University of Dhaka
Department of Theoretical Physics
Final Examination 2016
Course No. : 565
Course Name: Quantum Mechanics

Answer any five questions
Full marks: 70 Time: 3 hours

1. **(a)** Let $\hat{U}$ be a unitary operator. Consider the eigenvalue equation

$$\hat{U}|\lambda\rangle = \lambda|\lambda\rangle.$$

Prove that λ is of the form $e^{i\theta}$ with θ real. Show also that if $\lambda \neq \mu$, then $\langle\lambda|\mu\rangle = 0$. **(2+2)**.

(b) For any two linear operators $\hat{A}$ and $\hat{B}$ prove Weyl's identity, i.e.,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}.$$

(5)

(c) Using the fundamental commutation relation $[\hat{x},\hat{p}] = i\hbar\hat{I}$ and the identity $q\delta'(q) = -\delta(q)$, show that

$$\langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p-p')$$

i.e., $\langle p|\hat{x} = i\hbar\frac{\partial}{\partial p}\langle p|$. Hence express the Schrödinger equation for a one-dimensional linear harmonic oscillator in momentum representation. **(3+2)**

2. **(a)** Prove that a complete basis of simultaneous eigenvectors of two Hermitian operators $\hat{A}$ and $\hat{B}$ can be found if and only if $[\hat{A},\hat{B}] = 0$. **(6)**

(b) Briefly discuss how we can label the basis states of a quantum mechanical system by the eigenvalues of a complete set of commuting observables. **(2)**

(c) Consider two observables $\hat{\Lambda}$ and $\hat{\Omega}$ on $V^3(R)$ (three-dimensional real vector space) such that $[\hat{\Lambda},\hat{\Omega}] = 0$. The eigenvectors of $\hat{\Omega}$ are ω_1, ω_2 and ω_3 and they are distinct. The eigenvalues of $\hat{\Lambda}$ are λ_1, λ_2 and λ_3 where $\lambda_1 = \lambda_2 = \lambda$. The common orthonormal eigenvectors of the two operators are $|\omega_3\lambda_3\rangle$, $|\omega_1\lambda\rangle$ and $|\omega_2\lambda\rangle$, which we use as the basis. Consider a normalized state

$$|\psi\rangle = \alpha|\omega_3\lambda_3\rangle + \beta|\omega_1\lambda\rangle + \gamma|\omega_2\lambda\rangle$$

where α, β and γ are real numbers. Let us first measure $\hat{\Lambda}$ and immediately afterward we measure $\hat{\Omega}$.

(i) What is the probability that the result λ will obtained in the first measurement? If λ is actually obtained, what is the normalized state of the system immediately after the measurement? **(2)**

(ii) Next we measure $\hat{\Omega}$. What is the probability that the measurement will yield ω_1 and what is the state of the system immediately after the measurement if ω_1 is obtained? **(2)**

(iii) What is the probability $P(\lambda, \omega_1)$ for obtaining λ followed by ω_1. If we reversed the order of measurements what would be $P(\omega_1, \lambda)$? **(2)**

3. **(a)** Given that $\hat{A}$ and $\hat{B}$ are Hermitian operators and that

$$[\hat{A}, \hat{B}] = i\hat{C}$$

show that

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|.$$

Here, for instance, $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$ with $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$ and $\langle A^2\rangle = \langle\psi|\hat{A}^2|\psi\rangle$. **(6)**

(b) Show that the equality sign in the above generalized uncertainty relation holds if the state $|\psi\rangle$ in question satisfies

$$[\hat{A} - \langle A\rangle]|\psi\rangle = \lambda[\hat{B} - \langle B\rangle]|\psi\rangle$$

with λ purely imaginary. **(4)**

(c) A free particle moving in one dimension is initially (time $t = 0$) in the state $|\psi\rangle$ that satisfies the minimum uncertainty relation

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

Show that the initial momentum space wave function is given by

$$\langle p|\psi\rangle \equiv \tilde{\psi}(p) = N \exp\left[-\frac{1}{2\hbar a}(p - \langle p\rangle)^2\right] \exp\left(-\frac{i}{\hbar}\langle x\rangle p\right).$$

Here N is the normalization constant which you need not determine and a is a real quantity with $\lambda = -i/a$. **(4)**

4. An electron under the influence of a uniform magnetic field $\vec{B} = B\hat{\jmath}$ has its spin initially (i.e., at $t = 0$) pointing in the positive x-direction. That is, the initial state is an eigenstate of $\hat{S}_x = (\hbar/2)\sigma_x$ with eigenvalue $+\hbar/2$. Considering only the interaction of the magnetic dipole moment due to spin and the magnetic field, the Hamiltonian operator is given by

$$H = \frac{1}{2}\epsilon\sigma_y$$

where ϵ is real and has the dimension of energy.

(a) Find $|\chi(0)\rangle$, the initial spin state of the electron as a two-component column matrix in the basis in which S_z is diagonal. **(3)**

(b) Show that the time evolution operator $U(t)$ of the state vector can be written as

$$U(t) \equiv \exp\left(-\frac{i}{\hbar}tH\right) = \begin{bmatrix}\cos(\epsilon t/2\hbar) & -\sin(\epsilon t/2\hbar) \\ \sin(\epsilon t/2\hbar) & \cos(\epsilon t/2\hbar)\end{bmatrix}$$

(4)

(c) Hence or oherwise, find $|\chi(t)\rangle$ for $t > 0$. **(2)**

(d) Show that the probability of finding the electron with its spin pointing in the positive z-direction is

$$p = \frac{1}{2}\left(1 - \sin\frac{\epsilon t}{\hbar}\right).$$

(2)

(e) Hence, or otherwise, find the expectation value $\langle S_z\rangle$. Does it depend on time t? Explain. **(3)**

Hint: Pauli matrices are

$$\sigma_x = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

5. **(a)** Consider two angular momentum operators $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$. The total angular momentum operator is defined by $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$. Show that the allowed quantum number j associated with the square of the total angular momentum $\hat{\vec{J}}^2$ is given by

$$j = j_1 + j_2, j_1 + j_2 - 1, \cdots, |j_1 - j_2|$$

where j_i is the quantum number associated with the square of the angular momentum $\hat{\vec{J}}_i$, $i = 1, 2$. **(8)**

(b) Show that the Clebsch-Gordan coefficients have the following property

$$\langle j_1\, j_2\, m_1\, m_2 | jm \rangle = 0 \;\text{ unless } m_1 + m_2 = m.$$

Deduce the orthogonality properties of the CG coefficients. **(2+4)**

6. **(a)** A system is described by the time-independent Hamiltonian of the form

$$H = H_0 + V$$

where H_0 is the unperturbed part of the Hamiltonian whose eigenstates and eigenvalues are known and V is the perturbing potential. Using time-independent perturbation theory, find the corrections to the energy of a nondegenerate level of H_0 up to second order in V. **(7)**

(b) Calculate the shift in the ground-state energy of the hydrogen atom in the first order of perturbation if the proton is considered a uniform sphere of radius R instead of a point charge. You can take $R << a_0$ where a_0 is the Bohr radius.

Ground-state wave function of the hydrogen atom considering the proton to be a point particle is

$$\psi_{1s} = \frac{2}{\sqrt{4\pi a_0^3}}\, e^{-r/a_0} \;\; (\text{a}_0 \text{ is the Bohr radius}).$$

(7)

7. **(a)** Show that in the WKB approximation, the two linearly independent solutions of the time independent Schrödinger equation in one dimension can be written in the form

$$\psi_{\pm}(x) = \frac{1}{\sqrt{|p(x)|}} \exp\left(\pm \frac{i}{\hbar} \int^x p(x')dx' \right)$$

Discuss under what conditions WKB approximation is expected to be valid. **(8)**

(b) Using WKB approximation, find the bound state energy eigenvalues of a particle of mass m in the potential

$$V(x) = \begin{cases} Fx & \text{for } x > 0 \quad (F > 0) \\ \infty & \text{for } x \leq 0. \end{cases}$$

(6)

8. **(a)** A spinless nonrelativistic particle of mass m is elastically scattered from a fixed potential $V(r)$. The radial wave function $R_l(r)$ of the particle in the l^{th} partial wave is 'normalized' such that

$$R_l(r) \underset{r\to\infty}{\sim} \frac{\sin(kr - l\pi/2 + \delta_l)}{kr}$$

where δ_l is the phase shift. By considering the radial Schrödinger equations with and without the potential, show that

$$\sin\delta_l = -k\int_0^\infty r^2 j_l(kr)U(r)R_l(r)dr$$

where $U(r) = \frac{2m}{\hbar^2}V(r)$. Hence derive the Born-approximation formula for the phase shift in the case of a weak potential. Explain with the help of a diagram why an attractive (negative) potential produces a positive phase shift. **(5+1+2)**

(b) In the Born approximation, find the scattering amplitude and the differential cross section for elastic scattering of a particle from the screened Coulomb potential

$$V(r) = \frac{Ze^2}{r}e^{-r/a} \ .$$

Taking appropriate limit of your answer find also the Rutherford formula for differential cross section in case of scattering from a point Coulomb potential. **(4+2)**

End of questions