University of Dhaka
Department of Theoretical Physics
Final Examinations (session 2015 -2016)
Course No. : 505
Course Name: Quantum Mechanics

Answer any five questions
Full marks: 70 Time: 3 hours

1. The three questions **(a), (b)** and **(c)** are independent.

   (a) Using the fundamental commutation relation $[\hat{x}, \hat{p}] = i\hbar\hat{I}$ show that
   $$\langle x|\hat{p}|x'\rangle = -i\hbar\delta'(x - x')$$
   where the prime denotes derivative of the delta function with respect to its arguments. Hence show
   $$\langle x|\hat{p} = -i\hbar\frac{\partial}{\partial x}\langle x| \ .$$
   (4)

   (b) The translation operator $\hat{T}(a)$ is defined as
   $$\hat{T}(a)|x\rangle = |x + a\rangle \ .$$
   Show that $\hat{T}(a)$ is a unitary operator and also that
   $$\hat{T}(a) = e^{-ia\hat{p}/\hbar} \, ,$$
   where $\hat{p}$ is the momentum operator. **(4)**

   (c) Consider a two-dimensional vector space spanned by two orthonormal state vectors $|0\rangle$ and $|1\rangle$. An operator $\hat{A}$ is expressed as
   $$\hat{A} = |0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| - |1\rangle\langle 1| \ .$$
   (i) Is the operator Hermitian? **(2)**
   (ii) Find the eigenvalues and eigenvectors of $\hat{A}$ **(4)**

2. The Hamiltonian operator of a one-dimensional harmonic oscillator is $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$. Two non-Hermitain operators are also defined:

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) . \end{aligned}$$

Let $|0\rangle$ and $|1\rangle$ represent the energy eigenstates of the ground level and the first excited state, respectively, of the oscillator.

(a) Using the action of the lowering operator $\hat{a}$ on $|0\rangle$, calculate the normalized wave function of the ground state in position representation, i.e., $\langle x|0\rangle$. **(4)**

(b) Using the raising operator $\hat{a}^\dagger$ on $|0\rangle$, calculate the normalized wave function of the first excited state in position representation. **(4)**

(c) Let $|n\rangle$ be the energy eigenstate of the oscillator corresponding to the n^{th} level ($n \geq 2$). Find the uncertainty product $\Delta x \Delta p$ for the state. **(6)**

Given: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$

3. **(a)** Show that the time evolution operator $\hat{T}(t, t_0)$ for the state kets $|\psi(t)\rangle$ in the Schrödinger picture satisfies the equation

$$i\hbar \frac{\partial}{\partial t}\hat{T}(t, t_0) = \hat{H}(t)\hat{T}(t, t_0) .$$

Convert the above differential equation into an integral equation incorporating the appropriate boundary condition for $\hat{T}(t, t_0)$ and derive the Dyson series for $\hat{T}(t, t_0)$ in terms of the time-ordered products of $\hat{H}(t)$ if $\hat{H}$ at two different instants of time do not commute. **(2+4)**

(b) For a conservative system (i.e., H is time independent) show that the time evolution of an operator in the Heisenberg picture is given by

$$i\hbar \frac{dA_{\text{H}}(t)}{dt} = [A_{\text{H}}(t), H]$$

where the the operator in the Schrödinger picture is time-independent. **(4)**

(c) The Hamiltonian of a particle of mass m moving under the influence of gravity is given by

$$\hat{H} = \frac{\hat{p}_{\mathrm{z}}^2}{2m} + mg\hat{z}$$

where g is the acceleration of free fall. Find $\hat{z}_{\mathrm{H}}(t)$ as a function of time for the initial condition $\hat{z}_{\mathrm{H}}(0) = z_0$ and $\hat{p}_{z,H}(0) = 0$. **(4)**

4. **(a)** Given that $|a, b\rangle$ is a simultaneous eigenket of J^2 and J_z with eigenvalues a and b, respectively, prove that $J_{\pm}|a, b\rangle$ are simultaneous eigenkets of J^2 and J_z with eigenvalues a and $b \pm \hbar$, where $J_{\pm} = (J_x \pm iJ_y)$. Give a physical interpretation of $J_{\pm}$. **(6)**

   (b) Find all the eigenstates for a system of two particles each having angular momentum $j_1 = j_2 = 1/2$ and calculate the corresponding Clebsch-Gordan coefficients. **(8)**

5. **(a)** If a system is rotated by an angle ϕ (in the positive sense), the spin state $|\alpha\rangle$ of the system changes as

$$|\alpha\rangle_{\mathrm{R}} = e^{-i\hat{S}_z\phi/\hbar}|\alpha\rangle\ .$$

   Show that under the above transformation the expectation values $\langle\hat{S}_x\rangle$, $\langle\hat{S}_y\rangle$ and $\langle\hat{S}_z\rangle$ transform like the components of a classical vector. **(5)**

   (b) Consider an electron with its space degrees of freedom suppressed. If the electron is subjected to an external, uniform and time-independent magnetic field of flux density B pointed along the z-axis, show that the spin of the electron precesses around the z-axis with an angular frequency $\omega = eB/m$ where e is the magnitude of charge of the electron and m is its mass. **(5)**

   (c) Suppose that at $t = 0$ the spin of the electron is aligned along the x-axis, i.e., the initial spin state is

$$|\alpha, 0\rangle = |S_x+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)\ ,$$

   where $|+\rangle$ and $|-\rangle$ are, respectively, the eigenstates of $\hat{S}_z$ with eigenvalues $\hbar/2$ and $-\hbar/2$. Show that the probability of getting $\hbar/2$ in a measurement of $\hat{S}_x$ at time t is $\cos^2(\omega t/2)$ **(4)**

6. **(a)** Show that for any normalized state $|\psi\rangle$ the following inequality holds:

$$\langle\psi|\hat{H}|\psi\rangle \geq E_0 \,,$$

where E_0 is the ground-state energy (i.e., the lowest energy eigenvalue). Also show that the equality holds only for the case where $|\psi\rangle$ is the actual ground-state eigenket of $\hat{H}$. Hence discuss the variational method for estimating the ground-state energy of a quantum system. Also discuss how this method can be adapted to estimate the energy of an excited state. **(3+1+2+2)**

(b) We would like to estimate the energy of the first excited state of a one-dimensional harmonic oscillator using the trial wave function

$$\psi(x) = Bxe^{-\beta x^2/2} \,,$$

where B is the normalization constant and β is the variational parameter.

(i) The choice $Be^{-\beta x^2/2}$ for the trial wave function would be unsuitable for this problem. Discuss why. **(2)**

(ii) Using the correctly chosen trial wave function, estimate the energy of the first excited state of the oscillator. **(4)**

Given:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\beta x^2} dx = \frac{1 \times 2 \times 3 \times 3 \cdots \times (2n-1)}{2^n \beta^n} \sqrt{\frac{\pi}{\beta}}$$

7. **(a)** A system is described by the time-independent Hamiltonian of the form

$$H = H_0 + V$$

where H_0 is the unperturbed part of the Hamiltonian whose eigenstates and eigenvalues are known and V is the perturbing potential. Find expressions for the first-order corrections to the energy and to the wave function of a non-degenerate level of H_0. **(8)**

(b) Calculate the shift in the ground-state energy of the hydrogen atom if the proton is considered a uniform sphere of radius R instead of a point charge. You can take $R << a_0$ where a_0 is the Bohr radius. **(6)**

Ground-state wave function of the hydrogen atom considering the proton to be a point particle is:
$\psi_{1s} = \frac{2}{\sqrt{4\pi} a_0^{3/2}} e^{-r/a_0}$ (a_0 is the Bohr radius).

8. Questions **(a)** and **(b)** are independent of each other.

(a) A spin-less particle of mass m is elastically scattered from a fixed potential. The radial wave function $R_l(r)$ of the particle in the l^{th} partial wave is 'normalized' such that as r tends to infinity

$$R_l(r) \sim \frac{\sin(kr - l\pi/2 + \delta_l)}{kr}$$

where δ_l is the phase shift. Show that

$$\sin\delta_l = -k\int_0^\infty r^2 j_l(kr)U(r)R_l(r)dr$$

where $U(r) = \frac{2m}{\hbar^2}V(r)$. Hence derive the Born-approximation formula for the phase shift in the case of a weak potential. Explain with the help of a diagram why an attractive (negative) potential produces a positive phase shift. **(5+1+2)**

(b) In the Born approximation, find the scattering amplitude and the differential cross section for elastic scattering of a particle from the screened Coulomb potential

$$V(r) = \frac{Ze^2}{r}e^{-r/a}\ .$$

Taking appropriate limit of your answer find also the Rutherford formula for scattering from a point Coulomb potential. **(4+2)**

End of questions