A Separation Principle for Linear Switching Systems and Parametrization of All Stabilizing Controllers

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***Abstract*—In this paper, we investigate the problem of designing a switching compensator for a plant switching amongst a (finite) family of given configurations (A_i, B_i, C_i). We assume that switching is uncontrolled, namely governed by some arbitrary switching rule, and that the controller has the information of the current configuration i.**

As a first result, we provide necessary and sufficient conditions for the existence of a switching compensator such that the closed-loop plant is stable under arbitrary switching. These conditions are based on a separation principle, precisely, the switching stabilizing control can be achieved by separately designing an observer and an estimated state (dynamic) compensator. These conditions are associated with (non-quadratic) Lyapunov functions. In the quadratic framework, similar conditions can be given in terms of LMIs which provide a switching controller which has the same order of the plant.

As a second result, we furnish a characterization of all the stabilizing switching compensators for such switching plants. We show that, if the necessary and sufficient conditions are satisfied then, given *any arbitrary family* of compensators $\mathcal{K}_i(s)$, each one stabilizing the corresponding LTI plant (A_i, B_i, C_i) for fixed i, there exist suitable realizations for each of these compensators which assure stability under arbitrary switching.

***Index Terms*—Lyapunov functions, separation principle, switching systems, Youla–Kucera parametrization.**

I. Introduction

SYSTEMS including both logic and continuous variables, the so called hybrid systems, are currently considered a main stream topic as it can be seen from the considerable number of contributions (see for instance [1]–[4]). In particular, the so called switching systems, are relevant in many applications and are intensively considered in control theory for two basic reasons.

First, switching is a phenomenon that naturally occurs in several plants that can change suddenly their configuration and an efficient control design must take into account this fact. Basically, determining a single compensator which stabilizes a switching plant can be regarded as a robust design problem and faced with existing techniques [5], [6]. The most efficient techniques are perhaps those based on the Lyapunov approach [7]–[9]. In particular, those based on quadratic functions have been successful because of the development of efficient tools based on LMIs [10]. An interesting case is that in which the compensator is informed on-line (not in the design stage) of the plant configuration. This is basically a gain-scheduling problem [11], for which Lyapunov theory has been shown to be successful [12]–[15].

The second reason of the intense investigation of switching systems is that, even in the case of a single plant, considerable advantages in terms of performances can be achieved by properly switching among compensators. In this case, switching is not imposed by nature, but artificially introduced by the designer. The consequent benefit is well established and indeed switching techniques have been involved in adaptive schemes [16]–[18], supervisory control [19], [20], reset design [21] and robust synthesis [22].

In dealing with switching compensators, a fundamental issue is how to guarantee stability. In a recent paper [23] the following essential result has been proved. Given a single linear plant and a family of linear stabilizing compensators, there always exist (possibly non-minimal) realizations for all of them which assure global stability under arbitrary switching. This result is based on a proper formulation of the problem based on the Youla-Kucera parametrization [24], [25] of all stabilizing compensators. The key idea is to show that one can solve the problem, basically, by switching among Youla-Kucera parameters. A key point is that the realization of the Youla-Kucera parameters cannot be arbitrary, but suitably constructed.

The main idea of the present paper is to consider at the same time both the mentioned aspects: controlling a switching linear plant by means of a switching linear controller. We assume that plant switching is arbitrary while the compensator commutations are commanded by the plant. Our basic question is the following: given a switching plant, under which conditions there exists a switching compensator which stabilizes the plant under arbitrary switching? This issue was pointed out as an open problem in [23]. Under the assumption that the instantaneous exact knowledge of the current plant configuration is available on-line to the compensator, without delay, we provide the following main results.

- Necessary and sufficient stabilizability conditions are given. These are supported by polyhedral Lyapunov functions and are based on a separation principle. The controller is derived by designing an (extended) observer

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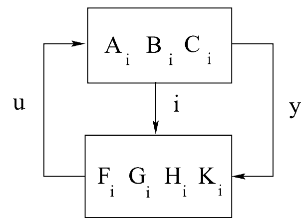

Fig. 1. The switching control.

and a (dynamic) state feedback, although we cannot provide bounds for the compensator order.

- The mentioned conditions are constructive, but computationally demanding. If we strengthen our requirements to quadratic stabilizability, then the necessary and sufficient conditions are expressed in terms of LMIs. We show that the compensator may have the same order of the plant.
- Once the necessary and sufficient conditions are assured, we can parametrize the set of all linear switching stabilizing (or quadratically stabilizing) compensators for the switching plant.

The results have several implications as well as applications. For instance, the complete parametrization is given in a form which is suitable for optimal design, since the closed-loop map is shown to be an affine function of the Youla-Kucera parameter, the natural extension of the standard linear time-invariant theory. We will consider not only the pure stabilizability property, but the contractive design, precisely the goal of assuring a certain "speed of convergence". We will investigate on what we call the paradox of the "zero transfer functions compensator". Given a system (which satisfies the assumptions) which is (Hurwitz/Schur) stable in any fixed configuration, but may be destabilized by switching, we can assure switching stability by means of a compensator with the (surprising) property of having zero transfer function for each fixed configuration. The explanation of this paradox is quite intriguing. Precisely the switching compensator *realized by the proposed technique*[1] is such that *its observable and reachable subsystems interact only during switching*. We propose a "switching manager" control as an application of this paradox.

The paper is organized as follows. After the formulation of the problem in Sections II, the main results are all stated in Section III without proofs, which are given later in Section IV. These proofs are essentially based on previous results on non-quadratic Lyapunov functions (see [26]–[29] and [9] for a survey), on generalized observers [30], [31] and duality properties between observer and state feedback design [15]. Numerical details for the computation of non-quadratic Lyapunov functions are reported in the Appendix. In the quadratic stabilization case the results are based on standard LMI techniques [10]. The parametrization of all stabilizing compensators is achieved by generalizing ideas described in [5] (see also [6]). The implications are described in Section V and we propose an illustrating example in Section VI. We finally discuss the results in Section VII.

[1]Obviously the property is not true for arbitrary realizations.

II. Definitions and Problem Statement

Consider the time-varying system

$$\begin{aligned}\delta x(t) &= A_i x(t) + B_i u(t)\\ y(t) &= C_i x(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. δ represents the derivative in the continuous-time case and the one-step shift operator $\delta x(t) = x(t+1)$ in the discrete-time case. We assume that the plant matrices can switch arbitrarily, precisely that

$$i = i(t) \in \mathcal{I} = \{1, 2, \ldots, r\}$$

and that for each i the plant (A_i, B_i, C_i) is stabilizable. For the simple notations, we have dropped the time t from the index i with the understanding that $(A_i, B_i, C_i) = (A_{i(t)}, B_{i(t)}, C_{i(t)})$. For this system, we consider the class of linear switching controllers (see Fig. 1)

$$\begin{aligned}\delta z(t) &= F_i z(t) + G_i y(t)\\ u(t) &= H_i z(t) + K_i y(t)\end{aligned} \tag{2}$$

where, again, $i = i(t) \in \mathcal{I}$, and $(F_i, G_i, H_i, K_i) = (F_{i(t)}, G_{i(t)}, H_{i(t)}, K_{i(t)})$. The following assumptions will be considered.

Assumption 1: Non-Zenoness: The number of switching instants is finite on every finite interval (although it may be arbitrarily large in the continuous-time case, i.e. we assume zero dwell time). This assumption is implicit in the discrete-time case.

Assumption 2: Zero Delay: There is no delay in the communication between the plant and the controller, which, at time t, knows the current $y(t)$ and configuration $i(t)$.

Assumption 1 is not an essential restriction and avoids well-posedness issues, we will comment on it later on. Conversely, Assumption 2, may be a restriction in practice, but fairly acceptable in most plants.

The closed-loop system matrix achieved from (1) and (2) becomes

$$A_i^{cl} = \begin{bmatrix} A_i + B_i K_i C_i & B_i H_i \\ G_i C_i & F_i \end{bmatrix}. \tag{3}$$

For this system (or any arbitrary switching system) we adopt these definitions.

Definition 2.1: The system governed by matrices A_i^{cl}, $i(t) \in \mathcal{I}$ is *Hurwitz (Schur) stable* if, for any fixed value i, its eigenvalues have negative real parts (respectively modulus less than one).

Definition 2.2: The system governed by the family of matrices A_i^{cl} is *switching stable* if it is asymptotically stable for any switching signal $i(t) \in \mathcal{I}$.

In the sequel, when we will talk about "stability", we will always refer to "switching stability".

Definition 2.3: The system governed by the matrices A_i^{cl}, $i(t) \in \mathcal{I}$ is *quadratically stable* if these matrices share a common quadratic Lyapunov function.

It is well established that the three definitions are not equivalent, precisely quadratic stability implies switching stability which implies Hurwitz stability [8] (we remind that we assumed zero dwell time). In a Lyapunov framework, switching stability is equivalent to the existence of a Lyapunov function which is a polyhedral norm (see [27]–[29] and [26]). We will use this fact later.

The next two problems are addressed in this paper.

Problem 1: Given the switching plant represented by (1), does there exist a family of matrices (F_i, G_i, H_i, K_i), $i \in \mathcal{I}$ such that the system governed by (3) is switching stable?

Once the previous problem has received a "yes" answer, the next question is in order.

Problem 2: Given a set of transfer functions $\mathcal{K}_i(s)$ assuring that the ith closed-loop system is Hurwitz (respectively Schur), namely stable for fixed i, does there exist realizations for the $\mathcal{K}_i(s)$ such that the system is switching stable?

In the next section we come up with a necessary and sufficient condition for Problem 1 and with an "always yes" reply to the question of Problem 1.

III. Main Results

A. Necessary and Sufficient Stabilizability Conditions

To state our results, we need a technical definition. Given a square matrix P, $\|P\|_1$ and $\|P\|_\infty$ denote the standard induced matrix norms with respect to the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms for vectors.

Definition 3.1: The square matrix M is of class $\mathcal{H}_1$ if there exists $\tau > 0$ such that $\|I+\tau M\|_1 < 1$. It is of class $\mathcal{H}_\infty$ if there exists $\tau > 0$ such that $\|I+\tau M\|_\infty < 1$.

The classes $\mathcal{H}_1$ and $\mathcal{H}_\infty$ are introduced to state the continuous-time conditions. These are associated to existing algorithms based on the Euler auxiliary system (see [9] for details). The following result holds.

Theorem 3.1: The following two statements are equivalent for continuous-time (resp. discrete-time) systems.

i) There exists a linear switching compensator (2) for the switching plant (1) which assures switching stability to the closed-loop system.

ii) There exist $\mu\times\mu$ matrices $P_i \in \mathcal{H}_1$, $\nu\times\nu$ matrices $Q_i \in \mathcal{H}_\infty$, (respectively matrices $\|P_i\|_1 < 1$, $\|Q_i\|_\infty < 1$), $m\times\mu$ matrices U_i, $p\times\nu$ matrices L_i, and there exist a $n\times\mu$ matrix X and a $\nu\times n$ matrix R, of full row rank and full column rank respectively, such that

$$A_iX + B_iU_i = XP_i \tag{4}$$

$$RA_i + L_iC_i = Q_iR. \tag{5}$$

Corollary 3.1: If the necessary and sufficient conditions are satisfied, then a stabilizing compensator is given by

$$\delta w(t) = Q_iw(t) - L_iy(t) + RB_iu(t) \tag{6}$$

$$\hat{x}(t) = Mw(t) \tag{7}$$

$$\delta z(t) = F_iz(t) + G_i\hat{x}(t) \tag{8}$$

$$u(t) = H_iz(t) + K_i\hat{x}(t) + v(t) \tag{9}$$

where $v(t) = 0$ (the reason of introducing this dummy signal will become clear later). The new matrix M is any left inverse of R

$$MR = I$$

while (F_i, G_i, H_i, K_i) can be computed as

$$\begin{bmatrix} K_i & H_i \\ G_i & F_i \end{bmatrix} = \begin{bmatrix} U_i \\ V_i \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \tag{10}$$

where Z is any complement of X which makes the square matrix invertible and

$$V_i = ZP_i$$

The compensator is of order $\nu + \mu - n$.

Remark 3.1: The compensator has a separation structure. Indeed, we will see that $\hat{x}(t)$ estimates asymptotically $x(t)$ since $\|Rx - w\|_\infty \to 0$. Conversely, the dynamic compensator having z as state variable is a dynamic state feedback stabilizing compensator. The state feedback condition (4) was previously given in [32].

Unfortunately, the computation of the solution of (4) and (5) may be non-trivial. Indeed, (4), (5) are bilinear and therefore they cannot be easily solved for fixed dimensions ν and μ of R and X (which is equivalent to fixing the compensator complexity). However, they can be solved by means of known iterative procedures to determine polyhedral Lyapunov functions [9], [33], [34]. This issue will be considered in Appendix A. The problem is that there is no upper bound for the order of the compensator, because there are no bounds on ν and μ which depend on the system data, say on the matrices A_i, B_i and C_i. The unlimited complexity is a price we have to pay to assure non-quadratic stabilization. This is a well known issue supported by the fact that even checking switching stability (a special case of our problem with $B_i = 0$ and $C_i = 0$) without resorting to quadratic Lyapunov functions is an undecidable problem [35].

B. The Quadratic Stabilization Case

If we strengthen our requirements, invoking quadratic stabilizability, the next theorem holds.

Theorem 3.2: The following two statements are equivalent in the continuous-time case.

i) There exists a linear switching compensator (2) for the switching plant (1) assuring switching quadratic stability to the closed-loop system.

ii) There exist $n\times n$ positive definite symmetric matrices P and Q, $m\times n$ matrices U_i and $n\times p$ matrices Y_i such that

$$PA_i^T + A_iP + B_iU_i + U_i^TB_i^T < 0 \tag{11}$$

$$A_i^TQ + QA_i + Y_iC_i + C_i^TY_i^T < 0. \tag{12}$$

In the discrete-time case the LMIs are different, precisely, the following holds.

Theorem 3.3: The following two statements are equivalent in the discrete-time case.

i) There exists a linear switching compensator (2) for the switching plant (1) assuring switching quadratic stability to the closed-loop system.

ii) There exist $n \times n$ symmetric positive definite matrices P and Q, and $m \times n$ matrices U_i and $n \times p$ matrices Y_i such that

$$\begin{bmatrix} P & (A_iP + B_iU_i)^T \\ A_iP + B_iU_i & P \end{bmatrix} > 0 \tag{13}$$

$$\begin{bmatrix} Q & (QA_i + Y_iC_i)^T \\ QA_i + Y_iC_i & Q \end{bmatrix} > 0. \tag{14}$$

Corollary 3.2: If the necessary and sufficient conditions are satisfied, then a stabilizing compensator is given by

$$\begin{aligned} \delta\hat{x}(t) &= (A_i + L_iC_i + B_iJ_i)\hat{x}(t) - L_iy(t) + B_iv(t) \\ u(t) &= J_i\hat{x}(t) + v(t) \end{aligned} \tag{15}$$

with $v(t) = 0$ (again this signal will be used later), and where

$$J_i = U_iP^{-1} \quad \text{and} \quad L_i = Q^{-1}Y_i$$

where P and Q are the symmetric matrices defined in (11) and (12) (or by (13) and (14) in the discrete-time case).

Remark 3.2: This compensator has also an observer-based structure. It is of order n, and thus of fixed complexity. This shows that, for switching systems, quadratic stabilizability is equivalent to quadratic stabilizability by means of a compensator of the same order of the plant.

Note that (11), (12) and (13), (14) are LMIs, thus easily solvable. We stress that this kind of conditions are known in the LMI literature for both state feedback and observer design [12], [13], [36]. They have been proposed for instance for LPV systems [12] (see also [10]). In [12] when the LMIs are stated (Th. 4.3) it is assumed that B and C are certain matrices. This is a critical assumption in the LPV case but not an issue in the switching case. The conditions based on LMIs and quadratic functions lead to efficient algorithms but they are conservative. Indeed, there are switching stable systems which do not admit quadratic Lyapunov functions. Less conservative results can be achieved if one considers synthesis results based on parameter-dependent Lyapunov functions [14], [37], [38].

C. The Set of All Stabilizing Compensators

In this section, we consider the problem of parametrizing all the switching compensators which can be associated with a switching plant. An efficient parametrization setup is achieved by means of an observer-based pre-compensator and an input injection [5] (see also [6]). We adapt such a structure (which can be derived if the provided stabilizability conditions are satisfied) to switching plants. Once the pre-compensator is determined, the free parameter is a proper stable transfer function which must be properly realized, in agreement with the results presented in [23] for the case of a single plant.

Henceforth, we will always assume stabilizability conditions (quadratic stabilizability) are satisfied. The main result of this subsection is simply stated as follows.

Theorem 3.4: Assume that the necessary and sufficient conditions for switching stabilizability of Theorem 3.1 (switching quadratic stabilizability of Theorem 3.2 or Theorem 3.3) are satisfied. Then, given any arbitrary family of transfer functions $\mathcal{K}_i(s)$, $i = 1, \ldots, r$, each stabilizing the ith plant, there exists a switching compensator of the form (2) such that $H_i(sI - F_i)^{-1}G_i + K_i = \mathcal{K}_i(s)$ and such that the closed-loop system is switching stable (switching quadratically stable).

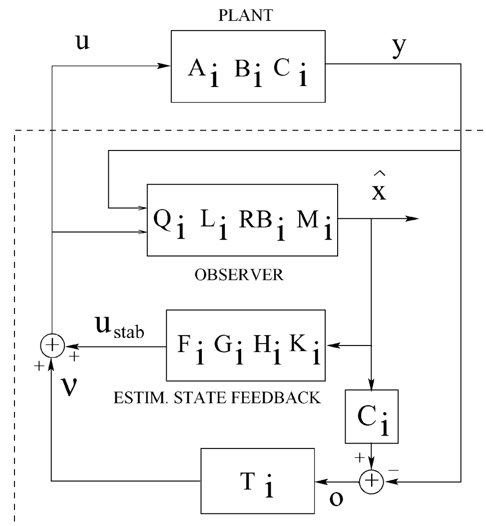

Fig. 2. The observer-based compensator structure.

The realization of such compensator $\mathcal{K}_i(s)$ is illustrated in Fig. 2. More precisely, consider the observer-based compensator (6)–(9) or (15) and, instead of assuming $v \equiv 0$, take

$$v(s) = T_i(s)\left(\hat{y}(s) - y(s)\right)$$

where $\hat{y}(t) = C_i\hat{x}(t)$ is the estimated output, namely

$$u(s) = u_{stab}(s) + v(s) = u_{stab}(s) + T_i(s)\left(C_i\hat{x}(s) - y(s)\right). \tag{16}$$

In other words, u_{stab} is derived by means of the feedback (6)–(9) (or (15)), and $T_i(s)$ is a stable transfer function (the Youla-Kucera parameter [24], [25]). Note that the structure in Fig. 2 is valid for both types of observer-based compensators (indeed, (6), (7) parametrize all types of observers for fixed i [30], [31]) including (15) as special case with $Q_i = A_i + L_iC_i$, $R = I$ and $M = I$.

The transfer function $T_i(s)$ can be selected in such a way that the resulting compensator transfer function is the desired one, $\mathcal{K}_i(s)$.[2] The only problem with $T_i(s)$ is its implementation, which cannot be arbitrary. In this case it is sufficient to exploit the idea of [23] and realize $T_i(s)$ as

$$T_i(s) = H_i^{(T)}\left(sI - F_i^{(T)}\right)^{-1}G_i^{(T)} + K_i^{(T)}$$

in such a way that the family $F_i^{(T)}$ is switching stable. This results in switching stability and transfer function matching for any i. The procedure for control synthesis is the following

Procedure 3.1: Given $\mathcal{K}_i(s)$ $i = 1, \ldots, r$, each stabilizing (A_i, B_i, C_i), perform the following operations.

1) Check if the switching plant (A_i, B_i, C_i) satisfies the necessary and sufficient conditions and synthesize any stabilizing control of the form (6)–(9) (or (15)).

[2] In the case of a standard observer (15) this result is well established [5], but it will require some attention to deal with extended observers (6)–(9).

2) Select the free stable parameter $T_i(s)$ in such a way that the ith compensator has transfer function $\mathcal{K}_i(s)$. This is always possible according to Lemma 10.2 in [5].
3) Select a Hurwitz (Schur) realization for each $T_i(s)$. Make all these realizations of the same order, possibly adding dummy non reachable and non-observable asymptotically stable dynamics:

$$T_i(s) = \hat{H}_i^{(T)} \left(sI - \hat{F}_i^{(T)}\right)^{-1} \hat{G}_i^{(T)} + \hat{K}_i^{(T)}$$

4) Find a realization $(F_i^{(T)}, G_i^{(T)}, H_i^{(T)}, K_i^{(T)})$ for $T_i(s)$ which is switching-stable as follows. Take the previous realizations $(\hat{F}_i^{(T)}, \hat{G}_i^{(T)}, \hat{H}_i^{(T)}, \hat{K}_i^{(T)})$ and apply a transformation to each of them in such a way that all the state matrices $F_i^{(T)}$ share $\|\cdot\|_2^2$, the square of the Euclidean norm, as a Lyapunov function. This can be done by solving the Lyapunov equations

$$\hat{F}_i^{(T)^T} \Pi_i + \Pi_i \hat{F}_i^{(T)} = -I$$

(positive definite solution Π_i exist since $\hat{F}_i^{(T)}$ is stable by construction). In the discrete-time, use the analogous Lyapunov equations. Denote by Ω_i the positive square root of Π_i, say such that $\Pi_i = \Omega_i^2$. Apply the transformation [23] (an alternative is to use proper reset maps)

$$\begin{aligned} F_i^{(T)} &= \Omega_i \hat{F}_i^{(T)} \Omega_i^{-1}, \quad G_i^{(T)} = \Omega_i \hat{G}_i^{(T)}, \\ H_i^{(T)} &= \hat{H}_i^{(T)} \Omega_i^{-1}, \quad K_i^{(T)} = \hat{K}_i^{(T)}. \end{aligned}$$

5) Realize the compensator as in Fig. 2.

The next corollary formalizes the fact that, if we are seeking a single compensator transfer function for all plants, our parametrization works as well.

Corollary 3.3: Assume that the stabilizability conditions are satisfied. Then a single compensator $C(s)$ stabilizes the plant (under switching) if and only if it can be realized as in Fig. 2 with proper T_i (suitably realized). Moreover, if there exists $C(s)$ such that all the closed-loop systems are Hurwitz (Schur) stable, then there exist proper realizations for $C(s)$ such that the overall system is switching stable.

Remark 3.3: Clearly, we have no guarantee that a single realization of a compensator which assures Hurwitz (Schur) stability preserves stability also under switching. This property becomes true under suitable and, in general, different realizations of such a compensator.

IV. Proofs of the Results

To prove the results we need several preliminaries. The first lemma is a key point.

Lemma 4.1: The following statements are equivalent for continuous-time (discrete-time) systems.

- The system

$$\delta x(t) = A_i x(t), \quad i = 1, \ldots, r$$

is (switching) stable.
- There exist a full row rank matrix X and r matrices $P_i \in \mathcal{H}_1$ ($\|P_i\|_1 < 1$ in the discrete-time case) such that

$$A_i X = X P_i$$

(equivalently, the norm $\|x\|_{X,1} \doteq \min\{\|p\|_1 : x = Xp\}$ is a polyhedral Lyapunov function)
- There exist a full column rank matrix R and r matrices $Q_i \in \mathcal{H}_\infty$ ($\|Q_i\|_\infty < 1$, in the discrete-time case) such that

$$R A_i = Q_i R$$

(equivalently, the norm $\|x\|_{R,\infty} \doteq \|Rx\|_\infty$ is a polyhedral Lyapunov function).

Proof: The Lemma is nothing else that a re-statement of well-known results due to [27], [28] and to [26], [29]. Indeed, the stability of the switching system is equivalent to the robust stability of the corresponding LPV system (see [27], Th. 5 or [26] Part I, Th. 1)

$$\delta x(t) = \left[\sum_{i=1}^{r} A_i w_i(t)\right] x(t), \quad \sum_{i=1}^{r} w_i(t) = 1, \quad w_i(t) \geq 0.$$

Precisely, the first equation is eq. (8) in [29] Part III, The second equation is its dual. As far as the discrete-time equations are concerned, the first is given in [26] (see Part II, Theorem 1), the second (the dual) can be derived by the dual system $\delta x = A_i^T x$ which is switching stable if and only if $\delta x = A_i x$ is such ([26], Part I, Th. 3). Duality between the two equations has been also investigated in [15]. ■

Remark 4.1: As a special case of the lemma with $X = I$ or $R = I$, a switching system governed by the matrices $P_i \in \mathcal{H}_1$ or $Q_i \in \mathcal{H}_\infty$ (resp. $\|P_i\|_1 < 1$ or $\|Q_i\|_\infty < 1$, in the discrete-time case) is stable, because it admits $\|\cdot\|_1$ or $\|\cdot\|_\infty$ as a Lyapunov function. Note that in general we know that X has at least as many columns as rows and that R has at least as many rows as columns, but no upper bound can be established.

A. Proof of Theorem 3.1 and Corollary 3.1

Proof: **i) ⇒ ii) (or iii) for discrete time systems)**. Assume that (1) is stabilized by (2). Then, in view of Lemma 4.1, the following set of equations

$$A_i^{cl} X = \begin{bmatrix} A_i + B_i K_i C_i & B_i H_i \\ G_i C_i & F_i \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} P_i \quad (17)$$

hold for $i = 1, \ldots, r$, with $P_i \in \mathcal{H}_1$ (or $\|P_i\|_1 < 1$, in the discrete-time case). Here the full row rank matrix X has been partitioned according to the partition of A_i^{cl}. If we take the upper-left block we have

$$A_i X_1 + B_i K_i C_i X_1 + B_i H_i X_2 = A_i X_1 + B_i U_i = X_1 P_i$$

with $U_i = K_i C_i X_1 + H_i X_2$. Since X is full row rank, so is X_1. This proves (4). The dual (5) can be proved exactly in the same way starting from

$$[R_1 \; R_2] A_i^{cl} = Q_i [R_1 \; R_2] \quad (18)$$

with $Q_i \in \mathcal{H}_\infty$ ($\|Q_i\|_\infty < 1$, in the discrete-time case).

ii) (or iii) for discrete time systems) $\Rightarrow$ i). To prove the sufficiency, the first step is to show that, if (5) holds, then the first two (6), (7) of the proposed compensator represent an extended observer [30], [31] for the system.[3] Multiply the plant state equation by R and subtract (6)

$$\begin{aligned}\delta\left(Rx(t)-w(t)\right) &= RA_ix(t)+RB_iu(t)-Q_iw(t)\\ &\quad +L_iy(t)-RB_iu(t)\\ &= RA_ix(t)-Q_iw(t)+L_iC_ix(t)\\ &= Q_i\left(Rx(t)-w(t)\right).\end{aligned}$$

Since $Q_i \in \mathcal{H}_\infty$ (or $\|Q_i\|_\infty < 1$) the corresponding system is stable (see Remark 4.1) and therefore $\|Rx(t)-w(t)\| \to 0$ and, in view of (7),

$$M\left[Rx(t)-w(t)\right] = x(t)-\hat{x}(t) \to 0.$$

As a second step, we show that if (4) holds, then the (8), (9) represent a stabilizing state feedback compensator (if we replace $\hat{x}(t)$ by $x(t)$). Indeed, the closed-loop matrix satisfies the equation

$$\begin{bmatrix} A_i+B_iK_i & B_iH_i\\ G_i & F_i\end{bmatrix}\begin{bmatrix} X\\ Z\end{bmatrix} = \begin{bmatrix} X\\ Z\end{bmatrix}P_i$$

as it can be immediately verified from (10). Since, by construction, $[X^T\ Z^T]^T$ is invertible, in view of Lemma 4.1, the state feedback is stabilizing.

To prove closed-loop switching stability, consider the new variable

$$e(t)=w(t)-Rx(t), \quad \text{so that} \quad \hat{x}(t)-x(t)=Me(t) \tag{19}$$

Assuming $x(t)$, $z(t)$ and $e(t)$ as state variables, we get the following overall closed-loop matrix

$$\begin{bmatrix} A_i+B_iK_i & B_iH_i & B_iK_iM\\ G_i & F_i & G_iM\\ 0 & 0 & Q_i\end{bmatrix}. \tag{20}$$

The corresponding block-triangular switching system is stable if and only if its diagonal blocks are switching stable ([29], Part I). The first diagonal block (which comes from the "state feedback") has been just proved to be switching stable. The "error system" governed by Q_i is also stable (see Remark 4.1) so the proof is completed. ∎

B. *Proof of Theorem 3.2*

We give a formal proof of Theorem 3.2 in the continuous-time case only. The proof of Theorem 3.3, the discrete-time version, is similar.

Proof: **i) $\Rightarrow$ ii)** If (1) is quadratically stabilized by (2), then by definition there exists a symmetric positive definite matrix P such that $(A_i^{cl})P + P(A_i^{cl})^T < 0$. After a proper partition, we write

$$\begin{bmatrix} A_i+B_iK_iC_i & B_iH_i\\ G_iC_i & F_i\end{bmatrix}\begin{bmatrix} P_{11} & P_{12}\\ P_{12}^T & P_{22}\end{bmatrix} + \begin{bmatrix} P_{11} & P_{12}\\ P_{12}^T & P_{22}\end{bmatrix}\begin{bmatrix} A_i+B_iK_iC_i & B_iH_i\\ G_iC_i & F_i\end{bmatrix}^T < 0.$$

[3]This was shown in [15] in the LPV context, with B and C constant matrices, and it is reported here for completeness.

If we take the left upper block of the previous expression, denoting by $U_i \doteq K_iC_iP_{11} + H_iP_{12}^T$, we have

$$\begin{aligned}&(A_i+B_iK_iC_i)P_{11}+(B_iH_i)P_{12}^T+P_{11}(A_i+B_iK_iC_i)^T+\\ &\quad P_{12}(B_iH_i)^T = A_iP_{11}+P_{11}A_i^T+B_iU_i+U_i^TB_i<0\end{aligned}$$

which is (11). The dual (12) is derived exactly in the same way.

ii) $\Rightarrow$ i). Assume that (11) and (12) hold and take the linear gains $J_i = U_iP^{-1}$ and $L_i = Q^{-1}Y_i$. Consider now the observer-based feedback compensator (15) and consider the variables $x_1(t) = x(t)$ and $x_2(t) = \hat{x}(t) - x(t)$ to achieve the closed-loop system matrix

$$\begin{bmatrix} A_i+B_iJ_i & B_iJ_i\\ 0 & A_i+L_iC_i\end{bmatrix}.$$

Both diagonal blocks are quadratically stable with Lyapunov matrices given by P and Q, respectively, since

$$\begin{aligned}&P(A_i+B_iJ_i)^T+(A_i+B_iJ_i)P\\ &\quad = PA_i^T+U_i^TB_i^T+A_iP+B_iU_i<0\\ &(A_i+L_iC_i)^TQ+Q(A_i+L_iC_i)\\ &\quad = A_i^TQ+C_i^TY_i^T+QA_i+Y_iC_i<0\end{aligned}.$$

It is then trivial to prove that this system is quadratically stable since, by the previous inequalities,

$$\begin{bmatrix} P & 0\\ 0 & \theta Q^{-1}\end{bmatrix}\begin{bmatrix} A_i+B_iJ_i & B_iJ_i\\ 0 & A_i+L_iC_i\end{bmatrix}^T + \begin{bmatrix} A_i+B_iJ_i & B_iJ_i\\ 0 & A_i+L_iC_i\end{bmatrix}\begin{bmatrix} P & 0\\ 0 & \theta Q^{-1}\end{bmatrix} < 0$$

provided that $\theta > 0$ is large enough. ∎

C. *Proof of Theorem 3.4*

The proof of this result is quite technical in the general case and much simpler in the quadratic stability framework since it involves tools which are available on most (advanced) books. Therefore, we first consider the quadratic case. Let us first recall the following fundamental lemma, which gives an observer based interpretation of the Youla-Kucera parametrization of all stabilizing compensators.

Lemma 4.2: [5] Given a stabilizable plant (A, B, C), the set of all stabilizing linear compensators is given by

$$\begin{aligned}\delta\hat{x}(t) &= (A+LC+BJ)\hat{x}(t)-Ly(t)+Bv(t)\\ u(t) &= J\hat{x}(t)+v(t)\\ o(t) &= C\left(\hat{x}(t)-x(t)\right)\\ v(s) &= T(s)o(s)\end{aligned} \tag{21}$$

where $A+BJ$ and $A+LC$ are Hurwitz (Schur) and $T(s)$ is a suitable stable transfer function.

Proof: (Theorem 3.4, the Quadratic Stability Case): The proof of Theorem 3.4 under the quadratic stabilizability conditions of Theorem 3.2 is easy. Indeed the structure of the compensator (15) is exactly that prescribed in Lemma 4.2. Therefore there exists a proper $T_i(s)$ such that the i-th transfer function is any $\mathcal{K}_i(s)$ which makes the i-th plant Hurwitz. The only remaining point is the determination of proper switching stable realizations for the family $T_i(s)$. In view of [23], such realizations can be determined according to Procedure 3.1, Step 4. Intuitively, if we take such a realization, then "$T_i(s)$ sees no feedback" since $o(t) \to 0$ no matter what $v(t)$ does, therefore the overall system remains switching stable. We will come back on this issue later to provide a formal proof. ■

The next lemma shows how the parametrization of [5], based on a Luenberger observer, can be extended if we adopt a generalized observer, as defined in [30], [31].

Lemma 4.3: Given a stabilizable plant (A, B, C), assume that there exist a Hurwitz (Schur) matrix Q, a matrix L, a full column rank matrix R, a full row rank matrix M such that

$$RA + LC = QR, \quad MR = I \tag{22}$$

and matrices F, G, H, K such that

$$u(s) = \left[H(sI - F)^{-1}G + K\right] x(s)$$

is a stabilizing state feedback compensator. Then

$$\begin{aligned} \delta w(t) &= Qw(t) - Ly(t) + RBu(t) &(23)\\ \hat{x}(t) &= Mw(t) &(24)\\ \delta z(t) &= Fz(t) + G\hat{x}(t) &(25)\\ u(t) &= Hz(t) + K\hat{x}(t) + v(t) &(26)\\ o(t) &= C\left(\hat{x}(t) - x(t)\right) &(27)\\ v(s) &= T(s)o(s) &(28) \end{aligned}$$

parametrize all the stabilizing compensators for (A, B, C).

Proof: The fact that any compensator of the form (23)–(28) is stabilizing is not proved here, since it will be proved in the more general (switching) case later. Thus we prove now that any given $\mathcal{K}(s)$ which stabilizes (A, B, C) can be written as in (23)–(28).

Augment (22) as follows

$$[R \quad S]\begin{bmatrix} A & \Phi \\ 0 & A_\eta \end{bmatrix} + L[C \quad 0] = Q[R \quad S] \tag{29}$$

where S is a complement of R (i.e. $[R\ S]$ is invertible) and

$$\begin{bmatrix} \Phi \\ A_\eta \end{bmatrix} = [R \quad S]^{-1}QS.$$

We need the following:

Claim 4.1: Matrix S can be taken in such a way that A_η is Hurwitz (Schur). The proof is given in Appendix B.

Therefore, assume that A_η is stable. Consider the following transformation[4]

$$w = [R \quad S]\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \end{bmatrix} \text{ or } \begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \end{bmatrix} = [R \quad S]^{-1}w$$

[4]We are putting "hats" on these variables because they will be shown to be estimators.

for (23) and (24) so as to achieve

$$\begin{bmatrix} \delta\hat{\varphi} \\ \delta\hat{\eta} \end{bmatrix} = [R \ \ S]^{-1}Q[R \ \ S]\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \end{bmatrix} - [R \ \ S]^{-1}Ly + [R \ \ S]^{-1}RBu$$

namely

$$\begin{bmatrix} \delta\hat{\varphi} \\ \delta\hat{\eta} \end{bmatrix} = \begin{bmatrix} A + \hat{L}_1C & \Phi \\ \hat{L}_2C & A_\eta \end{bmatrix}\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \end{bmatrix} - \begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 \end{bmatrix} y + \begin{bmatrix} B \\ 0 \end{bmatrix} u \tag{30}$$

$$\hat{x} = \hat{\varphi} + MS\hat{\eta} \tag{31}$$

where

$$\begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 \end{bmatrix} = [R \quad S]^{-1}L.$$

Now we apply the transformation to (25) and (26) to achieve (we use the dummy substitution $\hat{\zeta} = z$)

$$\begin{aligned} \delta\hat{\zeta} &= F\hat{\zeta} + G\hat{\varphi} + GMS\hat{\eta} &(32)\\ u &= H\hat{\zeta} + K\hat{\varphi} + KMS\hat{\eta} + v. &(33) \end{aligned}$$

The (30)–(33) represent an observer-based stabilizing compensator for the fictitiously extended plant

$$\begin{bmatrix} \delta\varphi \\ \delta\eta \\ \delta\zeta \end{bmatrix} = \underbrace{\begin{bmatrix} A & \Phi & 0 \\ 0 & A_\eta & 0 \\ 0 & 0 & A_\zeta \end{bmatrix}}_{A_{aug}}\begin{bmatrix} \varphi \\ \eta \\ \zeta \end{bmatrix} + \underbrace{\begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}}_{B_{aug}}\begin{bmatrix} u \\ \sigma \end{bmatrix} \tag{34}$$

$$y = \underbrace{[C \quad 0 \quad 0]}_{C_{aug}}\begin{bmatrix} \varphi \\ \eta \\ \zeta \end{bmatrix} \tag{35}$$

where A_ζ is an arbitrary stable matrix and σ a fictitious input. Let us now observe that the η and ζ dynamics are, respectively, unreachable and unobservable. Note also that in no way σ can stabilize/destabilize the system. It is apparent that $\mathcal{K}(s)$ stabilizes the original plant if and only if

$$\begin{bmatrix} \mathcal{K}(s) \\ 0 \end{bmatrix} \tag{36}$$

stabilizes (34), (35). Since, by construction, (34), (35) is stabilizable (the original (A, B, C) system is such, and the added unreachable/unobservable dynamics are asymptotically stable), the set of all stabilizing compensators, according to Lemma 4.2 applied to the extended plant, is given by

$$\begin{aligned} \begin{bmatrix} \delta\hat{\varphi} \\ \delta\hat{\eta} \\ \delta\hat{\zeta} \end{bmatrix} &= \begin{bmatrix} A & \Phi & 0 \\ 0 & A_\eta & 0 \\ 0 & 0 & A_\zeta \end{bmatrix}\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \\ \hat{\zeta} \end{bmatrix} \\ &+ \begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\underbrace{\begin{bmatrix} K & KMS & H \\ G & GMS & F - A_\zeta \end{bmatrix}}_{J_{aug}}\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \\ \hat{\zeta} \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 0 \end{bmatrix}}_{L_{aug}}[C\ 0\ 0]\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \\ \hat{\zeta} \end{bmatrix} - \begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 \\ 0 \end{bmatrix} y \end{aligned}$$

$$+\begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} v \\ \rho \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} u \\ \sigma \end{bmatrix} = \begin{bmatrix} K & KMS & H \\ G & GMS & F-A_\zeta \end{bmatrix}\begin{bmatrix} \hat{\varphi} \\ \hat{\eta} \\ \hat{\zeta} \end{bmatrix} + \begin{bmatrix} v \\ \rho \end{bmatrix} \tag{38}$$

$$\begin{bmatrix} v \\ \rho \end{bmatrix} = \begin{bmatrix} T(s) \\ \tilde{T}(s) \end{bmatrix} o(s) = \begin{bmatrix} T(s) \\ \tilde{T}(s) \end{bmatrix} [\hat{y}-y] \tag{39}$$

therefore the compensator (36) can be written in this form. The proof is now concluded if one notices that (37)–(39) is exactly of the form (30)–(33) (if one disregards the fictitious input σ).■

We are now in the position of proving Theorem 3.4.

Proof: (Theorem 3.4, Switching Stability): Consider the compensator (6)–(9) with the input injection (16).

Denote by $z^{(T)}$ the state of any switching stable realization for the (stable) transfer functions $T_i(s)$

$$\delta z^{(T)}(t) = F_i^{(T)} z^{(T)}(t) + G_i^{(T)} o(t), \tag{40}$$

$$v(t) = H_i^{(T)} z^{(T)}(t) + K_i^{(T)} o(t) \tag{41}$$

and consider the state variables $x(t)$, $z^{(T)}(t)$, $z(t)$, and $e(t) = w(t) - Rx(t)$. Without restriction, we assume that *all these realizations have the same dimension*, a condition which can be possibly met by adding stable redundant dynamics. Then, bearing in mind that $MR = I$, that from (19) $\hat{x} - x = Me$ and that from (27) $o = CMe$, we have that

$$\begin{aligned} \delta x &= A_i x + B_i u = A_i x + B_i K_i \hat{x} + B_i H_i z + B_i v \\ &= A_i x + B_i K_i x + B_i K_i M e + B_i H_i z \\ &\quad + B_i \left(K_i^{(T)} o + H_i^{(T)} z^{(T)}\right) \\ &= (A_i + B_i K_i) x + B_i \left(K_i + K_i^{(T)} C_i\right) Me \\ &\quad + B_i H_i z + B_i H_i^{(T)} z^{(T)} \end{aligned}$$

and that

$$\delta z = F_i z + G_i \hat{x} = F_i z + G_i x + G_i M e.$$

Since $\delta e = Q_i e$, we can cast the closed-loop system in the form $\delta x_{cl} = A_i^{cl} x_{cl}$ with $x_{cl} = [x^T z^T {z^{(T)}}^T e^T]^T$ and

$$A_i^{cl} = \begin{bmatrix} A_i + B_i K_i & B_i H_i & B_i H_i^{(T)} & B_i \left(K_i + K_i^{(T)} C_i\right) M \\ G_i & F_i & 0 & G_i M \\ 0 & 0 & F_i^{(T)} & G_i^{(T)} C_i M \\ 0 & 0 & 0 & Q_i \end{bmatrix}.$$

This matrix is in an upper block-triangular form and then the system is stable if and only if the diagonal blocks are such ([26], Part 1, Th. 1). The stability of the first block has been proved in Section IV-A (see (17)). The second block is stable because we took a switching stable realization for the Youla-Kucera parameters ($F_i^{(T)}$ is switching stable) and the subsystem governed by Q_i is switching stable (see Remark 4.1). Thus the proposed construction produces a switching stabilizing compensator for any switching stable realization (40), (41) of $T_i(s)$.

The proof is now over in view of Lemma 4.3. Indeed for fixed $i \in \mathcal{I}$, (23)–(28) represent any stabilizing compensator for the i-th plant, with proper functions $T_i(s)$. ■

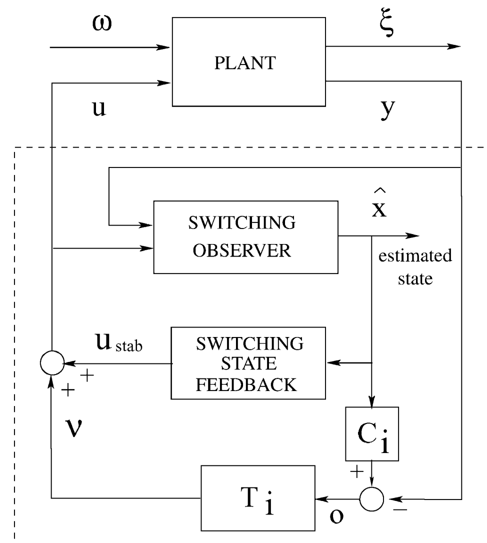

Fig. 3. The optimization setup.

We conclude the section pointing out that, since we eventually come up with a Lyapunov function, a generalization of Assumption 1 is possible, because as long as we can define generalized solution we could show that the corresponding Lyapunov derivative is negative almost everywhere.

V. Implications of the Results

A. Switching Systems and Optimization

Consider the problem of optimizing a set of plants

$$\begin{aligned} \delta x(t) &= A_i x(t) + B_i u(t) + B_i^{\omega} \omega(t) \\ y(t) &= C_i x(t) + D_i^{y,\omega} \omega(t) \\ \xi(t) &= E_i x(t) + D_i^{\xi,u} u(t) + D_i^{\xi,\omega} \omega(t) \end{aligned} \tag{42}$$

according to some criteria. Here $\omega(t)$ and $\xi(t)$ represent any disturbance-input and performance-output relevant to the problem. Assume that a family of compensators $\mathcal{K}_i(s)$ (optimal for fixed i) is designed. Then, in view of Theorem 3.4, we can combine them together according to the following proposition.

Proposition 5.1: If the switching system satisfies the necessary and sufficient conditions, then there exists a switching compensator (F_i, G_i, H_i, K_i) which

- achieves optimality for any fixed configuration i;
- assures stability under switching.

Once we have computed the observer-based pre-compensator (see Fig. 3) the i-th input output map is of the form

$$\xi(s) = \left[M_i^{\xi,\omega}(s) + M_i^{\xi,v}(s) T_i(s) M_i^{o,\omega}(s)\right] \omega(s)$$

where $M_i^{\xi,\omega}(s)$, $M_i^{\xi,v}(s)$ and $M_i^{o,\omega}(s)$ are the ω-to-ξ, v-to-ξ and ω-to-o transfer functions for the i-th configuration. This is an important feature, since it allows for the well-known Wiener-Hopf design [25] for the ith transfer function (which is affine with respect to T_i). Stability is eventually assured by the realization (40), (41).

As a final remark we stress an essential difference from the previous (inspiring) work [23]. Roughly, Hespanha and Morse idea is to show that, for a single plant you may construct *a single* pre-compensator and freely "switch among the Youla-Kucera parameters $T_i(s)$". In brief a single M instead of M_i is sufficient for a single plant. Here we extend the idea, but we cannot rely, in general, on a single pre-compensator.

B. Contractive Design

In practice, stability may be a goal which is not sufficient and one might require the overall switching system to satisfy certain performance requirements in terms of convergence speed. For a single plant this boils down to assigning the poles. The natural question is whether we can enforce a convergence speed *even under switching*. We consider for brevity the discrete-time case only. Assume that we wish to assure a speed of convergence λ, $0 \leq \lambda < 1$ to the system, namely

$$\|x(t)\| \leq \gamma \|x(0)\| \lambda^t.$$

Then we can work with the modified system

$$\begin{aligned} \delta x(t) &= \frac{A_i}{\lambda} x(t) + B_i u(t) \\ y(t) &= C_i x(t). \end{aligned} \tag{43}$$

For this switching system to admit a solution, the necessary and sufficient conditions (4) and (5) must hold

$$A_i X + B_i(\lambda U_i) = X(\lambda P_i) \tag{44}$$

$$R A_i + (\lambda L_i) C_i = (\lambda Q_i) R. \tag{45}$$

If the above are satisfied, it is possible to set $\lambda U_i = \hat{U}_i$, $\lambda L_i = \hat{L}_i$ and, most importantly

$$\hat{P}_i = \lambda P_i, \quad \text{and} \quad \hat{Q}_i = \lambda Q_i.$$

It is immediate to see that, since $\|Q_i\|_\infty < 1$ and $\|P_i\|_1 < 1$, then $\|\hat{Q}_i\|_\infty < \lambda$ and $\|\hat{P}_i\|_1 < \lambda$. These new matrices are responsible for the convergence speed of the overall switching system (even under switching) and this assures a speed of convergence λ with the pre-compensator (6)–(9). It is not difficult to see that, given any choice of $T_i(z)$ with all the poles in the open λ-disk, it is possible to find proper realizations which are λ-switching stable[5] and thus the overall realization of the compensator+Youla-Kucera parameter assures a global λ-convergence. Working with the modified system is basically equivalent to fix λ in the procedure to determine the polyhedral function, reported in Appendix A.

C. The Zero Transfer Functions Paradox

Assume that we are given a plant composed by a (finite) family of switching systems which are Hurwitz (Schur), but not switching stable. According to Theorem 3.4, if this plant is switching stabilizable, then we can apply a switching compensator such that the system is switching stabilized, but, at the same time, for any fixed i the compensator transfer function is $\mathcal{K}_i(s) \equiv 0$. A potential application of this property is what we call the switching manager (Fig. 4), a device which leaves the plant uncontrolled as long as it remains on a fixed configuration

[5]i.e. the associated switching system assures a convergence speed λ.

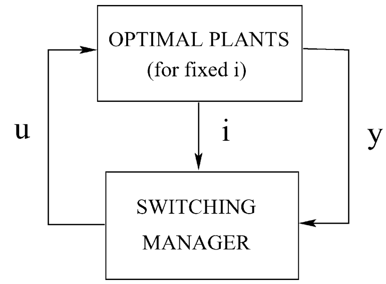

Fig. 4. The switching manager: a switching stabilizing control having 0 transfer function for each i.

(for instance because optimal compensators are already applied for each i) and produces a control action only under switching.

Let (A_i, B_i, C_i) be a continuous-time switching plant. For the simple exposition, let us assume quadratic stabilizability (the property holds also in the general case). Let J_i and L_i be the linear gains obtained as in Corollary 3.2. A switching stabilizing observer-based compensator is the following:

$$\begin{aligned} \dot{\hat{x}} &= A_i \hat{x} + L_i(C_i \hat{x} - y) + B_i u \\ u &= J_i \hat{x} + v \\ \dot{z} &= \Omega_i A_i \Omega_i^{-1} z + \Omega_i L_i (C_i \hat{x} - y) \\ v &= -J_i \Omega_i^{-1} z \end{aligned} \tag{46}$$

where Ω_i is the transformation previously introduced (chosen in such a way that $\Omega_i A_i \Omega_i^{-1}$ is switching stable and admits $z^T z$ as a common Lyapunov function). Let us write it as

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i + B_i J_i + L_i C_i & -B_i J_i \Omega_i^{-1} \\ \Omega_i L_i C_i & \Omega_i A_i \Omega_i^{-1} \end{bmatrix} \begin{bmatrix} \hat{x} \\ z \end{bmatrix} + \begin{bmatrix} -L_i \\ -\Omega_i L_i \end{bmatrix} y$$

$$u = \begin{bmatrix} J_i & -J_i \Omega_i^{-1} \end{bmatrix} \begin{bmatrix} \hat{x} \\ z \end{bmatrix}.$$

It is immediate to see that (for any choice of Ω_i) the transfer function from y to u is zero. Indeed, consider the following change of variables

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{x} - \Omega_i^{-1} z \end{bmatrix}, \quad \begin{bmatrix} \hat{x} \\ z \end{bmatrix} = \begin{bmatrix} q_1 \\ \Omega_i (q_1 - q_2) \end{bmatrix}$$

to get

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} A_i + L_i C_i & B_i J_i \\ 0 & A_i + B_i J_i \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} -L_i \\ 0 \end{bmatrix} y$$

$$u = [0 \quad J_i] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

This compensator has 0-transfer function, for fixed i, as it can be verified by direct calculation (the system Markov parameters are all 0). However, to better explain the phenomenon, we let the reader notice that the two components q_1 and q_2 of the vector q are unobservable and unreachable respectively. From a geometric point of view, the reachable subspace is included in the q_1-subspace and thus in the unobservable subspace. Since the transformation depends on i, this property is clearly true for fixed i only and we will see the consequences soon.

To check the overall stability consider the state vector composed by x, $e = \hat{x} - x$ and z. The resulting equations become

$$\begin{bmatrix} \dot{x} \\ \dot{e} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i + B_i J_i & B_i J_i & -B_i J_i \Omega_i^{-1} \\ 0 & A_i + L_i C_i & 0 \\ 0 & \Omega_i L_i C_i & \Omega_i A_i \Omega_i^{-1} \end{bmatrix} \begin{bmatrix} x \\ e \\ z \end{bmatrix}$$

$$y = [C_i \quad 0 \quad 0] \begin{bmatrix} x \\ e \\ z \end{bmatrix}.$$

Given its upper block-triangular structure, the state matrix is switching stable.

As a simple example, consider the system with two vertices

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2+\gamma & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [1 \quad 0] x \tag{47}$$

where $\gamma \in \{-1, 1\}$. Denote by A_1 and A_2 the values of the state matrix for $\gamma = -1$ and 1, respectively. Although each A_i is Hurwitz, the time-varying system governed by the switch rule $\gamma(t) = sign[x_1(t)x_2(t)]$ (x_i represents the state component) is unstable (see [8] for details). The phase portrait of the free evolution with the adopted switching rule and $x(0) = [0\ 1]^T$ is reported in Fig. 5 (top). By means of the next observer and state feedback gains

$$L_1 = L_2 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad J_1 = J_2 = [0 \quad -10]$$

switching stability is assured. A switching regulator can be realized in such a way that its transfer function is zero for each i which is given by (46) with

$$\Omega_1 = \begin{bmatrix} 14.142 & 0.0075 \\ 0.0075 & 8.165 \end{bmatrix}, \quad \text{and} \quad \Omega_2 = \begin{bmatrix} 10.001 & 0.0245 \\ 0.025 & 10.00 \end{bmatrix}.$$

The state-space evolution of the closed-loop system is reported in Fig. 5 (bottom). Finally, the control input evolution (which becomes active only under switching) is reported in Fig. 6.

Although the fact that switching stability for stabilizable Hurwitz plants can be achieved with a compensator such that $\mathcal{K}_i(s) \equiv 0$, is a consequence of Theorem 3.4, we would like to give a simple explanation for this paradox. The proposed technique produces a compensator realization which is non-minimal. In fact, for fixed i, the realization reachable and observable subspaces have trivial intersection. The situation is different under switching since the actual "reachable" and "observable" subspace are "larger" than those of the single realizations (see [39], Section IV, for details). The "separation" between the reachable and observable subsystems is not true anymore and the interaction (possible under switching) implies that the input-output map is non-zero. This allows for a stabilizing feedback action.

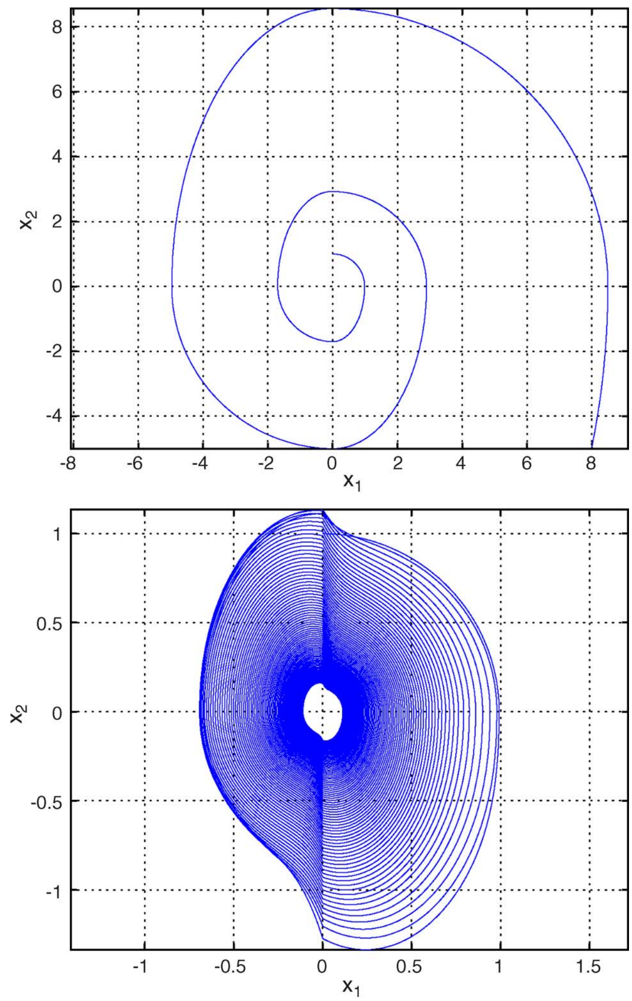

Fig. 5. Unstable state-space evolution of system (47) with switching rule $\gamma(t) = sign[x_1(t)x_2(t)]$ (top) and the corresponding evolution with the switching manager (bottom).

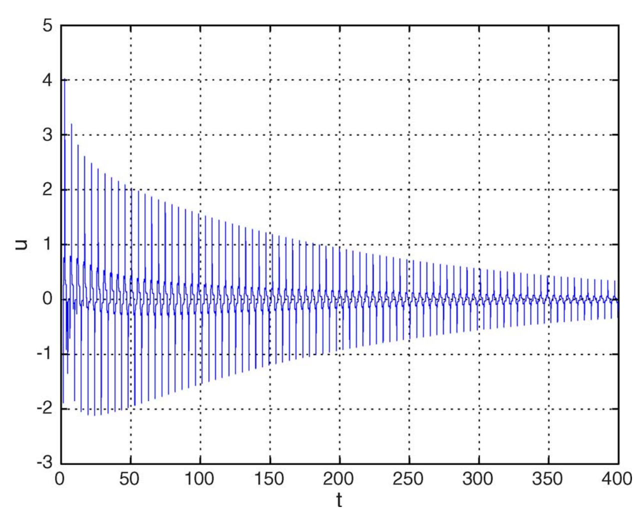

Fig. 6. Input evolution of system (47) in closed-loop.

VI. EXAMPLE

We present a very simple (academic) example to show that instability problems due to switching can arise in very simple cases. Consider the system in Fig. 7 where a fluid flows through two-tanks. The state values are the reservoir levels (with respect to their nominal values-the dashed lines). The flow between the two reservoirs is proportional to the difference of their levels. We assume that both flow control and level measurement can switch arbitrarily from the first tank (actuator 1 -sensor 1) to the second tank (actuator 2 -sensor 2). We assume that, if uncontrolled, the incoming and outgoing flows are constant and

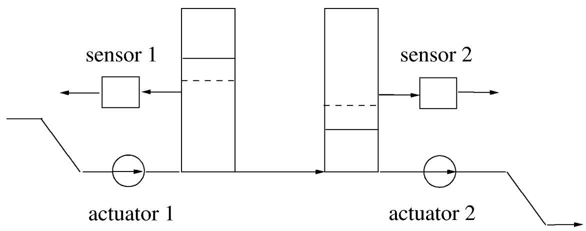

Fig. 7. The two-tank system.

equal to the nominal values. For this system the state matrix remains clearly unchanged while the input and output matrices can switch. We derive the next model with two vertices (including input and measurements noises)

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}x(t) + B_i u(t) + w_x(t)\\ y(t) &= C_i x(t) + w_y(t)\end{aligned}$$

where the ones in A are achieved by time-normalization and where the input and output (normalized) matrices are

$$\begin{aligned}B_1 &= \begin{bmatrix}1\\0\end{bmatrix}, \quad B_2 = \begin{bmatrix}0\\-1\end{bmatrix}\\ C_1 &= [0 \quad 1], \quad C_2 = [1 \quad 0].\end{aligned}$$

We computed the optimal LQ and Kalman gains $J_i^{(opt)}$ and $L_i^{(opt)}$ minimizing, respectively the cost

$$\int_0^\infty \left(1000x_1^2(t) + 10000x_2^2(t) + \frac{1}{10000}u^2(t)\right)dt$$

and the estimated state error covariance when the process noise w_x and output noise w_y (Gaussian white noises with zero mean) have covariance matrices

$$Q_{w_x} = \begin{bmatrix}1/10 & 0\\ 0 & 1/10\end{bmatrix} \quad Q_{w_y} = \frac{1}{1000}.$$

To enforce a certain convergence rate to the overall LQG controller, we shifted to the right the poles of the original system by a value $\alpha = 2$ (say, we solved the LQ and Kalman equations when the state matrix is $A + 2I$). The resulting optimal compensators $\mathcal{K}_i(s)$ are

$$\begin{aligned}\dot{q} &= \left(A + B_i J_i^{(opt)} + L_i^{(opt)} C_i\right)q - L_i^{(opt)} y\\ u &= J_i^{(opt)} q, \quad i = 1, 2\end{aligned}$$

with

$$\begin{aligned}J_1^{(opt)} &= [-3167.59 \quad -13655.68]\\ J_2^{(opt)} &= [20491.14 \quad 10003.05]\end{aligned}$$

and

$$L_1^{(opt)} = \begin{bmatrix}-27.64\\-13.50\end{bmatrix} \quad L_2^{(opt)} = \begin{bmatrix}-13.50\\-27.64\end{bmatrix}.$$

Unfortunately we have the following

Claim 6.1: If the controller are implemented directly, we achieve the following closed-loop matrices

$$\hat{A}_1^{cl} = \begin{bmatrix}-3168.59 & -13682.32 & 0 & 27.6436\\ 1 & -14.5014 & 0 & 13.5014\\ -3167.59 & -13655.69 & -1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}$$
$$\hat{A}_2^{cl} = \begin{bmatrix}-14.5015 & 1 & 13.5015 & 0\\ -20517.78 & -10004.04 & 27.6436 & 0\\ 0 & 0 & -1 & 1\\ -20491.14 & -10003.05 & 1 & -1\end{bmatrix}$$

and the resulting system is switching unstable.

The reader can easily verify the claim by computing the exponential matrices and checking that the product $exp[A_1^{cl}T]exp[A_2^{cl}T]$ for $T = 0.1$ is not Schur. Therefore, the continuous-time system diverges if we activate the two systems alternatively in sampling periods of length $T = 0.1$.

However, the above switching system passes the switching stabilizability test (even quadratic)[6] and then we can compute any compensator for each subsystem and then realize it according to the proposed results. We computed the stabilizing switching controllers guaranteeing a convergence rate $\alpha = 2$. As a first step, we solved the two (4) and (5) for the shifted systems and we derived the stabilizing controllers

$$\mathcal{K}_1^{stab} : \begin{bmatrix}\dot{w}\\u\\o\end{bmatrix} = \left[\begin{array}{cc|cc}-2 & 14.4 & 0 & -1\\ -10 & -24 & 20 & \frac{5}{6}\\ \hline 0 & -14.4 & 0 & 1\\ \frac{1}{2} & \frac{3}{5} & -1 & 0\end{array}\right]\begin{bmatrix}w\\y\\v\end{bmatrix}$$
$$\mathcal{K}_2^{stab} : \begin{bmatrix}\dot{w}\\u\\o\end{bmatrix} = \left[\begin{array}{cc|cc}-2 & -14.4 & 0 & -1\\ 10 & -24 & 20 & -\frac{5}{6}\\ \hline 0 & 14.4 & 0 & 1\\ -\frac{1}{2} & \frac{3}{5} & -1 & 0\end{array}\right]\begin{bmatrix}w\\y\\v\end{bmatrix}.$$

As a second step, we computed the Youla-Kucera parameters $T_i(s)$ (and we derived a switching stable realization of the two) in such a way that the lower linear fractional transformation [5] of the switching stabilizing controllers and $T_i(s)$, depicted in Fig. 8, results in the desired controllers $\mathcal{K}_i(s)$. The state-space realizations of the (switching stabilizing) regulators $\mathcal{K}_1$ and $\mathcal{K}_2$ are reported Table I. Fig. 9 depicts the time evolution of the system state and the switching signal.

VII. Conclusion

In this paper, we have given necessary and sufficient stabilizability conditions for the existence of a stabilizing and quadratically stabilizing switching linear compensators for a switching plant. We have seen that if these conditions are satisfied, no matter how we associate a family of compensators with a family of plants, then there exist realizations for which the closed-loop system is switching stable. We have shown how to derive these realizations. In this way we extended the results in [23], where

[6]We used the efficient code by M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming [web page and software]. http://www.stanford.edu/boyd/cvx, October 2007.

TABLE I
THE STABILIZING COMPENSATOR REALIZATIONS

$$\mathscr{K}_1 : \left[\begin{array}{c} \dot{q} \\ \hline u \end{array}\right] = \left[\begin{array}{cccccc|c} -2 & 14.4 & 40355 & 9368.1 & 385.55 & -10.812 & 0 \\ -10 & -24 & -33629 & -7806.8 & -321.29 & 9 & 20 \\ -3.825 & -4.59 & -3000.1 & -700.17 & -28.78 & 0.77 & 7.65 \\ 1.9684 & 2.3621 & -700.61 & -169.96 & -5.7987 & 0.094382 & -3.9369 \\ 0.23811 & 0.28573 & -31.484 & -9.823 & -2.9053 & 0.13585 & -0.47621 \\ -0.01011 & -0.012132 & 1.0243 & 0.2528 & 0.14111 & -12.101 & 0.02022 \\ \hline 0 & -14.399 & -40355 & -9368.1 & -385.55 & 10.812 & 0 \end{array}\right] \left[\begin{array}{c} q \\ \hline y \end{array}\right]$$

$$\mathscr{K}_2 : \left[\begin{array}{c} \dot{q} \\ \hline u \end{array}\right] = \left[\begin{array}{cccccc|c} -2 & -14.4 & -32338 & 100043 & 433.14 & -21.812 & 0 \\ 10 & -24 & -26948 & 83370 & 360.95 & -18.176 & 20 \\ 3.4642 & -4.157 & -947.27 & 2922.4 & 13.162 & -0.68434 & 6.9283 \\ -1.6446 & 1.9735 & 2922.5 & -9058.6 & -38.815 & 1.8882 & -3.2892 \\ 0.24837 & -0.29804 & 12.43 & -41.824 & -2.6076 & 0.25012 & 0.49673 \\ -0.0099683 & 0.011962 & -0.73804 & 2.2786 & 0.25167 & -12.096 & -0.019937 \\ \hline 0 & 14.4 & 32338 & -100043 & -433.14 & 21.812 & 0 \end{array}\right] \left[\begin{array}{c} q \\ \hline y \end{array}\right]$$

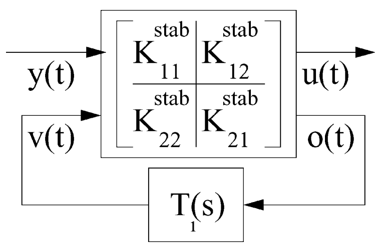

Fig. 8. Linear fractional transformation.

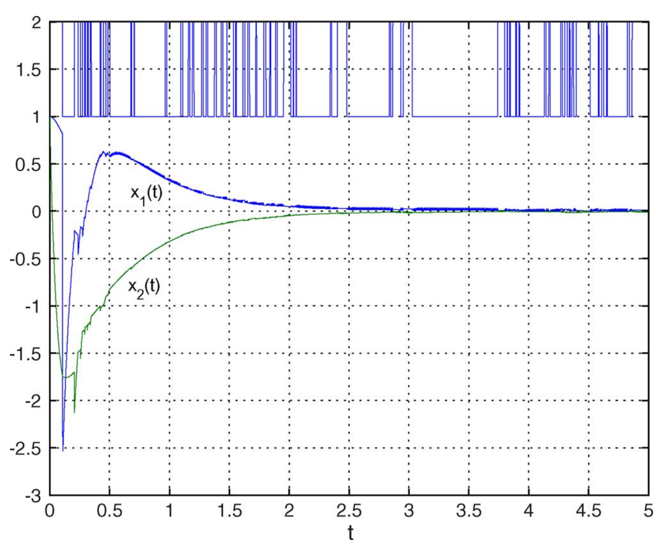

Fig. 9. The state components and switching signal (top) evolutions.

switching compensators are applied to a single linear plant. The results have several implications such as the "paradox of the zero transfer functions", with its application to the switching manager, the optimal Wiener-Hopf synthesis and the contractive design. The general stabilizability conditions suffer of the well known problem of the complexity of the required algorithms and of the complexity of the compensator. Conversely, if we resort to the quadratic stabilizability, efficient LMI algorithms are involved and the compensator has a complexity which is fixed a-priori.

The results are suitable for several extensions and we believe that there are several interesting issues to be investigated.

In the formulation of the problem we assumed that switching is uncontrolled and that the compensator switches accordingly, with full information and no delay. It would be interesting to investigate the case of delayed information or the case in which the controller has to identify the current system configuration. For the latter problem the observer-based structure of the proposed compensator seems to be promising [36], [40]. In this work we have not considered the case in which the plant is switched, namely the discrete variable is a control signal (see for instance [39]). However, we believe that the presented results could be successfully applied to the switched case in the sense that the scheme can be used to enforce stability jointly with another scheme which controls switching to optimize performances. Another issue not currently investigated is whether the current results can be extended to LPV systems rather than switching systems. We conjecture that as long as B and C are constant and known matrices, results along these line can be established. Finally we would like to point out that we have seen how to optimize the performances in each configuration while assuring switching stability. The overall optimality problem is a non-trivial problem left to future investigation. The ideas proposed in the recent work [41] seem to be promising in this sense.

APPENDIX A
SOLUTION OF (4) AND (5)

Equation (4) can be solved as follows in the discrete-time case.

Procedure A.1: Iterative Solution of (4) for the Determination of X, U_i and P_i:

1) Fix a positive $\lambda < 1$, and $\epsilon > 0$. Set $k = 0$. Fix any arbitrary symmetric polytope $\mathcal{S} = \mathcal{S}^{(0)}$ including the origin in its interior.
2) Consider the plane representation $\mathcal{S}^{(k)} = \{x : \Phi^k x \leq \bar{1}\}$, and consider the polyhedral sets
$$\mathcal{R}_i^{(k)} = \left\{(x, u) : \Phi^k (A_i x + B_i u) \leq \lambda \bar{1}\right\}.$$
3) Consider the projection of these sets, i.e. the sets of all x for which there exists u such that $(x, u) \in \mathcal{R}_i^{(k)}$, and determine their plane representation
$$\mathcal{P}_i^{(k)} = \left\{x : \exists u : \Phi^k (A_i x + B_i u) \leq \bar{1}\right\} = \{x : \tilde{\Phi}^k x \leq \bar{1}\}.$$
4) Determine the intersection
$$\mathcal{P}^{(k)} = \bigcap_i \mathcal{P}_i^{(k)}.$$

It is easy to see that this set has this property: for all $x \in \mathcal{P}^{(k)}$ and any given i there exists $u_i(x)$ such that $A_i x + B_i u_i(x) \in \mathcal{S}^{(k)}$

5) Let

$$\mathcal{S}^{(k+1)} = \mathcal{P}^{(k)} \bigcap \mathcal{S}$$

(necessarily $\mathcal{S}^{(k+1)} \subseteq \mathcal{S}^{(k)}$).

6) If $\mathcal{S}^{(k)} \subseteq (\lambda+\epsilon)\mathcal{S}^{(k+1)}$ then stop iterating, the derived set $\mathcal{X} = \mathcal{S}^{(k)}$ is $\lambda+\epsilon$ contractive: go to step 8. Otherwise, set $k = k+1$ and go to step 2.
7) If $\mathcal{S}^{(k+1)} \subset int\mathcal{S}^{(k)}$ (the interior of $\mathcal{S}^{(k)}$) then stop iterating, since the required λ is too stringent. Augment λ (or/and ϵ and go to step 2.
8) Compute matrix X as the vertex representation of $\mathcal{X}$ namely X is the matrix whose columns are either v_k or $-v_k$, the vertices of $\mathcal{S}^{(k)}$. The order of the compensator is $\nu =$ number of columns of X.
9) For each i and for each column x_j of X determine the vector $u_{i,j}$ such that

$$\xi_{i,j} = A_i x_j + B_i u_{i,j} \in (\lambda+\epsilon)\mathcal{X}$$

(such a u exists because $\mathcal{X}$ is $\lambda+\epsilon$ contractive). Matrix U_i is formed by

$$U_i = [\, u_{i,1}\, u_{i,2} \ldots\, u_{i,\nu}]$$

10) Determine a vector $\pi_{i,j} : \xi_{i,j} = X\pi_{i,j}$, with $\|\pi_{i,j}\|_1 \leq \lambda+\epsilon$. Matrix P_i is formed by

$$P_i = [\pi_{i,1}\, \pi_{i,2} \ldots\, \pi_{i,\nu}].$$

By construction $\|P_i\|_1 \leq \lambda+\epsilon$.

It is known that if the system admits a λ-contractive set then this procedure converges in a finite number of steps [33] for any $\epsilon > 0$ thus we can tune the convergence requirements by selecting $\lambda < 1$.

As far as (5) (in discrete-time) is concerned, this is the dual of (4) [15] and therefore we can solve it by means of the same procedure just described after transposition

$$A_i^T R^T + C_i^T L_i^T = Q_i^T R^T.$$

Note that $\|Q_i\|_1 = \|Q_i^T\|_\infty$. Note also that the number of rows of the resulting R will be equal to μ.

As far as the continuous-time is concerned, we can just use the Euler Auxiliary System (EAS)

$$x(t+1) = [I + \tau A_i]x(t) + \tau B_i u(t), \quad y(t) = C_i x(t)$$

and compute X, U_i and P_i for the EAS. The procedure must be applied with τ "small enough" (see [9] for details). Note that if we solve the problem for the EAS with $\|\tilde{P}_1\|_1 \leq \lambda$ we have that (4) becomes

$$A_i X + B_i \frac{U_i}{\tau} = X\frac{P_i - I}{\tau} = X\tilde{P}_i$$

where $\tilde{P}_i$ is in the class $\mathcal{H}_1$. The dual procedure assures a $\tilde{Q}_i$ of the class $\mathcal{H}_\infty$. Finally we claim that assuring a convergence λ to the EAS $\|x(t)\| \leq \gamma\lambda^t\|x(0)\|$ implies a convergence $\beta > 0$ to the continuous-time system, precisely $\|x(t)\| \leq \gamma e^{\beta^t}\|x(0)\|$ [42]. The reader is referred to [9] for a deeper discussion.

APPENDIX B
PROOF OF THE CLAIM 4.1

Proof: Fix any arbitrary $\hat{S}$ with $[R\ \hat{S}]$ invertible and let $\Gamma \doteq [R\ \hat{S}]$. From (29) we get the equivalent equation

$$\Gamma^{-1}[R \quad S]\begin{bmatrix} A & \Phi \\ 0 & A_\eta \end{bmatrix} + \Gamma^{-1}L[C \quad 0] = \Gamma^{-1}Q\Gamma\Gamma^{-1}[R \quad S]$$

where S, Φ and A_η are to be found, and we write it as

$$\begin{bmatrix} I & \Sigma_1 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} A & \Phi \\ 0 & A_\eta \end{bmatrix} + \begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 \end{bmatrix}[C\ 0] = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix}\Gamma^{-1}[R \quad S]$$

where Σ_1 and Σ_2, which uniquely parametrize S, have now to be decided. Since $\hat{Q}$ is a stable matrix, necessarily the pair $(\hat{Q}_{22}, \hat{Q}_{21})$, intended as a state-input matrices of the second subsystem, is stabilizable. Then there exists $\tilde{K}$ such that $\hat{Q}_{22} + \hat{Q}_{21}\tilde{K}$is Hurwitz (Schur). Then the modified equation has solution with

$$A_\eta \doteq \hat{Q}_{22} + \hat{Q}_{21}\tilde{K}, \quad \Sigma_1 = \tilde{K}, \quad \Sigma_2 = I$$

and

$$\Phi = -\Sigma_1 A_\eta + \hat{Q}_{11}\Sigma_1 + \hat{Q}_{12}\Sigma_2.$$

Finally the required S is derived as

$$[R \quad S] = \Gamma\begin{bmatrix} I & \Sigma_1 \\ 0 & \Sigma_2 \end{bmatrix} = \Gamma\begin{bmatrix} I & \tilde{K} \\ 0 & I \end{bmatrix}$$

thus it is invertible. ■

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