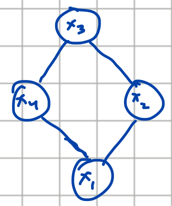
Q1. (see 2019 solutions - Q1.)

Q3.

a)



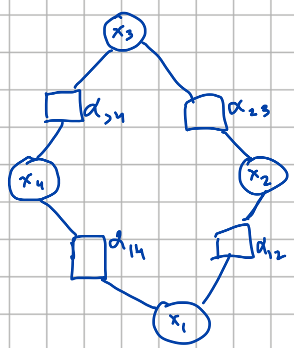
$p(x_1, x_2, x_3, x_4) = \frac{1}{Z} \psi_{1,2}(x_1, x_2)\, \psi_{2,3}(x_2, x_3)\, \psi_{3,4}(x_3, x_4)\, \psi_{1,4}(x_1, x_4)$

// Ans.

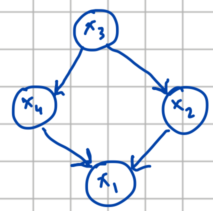
b) 1) None ?

2) $x_1 \perp\!\!\!\perp x_3 \mid \{x_2, x_4\}$
$x_2 \perp\!\!\!\perp x_4 \mid \{x_1, x_3\}$

c)



d)

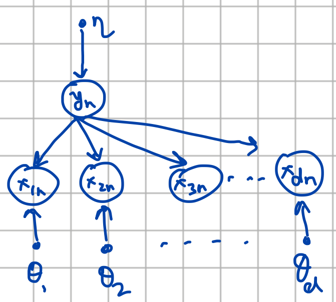


e) 1) None ?

2) $x_1 \perp\!\!\!\perp x_3 \mid \{x_2, x_4\}$
$x_2 \not\perp\!\!\!\perp x_4 \mid \{x_1, x_3\}$
$x_2 \perp\!\!\!\perp x_4 \mid \{x_3\}$

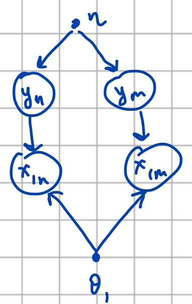
Q4.

a)



1) $x_{1n} \perp\!\!\!\perp x_{2n} \mid \phi$ — False — unblocked path through y_n

2) $x_{1n} \perp\!\!\!\perp x_{2n} \mid y_n$ — True — y_n blocks all paths from $x_{1n} \to x_{2n}$



Since η & μ are fixed given parameters:

3) $x_{1n} \perp\!\!\!\perp x_{1m} \mid \phi$ — True — no paths b/w them

4) $x_{1n} \perp\!\!\!\perp x_{1m} \mid \{y_n, y_m\}$ — True — no paths b/w them

b) $p(x, y \mid \theta, \eta) = \prod_{n=1}^{N} p(x_n, y_n \mid \theta, \eta) = \prod_{n=1}^{N} \prod_{d=1}^{D} p(x_{nd}, y_n \mid \theta_d, \vec{\eta})\, p(y_n \mid \eta)$

// Ans.

c) $p(y^*|x^*,\theta^*,\eta^*) = \frac{p(x^*|y^*,\theta^*,\eta^*)\,p(y^*|\eta)}{p(x^*)}$

d) $p(x_{dn}|y_n,\theta_d) = p(x_{dn}|y_n)\,p(y_n|\eta)$

$$L = \mathcal{N}(x_{dn}|\beta_d y_n+\gamma_d,\ \sigma_d^2)\ \mathrm{Bern}(y_n|\eta)$$

$$\log L = \underbrace{\log \mathcal{N}(x_{dn}|\beta_d y_n+\gamma_d,\ \sigma_d^2)}_{\text{const. w.r.t } \eta} + \log \mathrm{Bern}(y_n|\eta)$$

[maxim. likel. = max. log likel. (log is monotonic)]

$$\nabla_\eta \log L = 0 + \frac{\partial}{\partial\eta}\log\prod_{i=1}^{N}\eta^{y_n}(1-\eta)^{1-y_n}$$

$$0 = \sum_{n=1}^{N}\frac{y_n}{\eta} - \frac{1-y_n}{1-\eta}$$

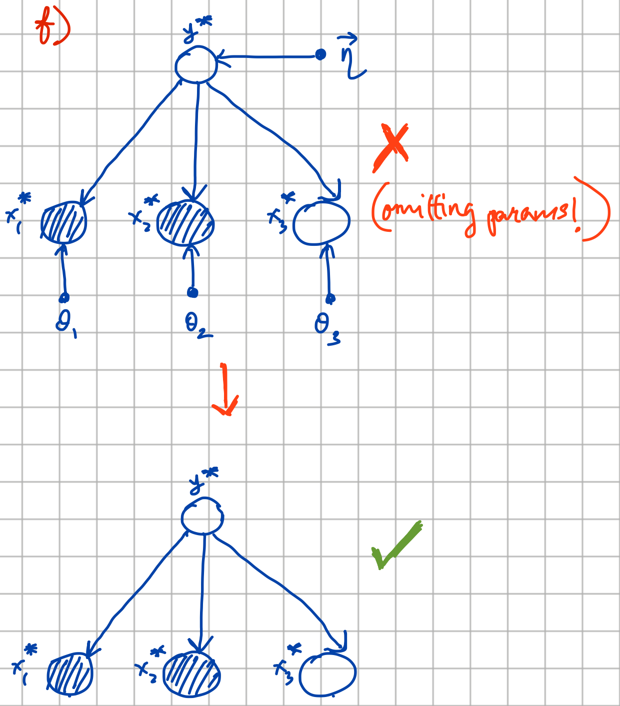
$$\sum_n y_n(1-\eta) = \sum_n \eta(1-y_n)$$

$$\sum_n y_n - N\eta = N\eta - \sum_n y_n$$

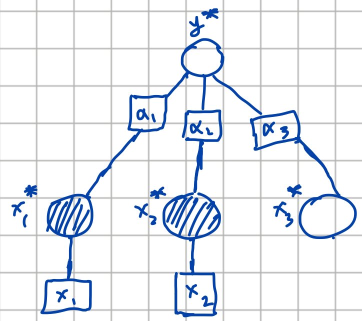
$$\boxed{\eta_{ML} = \frac{\sum_n y_n}{N}}$$ // Ans.

e) $\eta_{ML} = \frac{\sum_n y_n}{N} = \frac{1}{3}$ // Ans.

f)



g)



where,

$\alpha_1 = \mathcal{N}(x_{11}|\beta_1 y_1+\gamma_1,\ \sigma_1^2)$

$\alpha_2 = \mathcal{N}(x_{21}|\beta_2 y_1+\gamma_2,\ \sigma_2^2)$

$\alpha_3 = \mathcal{N}(x_{31}|\beta_3 y_1+\gamma_3,\ \sigma_3^2)$

$x_1 = \theta_1$

$x_2 = \theta_2$

$x_3 = \theta_3$

Q2.

a) exponential fam: $p(x|\eta) = h(x)\exp(\eta\, u(x) - A(\eta))$

$$\text{Bin.}(x|\pi) = \frac{K!}{x!(K-x)!} \cdot \exp\left(x \ln \pi + (K-x)\ln(1-\pi)\right)$$

$$= \frac{K!}{x!(K-x)!} \exp\left(\ln\left(\frac{\pi}{1-\pi}\right)x - \left(-K\ln(1-\pi)\right)\right)$$

Thus,

$$\eta = \ln \frac{\pi}{1-\pi} \Rightarrow e^{\eta} = \frac{\pi}{1-\pi} \Rightarrow \boxed{\pi = (1+e^{-\eta})^{-1}}$$

$$A(\eta) = -K\ln(1-\pi) = -K\ln\left(1 - \frac{1}{1+\frac{1}{e^{\eta}}}\right) = K\ln\left(1 - \frac{e^{\eta}}{1+e^{\eta}}\right)^{-1} = K\ln(1+e^{\eta})$$

$$u(x) = x$$

$$h(x) = \frac{K!}{x!(K-x)!}$$

Ans.

b) $$\mathbb{E}[x] = \frac{\partial}{\partial\eta} A(\eta) = \frac{\partial}{\partial\eta} K\ln(1+e^{\eta}) = \frac{Ke^{\eta}}{1+e^{\eta}}$$ Ans.

$$\text{Var}[x] = \frac{\partial^2}{\partial\eta^2}(A(\eta)) = \frac{\partial}{\partial\eta}\frac{Ke^{\eta}}{1+e^{\eta}} = K\left(\frac{e^{2\eta}}{(1+e^{\eta})^2} + \frac{e^{\eta}}{1+e^{\eta}}\right)$$

$$= \frac{Ke^{\eta}}{(1+e^{\eta})^2}$$ Ans.

c) Conjugate prior of Binomial is the Beta distrib.:

$$p(\pi|x) = p(x|\pi)\,p(\pi|a,b)$$

$$= \pi^{x}(1-\pi)^{K-x} \cdot \pi^{a-1}(1-\pi)^{b-1}$$

$$= \pi^{x+a-1} \cdot (1-\pi)^{(K-x)+b-1}$$

$$= \text{Beta}\left(\pi \,|\, a+x,\ (K-x)+b\right)$$

Ans.

(As the posterior is of the same functional form of the prior, conjugacy is proved.)

d) $$a \rightarrow \alpha + \sum_n x_n$$

$$b \rightarrow \left(NK - \sum_n x_n\right) + b$$

same as above but with iid assum. $\left(\prod_n [\dots]\right)$

Ans.