Lec -3

8.29. Given triangle $\mathcal{T}$ in the z plane with vertices at i, $1-i$, $1+i$, determine the triangle $\mathcal{T}'$ into which $\mathcal{T}$ is mapped under the transformations (a) $w = 3z + 4 - 2i$, (b) $w = iz + 2 - i$, (c) $w = 5e^{\pi i/3}z - 2 + 4i$. What is the relationship between $\mathcal{T}$ and $\mathcal{T}'$ in each case?

8.30. Sketch the region of the w plane into which the interior of triangle $\mathcal{T}$ of Problem 8.29 is mapped under the transformations (a) $w = z^2$, (b) $w = iz^2 + (2-i)z$, (c) $w = z + 1/z$.

8.31. (a) Show that by means of the transformation $w = 1/z$, the circle C given by $|z-3| = 5$ is mapped into the circle $|w + 3/16| = 5/16$. (b) Into what region is the interior of C mapped?

8.32. (a) Prove that under the transformation $w = (z-i)/(iz-1)$, the region $\text{Im}\{z\} \geq 0$ is mapped into the region $|w| \leq 1$. (b) Into what region is $\text{Im}\{z\} \leq 0$ mapped under the transformation?

P

8.33. (a) Show that the transformation $w = \frac{1}{2}(ze^{-\alpha} + z^{-1}e^{\alpha})$ where α is real, maps the interior of the circle $|z| = 1$ onto the exterior of an ellipse [see entry B-2 on page 253].

P

(b) Find the lengths of the major and minor axes of the ellipse in (a) and construct the ellipse.

8.34. Determine the equation of the curve in the w plane into which the straight line $x + y = 1$ is mapped under the transformations (a) $w = z^2$, (b) $w = 1/z$.

P

8.35. Show that $w = \{(1+z)/(1-z)\}^{2/3}$ maps the unit circle onto a wedge-shaped region and illustrate graphically.

8.36. (a) Show that the transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5$, $v = 2y - 3x - 4$.

(b) Determine the triangle in the uv plane into which triangle $\mathcal{T}$ of Problem 8.29 is mapped under the transformation in (a). Are the triangles similar?

8.37. Express the transformations (a) $u = 4x^2 - 8y$, $v = 8x - 4y^2$ and (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$ in the form $w = F(z, \bar{z})$.

8.29 (a) $w = 3z + 4 - 2i$

$= 3(x+iy) + 4 - 2i$

$u + iv = (3x+4) + (3y-2)i$

$u = 3x+4$

$v = 3y - 2$

$i \to (0,1) \to u = 4, v = 1$

$1-i \to (1,-1) \to u = 7, v = -5$

$1+i \to (1,1) \to u = 7, v = 1$

$\mathcal{T}' \to 4+i, 7-5i, 7+i$

(c) $w = 5e^{\pi i/3} z - 2 + 4i$

$= 5 \times (\sin 60^\circ + i \cos 60^\circ)(x+iy) - 2 + 4i$

$= 5 \times \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)(x+iy) - 2 + 4i$

$= \left(\frac{5\sqrt{3}}{2}x - \frac{5}{2}y - 2\right) + \left(\frac{5}{2}x + \frac{\sqrt{3}}{2}y + 4\right)i$

u v

$(0,1)$ $u = -\frac{9}{2}$, $v = \frac{\sqrt{3}}{2} + 4$

$(1,-1)$ $u = \frac{5\sqrt{3}}{2} + \frac{5}{2} - 2$, $v = \frac{5}{2} - \frac{\sqrt{3}}{2} + 4$

$(1,1)$ $u = \frac{5\sqrt{3}}{2} - \frac{5}{2} - 2$, $v = \frac{5}{2} + \frac{\sqrt{3}}{2} + 4$

8.30. Sketch the region of the w plane into which the interior of triangle T of Problem 8.29 is mapped under the transformations (a) $w = z^2$, (b) $w = iz^2 + (2-i)z$, (c) $w = z + 1/z$.

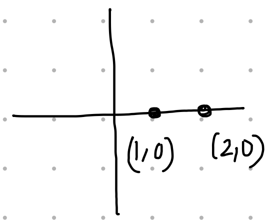
→ (a) $w = z^2 = z\bar{z} = (x+iy)(x-iy)$

$= x^2+y^2$

$(0,1) \to u = 1, v = 0$

$(1,-1) \to u = 2, v = 0$

$(1,1) \to u = 2, v = 0$



(b) $w = i(x^2+y^2) + (2-i)(x+iy)$

$= i(x^2+y^2) + (2x+y)$

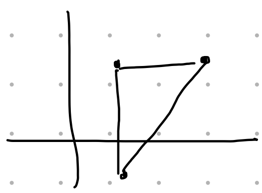
$+ (-x+2y)i$

$= \underbrace{(2x+y)}_{u} + \underbrace{(x^2-x+2y+y^2)}_{v}i$

$(0,1) \to u = 1, v = 3$

$(1,-1) \to u = 1, v = -1$

$(1,1) \to u = 3, v = 3$



(c) $w = z + \frac{1}{z}$

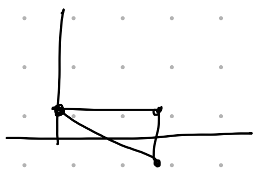
$= x+iy + \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$

$= \left(x + \frac{x}{x^2+y^2}\right) + \left(y - \frac{y}{x^2+y^2}\right)i$

$(0,1) \to u = 0, v = \frac{1}{2}$

$(1,-1) \to u = \frac{3}{2}, v = -\frac{1}{2}$

$(1,1) \to u = \frac{3}{2}, v = \frac{1}{2}$



8.31. (a) Show that by means of the transformation $w = 1/z$, the circle C given by $|z - 3| = 5$ is mapped into the circle $|w + 3/16| = 5/16$. (b) Into what region is the interior of C mapped?

$$w = \frac{1}{z} \rightarrow z = \frac{1}{w}$$

$$|z-3| = 5$$

$$\rightarrow |z-3|^2 = 5^2$$

$$\rightarrow (z-3)(\overline{z-3}) = 5^2$$

$$\rightarrow \left(\frac{1}{w}-3\right)\left(\overline{\frac{1}{w}-3}\right) = 5^2$$

$$\rightarrow \left(\frac{1}{w}-3\right)\left(\frac{1}{\overline{w}}-3\right) = 5^2$$

$$\rightarrow \frac{1}{w\overline{w}} - \frac{3}{\overline{w}} - \frac{3}{w} + 9 = 25$$

$$\rightarrow 1 - 3w - 3\overline{w} = 16w\overline{w}$$

$$\rightarrow 16w\overline{w} + 3w + 3\overline{w} = 1$$

$$\rightarrow w\overline{w} + \frac{3}{16}w + \frac{3}{16}\overline{w} + \frac{9}{256} - \frac{9}{256} = \frac{1}{16}$$

$$\rightarrow w\left(\overline{w} + \frac{3}{16}\right) + \frac{3}{16}\left(\overline{w} + \frac{3}{16}\right) = \frac{25}{256}$$

$$\rightarrow \left(\overline{w} + \frac{3}{16}\right)\left(w + \frac{3}{16}\right) = \frac{25}{256}$$

$$\rightarrow \left(w + \frac{3}{16}\right)^2 = \left(\frac{5}{16}\right)^2$$

$$\rightarrow \left|w + \frac{3}{16}\right| = \frac{5}{16}$$

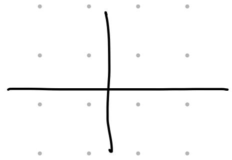
(b) $|w + 3/16| = 5/16$

$\rightarrow |(u + 3/16) + iv| = 5/16$

$\rightarrow$ $(-3/16, 0)$ center with radius $5/16 \rightarrow$ circle.

8.32. (a) Prove that under the transformation $w = (z - i)/(iz - 1)$, the region $\text{Im}\{z\} \geq 0$ is mapped into the region $|w| \leq 1$. (b) Into what region is $\text{Im}\{z\} \leq 0$ mapped under the transformation?

$\rightarrow \quad w = \dfrac{z - i}{iz - 1}$



$\text{Im}\, z \geq 0$

$(x, 0) \rightarrow \boxed{b > 0}$

$(0,0) \rightarrow \dfrac{-i}{-1} = i$

$(1,0) \rightarrow \dfrac{1-i}{i-1} = -1$

$\hookrightarrow 1$

$(-1,0) \rightarrow \dfrac{-1-i}{-i-1} = 1$

$\hookrightarrow -1$

$(\alpha, 0) \rightarrow \dfrac{1 - \frac{i}{z}}{i - \frac{1}{z}} \rightarrow \dfrac{1}{1} = 1$

$$(a,b) \to \frac{a+ib-i}{i(a+ib)-1} = \frac{a+(b-1)i}{-b+(a-1)i} = \frac{\sqrt{a^2+(b-1)^2}}{\sqrt{b^2+(a-1)^2}}$$

$$\frac{a^2+b^2+1-2b}{a^2+b^2+1-2a}$$

8.33. (a) Show that the transformation $w = \frac{1}{2}(ze^{-\alpha} + z^{-1}e^{\alpha})$ where α is real, maps the interior of the circle $|z| = 1$ onto the exterior of an ellipse [see entry B-2 on page 253].

(b) Find the lengths of the major and minor axes of the ellipse in (a) and construct the ellipse.

$\rightarrow$ $|z| = 1$

$(0,0) \rightarrow w = 0$

$(0,1) \rightarrow w = \frac{1}{2}\left[ie^{-\alpha} - ie^{\alpha}\right]$

$\hookrightarrow i$

$(0,-1) \rightarrow w = \frac{1}{2}\left[-ie^{-\alpha} + ie^{\alpha}\right]$

$(1,0) \rightarrow w = \frac{1}{2}\left[e^{-\alpha} + e^{\alpha}\right]$

$$(a,b) = \frac{1}{2}\left[(a+ib)e^{-\alpha} + \frac{1}{(a+ib)}e^{\alpha}\right]$$

$$= \frac{1}{2}\left[(a+ib)e^{-\alpha} + \frac{ae^{\alpha}}{a^2+b^2} - i\frac{b}{a^2+b^2}e^{\alpha}\right]$$

$$= \frac{1}{2}\left[\left(a e^{-\alpha} + \frac{a e^{\alpha}}{a^2+b^2}\right)\right.$$

$$\left. + \left(b e^{-\alpha} - \frac{b e^{\alpha}}{a^2+b^2}\right) i\right.$$

$$(a^2+b^2) e^{-2\alpha} + \frac{a^2+b^2}{(a^2+b^2)^2} (e^{\alpha})^2$$

$$+ 2 \cdot \frac{a^2}{a^2+b^2} - 2 \frac{b^2}{a^2+b^2}$$

8.34. Determine the equation of the curve in the w plane into which the straight line $x + y = 1$ is mapped under the transformations (a) $w = z^2$, (b) $w = 1/z$.

$\rightarrow$ (a) $w = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2xy}_{v}\, i$

$\rightarrow$ $u = x^2 - y^2 \longrightarrow u = (1-y)^2 - y^2$

$x+y=1$ $\quad = 1 + \cancel{y^2} - 2y - \cancel{y^2}$

$\rightarrow$ $v = 2xy$ $\quad \boxed{x = 1-y}$ $\quad = 1-2y$

$\boxed{y = \frac{1-u}{2}}$

$\rightarrow$ $v = 2(1-y)y$

$\frac{2-1+u}{2}$

$\frac{1+u}{2}$

$\rightarrow$ $v = 2\left[1 - \frac{1-u}{2}\right]\left[\frac{1-u}{2}\right]$

$\rightarrow$ $v = \cancel{2} \cdot \frac{1-u^2}{\cancel{4}2}$

$\rightarrow$ $v = \frac{1-u^2}{2} \longrightarrow u^2 + 2v - 1 = 0 \rightarrow$ $\boxed{\text{parabola}}$

(b) $w = \frac{1}{z}$ $\qquad x = \frac{u}{u^2+v^2}$, $y = -\frac{v}{u^2+v^2}$

$x + y = 1$

$u - v = u^2 + v^2$

$\left(u^2 - u + \frac{1}{4}\right) + \left(v^2 + v + \frac{1}{4}\right) = \frac{1}{2}$

$\rightarrow$ $\left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$

$\boxed{\text{circle}}$

8.35. Show that $w = \{(1+z)/(1-z)\}^{2/3}$ maps the unit circle onto a wedge-shaped region and illustrate graphically.

$\rightarrow$ $$w = \left\{ \frac{(1+z)}{(1-z)} \right\}^{2/3}$$

$\rightarrow$ $(1,0) \rightarrow \propto$

↳ 1

$(0,1) \rightarrow \left(\frac{1+i}{1-i}\right)^{2/3} = i^{2/3} = (-1)^{1/3} = -1$

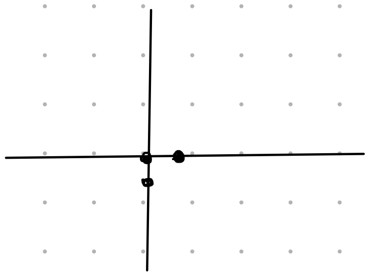
↳ i

$(0,-1)$ $\left(\frac{1-i}{1+i}\right)^{2/3} = -1$

↳ $-i$

$(0,-1)$ $\left(\frac{0}{2}\right)^{2/3} = 0$

$(0,0)$ $\frac{1}{1} = 1$



$$w = \left(\frac{1+x+iy}{1-x-iy}\right)^{2/3}$$

$$w = \left(\frac{(1+x)+iy}{(1-x)\ -iy}\right)^{2/3}$$

$$= \left\{\frac{((1+x)+iy)^2}{(1-x)^2+y^2}\right\}^{2/3}$$

$$= \left\{\frac{(1+x)^2-y^2\ -2(1+x)\cdot iy}{(1-x)^2+y^2}\right\}^{2/3}$$

$=$

8.36. (a) Show that the transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5$, $v = 2y - 3x - 4$.

(b) Determine the triangle in the uv plane into which triangle $\mathcal{T}$ of Problem 8.29 is mapped under the transformation in (a). Are the triangles similar?

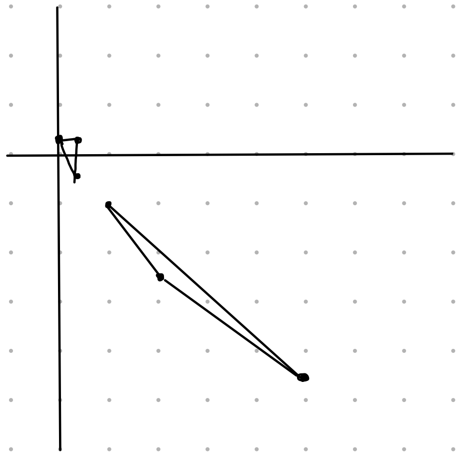
→ (a) $W = 2(x+iy) - 3i(x-iy) + 5 - 4i$

$= (2x - 3y + 5) + (2y - 3x - 4)i$

(b) $i \to (0,1) \to u = 2,\ v = -2$

$1-i \to (1,-1) \to u = 10,\ v = -9$

$1+i \to (1,1) \to u = 4,\ v = -5$



$$\frac{(i) - (1-i)}{(2-2i) - (10-9i)} = \frac{-1+2i}{-8+7i}$$

$$\frac{(1-i) - (1+i)}{(10-9i) - (4-5i)} = \frac{-2i}{6-4i}$$

ratio not same, so triagles not similar.

8.37. Express the transformations (a) $u = 4x^2 - 8y$, $v = 8x - 4y^2$ and (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$ in the form $w = F(z, \bar{z})$.

→ (a) $w = u + iv = (4x^2 - 8y) + i(8x - 4y^2)$

$= (z+\bar{z})^2 + 4\cdot(z-\bar{z})i$

$+ 4\cdot(z+\bar{z})i - (z-\bar{z})^2 i^2 \cdot i$

$x+iy = z$
$x-iy = \bar{z}$

$2x = z+\bar{z}$

$2yi = z-\bar{z}$

$2y = -(z-\bar{z})i$

$$= (z+\bar{z})^2 + 4(z-\cancel{\bar{z}}+z+\cancel{\bar{z}})i + (z-\bar{z})^2 i$$

$$= (z+\bar{z})^2 + (z-\bar{z})^2 i + 8iz$$

(b) $u = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\rightarrow u = z^3$

$x + iy = z$

$\rightarrow z^3 = x^3 + i^3y^3 + 3x^2 \cdot iy + 3x \cdot i^2y^2$

$= x^3 - y^3 i + 3x^2yi - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

lec-5

Example 1

Prove that the function $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Hence find a function $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is analytic.

$\rightarrow$ u harmonic if satisfies $u_{xx} + u_{yy} = 0$

$$u_{xx} = \frac{\partial}{\partial x}(2x - 2y - 2) \qquad u_{yy} = \frac{\partial}{\partial y}(-2y - 2x + 3)$$

$$= 2 \qquad\qquad = -2$$

$\because u_{xx} + u_{yy} = 0 \quad \therefore$ u is harmonic

$$u_x = 2x - 2y - 2 = v_y$$

$$u_y = -2y - 2x + 3 = -v_x$$

$$v_y = \frac{\partial v}{\partial y} = 2x - 2y - 2$$

$\rightarrow \int \partial v = \int (2x - 2y - 2)\, dy + F(x)$

$\rightarrow v = 2x \cdot y - 2 \cdot \frac{y^2}{2} - 2 \cdot y + F(x)$

$\rightarrow \frac{\partial v}{\partial x} = 2y - 0 - 0 + F'(x) = 2y + 2x - 3$

$$F'(x) = 2x-3$$

$$\rightarrow F(x) = x^2 - 3x + c$$

$$v = 2xy - y^2 - 2y + x^2 - 3x + c$$

Example 2

Prove that the function $u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic. Find a function $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is analytic and write $f(z)$ in terms of z.

$$\rightarrow \quad u_x = e^{-x}(\sin y - 0) + (x\sin y - y\cos y)(-e^{-x})$$

$$u_x = e^{-x}\sin y - u$$

$$u_{xx} = -e^{-x}\sin y - u_x = -e^{-x}\sin y - e^{-x}\sin y + u$$

$$= -2e^{-x}\sin y + u$$

$$u_y = e^{-x}\left[x\cdot\cos y - \{y(-\sin y) + \cos y \cdot 1\}\right]$$

$$u_y = e^{-x}.x\cos y - e^{-x}(\cos y - y\sin y)$$

$$u_{yy} = e^{-x}x\cdot(-\sin y) - e^{-x}\left[-\sin y - \{y\cdot\cos y + \sin y\cdot 1\}\right]$$

$$= \underbrace{-e^{-x}x\sin y} + e^{-x}\sin y + \underbrace{e^{-x}y\cos y} + e^{-x}\sin y$$

$$= 2e^{-x}\sin y - u$$

$$u_{xx}+u_{yy} = -2e^{-x}\sin y + u + 2e^{-x}\sin y - u = 0$$

analytic $\rightarrow$ $u_x = v_y$ $\qquad$ $u_y = -v_x$

$$v_y = u_x = e^{-x}\sin y - u = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y)$$

$$v = e^{-x}\int \sin y\,dy - e^{-x}x\int \sin y\,dy + e^{-x}\int (y\cos y)dy + F(x)$$

$$= e^{-x}(-\cos y) - xe^{-x}(-\cos y) + e^{-x}\left[y\sin y - \int \sin y\,dy\right] + F(x)$$

$$v = \cancel{-e^{-x}\cos y} + e^{-x}x\cos y + e^{-x}y\sin y + \cancel{e^{-x}\cos y} + F(x)$$

$$v = e^{-x}x\cos y + e^{-x}y\sin y + F(x)$$

$\rightarrow$ $v_x = \cos y(-xe^{-x}+e^{-x}) + y\sin y(-e^{-x}) + F'(x)$

$\rightarrow$ $-u_y = -e^{-x}x\cos y + e^{-x}\cos y - e^{-x}y\sin y + F'(x)$

$$- e^{-x}\cos y - e^{-x}(x\cos y + y\sin y) + F'(x)$$

$$= \underbrace{-e^{-x}\cdot x\cos y} + \underbrace{e^{-x}}(\cos y - \underbrace{y\sin y})$$

$$F'(x) = 0$$

$$\rightarrow F(x) = c$$

$$v = -e^{-x}(x\cos y + y\sin y) + c$$

$$f(z) = w = u + iv$$

$$= e^{-x}(x\sin y - y\cos y) + i\{e^{-x}(x\cos y + y\sin y) + c\}$$

$$= e^{-x}(x\sin y - y\cos y) + i(e^{-x}(x\cos y + y\sin y)) + ic$$

$$= e^{-x}x[\sin y + i\cos y] - e^{-x}y(\cos y - i\sin y) + ic$$

$$= ixe^{-x}[\cos y - i\sin y] - e^{-x}y[\cos y - i\sin y] + ic$$

$$= [\cos y - i\sin y][e^{-x}][ix - y] + ic$$

$$= i\cdot e^{-iy}\cdot e^{-x}(x + iy) + ic$$

$$= i\cdot e^{-(x+iy)}(x+iy) + ic = i\cdot e^{-z}\cdot z + ic$$

Example 3

(a) Prove that $u(x,y) = \dfrac{1}{2}\ln(x^2 + y^2)$ is a harmonic function.

(b) Find $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is an analytic function.

(c) Write $f(z)$ in terms of z.

→ (a) $u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2}$

$$u_{xx} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(1) - (y)(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{x^2+y^2}$$

$\therefore u_{xx} + u_{yy} = 0$ $\quad\therefore u$ is harmonic.

(b) $u_x = v_y = \frac{x}{x^2+y^2}$

$$\rightarrow v = \int \frac{x}{x^2+y^2}\,dy + F(x)$$

$$= x \int \frac{dy}{x^2+y^2} + F(x) = \cancel{x} \cdot \frac{1}{\cancel{x}} \tan^{-1}\left(y/x\right) + F(x)$$

$$\rightarrow V_x = \frac{d}{dx}\left(\tan^{-1}\left(y/x\right)\right) + F'(x)$$

$$\rightarrow -\frac{y}{x^2+y^2} = \frac{1}{1+\frac{y^2}{x^2}} \cdot y\left(-\frac{1}{x^2}\right) + F'(x)$$

$$= \frac{-y}{x^2+y^2} + F'(x)$$

$$\therefore F'(x) = 0 \quad \rightarrow F(x) = c$$

$$V = \tan^{-1}(y/x) + c$$

$$f(z) = \frac{1}{2}\ln(x^2+y^2) + i\left(\tan^{-1} y/x + c\right)$$

(c)

$$z = re^{i\theta}$$

$$\rightarrow \ln z = \ln r + i\theta \qquad r = |z| = \sqrt{x^2+y^2}$$

$$\rightarrow \ln z = \ln|z| + i\theta \qquad \theta = \tan^{-1} y/x$$

$$\rightarrow i\theta = \ln z - \ln|z| = \ln\left(\frac{z}{|z|}\right)$$

$$\boxed{\therefore i\theta = \ln\left(\frac{z}{|z|}\right)}$$

$$f(z) = \ln\sqrt{x^2+y^2} + i\left(\tan^{-1} y/x + c\right)$$

$$= \ln|z| + i\tan^{-1} y/x + ic$$

$$= \ln|z| + i\theta + ic$$

$$= \ln|z| + \ln\left(\frac{z}{|z|}\right) + ic$$

$$= \ln\left(|z| \cdot \frac{z}{|z|}\right) + ic$$

$$= \ln z + ic$$

random integration:

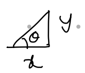
$$\int \frac{y^2-x^2}{(x^2+y^2)^2}\,dy = \int \frac{y^2+x^2-2x^2}{(x^2+y^2)^2}\,dy$$

$$= \int \frac{dy}{x^2+y^2} - 2x^2 \int \frac{1}{(x^2+y^2)^2}\,dy$$

$$= \frac{1}{x}\tan^{-1}(y/x) - 2x^2 \int \frac{x\sec^2 d\theta}{(x^2\tan^2\theta + x^2)^2}$$

$y = x\tan\theta$

$\rightarrow dy = x \cdot \sec^2\theta\, d\theta$



$$= \frac{1}{x}\tan^{-1}(y/x) - \frac{2x^2 \cdot x}{x^4} \int \frac{\sec^2 d\theta}{(1+\tan^2\theta)^2}$$

$$= \frac{1}{x}\tan^{-1}(y/x) - \frac{2}{x} \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \frac{1}{x}\tan^{-1}(y/x) - \frac{2}{x} \int \cos^2\theta\, d\theta$$

$$= \frac{1}{x}\tan^{-1}(y/x) - \frac{1}{x} \int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{x}\tan^{-1}(y/x) - \frac{1}{x}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{1}{x}\tan^{-1}(y/x) - \frac{1}{x}\left[\tan^{-1}(y/x) + \frac{1}{\cancel{2}}\cancel{2}\frac{y}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}}\right] + c$$

$$= -\frac{1}{\cancel{x}} \cdot \frac{\cancel{x}y}{x^2+y^2} + c$$

$$= -\frac{y}{x^2+y^2} + c$$

Example 4

If $f(z)=e^{\bar{z}}$, then $f'(z)$ does not exist at any point.

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