BOOLEAN ALGEBRAS

- IMPORTANT CLASS OF ALGEBRAS EXTENSIVELY USED FOR MANY PURPOSES
- BASIS OF THE *SWITCHING ALGEBRA* (SA) USED FOR FORMAL TREATMENT OF SWITCHING CIRCUITS:
  – TRANSFORMATION OF SWITCHING EXPRESSIONS
  – IDENTITIES FROM BA ENABLE GRAPHICAL AND TABULAR TECHNIQUES FOR MINIMIZATION OF SWITCHING EXPRESSIONS LEADING TO SIMPLER/FASTER CIRCUITS

DEFINITION OF BOOLEAN ALGEBRA

A BOOLEAN ALGEBRA IS A TUPLE $\{B, +, \cdot\}$:

- B IS A SET OF ELEMENTS;
- $+$ AND $\cdot$ ARE BINARY OPERATIONS APPLIED OVER THE ELEMENTS OF B,

SATISFYING THE FOLLOWING POSTULATES:

P1: If $a, b \in B$, then

$$\text{(i)} \quad a + b = b + a$$
$$\text{(ii)} \quad a \cdot b = b \cdot a$$

That is, $+$ and $\cdot$ are COMMUTATIVE.

P2: If $a, b, c \in B$, then

$$\text{(i)} \quad a + (b \cdot c) = (a + b) \cdot (a + c)$$
$$\text{(ii)} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

P3: THE SET B HAS TWO DISTINCT *IDENTITY ELEMENTS*, DENOTED AS 0 AND 1, SUCH THAT FOR EVERY ELEMENT IN B

$$\text{(i)}\quad 0 + a = a + 0 = a$$
$$\text{(ii)}\quad 1 \cdot a = a \cdot 1 = a$$

THE ELEMENTS 0 AND 1 ARE CALLED THE *ADDITIVE IDENTITY ELEMENT* AND THE *MULTIPLICATIVE IDENTITY ELEMENT*, respectively.

P4: FOR EVERY ELEMENT $a \in B$ THERE EXISTS AN ELEMENT a', CALLED THE *COMPLEMENT* OF a, SUCH THAT

$$\text{(i)}\quad a + a' = 1$$
$$\text{(ii)}\quad a \cdot a' = 0$$

COMMENTS

- SYMBOLS $+$ AND $\cdot$ ARE NOT THE ARITHMETIC ADDITION AND MULTIPLICATION SYMBOLS

  For convenience $+$ and $\cdot$ called "plus" and "times"

  Expressions $a + b$ and $a \cdot b$ called "sum" and "product,"

  $+$ and $\cdot$ also called "OR" and "AND"

- THE ELEMENTS OF THE SET B ARE CALLED *CONSTANTS:* 0 AND 1 ARE CONSTANTS

- SYMBOLS REPRESENTING ARBITRARY ELEMENTS OF B ARE *VARIABLES:* a, b and c in the postulates are variables.

- *PRECEDENCE ORDERING* DEFINED ON THE OPERATORS: $\cdot$ HAS PRECEDENCE OVER $+$.

$$a + (b \cdot c) \quad \text{can be written as} \quad a + bc$$

- SYMBOLS $a, b, c, \ldots$ IN THEOREMS AND POSTULATES ARE *GENERIC VARIABLES*: THEY CAN BE SUBSTITUTED BY
  - COMPLEMENTED VARIABLES
  - EXPRESSIONS (FORMULAS)

SWITCHING ALGEBRA

- SERVES THE SAME ROLE FOR SWITCHING FUNCTIONS AS THE ORDINARY ALGEBRA DOES FOR ARITHMETIC FUNCTIONS
- SA: SET OF TWO ELEMENTS $B = \{0, 1\}$, AND TWO OPERATIONS AND AND OR:

AND	0	1
0	0	0
1	0	1

OR	0	1
0	0	1
1	1	1

- THESE OPERATIONS ARE USED TO EVALUATE SWITCHING EXPRESSIONS

Theorem 1 *THE SWITCHING ALGEBRA IS A BOOLEAN ALGEBRA*.

PROOF ...

SHOW THAT SA SATISFIES POSTULATES OF BA: FOR EXAMPLE

[P2:] DISTRIBUTIVITY of (+) and $(\cdot)$ shown by *perfect induction:*

abc	$a + bc$	$(a + b)(a + c)$
0 0 0	0	0
0 0 1	0	0
0 1 0	0	0
0 1 1	1	1
1 0 0	1	1
1 0 1	1	1
1 1 0	1	1
1 1 1	1	1

Since $a + bc = (a + b)(a + c)$ for all cases, P2(i) is satisfied. Show that P2(ii) is also satisfied.

IMPORTANT THEOREMS IN BOOLEAN ALGEBRA

Theorem 2 *PRINCIPLE OF DUALITY.*
EVERY ALGEBRAIC IDENTITY DEDUCIBLE FROM THE POSTULATES OF A BOOLEAN ALGEBRA REMAINS VALID IF

- *the operations* $+$ *and* $\cdot$ *are interchanged throughout; and*
- *the identity elements 0 and 1 are also interchanged throughout.*

- This theorem is useful because it reduces the number of different theorems that must be proven: every theorem has its dual.

Theorem 3 *EVERY ELEMENT IN B HAS A UNIQUE COMPLEMENT.*

Proof: Let $a \in B$; let us assume that a'_1 and a'_2 are both complements of a. Then, using the postulates we can perform the following transformations:

$$
\begin{array}{rlll}
a'_1 & = a'_1 \cdot 1 & \text{by P3(ii)} & \text{(identity)} \\
& = a'_1 \cdot (a + a'_2) & \text{by hypothesis} & (a'_2 \text{ is the complement of } a) \\
& = a'_1 \cdot a + a'_1 \cdot a'_2 & \text{by P2(ii)} & \text{(distributivity)} \\
& = a \cdot a'_1 + a'_1 \cdot a'_2 & \text{by P1(ii)} & \text{(commutativity)} \\
& = 0 + a'_1 \cdot a'_2 & \text{by hypothesis} & (a'_1 \text{ is complement of } a) \\
& = a'_1 \cdot a'_2 & \text{by P3(i)} & \text{(identity)}
\end{array}
$$

Changing the index 1 for 2 and vice versa, and repeating all steps for a'_2 we get

$$
\begin{array}{rll}
a'_2 & = a'_2 \cdot a'_1 & \\
& = a'_1 \cdot a'_2 & \text{by P1(ii)}
\end{array}
$$

and therefore $a'_2 = a'_1$. ∎

The uniqueness of the complement of an element allows considering $'$ as a unary operation called *complementation*.

Theorem 4 *For any $a \in B$:*

$$(1)\quad a + 1 = 1$$
$$(2)\quad a \cdot 0 = 0$$

Proof: Using the postulates, we can perform the following transformations:

Case (1):

$$
\begin{aligned}
a + 1 &= 1 \cdot (a + 1) && \text{P3 (ii)} \\
&= (a + a') \cdot (a + 1) && \text{P4 (i)} \\
&= a + (a' \cdot 1) && \text{P2 (i)} \\
&= a + a' && \text{P3 (ii)} \\
&= 1 && \text{P4 (i)}
\end{aligned}
$$

by

Case (2):

$$
\begin{aligned}
a \cdot 0 &= 0 + (a \cdot 0) && \text{P3 (i)} \\
&= (a \cdot a') + (a \cdot 0) && \text{P4 (ii)} \\
&= a \cdot (a' + 0) && \text{P2 (ii)} \\
&= a \cdot a' && \text{P3 (i)} \\
&= 0 && \text{P4 (ii)}
\end{aligned}
$$

by

(2) can also be proven by means of (1) and the principle of duality.

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Theorem 5 *THE COMPLEMENT OF THE ELEMENT 1 IS 0, AND VICE VERSA. That is,*

$$\begin{aligned} (1)\ \ 0' &= 1 \\ (2)\ \ 1' &= 0 \end{aligned}$$

Proof: By Theorem 4,

$$0 + 1 = 1 \qquad \text{and}$$
$$0 \cdot 1 = 0$$

Since, by Theorem 3, the complement of an element is unique, Theorem 5 follows. ∎

Theorem 6 *IDEMPOTENT LAW.*

For every $a \in B$

$$(1) \quad a + a = a$$
$$(2) \quad a \cdot a = a$$

Proof:

$$\begin{array}{lll} & & \text{by} \\ (1):\ a + a & = (a + a) \cdot 1 & \text{P3 (ii)} \\ & = (a + a) \cdot (a + a') & \text{P4 (i)} \\ & = (a + (a \cdot a')) & \text{P2 (i)} \\ & = a + 0 & \text{P4 (ii)} \\ & = a & \text{P3 (i)} \\ & & \\ (2): & & \text{duality} \end{array}$$

■

Theorem 7 *INVOLUTION LAW.*

For every $a \in B$,

$$(a')' = a$$

Proof: From the definition of complement $(a')'$ and a are both complements of a'. But, by Theorem 3, the complement of an element is unique, which proves the theorem. ∎

Theorem 8 *ABSORPTION LAW.* *For every pair of elements $a, b \in B$,*

$$
\begin{aligned}
&(1)\ \ a + a \cdot b = a \\
&(2)\ \ a \cdot (a + b) = a
\end{aligned}
$$

Proof:

$$
\begin{array}{lll}
 & & \text{by} \\
(1)\text{: } a + ab & = a \cdot 1 + ab & \text{P3 (ii)} \\
 & = a(1 + b) & \text{P2 (ii)} \\
 & = a(b + 1) & \text{P1 (i)} \\
 & = a \cdot 1 & \text{Theorem 4 (1)} \\
 & = a & \text{P3 (ii)} \\
 & & \\
(2) & & \text{duality}
\end{array}
$$

■

Theorem 9 *FOR EVERY PAIR OF ELEMENTS* $a, b \in B$,

$$(1)\ \ a + a'b = a + b$$
$$(2)\ \ a(a' + b) = ab$$

Proof:

$$\begin{array}{llll} & & & \text{by} \\ \text{(1):} & a + a'b & = (a + a')(a + b) & \text{P2 (i)} \\ & & = 1 \cdot (a + b) & \text{P4 (i)} \\ & & = a + b & \text{P3 (ii)} \end{array}$$

(2): duality

■

ASSOCIATIVITY OF OR and AND

Theorem 10 *In a Boolean algebra, each of the binary operations (+) and ($\cdot$) is associative. That is, for every $a, b, c \in B$,*

$$\begin{array}{ll} (1) & a + (b + c) = (a + b) + c \\ (2) & a(bc) = (ab)c \end{array}$$

Corollary 1

1. *The order in applying the + operator among n elements does not matter. For example*

$$\begin{aligned} a + (b + (c + (d + e))) &= (((a + b) + c) + d) + e \\ &= a + ((b + c) + d)) + e \\ &= a + b + c + d + e \end{aligned}$$

2. *The order in applying the $\cdot$ operator among n elements does not matter.*

DeMorgan's Law

Theorem 11 *For every pair of elements $a, b \in B$:*

$$(1)\quad (a + b)' = a'b'$$
$$(2)\quad (ab)' = a' + b'$$

By duality,

$$(a \cdot b)' = a' + b'$$

Theorem 12 *GENERALIZED DeMorgan's LAW.*

Let $\{a, b, \ldots, c, d\}$ be a set of elements in a Boolean algebra. Then, the following identities hold:

$$(1) \quad (a + b \ldots + c + d)' = a'b' \ldots c'd'$$
$$(2) \quad (ab \ldots cd)' = a' + b' + \ldots + c' + d'$$

DeMorgan's theorems useful in manipulating switching expressions:

Example: find the complement of a switching expression

$$\begin{aligned} [(a + b')(c' + d') + (f' + g)']' &= [(a + b')(c' + d')]'[(f' + g)']' \\ &= [(a + b')' + (c' + d')'](f' + g) \\ &= (a'b + cd)(f' + g) \end{aligned}$$

OTHER EXAMPLES OF BOOLEAN ALGEBRAS

ALGEBRA OF SETS. The elements of B: set of all subsets of a set S

($P(S)$). The operations: set-union ($\cup$) and set-intersection ($\cap$).

$$M = (P(S), \cup, \cap)$$

ALGEBRA OF LOGIC (PROPOSITIONAL CALCULUS). The elements B: T and F (true and false). The operations: LOGICAL AND and LOGICAL OR.

This algebra is isomorphic with the switching algebra.