ML-2

Homework Assigment - 1

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Q1.

From linearity of expectations, we have :

$$\mathbb{E}[y] = \mathbb{E}[x+z]$$
$$= \mathbb{E}[x] + \mathbb{E}[z] \Rightarrow \boxed{\mu_y = \mu_x + \mu_z}$$

In case of multi variate distributions : (x & z dependent)

$$\text{cov}[y] = \mathbb{E}[yy^T] - \mathbb{E}[y]\mathbb{E}[y^T]$$
$$= \mathbb{E}[(x+z)(x+z)^T] - \mathbb{E}[x+z]\mathbb{E}[(x+z)^T] \quad \text{——— (1)}$$
$$= \mathbb{E}[xx^T + xz^T + zx^T + zz^T] - (\mathbb{E}[x]+\mathbb{E}[z])(\mathbb{E}[x^T]+\mathbb{E}[z^T])$$
$$= \underbrace{\mathbb{E}[xx^T] - \mu_x\mu_x^T} + \underbrace{\mathbb{E}[zz^T] - \mu_z\mu_z^T} + \underbrace{\mathbb{E}[zx^T] - \mu_z\mu_x^T + \mathbb{E}[xz^T] - \mu_x\mu_z^T}$$
$$= \text{var}[x] + \text{var}[z] + 2\,\text{cov}[x,z]$$

// Ans.

In case of mult variate distributions : (x & z independent)

we can expand (1) as :

$$\text{cov}[y] = \mathbb{E}[xx^T] + \mathbb{E}[x]\mathbb{E}[z^T] + \mathbb{E}[z]\mathbb{E}[x^T] + \mathbb{E}[zz^T] - (\mathbb{E}[x]+\mathbb{E}[z])(\mathbb{E}[x^T]+\mathbb{E}[z^T])$$
$$= \underbrace{\mathbb{E}[xx^T] - \mathbb{E}[x]\mathbb{E}[x^T]} + \underbrace{\mathbb{E}[zz^T] - \mathbb{E}[z]\mathbb{E}[z^T]}$$
$$= \Sigma_x + \Sigma_z$$

// Ans.

Q2.

(1)
$$p(X|\mu,\Sigma) = \prod_{i=1}^{N} p(x_i|\mu,\Sigma) = \prod_{i=1}^{N} \mathcal{N}(x_i|\mu,\Sigma) = \prod_{i=1}^{N} \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x_i-\mu)^T\Sigma^{-1}(x-\mu_i)\right)$$

// Ans.

(2)
$$p(\mu|X,\Sigma,\mu_0,\Sigma_0) = \frac{p(X|\mu,\Sigma)\cdot p(\mu|\mu_0,\Sigma_0)}{P(X)} = \frac{\prod_{i=1}^{N}\mathcal{N}(x_i|\mu,\Sigma)\cdot\mathcal{N}(\mu|\mu_0,\Sigma_0)}{P(X)}$$

where, $P(X) = \int p(X|\mu,\Sigma)\cdot p(\mu|\mu_0,\Sigma_0)$

// Ans.

(3)
$$p(\mu|X,\Sigma,\mu_0,\Sigma_0) = \frac{\prod_{i=1}^{N}\mathcal{N}(x_i|\mu,\Sigma)\cdot\mathcal{N}(\mu|\mu_0,\Sigma_0)}{p(X)}$$

Focussing on just the 'exp' part of $\mathcal{N}(\cdot)$,

$$= \exp\left(-\frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu) + (\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\right)$$

On expanding the terms we get,

$$= \exp\left(\frac{-1}{2}\left(\underbrace{\sum_{i=1}^{N} x_i^T \Sigma^{-1} x_i}_{\text{constant}} - 2\mu^T\Sigma^{-1}\underbrace{\sum_{i=1}^{N} x_i}_{= N\bar{x}\ \left(\bar{x} = \frac{1}{N}\sum x_i\right)} + N\mu^T\Sigma^{-1}\mu\right) + \left(\mu^T\Sigma_0^{-1}\mu - 2\mu_0^T\Sigma_0^{-1}\mu + \underbrace{\mu_0^T\Sigma_0^{-1}\mu_0}_{\text{constant}}\right)\right)$$

$$= \exp\left(\mu^T(N\Sigma^{-1} + \Sigma_0^{-1})\mu - \mu^T(N\Sigma^{-1}\bar{x} + \Sigma_0^{-1}\mu_0) - (N\Sigma^{-1}\bar{x} + \Sigma_0^{-1}\mu_0)^T\mu\right)$$

Let $A = N\Sigma^{-1} + \Sigma_0^{-1}$ & $B = N\Sigma^{-1}\bar{x} + \Sigma_0^{-1}\mu_0$

$$= \exp\left(\mu^T A\mu - \mu^T B - B^T\mu\right)$$

$$= \exp\left(\mu^T A\mu - \mu^T B - B^T\mu + \underbrace{B^T A^{-1} B}_{\text{indep. of } \mu}\right)$$

$$= \exp\left(\mu^T \underbrace{A}_{\text{covariance}}\mu - \mu^T A \underbrace{A^{-1}B}_{\text{mean}} - B^T A^{-1}A\mu + B^T A^{-1} A A^{-1} B\right) \qquad \left[\text{using } I = AA^{-1} = A^{-1}A\right]$$

since A is symmetric.

Thus, using completing the square we get,

$$\Sigma_n = A^{-1}, \quad \mu_N = A^{-1}B$$

ie,
$$\boxed{\begin{aligned} \Sigma_n &= (N\Sigma^{-1} + \Sigma_0^{-1})^{-1} \\ \mu_N &= \Sigma_n(N\Sigma^{-1}\bar{x} + \Sigma_0^{-1}\mu_0) \end{aligned}}$$

/// ANS.

4) μ_{MAP} maximizes the posterior, thus:

$$\mu_{MAP} = \underset{\mu}{\operatorname{argmax}}\ p(\mu | X, \Sigma, \mu_0, \Sigma_0)$$

$$= \underset{\mu}{\operatorname{argmax}}\ \frac{p(X|\mu,\Sigma)\, p(\mu|\mu_0,\Sigma_0)}{P(X|\Sigma)}$$

$$= \underset{\mu}{\operatorname{argmax}}\left[\log p(X|\mu,\Sigma) + \log p(\mu|\mu_0,\Sigma_0) - \underbrace{\log p(X|\Sigma)}_{}\right]$$

does not depend on μ, so drop.

Differentiating and setting to 0:

$$\log p(\mu|X,\Sigma,\mu_0,\Sigma_0) = \sum_{i=1}^{N}\log \mathcal{N}(x_i|\mu,\Sigma) + \sum_{i=1}^{N}\log \mathcal{N}(\mu|\mu_0,\Sigma_0)$$

$$= \log\left(\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\right) + \sum_{i=1}^{N}\frac{-1}{2}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu) + \log\left(\frac{1}{(2\pi)^{D/2}|\Sigma_0|^{1/2}}\right) + \frac{-1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)$$

or,
$$\frac{\partial}{\partial\mu}\log p(\mu|X,\Sigma,\mu_0,\Sigma_0) = -\Sigma^{-1}\sum_{i=1}^{N}(x_i-\mu) - \Sigma_0^{-1}(\mu-\mu_0)$$

[constant terms drop off]

$$-\Sigma^{-1}\sum_{i=1}^{N} x_i - N\Sigma^{-1}\mu - \Sigma_0^{-1}\mu + \Sigma_0^{-1}\mu_0 = 0$$ [expanding]

$$\mu\left(N\Sigma^{-1} + \Sigma_0^{-1}\right) = \Sigma^{-1}\sum_{i=1}^{N} x_i + \Sigma_0^{-1}\mu_0$$ [collecting terms]

$$\therefore \boxed{\mu_{MAP} = \frac{\Sigma^{-1}\sum_{i=1}^{N} x_i + \Sigma_0^{-1}\mu_0}{N\Sigma^{-1} + \Sigma_0^{-1}}}$$

Ans.

Q3.

1. Since all the observed variables have given 'heads', the MLE will assign it a probability of 1. As a result, probability of observing 'tails' = 0.
ie, $\mu = 1$
Hence, probability of observing heads in the next toss = 1

Ans.

Mathematically we can see that as:

likelihood : $p(x|\mu) = \prod_{i=1}^{N} \mu^{x_i}(1-\mu)^{1-x_i}$

$$\ln p(x|\mu) = \sum_{i=1}^{N} x_i \ln\mu + (1-x_i)\ln(1-\mu)$$

Differentiating w.r.t μ & setting to 0:

$$\frac{\partial}{\partial\mu}\left(\ln p(x|\mu)\right) = \sum_{i=1}^{N} \frac{x_i}{\mu} - \frac{1-x_i}{1-\mu} = 0$$

$$\Rightarrow \frac{1}{\mu(1-\mu)} \sum_{i=1}^{N} x_i(1-\mu) - (1-x_i)\mu = 0$$

$$\Rightarrow \sum_{i=1}^{N} (x_i - \mu) = 0$$

$$\Rightarrow \frac{1}{N}\sum_{i=1}^{N} x_i = \mu$$

In our case, $N = 3$ and $x = \{x_1, x_2, x_3\} = \{1, 1, 1\}$

$$\therefore \mu = \frac{1}{3}\cdot 3 = 1$$

Ans.

2. Posterior distr. of μ under a Beta prior is:

$$p(\mu|x,a,b) \propto p(x|\mu,a,b)\, p(\mu|a,b)$$

$$= \underbrace{\frac{\Gamma(m+a+l+b)}{\Gamma(m+a)\,\Gamma(l+b)}}_{\substack{\text{normalizing factor} \\ = C}} \underbrace{\mu^{a-1}(1-\mu)^{b-1}}_{\text{prior}} \underbrace{\prod_{i=1}^{N} \mu^{x_i}(1-\mu)^{1-x_i}}_{\text{likelihood}}$$

$$\ln p(\mu|x,a,b) = c + (a-1)\ln\mu + (b-1)\ln(1-\mu) + \sum_{i=1}^{N} x_i \ln\mu + (1-x_i)\ln(1-\mu)$$

Differentiating w.r.t μ & setting to 0 :

$$\frac{\partial}{\partial\mu}\ln p(\mu|x,a,b) = c + \frac{a-1}{\mu} + \frac{b-1}{1-\mu} + \sum_{i=1}^{N}\frac{x_i}{\mu} + \frac{(1-x_i)}{1-\mu} = 0$$

$$\Rightarrow (a-1)(1-\mu) + (b-1)\mu + \sum_{i=1}^{N} x_i(1-\mu) + (1-x_i)\mu = 0$$

$$\Rightarrow (a-1)(1-\mu) + (b-1)\mu + \sum_{i=1}^{N}(x_i - \mu) = 0$$

$$\Rightarrow a - a\mu - 1 + \cancel{\mu} + b\mu - \cancel{\mu} + 3 - \mu = 0$$

$$\Rightarrow \mu_{MAP} = \frac{a+2}{a+b+1}$$

// Ans.

3

$$\mu_{MLE} = \frac{m}{m+l}$$

prior mean $= \dfrac{a}{a+b}$ [i.e, $E[\mu]$ of beta distribution]

posterior mean $= \dfrac{m+a}{m+a+l+b}$ $\left[E[\mu|D] = \int_0^1 \mu\, p(\mu|D)\, d\mu\right]$

To show that posterior mean lies between μ_{MLE} & prior mean, we can express the former as a linear combination of the latter 2 (i.e, $\alpha\mu_{MLE} + (1-\alpha)\mu_{prior} = \mu_{post}$, with constraint : $0 \le \alpha \le 1$).

$$\frac{m+a}{m+l+a+b} = \frac{m}{m+l+a+b} + \frac{a}{m+l+a+b}$$

$$= \frac{m+l}{m+l}\cdot\frac{m}{m+l+a+b} + \frac{a+b}{a+b}\cdot\frac{a}{m+l+a+b}$$

$$= \underbrace{\frac{m}{m+l}}_{\mu_{MLE}}\cdot\underbrace{\frac{m+l}{m+l+a+b}}_{\alpha} + \underbrace{\frac{a}{a+b}}_{\mu_{prior}}\cdot\underbrace{\frac{a+b}{m+l+a+b}}_{1-\alpha}$$ (Switching terms) —— (i)

or, $\mu_{posterior} = \alpha\,\mu_{MLE} + (1-\alpha)\,\mu_{prior}$ // Proved

NOTE : where, $X = m+l$ and $Y = a+b$

$$\therefore \alpha = \frac{X}{X+Y} \Rightarrow (1-\alpha) = 1 - \frac{X}{X+Y} = \frac{Y}{X+Y}$$

Hence, we see that 'α' is a weighted mean of X & Y.
Thus, 'α' satisfies : $0 \le \alpha \le 1$

And thus, by convex combination property the objective is proved // Ans.

Q4.

1

general form of exponential family.

$$p(x|\eta) = h(x)\, g(\eta) \exp\{\eta^T u(x)\}$$

We will try to reduce the following distributions to this form

1) $\text{Pois}(k|\lambda) = \dfrac{\lambda^k e^{-\lambda}}{k!} = \dfrac{1}{k!} \cdot e^{-\lambda} \cdot \exp(k \cdot \log \lambda)$

Hence, $h(k) = 1/k!$, $\eta = \log\lambda$ $\rightarrow \lambda = e^{\eta}$, $u(k) = k$, $g(\eta) = e^{-\lambda} = e^{-e^{\eta}}$

// Ans.

2) $\text{Gam}(\tau|a,b) = \dfrac{1}{\Gamma(a)} b^a \tau^{a-1} e^{-b\tau}$

$$= \frac{b^a}{\Gamma(a)} e^{(a-1)\log\tau - b\tau}$$

Hence, $\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}a-1\\ -b\end{pmatrix}$, $\begin{pmatrix}u_1(\tau)\\ u_2(\tau)\end{pmatrix} = \begin{pmatrix}\log\tau\\ \tau\end{pmatrix}$

$h(\tau) = 1$, $\begin{pmatrix}g(\eta_1)\\ g(\eta_2)\end{pmatrix} = \begin{pmatrix}b^a\\ \Gamma(a)^{-1}\end{pmatrix}$

// Ans.

3) $\text{Cauchy}(x|\gamma,\mu)$ is not part of the exponential family. This is because its mean is undefined which violates the required property of exponential family.

// Ans.

We can show this by calculating the $E[x]$:

$$E[x|1,0] = \int_{-\infty}^{\infty} \frac{x}{(1+x^2)\pi}\, dx$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{x}{(1+x^2)}\, dx$$

$$= \frac{1}{\pi} \log(1+x^2)\Big|_0^{\infty}$$

$$= \infty$$

//

Since for at least one set of params the expectation is undefined, it is not part of the exponential family.

// Proved

4) $\text{VonMises}(x|\kappa,\mu) = \dfrac{1}{2\pi I_0(\kappa)} e^{\kappa\cos(x-\mu)}$

$$= \frac{1}{2\pi I_0(\kappa)} e^{\kappa\cos x\cos\mu + \kappa\sin x\sin\mu}$$

Hence, $\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}\kappa\cos\mu\\ \kappa\sin\mu\end{pmatrix}$, $\begin{pmatrix}u_1(x)\\ u_2(x)\end{pmatrix} = \begin{pmatrix}\cos x\\ \sin x\end{pmatrix}$,

$h(x) = 1$, $g(\eta) = \dfrac{1}{2\pi I_0(\kappa)}$

// Ans.

$\boxed{2}$

1) Poisson : (Using Bishop 2.226)

$$
\begin{aligned}
E[k] = E[u(x)] &= -\nabla_\eta \ln g(\eta) \\
&= -\nabla_\eta \ln\left(e^{-e^{\eta}}\right) \\
&= +\nabla_\eta \left(e^{\eta}\right) \\
&= \lambda \quad \text{Ans.}
\end{aligned}
$$

$$
\begin{aligned}
E[k^2] = E[u^2(x)] &= -\nabla_\eta^2 \ln g(\eta) \\
&= \nabla_\eta^2 \left(e^{\eta}\right) \\
&= e^{\eta} \\
&= \lambda \quad \text{Ans.}
\end{aligned}
$$

2) Gamma : (Using Bishop 2.226)

$$
\begin{aligned}
E[\tau] = E[u_2(x)] &= -\nabla_{\eta_2} \ln g(\eta) \\
&= -\nabla_{\eta_2} \ln \frac{(-\eta_2)^{\eta_1+1}}{\Gamma(\eta_1+1)} \\
&= -\nabla_{\eta_2} (\eta_1+1)\ln(-\eta_2) + c \\
&= \frac{\eta_1+1}{-\eta_2} = \frac{a}{b} \quad \text{Ans.}
\end{aligned}
$$

NOTE: $u(\tau) = (\log \tau, \tau)$ we need to calculate $E[\tau]$, hence we use η_2 in our calculation as it is straight forward.

$$
\begin{aligned}
E[\tau^2] &= E[u_2^2(x)] \\
&= -\nabla_{\eta_2}^2 \ln g(\eta) \\
&= -(\eta_1+1)\nabla_{\eta_2}^2 \ln(-\eta_2) \\
&= +(\eta_1+1)\nabla_{\eta_2}(-\eta_2)^{-1} \\
&= \frac{\eta_1+1}{\eta_2^2} = \frac{a}{b^2} \quad \text{Ans.}
\end{aligned}
$$

$\boxed{3}$

Poisson distribution is part of exponential family and hence has a conjugate prior, which is the Gamma Distribution :

Likelihood = Poisson $\quad p(X|\lambda) = \dfrac{\lambda^x e^{-\lambda}}{X!} = \displaystyle\prod_{i=1}^{N} \frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!}$

prior = Gamma $\quad p(\lambda|a,b) = \dfrac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}$

posterior = $\dfrac{\text{Likelihood} \times \text{prior}}{\text{evidence}}$ or, $\quad p(\lambda|x,a,b) = \dfrac{p(x|\lambda)\cdot p(\lambda|a,b)}{p(x)}$

$$
= \frac{\displaystyle\prod_{i=1}^{N} \frac{\lambda^{x_i}\cdot e^{-\lambda}}{x_i!} \cdot \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}}{\displaystyle\int_\lambda \prod_{i=1}^{N} \frac{\lambda^{x_i}\cdot e^{-\lambda}}{x_i!} \cdot \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}}
$$

$$
= \frac{\lambda^{(a-1)+\sum_{i=1}^{N} x_i} \cdot e^{-(N+b)\lambda}}{\displaystyle\int_\lambda \lambda^{(a-1)+\sum x_i} \cdot e^{-(N+b)\lambda}} \cdot \frac{\cancel{\prod_{i=1}^{N} x_i!\,\Gamma(a)}}{\cancel{\prod_{i=1}^{N} x_i!\,\Gamma(a)}}
$$

$$
= \frac{\lambda^{(a-1)+\sum_{i=1}^{N} x_i} \cdot e^{-(N+b)\lambda}}{\Gamma\left(a+\sum_{i=1}^{N} x_i\right)}
$$

Hence, posterior is of the same functional form as the prior (conjugacy)

$\therefore$ Gamma distribution is the conjugate prior of Poisson.

Ans.

$$
= \text{Gamma}\left(\lambda \,\middle|\, a+\sum_{i=1}^{N} x_i,\; N+b\right)
$$

Q5.

mean :

student-t distribution (with the constant term expressed as 'c') :

$$f(x) = c\left(1+\frac{xx^T}{n}\right)^{-\frac{(n+1)}{2}} \qquad \text{where} \qquad c = \frac{1}{\sqrt{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}$$

By definition :

$$E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

$$= \int_{-\infty}^{0} x\, f_x(x)\, dx + \int_{0}^{\infty} x\, f_x(x)\, dx$$

using change of variable : $t = -x \rightarrow dt = -dx$ for first integral

$$= -\int_{\infty}^{0} (-t)\, f_x(-t)\, dt + \int_{0}^{\infty} x\, f_x(x)\, dx$$

$$= \int_{\infty}^{0} t\, f_x(-t)\, dt + \int_{0}^{\infty} x\, f_x(x)\, dx$$

$$= -\int_{0}^{\infty} t\, f_x(-t)\, dt + \int_{0}^{\infty} x\, f_x(x)\, dx \qquad \text{[exchanging bounds of integral]}$$

$$= -\int_{0}^{\infty} x\, f(-x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx \qquad \text{[Trivial change of variable : } x = t\text{]}$$

$$= -\int_{0}^{\infty} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx \qquad [\because f(x) \text{ is an even func., thus } f(x) = f(-x)]$$

$$= 0$$

$$\therefore \boxed{E[x] = 0}$$ Ans.

variance :

By definition :

$$var[x] = E[xx^T] - E[x]^2$$

where, we have derived above that $E[x] = 0$. $\therefore E[x]^2 = 0$

$$E[x^2] = \int_{-\infty}^{\infty} xx^T f_x(x)\, dx$$

$$= \int_{-\infty}^{0} xx^T f_x(x)\, dx + \int_{0}^{\infty} xx^T f_x(x)\, dx$$

$$= -\int_{\infty}^{0} tt^T f_x(-t)\, dt + \int_{0}^{\infty} xx^T f_x(x)\, dx \qquad \left[\text{change of variable : } x = -t \Rightarrow dx = -dt\right]$$

$$= \int_0^\infty t t^T f_x(-t)\, dt + \int_0^\infty x x^T f_x(x)\, dx$$

$$= \int_0^\infty x x^T f_x(-x)\, dx + \int_0^\infty x x^T f_x(x)\, dx \qquad [\text{change of variable} : x = t]$$

$$= 2 \int_0^\infty x x^T f(x)\, dx \qquad [\because f(x) \text{ is an even func., thus } f(x) = f(-x)]$$

Substituting the definition of $f(x)$:

$$= 2c \int_0^\infty x x^T \left(1 + \frac{x x^T}{n}\right)^{-\frac{(n+1)}{2}} dx \quad \text{———} \quad ①$$

Using change of variables : $t = \left(1 + \frac{x x^T}{n}\right)^{-1}$

$$1 + \frac{x^2}{n} = \frac{1}{t} \Rightarrow x x^T = n\left(\frac{1}{t} - 1\right)$$

$$\underline{x = \sqrt{n}\left(\frac{1}{t} - 1\right)^{1/2}} \quad \text{———} \quad ②$$

Thus, $\frac{dx}{dt} = -\frac{\sqrt{n}}{2} t^{-\frac{3}{2}} \Rightarrow \underline{dx = -\frac{\sqrt{n}}{2} t^{-3/2}\, dt} \quad \text{———} \quad ③$

$$= n\sqrt{n}\, c \int_0^1 t^{n/2 - 2} (1 - t)^{1/2}\, dt \qquad [\text{rewriting ① using ② \& ③}]$$

$$= n \cancel{\sqrt{n}} \cdot \frac{1}{\cancel{\sqrt{n}}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_0^1 t^{n/2 - 2} \cdot (1 - t)^{1/2}\, dt \qquad [\text{Substituting } c]$$

Here, the part inside integral is of the form of gamma function :

$$\int_0^1 x^{p-1} (1 - x)^{q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

where, $p = \frac{n}{2} - 1$, $q = \frac{3}{2}$

Thus,

$$= n \cdot \frac{\cancel{\Gamma\left(\frac{n+1}{2}\right)}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{3}{2}\right)}{\cancel{\Gamma\left(\frac{n+1}{2}\right)}}$$

$$= \frac{n\, \Gamma\left(\frac{n}{2} - 1\right) \Gamma\left(\frac{1}{2} + 1\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}$$

$$= n \frac{\cancel{\Gamma\left(\frac{1}{2}\right)} \frac{1}{2} \cancel{\Gamma\left(\frac{n}{2}\right)} \frac{2}{n-2}}{\cancel{\Gamma\left(\frac{n}{2}\right)} \cancel{\Gamma\left(\frac{1}{2}\right)}} \qquad [\text{using } \Gamma(x) = \Gamma(x-1)(x-1)]$$

$$\therefore E[x^2] = \frac{n}{n-2}$$

$$\Rightarrow \quad \mathrm{var}[x] = E[xx^T] - E[x]^2$$

$$= \frac{n}{n-2} - 0$$

$$\therefore \boxed{\mathrm{var}[x] = \frac{n}{n-2}}$$

// Ans.