

ML - 2

Homework Assignment - 1

Shantanu Chandra
(12048461)

Q1.

From linearity of expectations, we have :

$$\begin{aligned} E[y] &= E[x+z] \\ &= E[x] + E[z] \Rightarrow \boxed{\mu_y = \mu_x + \mu_z} \end{aligned}$$

In case of multi-variate distributions : ($x \& z$ dependent)

$$\begin{aligned} \text{cov}[y] &= E[yy^T] - E[y]E[y^T] \\ &= E[(x+z)(x+z)^T] - E[x+z]E[(x+z)^T] \\ &= E[xx^T + xz^T + zx^T + zz^T] - (E[x] + E[z])(E[x^T] + E[z^T]) \\ &= \boxed{E[xx^T] - \mu_x \mu_x^T + E[zz^T] - \mu_z \mu_z^T + E[zx^T] - \mu_z \mu_x^T + E[xz^T] - \mu_x \mu_z^T} \\ &= \text{var}[x] + \text{var}[z] + 2 \text{cov}[x, z] \\ &\quad // Ans. \end{aligned} \quad \textcircled{1}$$

In case of multi-variate distributions : ($x \& z$ independent)

we can expand $\textcircled{1}$ as :

$$\begin{aligned} \text{cov}[y] &= E[xx^T] + E[x]E[z^T] + E[z]E[x^T] + E[zz^T] - (E[x] + E[z])(E[x^T] + E[z^T]) \\ &= \boxed{E[xx^T] - E[x]E[x^T]} + \boxed{E[zz^T] - E[z]E[z^T]} \\ &= \Sigma_x + \Sigma_z \\ &\quad // Ans. \end{aligned}$$

Q2.

$$\textcircled{1} \quad p(x|\mu, \Sigma) = \prod_{i=1}^N p(x_i|\mu, \Sigma) = \prod_{i=1}^N N(x_i|\mu, \Sigma) = \prod_{i=1}^N \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} (x_i - \mu)^T \Sigma^{-1} (x_i - \mu)\right) \\ // Ans.$$

$$\textcircled{2} \quad p(\mu|x, \Sigma, \mu_0, \Sigma_0) = \frac{p(x|\mu, \Sigma) \cdot p(\mu|\mu_0, \Sigma_0)}{p(x)} = \frac{\prod_{i=1}^N N(x_i|\mu, \Sigma) \cdot N(\mu|\mu_0, \Sigma_0)}{p(x)} \text{ where, } p(x) = \int p(x|\mu, \Sigma) \cdot p(\mu|\mu_0, \Sigma_0) \\ // Ans.$$

$$\textcircled{3} \quad p(\mu|x, \Sigma, \mu_0, \Sigma_0) = \frac{\prod_{i=1}^N N(x_i|\mu, \Sigma) \cdot N(\mu|\mu_0, \Sigma_0)}{p(x)}$$

Focussing on just the 'exp' part of $N(\cdot)$,

$$= \exp\left(-\frac{1}{2} \sum_{i=1}^N (x_i - \mu)^T \Sigma^{-1} (x_i - \mu) + (\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)\right)$$

On expanding the terms we get,

$$= \exp\left(\frac{-1}{2} \underbrace{\left(\sum_{i=1}^N x_i^T \Sigma^{-1} x_i - 2 \mu^T \Sigma^{-1} \sum_{i=1}^N x_i + N \mu^T \Sigma^{-1} \mu\right)}_{\text{constant}} + \underbrace{\left(\mu^T \Sigma_0^{-1} \mu - 2 \mu^T \Sigma_0^{-1} \mu_0 + \mu_0^T \Sigma_0^{-1} \mu_0\right)}_{N \bar{x} (\bar{x} = \frac{1}{N} \sum x_i)}\right)$$

$$= \exp\left(\mu^T (N \Sigma^{-1} + \Sigma_0^{-1}) \mu - \mu^T (N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0) - (N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0)^T \mu\right)$$

$$\text{let } A = N \Sigma^{-1} + \Sigma_0^{-1} \quad \& \quad B = N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0$$

$$= \exp(\mu^T A \mu - \mu^T B - B^T \mu)$$

$$= \exp(\mu^T A \mu - \mu^T B - B^T \mu + \underbrace{B^T A^{-1} B}_{\text{indep. of } \mu})$$

$$= \exp(\underbrace{\mu^T A \mu}_{\text{covariance}} - \underbrace{\mu^T A A^{-1} B}_{\text{mean}} - B^T A^{-1} B \mu + B^T A^{-1} A A^{-1} B)$$

[using $I = AA^{-1} = A^T A$
since A is symmetric.]

Thus, using completing the square we get,

$$\Sigma_n = A^{-1}, \quad \mu_n = A^{-1} B$$

$$\text{ie, } \boxed{\begin{aligned} \Sigma_n &= (N \Sigma^{-1} + \Sigma_0^{-1})^{-1} \\ \mu_n &= \Sigma_n (N \Sigma^{-1} \bar{x} + \Sigma_0^{-1} \mu_0) \end{aligned}}$$

/// Ans.

(4)

μ_{MAP} maximizes the posterior, thus:

$$\mu_{MAP} = \underset{\mu}{\operatorname{argmax}} p(\mu | x, \varepsilon, \mu_0, \Sigma_0)$$

$$= \underset{\mu}{\operatorname{argmax}} \frac{p(x | \mu, \varepsilon) p(\mu | \mu_0, \Sigma_0)}{p(x | \varepsilon)}$$

$$= \underset{\mu}{\operatorname{argmax}} \left[\log p(x | \mu, \varepsilon) + \log p(\mu | \mu_0, \Sigma_0) - \underbrace{\log p(x | \varepsilon)}_{\text{does not depend on } \mu, \text{ so drop.}} \right]$$

Differentiating and setting to 0 :

$$\log p(\mu | x, \varepsilon, \mu_0, \Sigma_0) = \sum_{i=1}^N \log N(x_i | \mu, \varepsilon) + \sum_{i=1}^N \log N(\mu | \mu_0, \Sigma_0)$$

$$= \log\left(\frac{1}{(2\pi)^{D/2} |\varepsilon|^{D/2}}\right) + \sum_{i=1}^N \frac{-1}{2} (x_i - \mu)^T \varepsilon^{-1} (x_i - \mu) + \log\left(\frac{1}{(2\pi)^{D/2} |\Sigma_0|^{D/2}}\right) + \frac{-1}{2} (\mu - \mu_0)^T \varepsilon_0^{-1} (\mu - \mu_0)$$

$$\text{or, } \frac{\partial}{\partial \mu} \log p(\mu | x, \varepsilon, \mu_0, \Sigma_0) = -\varepsilon^{-1} \sum_{i=1}^N (x_i - \mu) - \varepsilon_0^{-1} (\mu - \mu_0) \quad [\text{constant terms drop off}]$$

$$-\sum \xi^{-1} \sum_{i=1}^N x_i - N \sum \xi^{-1} \mu - \sum \xi^{-1} \mu_0 = 0 \quad [\text{expanding}]$$

$$\mu (N \sum \xi^{-1} + \sum \xi^{-1}) = \sum \xi^{-1} \sum_{i=1}^N x_i + \sum \xi^{-1} \mu_0 \quad [\text{collecting terms}]$$

$$\therefore \boxed{\mu_{\text{MAP}} = \frac{\sum \xi^{-1} \sum_{i=1}^N x_i + \sum \xi^{-1} \mu_0}{N \sum \xi^{-1} + \sum \xi^{-1}}} \\ \equiv \text{Ans.}$$

Q3.

- ① Since all the observed variables have given 'heads', the MLE will assign it a probability of 1. As a result, probability of observing 'tails' = 0.

$$\text{i.e., } \mu = 1$$

Hence, probability of observing heads in the next toss = 1
 $\equiv \text{Ans.}$

Mathematically we can see that as:

$$\text{likelihood: } p(x|\mu) = \prod_{i=1}^N \mu^{x_i} (1-\mu)^{1-x_i}$$

$$\ln p(x|\mu) = \sum_{i=1}^N x_i \ln \mu + (1-x_i) \ln (1-\mu)$$

Differentiating w.r.t μ & setting to 0:

$$\frac{\partial}{\partial \mu} (\ln p(x|\mu)) = \sum_{i=1}^N \frac{x_i}{\mu} - \frac{1-x_i}{1-\mu} = 0$$

$$\Rightarrow \frac{1}{\mu(1-\mu)} \sum_{i=1}^N x_i (1-\mu) - (1-x_i) \mu = 0$$

$$\Rightarrow \sum_{i=1}^N (x_i - \mu) = 0$$

$$\Rightarrow \frac{1}{N} \sum_{i=1}^N x_i = \mu$$

In our case, $N=3$ and $x=\{x_1, x_2, x_3\} = \{1, 1, 1\}$

$$\therefore \mu = \frac{1}{3} \cdot 3 = 1$$

$\equiv \text{Ans.}$

- ② Posterior dist. of μ under a Beta prior is:

$$p(\mu|x, a, b) \propto p(x|\mu, a, b) p(\mu|a, b)$$

$$= \frac{\Gamma(m+a+b)}{\Gamma(m+a) \Gamma(b)} \underbrace{\mu^{a-1} (1-\mu)^{b-1}}_{\text{prior}} \underbrace{\prod_{i=1}^N \mu^{x_i} (1-\mu)^{1-x_i}}_{\text{likelihood}}$$

normalizing factor = C

$$\ln p(\mu | x, a, b) = c + (a-1)\ln \mu + (b-1)\ln(1-\mu) + \sum_{i=1}^N x_i \ln \mu + (1-x_i) \ln(1-\mu)$$

Differentiating w.r.t μ & setting to 0:

$$\frac{\partial}{\partial \mu} \ln p(\mu | x, a, b) = c + \frac{a-1}{\mu} + \frac{b-1}{1-\mu} + \sum_{i=1}^N \frac{x_i}{\mu} + \frac{(1-x_i)}{1-\mu} = 0$$

$$\Rightarrow (a-1)(1-\mu) + (b-1)\mu + \sum_{i=1}^N x_i(1-\mu) + (1-x_i)\mu = 0$$

$$\Rightarrow (a-1)(1-\mu) + (b-1)\mu + \sum_{i=1}^N (x_i - \mu) = 0$$

$$\Rightarrow a - a\mu - 1 + \mu + b\mu - \mu + 3 - \mu = 0$$

$$\Rightarrow \mu_{\text{MAP}} = \frac{a+2}{a+b+1}$$

$\not\equiv \text{Ans.}$

Note, in case of Bayesian approach to the problem, using Bishop 2.20 ($m=3, l=0$):

$$\mu_{\text{posterior}} = \frac{m+a}{m+a+l+b} = \frac{3+a}{3+a+b}$$

$\not\equiv \text{Ans.}$

$$(3) \quad \mu_{\text{MLE}} = \frac{m}{m+l}$$

$$\text{prior mean} = \frac{a}{a+b} \quad [\text{i.e., } E[x_i] \text{ of beta distribution}]$$

$$\text{posterior mean} = \frac{m+a}{m+a+l+b} \quad \left[E[\mu|D] = \int_0^1 \mu p(\mu|D) d\mu \right]$$

To show that posterior mean lies between μ_{MLE} & prior mean, we can express the former as a linear combination of the latter i.e., $\alpha \mu_{\text{MLE}} + (1-\alpha) \mu_{\text{prior}} = \mu_{\text{post}}$, with constraint: $0 \leq \alpha \leq 1$.

$$\begin{aligned} \frac{m+a}{m+l+a+b} &= \frac{\frac{m}{m+l+a+b} + \frac{a}{m+l+a+b}}{\frac{m+l}{m+l+a+b} + \frac{a+b}{m+l+a+b}} \\ &= \frac{m+l}{m+l+a+b} \cdot \frac{m}{m+l+a+b} + \frac{a+b}{m+l+a+b} \cdot \frac{a}{m+l+a+b} \\ &= \underbrace{\frac{m}{m+l}}_{\mu_{\text{MLE}}} \cdot \underbrace{\frac{m+l}{m+l+a+b}}_{\alpha} + \underbrace{\frac{a}{a+b}}_{\mu_{\text{prior}}} \cdot \underbrace{\frac{a+b}{m+l+a+b}}_{1-\alpha} \end{aligned}$$

(Switching terms) — ①

$$\text{or, } \mu_{\text{posterior}} = \underline{\alpha \mu_{\text{MLE}}} + \underline{(1-\alpha) \mu_{\text{prior}}} \not\equiv \text{Proved}$$

NOTE: where, $X = m+l$ and $Y = a+b$

$$\therefore \alpha = \frac{X}{X+Y} \Rightarrow (1-\alpha) = \frac{Y}{X+Y} = \frac{Y}{X+Y}$$

Hence, we see that ' α ' is a weighted mean of X & Y .

Thus, ' α ' satisfies: $0 \leq \alpha \leq 1$

And thus, by convex combination property the objective is proved $\not\equiv \text{Ans.}$

Q4.

II

General form of exponential family.

$$p(x|n) = h(x) g(\eta) \exp\{\eta^T u(x)\}, \text{ sufficient stats} = \sum_{i=1}^n u_i(x)$$

We will try to reduce the following distributions to this form

1) Pois($k|\lambda$) = $\frac{\lambda^k e^{-\lambda}}{k!} = \frac{1}{k!} \cdot \lambda^k \cdot \exp(k \log \lambda)$

Hence, $h(k) = 1/k!$, $\eta = \log \lambda \rightarrow \eta = \lambda$, $u(k) = k$, $g(\eta) = \frac{e^{-\lambda}}{\lambda^k}$, sufficient stats = $\frac{1}{N} \sum_{i=1}^N k_i$

Ans.

2) Gamma(α, β) = $\frac{1}{\Gamma(\alpha)} b^{\alpha-1} e^{-bx}$

$$= \frac{b^\alpha}{\Gamma(\alpha)} e^{(\alpha-1)\log b - bx}$$

Hence, $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \alpha-1 \\ -b \end{pmatrix}$, $\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} \log b \\ x \end{pmatrix}$, sufficient stats = $\frac{1}{N} \sum_{i=1}^N \left(\log x_i \right)$

$h(\eta) = 1$, $\begin{pmatrix} g(\eta_1) \\ g(\eta_2) \end{pmatrix} = \begin{pmatrix} b^\alpha \\ \Gamma(\alpha) \end{pmatrix}$

Ans.

3) Cauchy($x|\mu, \sigma$) is not part of the exponential family. This is because its mean is undefined which violates the required property of exponential family.

Ans.

We can show this by calculating the $E[x]$:

$$E[x|\mu, \sigma] = \int_{-\infty}^{\infty} \frac{x}{(1+x^2)\pi} dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{x}{(1+x^2)} dx$$

$$= \frac{1}{\pi} \left[\log(1+x^2) \right]_0^{\infty}$$

$$= \infty$$

Since for at least one set of params the expectation is undefined, it is not part of the exponential family.

proved

4) NonMises($x|\kappa, \mu$) = $\frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(x-\mu)}$

$$= \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos x \cos \mu + \kappa \sin x \sin \mu}$$

Hence, $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \kappa \cos \mu \\ \kappa \sin \mu \end{pmatrix}$, $\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$, sufficient stats = $\frac{1}{N} \sum_{i=1}^N (\cos x_i)$

$h(x) = 1$, $g(\eta) = \frac{1}{2\pi I_0(\kappa)}$

Ans.

2

1) Poisson : (Using Bishop 2.226)

$$\begin{aligned} E[\lambda] &= E[u(x)] = -\nabla_{\eta} \ln g(\eta) \\ &= -\nabla_{\eta} \ln (\bar{e}^{\eta}) \\ &= +\nabla_{\eta} (\bar{e}^{\eta}) \\ &= \lambda \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} E[\lambda^2] &= E[u^2(x)] = -\nabla_{\eta}^2 \ln g(\eta) \\ &= \nabla_{\eta}^2 (\bar{e}^{\eta}) \\ &= \bar{e}^{2\eta} \\ &= \lambda^2 \quad \text{Ans.} \end{aligned}$$

2) Gamma : (Using Bishop 2.226)

$$\begin{aligned} E[\tau] &= E[u_2(x)] = -\nabla_{\eta_2} \ln g(\eta) \\ &= -\nabla_{\eta_2} \ln \frac{\eta_1^{n+1}}{\Gamma(n+1)} \\ &= -\nabla_{\eta_2} (\eta_1^{n+1}) \ln(\eta_2) + c \\ &= \frac{n+1}{\eta_2} = \frac{a}{b} \quad \text{Ans.} \end{aligned}$$

NOTE:
 $u(\tau) = (\log \tau, \tau)$
we need to calculate
 $E[\tau]$, hence we use
 η_2 in our calculation
as it is straight forward.

$$\begin{aligned} E[\tau^2] &= E[u_2^2(x)] \\ &= -\nabla_{\eta_2}^2 \ln g(\eta) \\ &= -(n+1) \nabla_{\eta_2}^2 \ln(-\eta_2) \\ &= +(n+1) \nabla_{\eta_2} (\eta_2)^{-1} \\ &= \frac{n+1}{\eta_2^2} = \frac{a}{b^2} \quad \text{Ans.} \end{aligned}$$

3

Poisson distribution is part of exponential family and hence has a conjugate prior, which is the gamma distribution :

$$\text{Likelihood} = \text{Poisson} \quad p(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$\text{prior} = \text{gamma} \quad p(\lambda|a, b) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}$$

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}} \text{ or, } p(\lambda|x, a, b) = \frac{p(x|\lambda) \cdot p(\lambda|a, b)}{p(x)}$$

$$\begin{aligned} &= \frac{\prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \cdot \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}}{\int \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \cdot \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} d\lambda} \\ &= \frac{\lambda^{(a-1)+\sum x_i} e^{-(n+b)\lambda}}{\int \lambda^{(a-1)+\sum x_i} e^{-(n+b)\lambda} d\lambda} \cdot \frac{\cdot \prod_{i=1}^n x_i! \Gamma(a)}{\prod_{i=1}^n \lambda^{x_i} \Gamma(a)} \\ &= \frac{\lambda^{(a-1)+\sum x_i} e^{-(n+b)\lambda}}{\prod_{i=1}^n \lambda^{x_i} \Gamma(a)} \end{aligned}$$

Hence, posterior is of the same functional form as the prior (conjugacy) \therefore gamma distribution is the conjugate prior of Poisson.Ans.= gamma ($\lambda | a + \sum_{i=1}^n x_i, n+b$)Ans.

Q5.

mean :student-t distribution (with the constant term expressed as ' ζ ') :

$$f(x) = c \left(1 + \frac{xx^T}{n}\right)^{-\frac{(n+1)}{2}} \quad \text{where } c = \frac{1}{\sqrt{n}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{n}{2})}$$

By definition :

$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^0 x f(x) dx + \int_0^{\infty} x f(x) dx \end{aligned}$$

using change of variable : $t = -x$ for first integral $\rightarrow dt = -dx$

$$= - \int_0^{\infty} (-t) f_x(-t) dt + \int_0^{\infty} x f(x) dx$$

$$= \int_0^{\infty} t f_x(-t) dt + \int_0^{\infty} x f(x) dx$$

$$= - \int_0^{\infty} t f_x(-t) dt + \int_0^{\infty} x f(x) dx \quad [\text{exchanging bounds of integral}]$$

$$= - \int_0^{\infty} x f(-x) dx + \int_0^{\infty} x f(x) dx \quad [\text{trivial change of variable : } x = t]$$

$$= - \int_0^{\infty} x f(x) dx + \int_0^{\infty} x f(x) dx \quad [\because f(x) \text{ is an even func., thus } f(x) = f(-x)]$$

$$= 0$$

$$\therefore E[x] = 0$$

Ans.

variance :

By definition :

$$\text{var}[x] = E[xx^T] - E[x]^2$$

where, we have derived above that $E[x] = 0$. $\therefore E[x]^2 = 0$

$$E[x^2] = \int_{-\infty}^{\infty} xx^T f(x) dx$$

$$= \int_{-\infty}^0 xx^T f(x) dx + \int_0^{\infty} xx^T f(x) dx$$

$$= - \int_0^{\infty} tt^T f_x(-t) dt + \int_0^{\infty} xx^T f(x) dx \quad [\text{change of variable : } x = -t \Rightarrow dx = -dt]$$

$$\begin{aligned}
&= \int_0^\infty xt^T f_{\frac{x}{t}}(-t) dt + \int_0^\infty t x^T f_{\frac{x}{t}}(x) dx \\
&= \int_0^\infty x x^T f_{\frac{x}{t}}(-x) dx + \int_0^\infty x x^T f_{\frac{x}{t}}(x) dx \quad [\text{change of variable : } x=t] \\
&= 2 \int_0^\infty x x^T f(x) dx \quad [\because f(x) \text{ is an even func., thus } f(x) = f(-x)]
\end{aligned}$$

Substituting the definition of $f(x)$:

$$= 2c \int_0^\infty x x^T \left(1 + \frac{x x^T}{n}\right)^{-\frac{(n+1)}{2}} dx \quad \text{--- (1)}$$

Using change of variables: $x = \left(1 + \frac{x x^T}{n}\right)^{-\frac{1}{2}}$

$$1 + \frac{x^2}{n} = \frac{1}{t} \Rightarrow x x^T = n \left(\frac{1}{t} - 1\right)$$

$$x = \sqrt{n} \left(\frac{1}{t} - 1\right)^{\frac{1}{2}} \quad \text{--- (2)}$$

Thus, $\frac{dx}{dt} = -\frac{\sqrt{n}}{2} t^{-\frac{3}{2}} \Rightarrow dx = -\frac{\sqrt{n}}{2} t^{-\frac{3}{2}} dt \quad \text{--- (3)}$

$$= n \sqrt{n} c \int_0^1 t^{\frac{n-2}{2}} (1-t)^{\frac{1}{2}} dt \quad [\text{rewriting (1) using (2) \& (3)}]$$

$$= n \sqrt{n} \cdot \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{n+1}{2})} \int_0^1 t^{\frac{n-2}{2}} (1-t)^{\frac{1}{2}} dt \quad [\text{substituting } c]$$

Here, the part inside integral is of the form of gamma function:

$$\int_0^1 x^{p-1} (1-x)^{q-1} = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}$$

where, $p = \frac{n-1}{2}$, $q = \frac{3}{2}$

Thus,

$$= n \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n-1}{2}) \Gamma(\frac{3}{2})}{\Gamma(\frac{n+1}{2})}$$

$$= \frac{n \Gamma(\frac{n-1}{2}) \Gamma(\frac{1}{2} + 1)}{\Gamma(\frac{1}{2}) \Gamma(\frac{n}{2})}$$

$$= n \frac{\Gamma(\frac{1}{2}) \frac{1}{2} \Gamma(\frac{n}{2}) \frac{2}{n-2}}{\Gamma(\frac{n}{2}) \Gamma(\frac{1}{2})} \quad [\text{using } \Gamma(x) = \Gamma(x-1)(x-1)]$$

$$\therefore E[x^2] = \frac{n}{n-2}$$

$$\Rightarrow \text{var}[x] = E[x^2] - E[x]^2$$
$$= \frac{n}{n-2} - 0$$
$$\therefore \boxed{\text{var}[x] = \frac{n}{n-2}}$$

= Ans.