

Homework 1

Instructors: Adeel Pervez, Pim de Haan, Noud de Kroon, David Zhang
 Email: a.a.pervez@uva.nl, pimdehaan@gmail.com, a.a.w.m.dekroon@uva.nl, w.d.zhang@uva.nl

You are allowed to discuss with your colleagues but you should write the answers in your own words. If you discuss with others, write down the name of your collaborators on top of the first page. No points will be deducted for collaborations. If we find similarities in solutions beyond the listed collaborations we will consider it as cheating. We will not accept any late submissions. The solutions to the previous homework set will be put on canvas by the end of the day of the hand-in date. \star denotes bonus exercise. You earn 1 point for solving each bonus exercise. All bonus points earned will be added to your total homework points.

Problem 1. (1 pt) Consider two random vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^n$ having Gaussian distribution $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$ and $p(\mathbf{z}) = \mathcal{N}(\mathbf{z}|\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z)$. Consider random vector $\mathbf{y} = \mathbf{x} + \mathbf{z}$. Derive mean and covariance of $p(\mathbf{y})$. What is the covariance of \mathbf{y} if you assume that \mathbf{x} and \mathbf{z} are independent?

Problem 2. ($0.5+0.5+1.5+0.5 = 3$ pts) Consider a D -dimensional Gaussian random variable \mathbf{x} with distribution $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in which the covariance $\boldsymbol{\Sigma}$ is known. Given a set of N i.i.d. observations $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$. Assume that $\mathbf{x}_i \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$. [Hint: you may directly use results from Bishop]

1. Write down the likelihood of the data $p(\mathcal{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$;
2. Write down the form of the posterior $p(\boldsymbol{\mu}|\mathcal{X}, \boldsymbol{\Sigma}, \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ (you do not need to normalize the probability distribution by calculating the evidence).
3. Show that $p(\boldsymbol{\mu}|\mathcal{X}, \boldsymbol{\Sigma}, \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ is a Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}|\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N)$ and find the values of $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$ (hint: use "completing the square")
4. Derive the maximum a posteriori solution for $\boldsymbol{\mu}$;

Problem 3. ($0.5 + 0.5 + 0.5 = 1.5$ pts) Tossing a biased coin with probability that it comes up heads is μ . [Hint: you may use results from Bishop] *Sec 2.1*

1. We toss the coin 3 times and it all comes up with heads. How likely is that in the next toss, the coin comes up with head according to MLE?
2. Suppose that the prior $\mu \sim \text{Beta}(\mu|a, b)$. What is the probability that the coin comes up with head in the 4th toss? *→ only distrib. That is in an interval [0, 1]
→ good for prob. priors*
3. Suppose that we observe m times that the coin lands heads and l times that it lands tails. Show that the posterior mean $E[\mu|\mathcal{D}]$ (see Bishop 2.19) lies between the prior mean and μ_{MLE} .

Problem 4. ($2 + 1 + 0.5 = 3.5$ pts) Consider the following distributions:

$$\text{(i)} \quad \text{Pois}(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\rightarrow P(x=1|\mathcal{D}) \Rightarrow \int P(x=1|\mathcal{D}, \mu) P(\mu) d\mu$$

$$\textcircled{3} \rightarrow E[\mu|\mathcal{D}]$$

✓ (ii) $\text{Gam}(\tau|a, b) = \frac{1}{\Gamma(a)} b^a \tau^{a-1} e^{-b\tau}$

✓ (iii) $\text{Cauchy}(x|\gamma, \mu) = \frac{1}{\pi\gamma} \frac{1}{1 + (\frac{x-\mu}{\gamma})^2}$

✓ (iv) $\text{vonMises}(x|\kappa, \mu) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(x-\mu)}$

Answer the following questions:

- ✓ 1. Are the above distributions members of an exponential family. If yes, then (a) cast them in exponential form (Bishop eq. 2.194) with a minimum numbers of parameters, (b) derive their sufficient statistics.
2. Derive the first moment about zero (i.e. the mean) and the second moment about the mean (i.e. the variance) of the distributions (i) and (ii).
- ✓ 3. Does the Poisson distribution have a conjugate prior? Derive the conjugate prior, if the answer is "yes".

Problem 5* (1 pt) Derive mean, covariance, and mode of multivariate Student's t-distribution.

Q1. Linearity of expectations : $\mu_y = \mu_x + \mu_z$

variance $[y] = E[y^2] - E[y]^2$

in case of multivariate : $E[yy^T] = E[y]E[y]^T$

$$E[(y - E[y])(y - E[y])^T] = \text{var}[y]$$

Q2. Completing the square :

$$\begin{aligned} & \exp\left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right) \\ &= \exp\left(-\frac{1}{2} \underbrace{x^T \Sigma^{-1} x}_{\text{const.}} + \underbrace{x^T \Sigma^{-1} \mu}_{\text{linear}} + \underbrace{\text{const.}}_{\text{const.}}\right) \\ & \quad \text{good term} \quad \text{mean} \quad \text{const.} \\ & \quad \text{const.} \quad \text{linear} \quad \text{const.} \\ & \quad \downarrow \text{const.} \quad \downarrow \text{mean} \end{aligned}$$

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Shantanu Chandra
(12048461)

Q1.

From linearity of expectations, we have :

$$\begin{aligned} E[y] &= E[x+z] \\ &= E[x] + E[z] \Rightarrow \boxed{\mu_y = \mu_x + \mu_z} \end{aligned}$$

In case of multi-variate distributions : ($x \& z$ dependent)

$$\begin{aligned} \text{cov}[y] &= E[yy^T] - E[y]E[y^T] \\ &= E[(x+z)(x+z)^T] - E[x+z]E[(x+z)^T] \\ &= E[xx^T + xz^T + zx^T + zz^T] - (E[x] + E[z])(E[x^T] + E[z^T]) \\ &= \boxed{E[xx^T] - \mu_x\mu_x^T + E[zz^T] - \mu_z\mu_z^T + E[zx^T] - \mu_z\mu_x^T + E[xz^T] - \mu_x\mu_z^T} \\ &= \text{var}[x] + \text{var}[z] + 2 \text{cov}[x, z] \\ &\quad // Ans. \end{aligned} \quad \textcircled{1}$$

In case of multi-variate distributions : ($x \& z$ independent)

we can expand $\textcircled{1}$ as :

$$\begin{aligned} \text{cov}[y] &= E[xx^T] + E[x]E[z^T] + E[z]E[x^T] + E[zz^T] - (E[x] + E[z])(E[x^T] + E[z^T]) \\ &= \frac{E[xx^T] - E[x]E[x^T]}{\Sigma_x + \Sigma_z} + \frac{E[zz^T] - E[z]E[z^T]}{\Sigma_z} \\ &\quad // Ans. \end{aligned}$$

Q2.

$$\textcircled{1} \quad p(x|\mu, \Sigma) = \prod_{i=1}^N p(x_i|\mu, \Sigma) = \prod_{i=1}^N N(x_i|\mu, \Sigma) = \prod_{i=1}^N \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} (x_i - \mu)^T \Sigma^{-1} (x_i - \mu)\right) \\ // Ans.$$

$$\textcircled{2} \quad p(\mu|x, \Sigma, \mu_0, \Sigma_0) = \frac{p(x|\mu, \Sigma) \cdot p(\mu|\mu_0, \Sigma_0)}{p(x)} = \frac{\prod_{i=1}^N N(x_i|\mu, \Sigma) \cdot N(\mu|\mu_0, \Sigma_0)}{p(x)} \text{ where, } p(x) = \int p(x|\mu, \Sigma) \cdot p(\mu|\mu_0, \Sigma_0) \\ // Ans.$$

$$\textcircled{3} \quad p(\mu|x, \Sigma, \mu_0, \Sigma_0) = \frac{\prod_{i=1}^N N(x_i|\mu, \Sigma) \cdot N(\mu|\mu_0, \Sigma_0)}{p(x)}$$

Focussing on just the 'exp' part of $N(\cdot)$,

$$= \exp\left(-\frac{1}{2} \sum_{i=1}^N (x_i - \mu)^T \Sigma^{-1} (x_i - \mu) + (\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)\right)$$

On expanding the terms we get,

$$= \exp\left(\frac{-1}{2} \underbrace{\left(\sum_{i=1}^N x_i^T \Sigma^{-1} x_i - 2 \mu^T \Sigma^{-1} \sum_{i=1}^N x_i + N \mu^T \Sigma^{-1} \mu\right)}_{\text{constant}} + \underbrace{\left(\mu^T \Sigma_0^{-1} \mu - 2 \mu_0^T \Sigma_0^{-1} \mu + \mu_0^T \Sigma_0^{-1} \mu_0\right)}_{\text{constant}}\right)$$

$$= \exp\left(\mu^T (N \Sigma^{-1} + \Sigma_0^{-1}) \mu - \mu^T (N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0) - (N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0)^T \mu\right)$$

$$\text{let } A = N \Sigma^{-1} + \Sigma_0^{-1} \quad \& \quad B = N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0$$

$$= \exp(\mu^T A \mu - \mu^T B - B^T \mu)$$

$$= \exp(\mu^T A \mu - \mu^T B - B^T \mu + \underbrace{B^T A^{-1} B}_{\text{indep. of } \mu})$$

$$= \exp(\mu^T A \mu - \mu^T A A^{-1} B - B^T A^{-1} A A^{-1} B) \quad [\text{using } I = A A^{-1} = A^{-1} A]$$

Thus, using completing the square we get,

$$\begin{aligned} \Sigma_n &= A^{-1}, \quad \mu_n = A^{-1} B \\ \text{ie, } \boxed{\Sigma_n = (N \Sigma^{-1} + \Sigma_0^{-1})^{-1}} \\ \boxed{\mu_n = \Sigma_n (N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0)} \\ \text{/// QED.} \end{aligned}$$

(4) μ_{MAP} maximizes the posterior, thus:

$$\mu_{MAP} = \underset{\mu}{\operatorname{argmax}} p(\mu | x, \varepsilon, \mu_0, \Sigma_0)$$

$$= \underset{\mu}{\operatorname{argmax}} \frac{p(x|\mu, \varepsilon) p(\mu|\mu_0, \Sigma_0)}{p(x|\varepsilon)}$$

$$= \underset{\mu}{\operatorname{argmax}} \left[\log p(x|\mu, \varepsilon) + \log p(\mu|\mu_0, \Sigma_0) - \underbrace{\log p(x|\varepsilon)}_{\text{does not depend on } \mu, \text{ so drop.}} \right]$$

Differentiating and setting to 0 :

$$\log p(\mu|x, \varepsilon, \mu_0, \Sigma_0) = \sum_{i=1}^N \log N(x_i|\mu, \varepsilon) + \sum_{i=1}^N \log N(\mu|\mu_0, \Sigma_0)$$

$$= \log\left(\frac{1}{(2\pi)^{D/2} |\varepsilon|^{D/2}}\right) + \sum_{i=1}^N \frac{-1}{2} (x_i - \mu)^T \varepsilon^{-1} (x_i - \mu) + \log\left(\frac{1}{(2\pi)^{D/2} |\Sigma_0|^{D/2}}\right) + \frac{-1}{2} (\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)$$

$$\text{or, } \frac{\partial}{\partial \mu} \log p(\mu|x, \varepsilon, \mu_0, \Sigma_0) = -\varepsilon^{-1} \sum_{i=1}^N (x_i - \mu) - \Sigma_0^{-1} (\mu - \mu_0) \quad [\text{constant terms drop off}]$$

$$-\sum \xi^{-1} \sum_{i=1}^N x_i - N \sum \mu - \sum \xi_0 \mu + \sum \xi_0 \mu_0 = 0 \quad [\text{expanding}]$$

$$\mu (N \sum \xi^{-1} + \xi_0^{-1}) = \sum \xi^{-1} \sum_{i=1}^N x_i + \sum \xi_0^{-1} \mu_0 \quad [\text{collecting terms}]$$

$$\therefore \boxed{\mu_{\text{MAP}} = \frac{\sum \xi^{-1} \sum_{i=1}^N x_i + \sum \xi_0^{-1} \mu_0}{N \sum \xi^{-1} + \xi_0^{-1}}} \\ \equiv \text{Ans.}$$

Q3.

① Since all the observed variables have given 'heads', the MLE will assign it a probability of 1. As a result, probability of observing 'tails' = 0.

i.e., $\mu = 1$

Hence, probability of observing heads in the next toss = 1
 $\equiv \text{Ans.}$

Mathematically we can see that as:

$$\text{likelihood: } p(x|\mu) = \prod_{i=1}^N \mu^{x_i} (1-\mu)^{1-x_i}$$

$$\ln p(x|\mu) = \sum_{i=1}^N x_i \ln \mu + (1-x_i) \ln (1-\mu)$$

Differentiating w.r.t μ & setting to 0:

$$\frac{\partial}{\partial \mu} (\ln p(x|\mu)) = \sum_{i=1}^N \frac{x_i}{\mu} - \frac{1-x_i}{1-\mu} = 0$$

$$\Rightarrow \frac{1}{\mu(1-\mu)} \sum_{i=1}^N x_i(1-\mu) - (1-x_i)\mu = 0$$

$$\Rightarrow \sum_{i=1}^N (x_i - \mu) = 0$$

$$\Rightarrow \frac{1}{N} \sum_{i=1}^N x_i = \mu$$

In our case, $N=3$ and $x=\{x_1, x_2, x_3\} = \{1, 1, 1\}$

$$\therefore \mu = \frac{1}{3} \cdot 3 = 1$$

$\equiv \text{Ans.}$

② Posterior dist. of μ under a Beta prior is:

$$p(\mu | x, a, b) \propto p(x | \mu, a, b) p(\mu | a, b)$$

$$= \underbrace{\frac{\Gamma(m+a+b)}{\Gamma(m+a)\Gamma(b)}}_{\text{normalizing factor}} \underbrace{\mu^{a-1} (1-\mu)^{b-1}}_{\text{prior}} \underbrace{\prod_{i=1}^N \mu^{x_i} (1-\mu)^{1-x_i}}_{\text{likelihood}}$$

$$\ln p(\mu | x, a, b) = c + (a-1) \ln \mu + (b-1) \ln(1-\mu) + \sum_{i=1}^N x_i \ln \mu + (1-x_i) \ln(1-\mu)$$

Differentiating w.r.t μ & setting to 0:

$$\frac{\partial}{\partial \mu} \ln p(\mu | x, a, b) = c + \frac{a-1}{\mu} + \frac{b-1}{1-\mu} + \sum_{i=1}^N \frac{x_i}{\mu} + \frac{(1-x_i)}{1-\mu} = 0$$

$$\Rightarrow (a-1)(1-\mu) + (b-1)\mu + \sum_{i=1}^N x_i(1-\mu) + (1-x_i)\mu = 0$$

$$\Rightarrow (a-1)(1-\mu) + (b-1)\mu + \sum_{i=1}^N (x_i - \mu) = 0$$

$$\Rightarrow a - a\mu - 1 + \mu + b\mu - \mu + 3 - \mu = 0$$

$$\Rightarrow \mu_{\text{MAP}} = \frac{a+2}{a+b+1}$$

\equiv AWD.

③ $\mu_{\text{MLE}} = 1 = \frac{m}{m+l}$

prior mean = $\frac{a}{a+b}$ [i.e., $E[\mu]$ of beta distribution]

posterior mean = $\frac{m+a}{m+a+l+b}$ $E[\mu|D] = \int_0^1 \mu p(\mu|D) d\mu$

$$\frac{a}{a+b} \leq \frac{m+a}{m+a+l+b} < \frac{m+a}{m+b} \quad [\because m, a, b, l \geq 0]$$

\equiv AWD.

Q4.

II

General form of exponential family.

$$p(x|\eta) = h(x) g(\eta) \exp\{\eta^T u(x)\}$$

We will try to reduce the following distributions to this form

1) Pois($k|\lambda$) = $\frac{\lambda^k e^{-\lambda}}{k!} = \frac{1}{k!} \cdot \lambda^k \cdot \exp(-\lambda)$

Hence, $h(k) = \frac{1}{k!}$, $\eta = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}$, $u(k) = k$, $g(\eta) = \frac{e^\lambda}{e^{-\lambda}}$
 \equiv Ans.

2) Gamma($\tau|a,b$) = $\frac{1}{\Gamma(a)} b^a \tau^{a-1} e^{-b\tau}$
 $= \frac{b^a}{\Gamma(a)} e^{(a-1)\log \tau - b\tau}$

Hence, $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} a-1 \\ -b \end{pmatrix}$, $\begin{pmatrix} u_1(\eta) \\ u_2(\eta) \end{pmatrix} = \begin{pmatrix} \log \tau \\ \tau \end{pmatrix}$
 $h(\tau) = 1$, $\begin{pmatrix} g(\eta_1) \\ g(\eta_2) \end{pmatrix} = \begin{pmatrix} b^a \\ \Gamma(a) \end{pmatrix}$
 \equiv Ans.

3) Cauchy($x|x,\mu$) is not part of the exponential family. This is because its mean is undefined which violates the required property of exponential family.
 \equiv Ans.

4) NonMises($x|\kappa, \mu$) = $\frac{1}{2\pi I_0(\kappa)} e^{\frac{\kappa \cos(x-\mu)}{2}}$
 $= \frac{1}{2\pi I_0(\kappa)} e^{\frac{\kappa \cos x \cos \mu + \kappa \sin x \sin \mu}{2}}$

Hence, $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \kappa \cos \mu \\ \kappa \sin \mu \end{pmatrix}$, $\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$,

$h(x) = 1$, $g(\eta) = \frac{1}{2\pi I_0(\kappa)}$
 \equiv Ans.

2

i) Poisson : (Using Bishop 2.226)

$$\begin{aligned} E[u(x)] &= -\nabla_\eta \ln g(\eta) \\ &= -\nabla_\eta \ln(e^\lambda) \\ &= +\nabla_\eta (\lambda) \\ &\equiv \text{Ans.} \end{aligned}$$

$$\begin{aligned} \text{Var}[u(x)] &= E^2[u(x)] - E[u(x)]^2 \\ &= -\nabla_\eta^2 \ln g(\eta) \\ &= \nabla_\eta^2 (\lambda) \\ &= e^\lambda \\ &= \lambda \quad \text{Ans.} \end{aligned}$$

2) gamma : (Using Bishop 2-226)

revise

$$\left. \begin{aligned} E[u_1(x)] &= -\nabla_{\eta_1} \ln g(\eta_1) \\ &= -\nabla_{\eta_1} \ln(b^a) \\ &= -\nabla_{\eta_1} a \ln b \\ &= -\nabla_{\eta_1} (\eta_1 + 1) \ln(-\eta_2) \\ &= -\ln \eta_1 \\ &\quad \text{Ans.} \end{aligned} \right\} \quad \left. \begin{aligned} E[u_2(x)] &= -\nabla_{\eta_2} \ln g(\eta_2) \\ &= -\nabla_{\eta_2} \ln(\Gamma(a)) \\ &= -\nabla_{\eta_2} \ln(\Gamma(\eta_1 + 1)) \\ &= 0 \\ &\quad \text{Ans.} \end{aligned} \right\}$$

3] Poisson distribution is part of exponential family and hence has a conjugate prior, which is the gamma Distribution :

$$\text{likelihood} = \text{Poisson} \quad p(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\text{prior} = \text{Beta} \quad p(\lambda|a, b) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}$$

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}} \text{ or, } p(\lambda|x, a, b) = \frac{p(x|\lambda) \cdot p(\lambda|a, b)}{p(x)}$$

$$\begin{aligned} &= \frac{\frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}}{\int \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}} \\ &= \frac{\lambda^{a+x-1} e^{-(b+1)\lambda}}{\int \lambda^{a+x-1} e^{-(b+1)\lambda}} \cdot \frac{x! \Gamma(a)}{\lambda^x \Gamma(a)} \\ &= \frac{\lambda^{a+x-1} e^{-(b+1)\lambda}}{\Gamma(a+x-1)} \\ &= \text{gamma}(\lambda | a+x-1, b+1) \end{aligned}$$

Hence, posterior is of the same form as the prior.

∴ gamma distribution is the conjugate prior of Poisson.
Ans.

Q5.

mean :student-t distribution (with the constant term expressed as ' ζ ') :

$$f(x) = c \left(1 + \frac{xx^T}{n}\right)^{-\frac{(n+1)}{2}} \quad \text{where } c = \frac{1}{\sqrt{n}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{n}{2})}$$

By definition :

$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^0 x f(x) dx + \int_0^{\infty} x f(x) dx \end{aligned}$$

using change of variable : $t = -x$ for first integral $\rightarrow dt = -dx$

$$= - \int_0^{\infty} (-t) f_x(-t) dt + \int_0^{\infty} x f(x) dx$$

$$= \int_0^{\infty} t f_x(-t) dt + \int_0^{\infty} x f(x) dx$$

$$= - \int_0^{\infty} t f_x(-t) dt + \int_0^{\infty} x f(x) dx$$

$$= - \int_0^{\infty} x f(-x) dx + \int_0^{\infty} x f(x) dx$$

$$= - \int_0^{\infty} x f(x) dx + \int_0^{\infty} x f(x) dx$$

[exchanging bounds of integral]

[trivial change of variable : $x = t$][$\because f(x)$ is an even func., thus $f(x) = f(-x)$]

$$= 0$$

$$\therefore E[x] = 0$$

Ans.

variance :

By definition :

$$\text{var}[x] = E[xx^T] - E[x]^2$$

where, we have derived above that $E[x] = 0$. $\therefore E[x]^2 = 0$

$$E[x^2] = \int_{-\infty}^{\infty} xx^T f(x) dx$$

$$= \int_{-\infty}^0 xx^T f(x) dx + \int_0^{\infty} xx^T f(x) dx$$

$$= - \int_0^{\infty} tt^T f_x(-t) dt + \int_0^{\infty} xx^T f(x) dx$$

[change of variable : $x = -t$]
 $\Rightarrow dx = -dt$

$$\begin{aligned}
&= \int_0^\infty xt^T f_x(-t) dt + \int_0^\infty t x^T f_x(x) dx \\
&= \int_0^\infty xx^T f_x(-x) dx + \int_0^\infty xx^T f_x(x) dx \quad [\text{change of variable : } x=t] \\
&= 2 \int_0^\infty xx^T f(x) dx \quad [\because f(x) \text{ is an even func., thus } f(x) = f(-x)]
\end{aligned}$$

Substituting the definition of $f(x)$:

$$= 2c \int_0^\infty xx^T \left(1 + \frac{xx^T}{n}\right)^{-\frac{(n+1)}{2}} dx$$

Using change of variables: $x = \left(1 + \frac{xx^T}{n}\right)^{-\frac{1}{2}}$

$$1 + \frac{x^2}{n} = \frac{1}{t} \Rightarrow xx^T = n(t-1)$$

$$x = \sqrt{n} \left(\frac{1}{t}-1\right)^{\frac{1}{2}}$$

$$\frac{dx}{dt} = -\frac{\sqrt{n}}{2} t^{-\frac{3}{2}} \Rightarrow dx = -\frac{\sqrt{n}}{2} t^{-\frac{3}{2}} dt$$

$$\begin{aligned}
&= n\sqrt{n} c \int_0^1 t^{\frac{n-2}{2}} (1-t)^{\frac{1}{2}} dt \\
&= n\sqrt{n} \cdot \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})} \int_0^1 t^{\frac{n-2}{2}} (1-t)^{\frac{1}{2}} dt \quad [\text{Substituting } c]
\end{aligned}$$

Here, the part inside integral is of the form of gamma function:

$$\int_0^1 x^{p-1} (1-x)^{q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\text{where, } p = \frac{n-1}{2}, \quad q = \frac{3}{2}$$

Thus,

$$= n \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{n+1}{2})}$$

$$= \frac{n \Gamma(\frac{n-1}{2}) \Gamma(\frac{1}{2} + 1)}{\Gamma(\frac{1}{2}) \Gamma(\frac{n}{2})}$$

$$= n \frac{\Gamma(\frac{1}{2}) \frac{1}{2} \Gamma(\frac{n}{2}) \frac{2}{n-2}}{\Gamma(\frac{n}{2}) \Gamma(\frac{1}{2})} \quad [\text{using } \Gamma(x) = \Gamma(x-1)(x-1)]$$

$$\therefore E[x^2] = \frac{n}{n-2}$$

$$\Rightarrow \text{var}[x] = E[x^2] - E[x]^2$$
$$= \frac{n}{n-2} - 0$$
$$\therefore \boxed{\text{var}[x] = \frac{n}{n-2}}$$

= Ans.