Introduction to Game Theory		Date:	*April 30, 2020*
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Using Game Theory And Mechanism Design for Group Project Assessment

1 Introduction

Increasingly, many courses have started adopting group projects as a means of assessment. They offer the ability for instructors to allot more difficult tasks to a larger group so that it becomes feasible to complete them. It also is supposed to teach students how to work in a team and how to properly manage group work, so that they are more prepared for the "real world".

However, a major issue that arises in such projects is how to assess how much each student has contributed and how to grade accordingly. Most of the time, the work done is not equally distributed across the members. Thus, for a fair scheme, we should give marks according to the work done by each person - this forms the basis for our social choice function. However, the problem becomes one of obtaining the true values of the contribution of each member. The simplest way to give marks would be to allot every one equal marks - however, this disadvantages the ones who work harder or care more about the grades, and this may unfairly advantage those who are not willing to put in the effort and just tag along with more interested or hardworking people. The alternative is to obtain the information about how much work was actually done by each person - which is not an easy task.

The problem as we are dealing with it is as follows:- In a group project, the total marks for the project have been awarded. Now, we need to award individual marks. Our goal is to design a mechanism such that each player reveals the actual work done by them. In this report, we first begin with the formal description of the problem, then some observations for it, and finally we propose some mechanisms for the same. We also provide a rough proof of why our mechanism is BNIC, and also provide a few examples. This is followed by some discussions regarding the drawbacks and possible pitfalls of using this mechanism.

The main reference was [Narahari, 2014], from which all the notation and definitions are used.

2 Formal Statement

We have the following elements in the game:-

- $N = \{1, \dots, n\}$:- The team members of the project.

- $u_i \in [0, M] \forall i \in N$:- The utility for player i, corresponding to the marks awarded to him, out of a maximum of M.

- $\forall i \in N \Theta(i) = [0, 1]$:- The type for each player, where $\theta(i) \in \Theta(i)$ is the fraction of work done by the player i. Note that here, there is a constraint on $\theta(i)$'s - the sum of $\theta(i)$'s should be 1, as the total work done is constant. Also, here, $\theta(i)$ is not exactly private - everyone knows the fraction of work done by everyone. The vector of actual work done in the project θ ($\theta = \{\theta(1), \theta(2), ..., \theta(N)\}$) is assumed to be common knowledge . Thus, the probability distribution ϕ_i is clear.

- $X = [0, M] \times [0, M] \times \ldots \times [0, M]$:- The set of all outcomes. Each $x \in X$ corresponds to an allocaion of marks such that x_i is the number of marks given to i^{th} player, out of a maximum of M.

Ideally, we wish to design a direct mechanism in which the the outcome, which is the marks allocation, is most in accordance to the fraction of the work done. Let $f : \{(x_1, x_2, \ldots, x_n) | 0 \leq x_i \leq 1, \sum_{i=1}^{n} x_i = 1\} \to X$ be our function which corresponds to the optimal mapping between the fraction of work done and the marks allotted(we discuss the nature of this function later). f is our social choice function. If $\theta' \in \Theta$ is the reported type, we want to design a mechanism such that the Nash equilibrium of the induced Bayesian game is when $\theta' = \theta$, where θ is the true type of the player.

3 Proposed Mechanism Design

Informally speaking, each of the members can be called individually and simultaneously to report the fraction of work done by each of the members according to him/her as a work distribution set. No prior information should be provided to them regarding the evaluation scheme . Hence we can assume here there that there was no cooperation among players while reporting their work distribution sets. Also the marks obtained by entire team for the project is known to the grader but hidden from the team . Now we intend to process the values reported by them and adjudge them marks accordingly on basis of their reported sets and the marks obtained by the team for the project .

The basic idea is as follows - we ask each of the people their fraction of the work done, and also the fraction of work done by everyone else. Based on this, we calculate the average of the reported works of each player, and assume that is the amount of work done by each. However, just this much is not sufficient - in this scheme, everyone would just say that they have done all the work. Thus, we introduce a penalty for lying - for any deviation in the reported values of the work done for all the people, we introduce some penalty. Therefore, lying would be harmful now - intuitively speaking, if everyone else is saying the truth and reporting correct fractions, then, if you lie, you would be the one facing the highest penalty, if the number of people who are reporting truthfully is greater than 1(i.e. total number of people in the group is greater than

2). Thus, lying would be harmful, assuming that everyone is saying the truth(a full worked out example is shown below), and thus, saying the truth is a Nash equilibrium.

More formally, we may say the following:-

- T:- The total team score i.e. the total marks given to the team's project
- M:- Maximum marks that can be awarded to a member
- θ_i:- The work distribution set, as reported by the i^{th} player. A vector in $[0,1]^n$
- μ_i:- The average value of the values reported i.e.

$$\mu_i = \frac{1}{n}\sum_{j=1}^{n}\theta_j(i)$$

- σ:- The deviations of each of the players

Now, in the mechanism we propose, the marks awarded i.e. the utilities would be given by

$$u_i(x) = \frac{T}{n} + f(\mu_i, \sigma_i)$$

where f is a function which takes as input the average work done by $i(\mu_i)$ as reported and the deviation in the reported values of $i(\sigma_i)$, and gives an output in the range $[-\frac{T}{n}, M - \frac{T}{n}]$ so that the utility has output in the range $[0, M]$. The function f also must satisfy some properties. If the deviation is 0, i.e. everyone reports the true amounts of work done, then the value of f must be positive if $\mu_i > \frac{1}{n}$, negative if $\mu_i < \frac{1}{n}$ and must be zero when $\mu_i = \frac{1}{n}$, corresponding to the cases where i does more, less or equal work to the case in which there would be an equal distribution of work. Also, the function must properly penalize deviations in the reporting so that lying is not an optimum strategy.

4 Bayesian-Nash Incentive Compatibility (BNIC)

We try to show how truth revelation is the best response for a member in expectation that rest of the members are also true.
Let i be the only player who diverts from truth and reports a misleading set of fractions of work done while all other players as an expectation report the true set of fractions of work done. Since this is a group project, we expect each of them exactly know how much work is done by which team member and this is common knowledge
.

- We have $\theta_j = \{\theta_j(1), \theta_j(2), ..., \theta_j(N)\} \quad \forall j \in N - \{i\}$
  Since all the members leaving i are reporting true, $\theta_j = \theta = \{\theta(1), \theta(2), ..., \theta(N)\} \quad \forall j \in N - \{i\}$

- For member i, we have $\theta_i = \{\theta_i(1), \theta_i(2), ..., \theta_i(N)\}$
  Let's say $\theta_i = \theta' = \{\theta'(1), \theta'(2), ..., \theta'(N)\}$

- Now according to our mechanism we estimate the means of work done by each player according to the sets provided by them .

$$\begin{aligned} \mu_j &= \frac{1}{N} \sum_k \theta_k(j) \\ &= \frac{1}{N}[(N-1) * \theta(j) + \theta'(j)] \end{aligned} \tag{1}$$

- Now we obtain the deviations for each of the members as follows :
  For the honest players, i.e, $\forall j \in N - \{i\}$,

$$\begin{aligned} \sigma_j &= \sqrt{\frac{1}{N} \sum_k (\mu_k - \theta(k))^2} \\ &= \sqrt{\frac{1}{N} \sum_k \{\frac{1}{N}[(N-1) * \theta(k) + \theta'(k)] - \theta(k)\}^2} \\ &= \sqrt{\frac{1}{N} \sum_k \{\frac{1}{N}[\theta'(k) - \theta(k)]\}^2} \\ &= \frac{1}{N\sqrt{N}} \sqrt{\sum_k [\theta'(k) - \theta(k)]^2} \end{aligned} \tag{2}$$

  For the dishonest player i,

$$\begin{aligned} \sigma_i &= \sqrt{\frac{1}{N} \sum_k (\mu_k - \theta'(k))^2} \\ &= \sqrt{\frac{1}{N} \sum_k \{\frac{1}{N}[(N-1) * \theta(k) + \theta'(k)] - \theta'(k)\}^2} \\ &= \sqrt{\frac{1}{N} \sum_k \{\frac{1}{N}[(N-1) * (\theta(k) - \theta'(k))]\}^2} \\ &= \frac{(N-1)}{N\sqrt{N}} \sqrt{\sum_k [\theta(k) - \theta'(k)]^2} \\ &= \frac{(N-1)}{N\sqrt{N}} \sqrt{\sum_k [\theta'(k) - \theta(k)]^2} \end{aligned} \tag{3}$$

- We can now actually see how large the deviation is for a dishonest member compared to an honest member.

$$\frac{\sigma_j}{\sigma_i} = \frac{1}{(N-1)} \quad \forall j \in N - \{i\} \tag{4}$$

Now in a group of size ≥ 3, we always have the dishonest member getting a higher deviation than the rest of the members.

Now if we can design a social choice function f that takes in μ_k and σ_k for a member k and penalises a member sufficiently with a higher deviation σ_k, we might be able to achieve an incentive compatible mechanism in expectation that rest of the members report true sets. Now we show how our mechanism obeys BNIC after exploring various possible candidates for social choice function in next section.

5 Possible Functions

The choice of the function f is an important factor to determine how our mechanism would work. In this section, some simple choices for the function are provided.

- **Function 1 :** The simplest function one can think of is a linear-like function, with the ends adjusted so that the function lies in the required range. For example,

$$f(\mu_i, \sigma_i) = \begin{cases} -\frac{T}{n} & a(\mu_i - \frac{1}{n}) + b\sigma_i \leq -\frac{T}{n} \\ a(\mu_i - \frac{1}{n}) + b\sigma_i & -\frac{T}{n} \leq a(\mu_i - \frac{1}{n}) + b\sigma_i \leq M - \frac{T}{n} \\ M - \frac{T}{n} & a(\mu_i - \frac{1}{n}) + b\sigma_i \geq M - \frac{T}{n} \end{cases}$$

  where $a > 0$ and $b < 0$ are constants. This function satisfies $f(\frac{1}{n}, 0) = 0$, and also penalizes false reporting of marks by adding a negative coefficient with σ_i. The coefficients a and b can be selected according to how much scaling is required - for example, for large a, if the contribution differs even slightly from the case where each person does average work, then the respective increase or decrease is large. Similarly, larger values of b imposes a larger penalty on false reporting. They can be tweaked according to the needs of the grader.

- **Function 2 :** Here we bring the notion of σ'_i which equals $\mu(\sigma) - \sigma_i$, where $\mu(\sigma)$ denotes the mean of deviations for each of the members.

$$f(\mu_i, \sigma_i) = \begin{cases} -\frac{T}{n} & a(\mu_i - \frac{1}{n}) + b\sigma'_i \leq -\frac{T}{n} \\ a(\mu_i - \frac{1}{n}) + b\sigma'_i & -\frac{T}{n} \leq a(\mu_i - \frac{1}{n}) + b\sigma'_i \leq M - \frac{T}{n} \\ M - \frac{T}{n} & a(\mu_i - \frac{1}{n}) + b\sigma'_i \geq M - \frac{T}{n} \end{cases}$$

  This function allows us to keep a and b constants > 0 and the unfairness added by any member is taken care of by σ'_i. Now just balancing the parameters a and b, the grader can decide to what factor he wants to give more priority. As per our sense of judgement, b should be greater than a since the deviations are generally of lesser order than the means in usual case , hence to punish the one who lies, b should be kept higher than a.

- **Function 3 :** The above given function provides a linear shape to the function, which may not be desired. Instead, something like a sigmoid might be better - where, to get higher marks, more work would need to be done, and similarly, to get zero marks, you would have to do very little work - the drops would not be constant, as opposed to the linear case. If such a scheme is required the following function can be used:-

$$f(\mu_i, \sigma_i) = (as(c(\mu_i - \frac{1}{n})) + bs(-c(\mu_i - \frac{1}{n}))) - \alpha\sigma_i$$

  where s is the sigmoid function:

$$s(x) = \frac{1}{1 + e^{-x}}$$

  and $a = M - \frac{T}{n}$, $b = -\frac{T}{n}$, and α,c are parameters(for large α, the penalty is larger, and it is small for small α).

6 Proof for BNIC

So far we have covered the intution behind BNIC, have learnt we are going to use means and standard deviations of the reported values to build a marks allotment mechanism according to use of any of the above mentioned functions. Note that there can exist certainly better function domains comapred to the ones we have mentioned above.
Now we are going to prove how with use of appropriate parameters, we can show that our mechanism is Bayesian Nash Incentive Compatible . We restrict ourselves to function 2 for further analysis, a likwise analysis can be extended for other functions as well .
We consider 2 cases below (continuation of section 4) when member i reports truthfully and when he doesn't in expectation that rest of the agents report truthfully.

$$f(\mu_i, \sigma_i) = \begin{cases} -\frac{T}{n} & a(\mu_i - \frac{1}{n}) + b\sigma_i' \leq -\frac{T}{n} \\ a(\mu_i - \frac{1}{n}) + b\sigma_i' & -\frac{T}{n} \leq a(\mu_i - \frac{1}{n}) + b\sigma_i' \leq M - \frac{T}{n} \\ M - \frac{T}{n} & a(\mu_i - \frac{1}{n}) + b\sigma_i' \geq M - \frac{T}{n} \end{cases}$$

$$a, b > 0$$

When i reports misleading values :

$$\mu_i = \frac{1}{N}[(N-1) * \theta(i) + \theta'(i)]$$

$$\sigma_i = \frac{(N-1)}{N\sqrt{N}} \sqrt{\sum_k [\theta'(k) - \theta(k)]^2}$$

$$\mu(\sigma) = \frac{2(N-1)}{N^2\sqrt{N}}\sqrt{\sum_k [\theta'(k) - \theta(k)]^2}$$

When i reports truthfully :

$$\mu_i = \theta(i)$$

$$\sigma_i = 0$$

$$\mu(\sigma) = 0$$

For showing Bayesian Incentive Comaptible, we need to show :

$$\mathbb{E}_{\theta_{-i}}\left[u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right)|\theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i\left(f\left(\theta'_i, \theta_{-i}\right), \theta_i\right)|\theta_i\right]$$

So let's compute

$$\mathbb{E}_{\theta_{-i}}\left[u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right)|\theta_i\right] - \mathbb{E}_{\theta_{-i}}\left[u_i\left(f\left(\theta'_i, \theta_{-i}\right), \theta_i\right)|\theta_i\right]$$

$$= \frac{a}{N}[\theta(i) - \theta'(i)] - \frac{bZ}{N^2\sqrt{N}}[2(N-1) - N(N-1)]$$

$$= \frac{bZ}{N^2\sqrt{N}}[(N-1)(N-2)] - \frac{a}{N}[\theta'(i) - \theta(i)]$$

where $Z = \sqrt{\sum_k [\theta'(k) - \theta(k)]^2}$ is necessarily a positive term.

Keynotes on the above result :

- $$\frac{bZ}{N^2\sqrt{N}}[(N-1)(N-2)]$$
  is necessarily positive, since $b, Z > 0$ and also the mechanism considers $N \geq 3$ as mentioned in section 4.

- $$\frac{a}{N}[\theta'(i) - \theta(i)]$$
  is positive when $\theta'(i) > \theta(i)$, i.e., the member i greedily reports a higher value of his own contribution than the truthful value . If he downgrades his own contribution the above is negative and hence the difference is necessarily positive and hence trivially reporting truth is best option. We are more interested in the other scenario, where the member i greedily reports a higher value of his own contribution.

- Now if the above 2 expressions are positive, we can always choose parameters b and a in a way to bring the difference always positive and hence satisfy our condition of truthfulness in expectation that others are truthful . b kept much higher than a satisfies our condition easily .

Hence we can adjust the parameters a, b to always have our mechanism as Bayesian Incentive Compatible.
We need to note here that we cannot keep b arbitrarily $>> a$, because though, in that case our condition of BNIC would be satisfied but our mechanism might not be fair in marks allotment. So further work can be done from here on where we bring in a $fairness$ function to check the fairness of marks allotment. Using the $fairness$ function we can actually run a whole lot of simulations to check which parameters fit best each of the above mentioned functions .

7 Examples

7.1

Consider the following set of truth values reported for a team of size 4.

	1	2	3	4
1	0.2	0.3	0.3	0.2
2	0.2	0.3	0.3	0.2
3	0.2	0.3	0.3	0.2
4	0.1	0.2	0.2	0.5

We look at the scenario here, where 4th member has likely provided misleading values whereas the other 3 have reported true values. Let us look at the means and deviation corresponding to these players.

	1	2	3	4
μ	0.175	0.275	0.275	0.275
σ	0.0433	0.0433	0.0433	0.1299

See by reporting false values, the 4th member has been able to get its mean adjusted to 2nd and 3rd player but his lie has probably been caught through deviations, as we see his deviation is 3 times the deviations of the other 3 members as we had witnessed in equation (4). Now we need to look through our functions, if this deviation is countered with the mean in each case to award fair marks to each candidate or not.

Shown below are the marks allotment where each member can be at most awarded 100 marks and the team has been awarded M out of 400. For various $M's$ the marks allotment is shown :
function 2 :

	1	2	3	4
200	46.83	56.83	56.83	39.51
246	58.33	68.33	68.33	51.01
291	69.58	79.58	79.58	62.26
333	80.08	90.08	90.08	72.76
372	89.83	99.83	99.83	82.51

We had set a as 100 and b as 200.

7.2

Consider the following set of truth values reported for a team of size 4.

	1	2	3	4
1	0.8	0.1	0.1	0.0
2	0.7	0.2	0.1	0.0
3	0.7	0.1	0.2	0.0
4	0.5	0.2	0.2	0.1

We look here at the above scenario, 1st member has mostly done the bulk of the work as the work distribution sets of each player suggest, while 2nd and 3rd members have seemed to put up below expected performance while 4th member has hardly been involved in the project. Now the question is how shall be the marks distribution in the above scheme, will the 4th member get any marks for the project, will the 1st member get punished heavily for reporting his biased value where he assumes to have done almost the entire project by himself ?

	1	2	3	4
μ	0.675	0.15	0.15	0.025
σ	0.0729	0.0395	0.0395	0.1015

The means and deviations corresponding to the project show up a bit that 4th member not only stayed out of the project but also misreported values of the other players and in doing so has struck the highest deviation. The 1st member as the mean suggests indeed did the entire work but has a slight deviation soaring up more than 2nd and 3rd members but let's see how the functions deal with the extraordinarily high share of work done by 1st member .

Shown below are the marks allotment where each member can be at most awarded 100 marks and the team has been awarded M out of 400. For various $M's$ the marks allotment is shown :
function 2 :

	1	2	3	4
200	90.60	44.77	44.77	19.86
246	100	56.27	26.27	31.36
291	100	67.52	67.52	42.61
333	100	78.02	78.02	53.11
372	100	82.77	82.77	62.86

We had set a as 100 and b as 200.

7.3

Consider the following set of truth values reported for a team of size 4.

	1	2	3	4
1	0.4	0.1	0.2	0.3
2	0.2	0.4	0.1	0.3
3	0.2	0.1	0.4	0.3
4	0.2	0.2	0.2	0.4

The general look of the data doesn't seem to hint at what would have been the actual work distribution in this case. It seems multiple members have provided misleading values because of which it gets tough to infer the actual work distribution. Let's see if getting the means and deviations help us getting a view of truth.

	1	2	3	4
μ	0.25	0.2	0.225	0.325
σ	0.0918	0.1212	0.1045	0.0468

As the means and deviations suggest, 2nd and 3rd member seem to have provided less contribution to the project and even have misreported the work distribution sets. While the 1st member seem to also have worked just a bit more than 2nd and 3rd members but seem to also have provided less manipulated values. It seems clear that the 4th player has actually the most contribution and is hence seeming to less defer from the actual values of work done. Let's see how the functions behave to award mark to each individual in this interesting case.

Shown below are the marks allotment where each member can be at most awarded 100 marks and the team has been awarded M out of 400. For various $M's$ the marks allotment is shown :
function 2 :

	1	2	3	4
200	49.85	38.98	44.80	66.37
246	61.35	50.48	56.30	77.87
291	72.60	61.73	67.55	89.12
333	83.10	72.23	78.05	99.62
372	92.85	81.98	87.80	100

We had set a as 100 and b as 200.

8 Additional Remarks

It must be kept in mind that every distribution of marks when everyone sends the same vector will result in a Nash Equilibrium. This is required by design as all possible distributions of work (with the above mentioned restrictions) can possibly occur.

Here comes the role of the mechanism designer to implement the mechanism in such a way that every one reports the true work distribution, not some other value (say 1/n). This has to be taken care of by appropriate *common knowledge*. While implementing the mechanism, every player must be told that filling true work distribution is common knowledge. Note that if while implementing the mechanism, if, for example, the players believe that filling equal values (i.e. 1/n) is common knowledge, then everyone will end up filling that and the whole mechanism will lose its purpose.

9 Conclusion

Here, we have provided a mechanism for the use case of assigning marks to a team project, based on the reported amount of work done by each person. We have designed the mechanism in such a way that saying the truth is always a better strategy in expectation that the others are also speaking the truth - that is, we expect that the mechanism is BNIC, and saying the truth is a Nash equilibrium. We also provided some example test cases, where we show how the mechanism performs. For more such examples, using different functions to assign the marks, we have written a simple program in C++ which can be found here.

Despite being a good mechanism which incentivizes telling the truth, the mechanism does have certain drawbacks. The first of it is that it does not take into account cooperation. For example, consider a 10 person project, where 1 person has done all the work. If the people know beforehand that the marks will be allotted using our scheme, then they may trick the mechanism - all of them may cooperate and report that they all have done equal the work whereas the one who has done all the work has done no work. Thus, even if the first person tells the truth, he would face huge penalty - it may only be beneficial to tell the truth when everyone else is also telling the truth. Similarly, people may co-operate to undermine someone, and "gang up" against them.

Of course, the other only works if the people know about the marking scheme before hand, and that they are allowed time to discuss before reporting their values. Hence, a proper way to implement this would be to not inform the before hand, and ask them to report all the values at once, thus not allowing them time to discuss. In such a scenario, the team members may realize that only speaking the truth is beneficial to them all.

References

[Narahari, 2014] Narahari, Y. (2014). *Game theory and mechanism design.* World Scientific Pub. Co.