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1. Generative Models, Naive Bayes Classifier [20 points]

a) $P(\text{water} = \text{cool} \mid \text{Play} = \text{yes}) = 1/3$
$P(\text{water} = \text{cool} \mid \text{Play} = \text{no}) = 0$

b) $P(\text{Play} = \text{yes} \mid \text{water} = \text{warm}) = 2/3$
$P(\text{Play} = \text{no} \mid \text{water} = \text{warm}) = 1/3$

c) $P(\text{Play} = \text{yes} \mid \text{Humid} = \text{high}) = 2/3$
$P(\text{Play} = \text{yes} \mid \text{Humid} = \text{normal}) = 1$

d) $P(\text{water} = \text{cool} \mid \text{Play} = \text{yes}) = 2/5$
$P(\text{water} = \text{cool} \mid \text{Play} = \text{no}) = 1/3$
[Laplace Smoothing]

2. Kernels [30 Points]

$K(x, z) = \phi(x)^T \phi(z)$ Assume $K_1(x,z)$ and $K_2(x,z)$ are two valid kernels

a) $K(x,z) = K_1(x,z) K_2(x,z)$

Need to show ① Symmetry ② Positive Semi-definite

① Symmetry

$$K(z,x) = \sum_{i=1}^{m} K_{i,1}(z,x) K_{i,2}(z,x)$$
$$= \sum_{i=1}^{m} K_{i,1}(x,z) K_{i,2}(x,z)$$
$$= K(x,z)$$

Thus since $K(z,x) = K(x,z)$, K is a symmetric matrix.

② Positive Semidefinite

Let $u \in \mathbb{R}^n$ be arbitrary. The gram matrix of K, denoted G has the property

$$G_{i,j} = K(x_i, z_j) = \sum_{i=1}^{m} K_{i,1}(z,x) K_{i,2}(z,x) = G_{1,1} G_{2,1} + \cdots + G_{1,m} G_{2,m}$$

$$u^T G u = u^T (G_{1,1} G_{2,1} + \cdots + G_{1,m} G_{2,m}) u = u^T G_{1,1} G_{2,1} u + \cdots + u^T G_{1,m} G_{2,m} u$$
$$= \sum_{i=1}^{m} u^T K_{1,i}(z,x) K_{2,i}(z,x) u$$

Since $u^T G_{1,i} u \geq 0$ and $u^T G_{2,i} u \geq 0$, then $u^T G_{1,m} G_{2,m} u \geq 0$

$\therefore K(x,z) = K_1(x,z) K_2(x,z)$ is a valid kernel function ■

(b) $K(x,z) = f_1(x)f_1(z) + f_2(x)f_2(z)$

Proof:

$K(z,x) = K_1(z,x) + K_2(z,x)$ // Addition of 2 Kernel Functions

$= K_1(x,z) + K_2(x,z)$

$= K(x,z)$ //Showing Symmetry

Thus, $K(z,x) = K(x,z)$.

// Showing Positive Semi-definite

Let $U \in \mathbb{R}^n$ be arbitrary. The Gram matrix of K, denoted by G has the property,

$$G_{i,j} = K(x_i, z_j) = \sum_{i=1}^{m} K_1(x_i, z_i) + K_2(x_j, z_j)$$

$$U^T G U = U^T(G_{1,1} + G_{2,1} + \cdots + G_{1,m} + G_{2,m})U$$
$$= U^T G_{1,1} U + U^T G_{2,1} U + \cdots + U^T G_{1,m} + U^T G_{2,m} U$$
$$= \sum_{i=1}^{m} U^T K_{1,i}(z,x) U + U^T K_{2,i}(z,x) U$$

Since $U^T K_{2,i}(z,x) U >= 0$ and $U^T K_{1,i}(z,x) U >= 0$, then $U^T K_{1,i}(z,x) U + U^T K_{2,i}(z,x) U >= 0$.

Next, Show $K(x,z) = f_1(x) f_1(z)$ is a valid Kernel. Need to find the φ function such that $K(x,z) = \varphi(x)^T \varphi(x)$.

$$K(x,z) = f_1(x) f_1(z) = I(x) I(z) = I(x)^T I(z) = \varphi(x)^T \varphi(x)$$

$\therefore$ $K(x,z) = f_1(x) f_1(z) + f_2(x) f_2(z)$ is a Valid Kernel. // ■

(c) $K(x,z) = \dfrac{K_1(x,z)}{\sqrt{K_1(x,x)\,K_1(z,z)}}$, where $K_1(x,x) > 0$ for any x.

① Denominator //Symmetry

$$K(z,x) = \sum_{i=1}^{m} K_{i,1}(x,x)\, K_{i,1}(z,z)$$

$$= \sum_{i=1}^{m} K_{i,1}(x,x)\, K_{i,1}(z,z) \quad \text{// swapping } x,x \text{ and } z,z$$

$$= K(x,z)$$

$\therefore$ The denominator is symmetric.

Since each polynomial term is made up of a division of a positive square root, making the term positive. It follows that $K(x,z) = f(K_1(x,z))$ where f is a polynomial with positive coefficients. Thus, since the denominator is also symmetric, $K(x,z)$ is a valid kernel function. ∎