Part 1: Warm Up:

$y'(x) = y(x)$, initial condition $y(x=0) = 1$

1.) Solve IVP of ODE analytically (Methods from ME 303)

① $y' - y = 0 \Rightarrow \frac{dy}{dx} - y = 0$

↳ Assume $y = e^{mx}$ is a solution $\Rightarrow y = e^{mx}$, $y' = m \cdot e^{mx}$

② Sub into ODE:

$me^{mx} - e^{mx} = 0$

$e^{mx}(m-1) = 0$

$m - 1 = 0$ (Aux Eqn)

③ Roots: $m = 1$

↳ $y_c = C_1 e^x$ (General Solution)

$y_p = 0$ $(F(x) = 0)$

④ Applying I.C.s: $(y(0) = 1)$

$1 = C_1 \cdot e^0 \Rightarrow 1 = C_1$

$\therefore$, $y = e^x$ //

2.) Solve using power series:

① $y'(x) - y(x) = 0$, $y(0) = 1$

↳ Assume $y = \sum_{n=0}^{\infty} C_n x^n$ as a solution $\Rightarrow y = \sum_{n=0}^{\infty} C_n x^n$, $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$

② Sub into ODE:

$\sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$ (Index Down higher order)

$\sum_{n=0}^{\infty} (n+1) C_{n+1} x^n = \sum_{n=0}^{\infty} C_n x^n$

③ Expand LHS & RHS:

$C_1 x^0 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + 5C_5 x^4 + \cdots = C_0 x^0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \cdots$

④ Compare Matching Coeff:

$C_0 = C_0 \Rightarrow$ arbitrary free parameter

$C_1 = C_0$

$$\left.\begin{aligned} 2C_2 = C_1 &\Rightarrow C_2 = \frac{C_1}{2} = \frac{1}{2} C_0 \\ 3C_3 = C_2 &\Rightarrow C_3 = \frac{1}{3} C_2 = \frac{1}{6} C_0 \\ 4C_4 = C_3 &\Rightarrow C_4 = \frac{1}{4} C_3 = \frac{1}{24} C_0 \\ 5C_5 = C_4 &\Rightarrow C_5 = \frac{1}{5} C_4 = \frac{1}{120} C_0 \end{aligned}\right\} \quad C_n = \frac{C_0}{n!}$$

⑤ Sub back into original ODE:

$y = C_0 + C_0 x + \frac{C_0}{2!} x^2 + \frac{C_0}{3!} x^3 + \frac{C_0}{4!} x^4 + \frac{C_0}{5!} x^5 + \cdots$

$y = C_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right]$

$y = C_0 \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow y = \underbrace{C_0}_{\text{arbitrary}} \cdot e^x$ //

⑥ Find particular solution of ODE using I.C.s:

① $y(0) = 1$

② $1 = C_0 + C_0(0) + \frac{C_0}{2!}0^2 + \frac{C_0}{3!}0^3 + \frac{C_0}{4!}0^4 + \frac{C_0}{5!}0^5 + \dots$

↳ $C_0 = 1$

③ $\therefore\ y = e^x$ //

4) Solve ODE Numerically using Forward's Euler Method: $\Delta x = 0.1, 0.05, 0.001$ (5 iterations):

① $y' - y = 0 \Rightarrow y' = y$ (rearrange so highest order on LHS) $\Rightarrow y(0) = 1,\ y'(0) = 1$

↳ $y' = F \rightarrow F = v$ ($\because v = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y$ in this ODE so use $v_i = Y_i$)

② Euler's Method:

$Y_{i+1} = Y_i + \Delta x \cdot F = Y_i + \Delta x Y_i$

③ Step 1-5 ($i = 0.1$):

$Y_1 = 1 + 0.1(1) = 1.1$

$Y_2 = 1.1 + 0.1(1.1) = 1.21$

$Y_3 = 1.21 + 0.1(1.21) = 1.331$

$Y_4 = 1.331 + 0.1(1.331) = 1.4641$

$Y_5 = 1.4641 + 0.1(1.4641) = 1.61051$

④ Step 1-5 ($i = 0.05$):

$Y_1 = 1 + 0.05(1) = 1.05$

$Y_2 = 1.05 + 0.05(1.05) = 1.1025$

$Y_3 = 1.1025 + 0.05(1.1025) = 1.157625$

$Y_4 = 1.157625 + 0.05(1.157625) = 1.21550625$

$Y_5 = 1.21550625 + 0.05(1.21550625) = 1.276281563$

⑤ Step 1-5 ($i = 0.001$):

$Y_1 = 1 + 0.001(1) = 1.001$

$Y_2 = 1.001 + 0.001(1.001) = 1.002001$

$Y_3 = 1.002001 + 0.001(1.002001) = 1.003003001$

$Y_4 = 1.003003001 + 0.01(1.003003001) = 1.004006004$

$Y_5 = 1.004006004 + 0.01(1.004006004) = 1.00501001$

⇒ As the step size (Δx) decreases, the Forward Euler's Method becomes more accurate compared to the exact solution e^x. As well, larger step sizes leaded to bigger errors but require fewer steps //

↳ $y_5 \doteq 1.61051$ @ $x = 0.5$, $e^{0.5} \doteq 1.64875$

↳ $y_5 \doteq 1.2768$ @ $x = 0.25$, $e^{0.25} \doteq 1.28403$

↳ $y_5 \doteq 1.00501$ @ $x = 0.005$, $e^{0.005} \doteq 1.00501252$