^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} + C$$

6. Solve $2xyy' = y^2 - 2x^3$, $y(1) = 2$

Solution: The given equation is $2xyy' = y^2 - 2x^3$

$$2xy\frac{dy}{dx} = y^2 - 2x^3$$

$$\frac{dy}{dx} = \frac{y^2}{2xy} - \frac{2x^3}{2xy}$$

$$\frac{dy}{dx} - \frac{y}{2x} = \frac{-x^2}{y}$$

$$\frac{dy}{dx} - \frac{y}{2x} = -x^2 y^{-1} \qquad (1)$$

This is bernoulli's equation

Here $n = -1$

Put $z = y^{1-n}$

$$z = y^2$$

$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$

Sub in (1) we get $\frac{1}{2y}\frac{dz}{dx} - \frac{y}{2x} = -x^2 y^{-1}$

$$\frac{dz}{dx} - \frac{y^2}{x} = -2x^2$$

$$\frac{dz}{dx} - \frac{z}{x} = -2x^2 \qquad (2)$$

It is of the form $\frac{dy}{dx} + py = Q$

Here $P = \frac{-1}{x}$ and $Q = -2x^2$

Now $\int P dx = \int \frac{-1}{x} dx$

$$\int P dx = -\log x$$

$$e^{\int P dx} = e^{-\log x}$$

$$= e^{\log x^{-1}}$$

Solution is $z e^{\int p dx} = \int Q\, e^{\int p dx} dx + C$

$$\therefore z e^{\log x^{-1}} = \int -2x^2 e^{\log x^{-1}} dx + C$$

$$z x^{-1} = -2\int x^2 x^{-1} dx + C$$

$$= -2\int \frac{x^2}{x} dx$$

$$= -2\int x\,dx$$

$$zx^{-1} = -2\frac{x^2}{2} + C$$

$$zx^{-1} = x^2 + C$$

$$\frac{y^2}{x} = x^2 + C \qquad (2)$$

Applying the initial condition $y(1) = 2$ in (3)

$$\frac{(2)^2}{1} = -1^2 + C$$

$$4 = -1 + C$$

$$\therefore C = 5$$

$$\therefore \text{Solution is } \frac{y^2}{x} + x^2 = 5$$

7. Solve $\frac{dy}{dx} - y\tan x = \frac{\sin x\cos^2 x}{y^2}$

Solution: The given equation is $\frac{dy}{dx} - y\tan x = \frac{\sin x\cos^2 x}{y^2}$ (1)

This is Bernoulli's Equation

Here $n = -2$

Put $z = y^{1-n}$ $z = y^3$

$$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$$

Sub in (1) we get $\frac{1}{3y^2}\frac{dz}{dx} - y\tan x = \frac{\sin x\cos^2 x}{y^2}$

$(\times 3y^2)$

$$\frac{dz}{dx} - 3y^3\tan x = 3\sin x\cos^2 x$$

$$\frac{dz}{dx} - 3z\tan x = 3\sin x\cos^2 x \qquad (2)$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = -3\tan x$ and $Q = 3\sin x\cos^2 x$

$$\text{Now } \int P\,dx = -3\int \tan x\,dx$$

$$= -3\log(\sec x)$$

$$= \log(\sec x)^{-3}$$

$$e^{\int P dx} = e^{\log(\sec x)^{-3}}$$

$$= \frac{1}{\sec^3 x}$$

$$= \cos^3 x$$

Solution is $ze^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$z\cos^3 x = \int 3\sin x \cos^2 x \cos^3 x dx \quad [t = \cos x \Rightarrow dt = -\sin x dx]$$

$$z\cos^3 x = 3\int \sin x \cos^5 x\, dx$$

$$= 3\int (t^5)(-dt)$$

$$= -3\int t^5 dt$$

$$= -\frac{3t^6}{6} + C$$

$$z\cos^3 x = \frac{-t^6}{2} + C$$

$$z\cos^3 x = \frac{-\cos^6 x}{2} + C$$

$$y^3\cos^3 x + \frac{\cos^6 x}{2} = C$$

$$\therefore\ y^3\cos^3 x + \frac{\cos^6 x}{2} = C$$

8. Solve $(xy^5 + y)dx = dy$

Solution: The given equation is $(xy^5 + y)\, dx = dy$

$$\frac{dy}{dx} = xy^5 + y \tag{1}$$

$$\frac{dy}{dx} - y = xy^5$$

This is Bernoulli's Equation

$$z = y^{1-n}$$

$$= y^{-4}$$

$$z = y^{-4}$$

$$\frac{dz}{dx} = -4y^{-5}\frac{dy}{dx}$$

Sub in (1) we get $\dfrac{-1}{4y^{-5}}\dfrac{dz}{dx} - y = xy^5$

$$\frac{dz}{dx} + 4yy^{-5} = -4xy^5y^{-5}$$

$$\frac{dz}{dx} + \frac{4}{y^4} = -4x$$

$$\frac{dz}{dx} + 4z = -4x$$

It is of the form $\dfrac{dy}{dx} + Py = Q$ (1)

Here $P = 4$ and $Q = -4x$

Now $\int P\,dx = \int 4\,dx$

$= 4x$

$e^{\int P\,dx} = e^{4x}$

Solution is $ze^{\int Pdx} = \int Qe^{\int Pdx}dx + C$

$$\left[u = x \quad dv = e^{4x}dx \quad u' = 1 \quad v = \frac{e^{4x}}{4} \quad v_1 = \frac{e^{4x}}{16} \right]$$

$$ze^{4x} = \int -4x\,e^{4x}dx$$

$$= -4\int xe^{4x}dx$$

$$= -4\left(x\frac{e^{4x}}{4} - \frac{e^{4x}}{16}\right) + C$$

$$= -4e^{4x}\left(\frac{x}{4} - \frac{1}{16}\right) + C$$

$$= \frac{-4e^{4x}}{4}\left(x - \frac{1}{4}\right) + C$$

$$= -e^{4x}\left(x - \frac{1}{4}\right) + C$$

$$= -xe^{4x} + \frac{e^{4x}}{4} + C$$

$$\therefore ze^{4x} = -xe^{4x} + \frac{e^{4x}}{4} + C$$

$$\therefore \frac{e^{4x}}{y^4} = -xe^{4x} + \frac{e^{4x}}{4} + C$$

9. Solve $\cos x\, dy = y(\sin x - y)dx$

Solution: The given equation is $\cos x\, dy = y(\sin x - y)\, dx$

$$\frac{dy}{dx} = \frac{y(\sin x - y)}{\cos x}$$

$$= \frac{y\sin x}{\cos x} - \frac{y^2}{\cos x}$$

$$\frac{dy}{dx} - \frac{y\sin x}{\cos x} = \frac{-y^2}{\cos x}$$

Here $n = 2$

This is Bernoulli's Equation

$$z = y^{1-n}$$

$$z = y^{-1}$$

$$\frac{dz}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

$$-y^2\frac{dz}{dx} - \frac{y\sin x}{\cos x} = \frac{-y^2}{\cos x}$$

$$\frac{dz}{dx} + \frac{y\sin x}{y^2\cos x} = \frac{y^2}{y^2\cos x}$$

$$\frac{dz}{dx} + \frac{\sin x}{y\cos x} = \frac{1}{\cos x}$$

$$\frac{dz}{dx} + \frac{\sin x}{\cos x}z = \frac{1}{\cos x} \qquad (1)$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \dfrac{\sin x}{\cos x} = \tan x$ and $Q = \dfrac{1}{\cos x} = \sec x$

Now $\int P dx = \int \tan x\, dx$

$$= \log(\sec x)$$

$$e^{\int P\,dx} = e^{\log(\sec x)}$$

$$e^{\int P\,dx} = \sec x$$

Solution is $ze^{\int P\,dx} = \int Qe^{\int p\,dx}dx + C$

$$z \sec x = \int \sec x \sec x \, dx$$
$$= \int \sec^2 x \, dx$$
$$z \sec x = \tan x + C$$
$$\frac{\sec x}{y} = \tan x + C$$
$$\therefore \frac{\sec x}{y} = \tan x + C$$

EXERCISE 4.5

1. Solve $\frac{dy}{dx} = x^3y^3 - xy.$
2. Solve $(y \log x - 1) y dx = x dy.$
3. Solve $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x..$
4. Solve $3\frac{dy}{dx} + \frac{2}{x+1} y = \frac{x^3}{y^2}.$
5. Solve $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}.$
6. Solve $(x^2y^3 + xy) dy = dx.$
7. Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$
8. Solve $(x - y^2) dx + 2xy dy = 0.$
9. Solve $x^2 y \frac{dy}{dx} = xy^2 - e^{-\frac{1}{x^3}}.$
10. Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}.$
11. Solve $(x^2y^2 + xy) dx = dy.$
12. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 \log x.$
13. Solve $\frac{dy}{dx} + 2y \tan x = y^2.$
14. Solve $x\frac{dy}{dx} + y = x^3y^4.$
15. Solve $x\frac{dy}{dx} + y = y^2x^3 \cos x.$

ANSWERS TO EXERCISE 4.5

1. $y^{-2}e^{-x^2} = e^{-x^2}(x^2 + 1) + c$
2. $\frac{1}{xy} = \frac{\log x}{x} + \frac{1}{x} + c$
3. $\frac{\sec^2 y}{y} = -\frac{\tan^3 x}{3} + c$
4. $y^2(x+1)^2 = \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + c$
5. $\sqrt{y} = -\frac{(1-x^2)}{3} + c(1-x^2)^{1/4}$
6. $\frac{1}{x} = -y^2 + 2 + ce^{-\frac{1}{2y^2}}$
7. $\tan y = \frac{x^2 - 1}{2} + ce^{-x^2}$
8. $\frac{y^2}{x} = -\log x + c$

9. $y^2 = x^2\left(-\frac{2}{3}e^{-\frac{1}{x^3}} + c\right)$

10. $\frac{e^{x^2}}{y^2} = 2x + c$

11. $\frac{1}{y} = -x^2 + 2 + ce^{-\frac{x^2}{2}}$

12. $\frac{1}{y} = -\frac{(\log x)^2}{2} + cx$

13. $\cos^2 x + \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)y = cy$

14. $\frac{1}{y^3} = -3x^3 \log x + cx^3$

15. $\frac{1}{xy} = -x\sin x - \cos x + c$

4.7 CLAIRAUT'S EQUATION

Definition: Clairaut's equation is a first order and first degree differential equation of the form $y = x.\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$ (or) $y = xp + f(p)$, where $p = \frac{dy}{dx}$. (1)

Complete Solution: The complete solution of the Clairaut's equation (1) is obtained by substituting $p = c$. The Complete solution is also called as General Solution.

Singular Solution: The singular Solution is obtained by solving (or) eliminating the parameter p from the following equations.

$$y = xp + f(p) \tag{2}$$

$$x + \frac{df}{dp} = 0 \tag{3}$$

Problems:

1. Solve $y = xy' - (y')^2$.

Solution: The given equation is $y = xy' - (y')^2$

$$y = x.\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

$$\therefore y = xp - p^2 \tag{1}$$

Put $p = c$ in (1)

$y = xc - c^2$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0 = x - 2p$$

$$x = 2p \Rightarrow p = \frac{x}{2}$$

Substituting $p = \frac{x}{2}$ in (1) we get

$$y = x\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2$$

$$y = \frac{x^2}{2} - \frac{x^2}{4} = \frac{2x^2 - x^2}{4} = \frac{x^2}{4}$$

$$4y = x^2$$

$\therefore x^2 = 4y$ is the required solution.

2. Solve $x\frac{dy}{dx} = y + \log\frac{dy}{dx}$.

Solution: The given equation is $x\frac{dy}{dx} = y + \log\frac{dy}{dx}$

$$y = x.\frac{dy}{dx} - \log\frac{dy}{dx}$$

$$\therefore y = xp - \log p \qquad (1)$$

Put $p = c$ in (1)

$\therefore y = xc - \log c$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0 = x - \frac{1}{p}$$

$$x = \frac{1}{p} \Rightarrow p = \frac{1}{x}$$

Substituting $p = \frac{1}{x}$ in (1) we get

$$y = x\left(\frac{1}{x}\right) - \log\left(\frac{1}{x}\right)$$

$$y = 1 - \log\left(\frac{1}{x}\right) = 1 - \log\left(x^{-1}\right)$$

$$y = 1 + \log x$$

$\therefore y = 1 + \log x$ is the required singular solution.

3. Solve $p = \log(px - y)$.

Solution: The given equation is $p = \log(px - y)$

$$e^p = px - y$$

$$\therefore y = px - e^p \quad (1)$$

Put $p = c$ in (1)

$\therefore\therefore y = xc - e^c$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0 = x - e^p$$
$$x = e^p \Rightarrow p = \log x$$

Substituting $p = \log x$ in (1) we get

$$y = x(\log x) - e^{\log x}$$
$$y = x(\log x) - x = x(\log x - 1)$$
$$y = x(\log x - 1)$$

$y = x(\log x - 1)$ is the required singular solution.

4. Solve $y = px + 2p^2$.

Solution: The given equation is $y = px + 2p^2$ (1)

Put $p = c$ in (1)

$y = cx + 2c^2$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0 = x + 4p$$
$$4p = -x \Rightarrow p = -\frac{x}{4}$$

Substituting $p = -\frac{x}{4}$ in (1) we get

$$y = x.\left(-\frac{x}{4}\right) + 2\left(-\frac{x}{4}\right)^2$$
$$y = -\frac{x^2}{4} + \frac{2x^2}{16} = -\frac{x^2}{4} + \frac{x^2}{8}$$
$$y = \frac{-2x^2 + x^2}{8} = \frac{-x^2}{8}$$
$$8y = -x^2 \Rightarrow x^2 = -8y$$

$x^2 = -8y$ is the required singular solution.

5. Solve $p = \ln(px - y)$.

Solution: The given equation is $p = \ln(px - y)$.

$$e^p = px - y$$

$$\therefore y = px - e^p \tag{1}$$

Put $p = c$ in (1)

$\therefore \therefore y = xc - e^c$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0 = x - e^p$$

$$x = e^p \Rightarrow p = \log x$$

Substituting $p = \log x$ in (1) we get

$$y = x(\log x) - e^{\log x}$$

$$y = x(\log x) - x = x(\log x - 1)$$

$$y = x(\log x - 1)$$

$y = x(\log x - 1)$ is the required singular solution.

6. Find the general solution and singular solution of the Clairaut's equation $p = \sin(y - xp)$.

Solution: The given equation is $p = \sin(y - xp)$.

$$y = xp + \sin^{-1} p \tag{1}$$

Put $p = c$ in (1)

$\therefore y = xc + \sin^{-1} c$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0 = x + \frac{1}{\sqrt{1-p^2}}$$

$$\frac{1}{\sqrt{1-p^2}} = -x$$

$$\frac{-1}{x} = \sqrt{1-p^2}$$

$$\frac{1}{x^2} = 1 - p^2$$

$$p^2 = 1 - \frac{1}{x^2}$$

$$p = \sqrt{1 - \frac{1}{x^2}}$$

Substituting $p=\sqrt{1-\frac{1}{x^2}}$ in (1) we get

$$y=x\left(\sqrt{1-\frac{1}{x^2}}\right)+\sin^{-1}\left(\sqrt{1-\frac{1}{x^2}}\right)$$

$$y=x\left(\frac{\sqrt{x^2-1}}{x}\right)+\sin^{-1}\left(\sqrt{\frac{x^2-1}{x^2}}\right)$$

$y=\left(\sqrt{x^2-1}\right)+\sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$ is the required singular solution.

7. Solve $y=px+\cos p$.

Solution: The given equation is $y=px+\cos p$ (1)

Put $p=c$ in (1)

$y=xc+\cos c$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (1) with respect to 'p'

$$0=x-\sin p$$

$$x=\sin p$$

$$p=\sin^{-1}x$$

Substituting $p=\sin^{-1}x$ in (1) we get

$$y=x.\sin^{-1}x+\cos\left(\sin^{-1}x\right)$$

$y=x\sin^{-1}x+\cos\left(\sin^{-1}x\right)$ is the required singular solution.

8. Solve $x^2(y-px)=yp^2$.

Solution: The given equation is $x^2(y-px)=yp^2$ (1)

Let $X=x^2, Y=y^2$

$dX=2xdx,\ dY=2ydy$

Now $P=\frac{dY}{dX}$

$$P=\frac{2ydy}{2xdx}$$

$$P=\frac{ydy}{xdx}$$

$$P=\frac{y}{x}\left(\frac{dy}{dx}\right)\Rightarrow P=\frac{y}{x}p$$

$$p=\frac{x}{y}P$$

Substituting $p=\frac{x}{y}P$ in (1)

$$x^2\left(y-\frac{x}{y}P.x\right)=y\left(\frac{x}{y}P\right)^2$$

$$x^2\left(y-\frac{x^2}{y}P\right)=y\frac{x^2}{y^2}P^2$$

$$y-\frac{x^2}{y}P=\frac{P^2}{y}$$

$$y\left(y-\frac{x^2}{y}P\right)=P^2$$

$$y^2-x^2P=P^2$$

$$Y-XP=P^2$$

$$\therefore Y=XP+P^2 \quad (2)$$

Put $P=c$ in (2)

$$Y=Xc+c^2$$

$y^2=cx^2+c^2$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (2) with respect to 'p'

$$0=X+2P$$

$$X=-2P$$

$$P=-\frac{X}{2}$$

Substituting $P=-\frac{X}{2}$ in (2) $Y=X\left(-\frac{X}{2}\right)+\left(-\frac{X}{2}\right)^2$

$$Y=-\frac{X^2}{2}+\frac{X^2}{4}=-\frac{X^2}{4}$$

$$4Y=-X^2$$

$$X^2=-4Y$$

$$(x)^2=-4(y)$$

$$x^4=-4y^2$$

$x^4=-4y^2$ is the required singular solution.

9. Solve $\sin y\cos^2 x=\cos^2 y\, p^2+\sin x\cos x\cos y\, p$.

Solution: The given equation is $\sin y\cos^2 x=\cos^2 y\, p^2+\sin x\cos x\cos y\, p$ (1)

Put $X=\sin x \quad Y=\sin y$

$dX=\cos x dx \quad dY=\cos y dy$

$$\text{Now } P = \frac{dY}{dX}$$

$$= \frac{\cos y\, dy}{\cos x\, dx}$$

$$= \frac{\cos y}{\cos x}\left(\frac{dy}{dx}\right)$$

$$P = \frac{\cos y}{\cos x} p$$

$$\therefore p = \frac{\cos x}{\cos y} P$$

Substituting $p = \frac{\cos x}{\cos y} P$ in (1) we get

$$\sin y \cos^2 x = \cos^2 y \left(\frac{\cos x}{\cos y} P\right)^2 + \sin x \cos x \cos y . \left(\frac{\cos x}{\cos y} P\right)$$

$$\sin y \cos^2 x = \cos^2 y . \frac{\cos^2 x}{\cos^2 y} P^2 + \sin x \cos x \cos y . \frac{\cos x}{\cos y} P$$

$$\sin y \cos^2 x = \cos^2 x\, P^2 + \sin x \cos^2 x\ P$$

Dividing by $\cos^2 x$ on both sides we get

$$\sin y = P^2 + P \sin x$$

$$Y = XP + P^2 \qquad (2)$$

Put $P = c$ in (2)

$$Y = Xc + c^2$$

$\sin y = c \sin x + c^2$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (2) with respect to 'p'

$$0 = X + 2P$$

$$2P = -X$$

$$P = -\frac{X}{2}$$

Substituting $P = -\frac{X}{2}$ in (2) we get

$$Y = X\left(-\frac{X}{2}\right) + \left(-\frac{X}{2}\right)^2$$

$$Y = -\frac{X^2}{2} + \frac{X^2}{4} = -\frac{X^2}{4}$$

$$4Y = -X^2$$

$$X^2 = -4Y$$

$$(\sin x)^2 = -4(\sin y)$$

$$\sin^2 x = -4 \sin y$$

$\sin^2 x = -4\sin y$ is the required singular Solution.

10. Solve $(px-y)(py+x)=2p$.

Solution: The given equation is $(px-y)(py+x)=2p$

$$p^2xy - py^2 + px^2 - xy = 2p \qquad (1)$$

$$\text{Put } X = x^2 \quad Y = y^2$$

$$dX = 2xdx \quad dY = 2ydy$$

$$\text{Now } P = \frac{dY}{dX}$$

$$P = \frac{2ydy}{2xdx}$$

$$P = \frac{ydy}{xdx}$$

$$P = \frac{y}{x}\left(\frac{dy}{dx}\right) \Rightarrow P = \frac{y}{x}p$$

$$\therefore p = \frac{x}{y}P$$

Substituting $p = \frac{x}{y}P$ in (1) we get

$$\left(\frac{x}{y}P\right)^2 xy - \frac{x}{y}P.y^2 + \frac{x}{y}P.x^2 - xy = 2\frac{x}{y}P$$

$$\frac{x^3}{y}P^2 - xyP + \frac{x^3}{y}P - xy = 2\frac{x}{y}P$$

Multiplying by $\frac{y}{x}$ we get

$$x^2P^2 - y^2P + x^2P - y^2 = 2P$$

$$XP^2 - YP + XP - Y = 2P$$

$$X\left(P^2+P\right) - 2P = Y(1+P)$$

$$\frac{X\left(P^2+P\right)-2P}{(1+P)} = Y$$

$$\frac{XP(P+1)}{(1+P)} - \frac{2P}{1+P} = Y$$

$$\therefore Y = XP - \frac{2P}{1+P} \qquad (2)$$

Put $P = c$ in (2)

$$Y = Xc - \frac{2c}{1+c}$$

$y^2 = x^2c - \frac{2c}{1+c}$ is the required complete solution of equation (1)

To find the Singular Solution:

Differentiating (2) with respect to 'p'

$$0 = X - \left(\frac{(1+P).2 - 2P.1}{(1+P)^2}\right)$$

$$0 = X - \left(\frac{2+2P-2P}{(1+P)^2}\right)$$

$$X = \frac{2}{(1+P)^2}$$

$$(1+P)^2 = \frac{2}{X} \Rightarrow 1+P = \sqrt{\frac{2}{X}}$$

$$1+P = \frac{\sqrt{2}}{\sqrt{x}}$$

$$P = \frac{\sqrt{2}}{\sqrt{x}} - 1$$

$$P = \frac{\sqrt{2}-\sqrt{x}}{\sqrt{x}}$$

Substituting $P = \frac{\sqrt{2}-\sqrt{x}}{\sqrt{x}}$ in (2)

$$Y = X\left(\frac{\sqrt{2}-\sqrt{x}}{\sqrt{x}}\right) - \frac{2\left(\frac{\sqrt{2}-\sqrt{x}}{\sqrt{x}}\right)}{1+\frac{\sqrt{2}-\sqrt{x}}{\sqrt{x}}}$$

$$= X\left(\frac{\sqrt{2}-\sqrt{x}}{\sqrt{x}}\right) - \frac{2(\sqrt{2}-\sqrt{x})}{\sqrt{2}}$$

$$Y = X\left(\sqrt{2x} - x\right)$$

EXERCISE 4.6

1. Solve $y = px + \frac{a}{p}$.
2. Solve $y = px + ap(1-p)$.
3. Solve $y = px + \sqrt{a^2p^2 + b^2}$.
4. Solve $y = px + p^3$.
5. Solve $(px^2 + y^2)(px + y) = (p+1)^2$.
6. Solve $y = 3px + 6p^2y^2$.

7. Solve $p^3x - p^2y - 1 = 0$.
8. Solve $p = \sin(y - px)$.
9. Solve $(y + px)^2 = x^2 p$.
10. Solve $x^2(y - px) = yp^2$.
11. Solve $e^{3x}(p-1) + p^3 e^{2y} = 0$.
12. Solve $(px - y)(py + x) = 2p$.

ANSWERS TO EXERCISE 4.6

1. $y = cx + \frac{a}{c}$
2. $y = cx + ac(1-c)$
3. $y = cx + \sqrt{a^2c^2 + b^2}$
4. $y = cx + c^3$
5.
6. $y^3 = cx + \frac{2}{3}c^2$
7. $y = cx - \frac{1}{c^2}$
8. $y = cx + \sin^{-1} c$
9. $xy = cx - c^2$
10. $e^y = ce^x + c^2$
11. $e^y = ce^x + c^2$
12. $y^2 = cx^2 - \frac{2c}{c+1}$

SHORT ANSWER QUESTIONS

1. When do you say that the first order differential equation is in variables separable form?

Ans: The equation of the form $F(x)G(y)dx + f(x)g(y)dy = 0$ (1) is called an equation with Variable separable (or) simply separable equation, because the variables x and y can be separated.

2. Solve $(1-x)dy + (1-x)dx = 0$.

Solution: The given equation is $(1-x)dy + (1-y)dx = 0$ (1)

$$(1-x)dy = -(1-y)dx$$

$$\frac{dy}{1-y} = -\frac{dx}{1-x}$$

Integrating on both sides

$$\int \frac{dy}{1-y} = -\int \frac{dx}{1-x}$$

$$-\log(1-y) = \log(1-x) + \log c_1$$

$$-\log(1-y) - \log(1-x) = \log c_1$$

$$-\log[(1-x)(1-y)] = \log c_1$$

$(1-x)(1-y) = c$ is the required Solution.

3. What do you mean by exact differential equation?
Ans: A differential equation which is obtained from its primitive by mere differentiation without any further operation is called exact differential equation.

4. State the necessary and sufficient condition for exactness.
Ans: The necessary and sufficient condition for a differential equation $Mdx + Ndy = 0$ is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

5. Determine whether the equation $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ is exact (or) not.

Solution: The given equation is $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

It is of the form $Mdx + Ndy = 0$

Here $M = (e^y + 1)\cos x$, $N = e^y \sin x$

Now $\frac{\partial M}{\partial y} = e^y \cos x$ and $\frac{\partial N}{\partial x} = e^y \cos x$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Hence the given equation is exact.

6. Write down the Solution of exact differential equation.
Ans: The Solution of the exact differential equation $Mdx + Ndy = 0$ is given by $\int M\,dx + \int (\text{Terms of } N \text{ not containg } x)\,dy = c$.

7. What do you mean by integrating factors?
Ans: A factor, which when multiplied to a non-exact differential equation makes it exact, is known as integrating factor.

8. Identify the Integrating factor of the differential equation $(x^2 + y^2 + x)dx + xy\,dy = 0$.
Solution: The given equation is $(x^2 + y^2 + x)dx + xydy = 0$.
The given equation is not exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - y) = \frac{1}{x}$

Since $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ is a function of x, the integrating factor is given by $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Thus integrating factor of given equation is x.

9. Determine the Integrating factor of the following differential equation. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$.

Solution: The given equation is $(x^2y^2 + xy + 1)dx + (x^2y^2 - xy + 1)dy = 0$

The given equation is not exact. (1)

Equation (1) is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$

Here $M = (x^2y^2 + xy + 1)y$, $N = (x^2y^2 - xy + 1)x$

Now $M.x - N.y = (x^2y^2 + xy + 1)y.x - (x^2y^2 - xy + 1)x.y$

$$= x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy = 2x^2y^2 \neq 0$$

Hence the integrating factor $= \frac{1}{M.x - N.y} = \frac{1}{2x^2y^2}$.

10. Define first order linear differential equation.

Ans: The differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are function of x only is called first order linear differential equation.

11. What do you meant by integrating factor of first order linear differential equation.

Ans: The value of the term $e^{\int P\,dx}$ is known as the integrating factor of first order linear differential equation.

10. Write the steps involved for solving first order linear differential equation.

Ans: The following are the steps for solving first order linear differential equation.

(i) First identify P and Q from the given differential equation.

(ii) Evaluate $\int P\,dx$

(iii) Determine the integrating factor $e^{\int P\,dx}$

(iv) Finally solution of given equation is obtained by solving $ye^{\int P\,dx} = \int Q e^{\int P\,dx} dx + c$

11. Find the integrating factor of $\frac{dy}{dx} + y\tan x = \cos^3 x$.

Solution: The given equation is $\frac{dy}{dx} + y\tan x = \cos^3 x$

Here $P = \tan x$

Now $\int P dx = \int \tan x\, dx = \log(\sec x)$

$e^{\int P\,dx} = e^{\log(\sec x)} = \sec x$

Thus integrating factor = $e^{\int P\,dx} = \sec x$

12. Solve $y' + y = \sin x$.

Solution: The given equation is $y' + y = \sin x$ **(1)**

Here $P = 1, Q = \sin x$

Now $\int P\,dx = \int dx = x$

$e^{\int P dx} = e^x$

The Solution of equation (1) is $ye^{\int P\,dx} = \int Q e^{\int P\,dx} dx + c$

$$ye^x = \int \sin x\, e^x dx$$

$$ye^x = \frac{e^x}{2}(\sin x - \cos x) + c$$

$\therefore y = \frac{(\sin x - \cos x)}{2} + ce^{-x}$ is the required Solution.

13. Write Bernoulli's equation.

Ans: A first order first degree differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where n is any constant other than 0 and 1 is called Bernoulli's equation.

14. Explain the method of finding solution to Bernoulli's equation.

Ans: The following are the steps involved in finding the solution to Bernoulli's equation.

Step – 1: Rewrite the given equation in standard Bernoulli's form. Identify n

Step – 2: Introduce a new variable $z = y^{1-n}$ and using it reduce the given equation to first order linear differential equation in z.

Step – 3: Solve the first order linear differential equation in z as before.

Step – 4: In the resulting solution replace z by y^{1-n}, we get the required Solution of given Bernoulli's equation.

15. Write the Clairaut's first order differential equation.

Ans: Clairaut's equation is a first order and first degree differential equation is of the form $y = x.\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$ (or) $y = xp + f(p)$, where $p = \frac{dy}{dx}$.

16. What do you mean by complete solution of Clairaut's equation?

Ans: The complete solution of the Clairaut's equation is obtained by substituting $p = c$. The Complete solution is also called as General solution.

17. Define general (or) singular solution of Clairaut's equation.

Ans: The general (or) singular solution is obtained by solving (or) eliminating the parameter p from the following equations.

$$y = xp + f(p) \tag{1}$$

$$x + \frac{df}{dp} = 0 \tag{2}$$

18. Rewrite the equation $(y - px)(p-1) = p$ into Clairaut's form.

Solution: The given equation is $(y - px)(p-1) = p$ (1)

$$(y - px) = \frac{p}{(p-1)}$$

$\Rightarrow y = px + \frac{p}{(p-1)}$ is the required Clairaut's form.

19. Find the complete integral of $p = \ln(px - y)$.

Solution: The given equation is $p = \ln(px - y)$

$$\Rightarrow e^p = px - y \quad \therefore y = px - e^p \tag{1}$$

Put $p = c$ in (1)

$\therefore \therefore y = xc - e^c$ is the required complete solution of equation (1)

20. Obtain the general Solution of $y = px + \sin^{-1} p$.

Solution: The given equation is $y = px + \sin^{-1} p$ (1)

Differentiating (1) with respect to 'p'

$$0 = x + \frac{1}{\sqrt{1-p^2}}$$

$$\frac{1}{\sqrt{1-p^2}} = -x$$

$$\frac{-1}{x} = \sqrt{1-p^2}$$

$$\frac{1}{x^2} = 1 - p^2$$

$$p^2 = 1 - \frac{1}{x^2}$$

$$p = \sqrt{1 - \frac{1}{x^2}}$$

Substituting $p=\sqrt{1-\frac{1}{x^2}}$ in (1) we get

$$y=x\left(\sqrt{1-\frac{1}{x^2}}\right)+\sin^{-1}\left(\sqrt{1-\frac{1}{x^2}}\right)$$

$$y=x\left(\frac{\sqrt{x^2-1}}{x}\right)+\sin^{-1}\left(\sqrt{\frac{x^2-1}{x^2}}\right)$$

$y=\left(\sqrt{x^2-1}\right)+\sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$ is the required general solution of (1).