University of Dhaka
Department of Theoretical Physics
Final Examination 2017
Course No. : TPG 565
Course Name: Quantum Mechanics - I

Answer any five questions
Full marks: 70 Time: 3 hours

1. **(a)** Discuss the postulates of quantum mechanics. **(5)**

   (b) Consider a two-state system with Hamiltonian

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

and another operator

$$S = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

(i) Write down the normalized eigenvectors $\psi_{1,2}$ of H. Show that the normalized vectors $\phi_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\phi_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are eigenstates of S. Find the corresponding eigenvalues and express $\phi_{1,2}$ in terms of $\psi_{1,2}$. Furthermore, find expressions for $\psi_{1,2}$ in terms of $\phi_{1,2}$ (this will be needed in the questions below). **(4)**

(ii) Now a sequence of measurements is performed on the system. In the following ignore the time evolution of the system and only take into account the collapse of the wave function after each measurement:
A beam of particles with energy E_1 is prepared. Write down the state of these particles. The beam is sent through a first apparatus that measures S: obtain the possible outcomes, their probabilities and the corresponding states after the measurement.
All particles that come out of the first apparatus are sent through a second apparatus that measures the energy of the particles. Find the probability to finding E_1 again. **(5)**

2. **(a)** Given that $\hat{A}$ and $\hat{B}$ are Hermitian operators and that

$$[\hat{A}, \hat{B}] = i\hat{C}$$

show that

$$\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle| .$$

Here, for instance, $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ with $\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle$ and $\langle A^2 \rangle = \langle \psi | \hat{A}^2 | \psi \rangle$. **(8)**

(b) Show that the equality sign in the above generalized uncertainty relation holds if the state $|\psi\rangle$ in question satisfies

$$[\hat{A} - \langle A \rangle] |\psi\rangle = \lambda [\hat{B} - \langle B \rangle] |\psi\rangle$$

with λ purely imaginary. **(6)**

3. The Hamiltonian operator of a one-dimensional harmonic oscillator is $\hat{H} = \hat{p}^2/2m + \frac{1}{2} m\omega^2 \hat{x}^2$. Two non-Hermitian operators are also defined:

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right) . \end{aligned}$$

Let $|0\rangle$ and $|1\rangle$ represent the energy eigenstates of the ground level and the first excited state, respectively, of the oscillator.

(a) Using the action of the lowering operator $\hat{a}$ on $|0\rangle$, calculate the normalized wave function of the ground state in position representation, i.e., $\langle x|0\rangle$. **(4)**

(b) Using the raising operator $\hat{a}^\dagger$ on $|0\rangle$, calculate the normalized wave function of the first excited state in position representation. **(4)**

(c) Let $|n\rangle$ be the energy eigenstate of the oscillator corresponding to the n^{th} level ($n \geq 2$). Find the uncertainty product $\Delta x \Delta p$ for the state. **(6)**

Given: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$

4. This question considers oscillations between electron neutrinos ν_e and muon neutrinos ν_μ. We assume that the neutrinos are so light that we can use the following relation between the energy E, momentum p and mass m:

$$E = \sqrt{p^2c^2 + m^2c^4} \approx pc + \frac{m^2c^4}{2pc}.$$

Let $\hat{H}$ be the Hamiltonian operator of a free neutrino with momentum p and let $|\nu_1\rangle$ and $|\nu_2\rangle$ the two eigenvectors of $\hat{H}$:

$$\hat{H}|\nu_j\rangle = E_j|\nu_j\rangle, \quad E_j = pc + \frac{m_j^2 c^4}{2pc}, \quad j = 1, 2.$$

Here m_1 and m_2 are the masses of the eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$. We assume that m_1 and m_2 are not equal. Neutrino oscillations are due to a quantum mechanical effect whereby detected neutrinos are neither in the state $|\nu_1\rangle$ nor $|\nu_2\rangle$, but instead linear combinations of these two states:

$$|\nu_e\rangle = |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta, \quad |\nu_\mu\rangle = -|\nu_1\rangle \sin\theta + |\nu_2\rangle \cos\theta.$$

Here θ is the so called mixing angle, which must be experimentally determined.

(a) At time $t = 0$ a neutrino in state $|\nu_e\rangle$ with momentum p is created. Calculate the state $|\nu(t)\rangle$ at time t in the basis $|\nu_1\rangle$ and $|\nu_2\rangle$, i.e., write $|\nu(t)\rangle$ as a linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$. **(2)**

(b) The likelihood, $P_e(t)$, that a neutrino at time t is found in state $|\nu_e\rangle$ is given by $P_e(t) = |\langle\nu_e|\nu(t)\rangle|^2$. Show that

$$P_e(t) = 1 - \sin^2 2\theta \sin^2\left(\frac{\pi ct}{L}\right)$$

with the corresponding oscillation length scale $L = \frac{4\pi\hbar p}{|\Delta m^2|c^2}$ and $\Delta m^2 = m_1^2 - m_2^2$. Plot $P_e(t)$ against t. **(8 + 2)**

(c) Calculate the oscillation length L for an energy $E \approx pc = 4$ MeV (average energy of reactor neutrinos) and a mass difference $\Delta m^2 c^4 = 10^{-4}$ eV^2. Given $\hbar c = 197$ MeV/fm. **(2)**

5. **(a)** Write down the commutation relations between the angular momentum operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$. Show that

$$[\hat{J}^2, \hat{J}_i] = 0\,,$$

for $i = x, y$ and z, where $\hat{J}^2$ is defined as $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$. **(2+2)**

(b) Choosing $\{\hat{J}^2, \hat{J}_z\}$ as a complete set of commuting observables, let the normalized simultaneous eigenvectors be $|\lambda m\rangle$ where

$$\begin{aligned} \hat{J}^2|\lambda m\rangle &= \lambda\hbar^2|\lambda m\rangle\,, \\ \hat{J}_z|\lambda m\rangle &= m\hbar|\lambda m\rangle\,. \end{aligned}$$

(i) Let $\hat{J}_\pm = \hat{J}_x \pm \hat{J}_y$. Show that $\hat{J}_\pm|\lambda m\rangle$ is an eigenvector of $\hat{J}^2$ with eigenvalue $\lambda\hbar^2$ and an eigenvector of $\hat{J}_z$ with eigenvalue $(m \pm 1)\hbar$. **(2)**

(ii) Prove that $\langle\lambda m|\hat{J}_+\hat{J}_-|\lambda m\rangle \geq 0$ and that $\langle\lambda m|\hat{J}_+\hat{J}_-|\lambda m\rangle \geq 0$. **(2)**

(iii) Using the results of **(i)** and **(ii)**, prove that the possible values of λ are $\lambda = j(j+1)$ where j is any non-negative half-integer: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.
You may use the formulas

$$\begin{aligned} \hat{J}_+\hat{J}_- &= \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z, \\ \hat{J}_-\hat{J}_+ &= \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z. \end{aligned}$$

What are the possible values of m for each j? **(6)**

6. An electron under the influence of a uniform magnetic field $\vec{B} = B\hat{j}$ has its spin initially (i.e., at $t = 0$) pointing in the positive x-direction. That is, the initial state is an eigenstate of $\hat{S}_x = (\hbar/2)\sigma_x$ with eigenvalue $+\hbar/2$. Considering only the interaction of the magnetic dipole moment due to spin and the magnetic field, the Hamiltonian operator is given by

$$H = \frac{1}{2}\epsilon\sigma_y$$

where ϵ is real and has the dimension of energy.

(a) Find $|\chi(0)\rangle$, the initial spin state of the electron as a two-component column matrix in the basis in which S_z is diagonal. **(3)**

(b) Show that the time evolution operator $U(t)$ of the state vector can be written as

$$U(t) \equiv \exp\left(-\frac{i}{\hbar}tH\right) \doteq \begin{bmatrix} \cos(\epsilon t/2\hbar) & -\sin(\epsilon t/2\hbar) \\ \sin(\epsilon t/2\hbar) & \cos(\epsilon t/2\hbar) \end{bmatrix}$$

(4)

(c) Hence or otherwise, find $|\chi(t)\rangle$ for $t > 0$. **(2)**

(d) Show that the probability of finding the electron with its spin pointing in the positive z-direction is

$$P(t) = \frac{1}{2}\left(1 - \sin\frac{\epsilon t}{\hbar}\right).$$

(2)

(e) Hence, or otherwise, find the expectation value $\langle S_z \rangle$. Does it depend on time t? Explain. **(3)**

Hint: Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i^2 = 1_{2\times 2}, \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

7. **(a)** Consider the elastic scattering of a spinless particle of mass m from a fixed potential. Derive the partial wave expansion of the scattering amplitude and hence work out the formula for the integrated elastic cross section. **(8)**

   You can assume without proof

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta).$$

   (b) Derive the phase shift in all partial waves for a particle of mass m being scattered from a hard sphere of radius a. Hence derive a formula for the elastic cross section in the high energy limit. **(6)**

8. The Lippmann-Schwinger equation in the basis independent form (operator form) is written as

$$|\psi^{\pm}\rangle = |\phi\rangle + G_0^{\pm}(E) V |\psi^{\pm}\rangle$$

where $G_0^{\pm}(E)$ are the "free" Green's operators (or Green's functions) defined as the inverse of the operators $E - H_0 \pm i\epsilon$, i.e.,

$$G_0^{\pm} = \frac{1}{E - H_0 \pm i\epsilon}.$$

(a) Let us define the "full" Green's operator $G^{\pm}(E)$ as

$$G^{\pm}(E) = \frac{1}{E - H \pm i\epsilon}.$$

(i) Prove the identities **(3)**

$$G^{\pm}(E) = G_0^{\pm}(E) + G_0^{\pm}(E) V G^{\pm}(E) = G_0^{\pm}(E) + G^{\pm}(E) V G_0^{\pm}(E).$$

(ii) Show that

$$|\psi^{\pm}\rangle = |\phi\rangle + G^{\pm}(E)V|\phi\rangle$$

is a formal solution of the Lippmann-Schwinger equation. **(3)**

(b) In the coordinate representation, the matrix elements of $G_0^{\pm}(E)$ are

$$\langle\vec{r}\,|G_0^{\pm}(E)|\vec{r}\,'\rangle = -\frac{2m}{\hbar^2}\,\frac{e^{\pm ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}.$$

Write down the integral equation (i.e., the Lippmann-Schwinger equation) for the wavefunction $\psi^{+}(\vec{r}\,)$ with outgoing boundary condition, and derive an exact integral expression for the scattering amplitude. What is the scattering amplitude in the Born approximation?

(6+2)

End of questions