LINEAR ALGEBRA ASSIGNMENT

Qns 1. $y = A + Bx + Cx^2$ through $(1,1), (2,-1), (3,1)$

substitute $\Rightarrow$ $A + B + C = 1$, $A + 2B + 4C = -1$, $A + 3B + 9C = 1$

Representation in augmented form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{array}{l} 2C = 4 \Rightarrow C = 2 \\ B + 3C = -2 \\ B = -8 \\ A = 7 \end{array}$$

$R_3 \to R_3 - 2R_1$

$\therefore$ Eq^n of parabola could be $y = 7 - 8x + 2x^2$

Qns 2.

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 28 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 5R_1 \\ R_4 \to R_4 - 5R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 + 2R_2 \end{array}$$

$$\Rightarrow \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad R_4 \to R_4 - 3R_3$$

$A = LU$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ -5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Ans 3. (i) Standard basis of $\mathbb{R}^3$ $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-2\end{pmatrix} = -1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} - 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(ii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2.$$

$\therefore$ Basis for C(A) is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

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Basis for null-space N(A) is

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

PV = u, v

FV = W

$v + w = 0$ $\qquad$ $u + 2v - w = 0$

$v = -w$ $\qquad$ $u - 3w = 0$

$u = 3w$

$$X = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3w \\ -w \\ w \end{bmatrix} = w \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for null-space $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

Basis for LNS : $N(A^T)$

$R_3 = R_3 + R_2 \Rightarrow R_3 - R_1 + R_2 \Rightarrow -R_1 + R_2 + R_3$

$\therefore$ Basis for LNS = $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(iii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

characteristic eq^n of T is $|T - \lambda I| = 0$

$$= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix}$$

$(1-\lambda)\left[(2+\lambda)(1-\lambda) - 1\right] - 2(-1) - 1\left(0 - (1-\lambda)\right) = 0$

$(1-\lambda)(\lambda + \lambda^2 - 3) = \lambda - 3$

$\cancel{\lambda} + \cancel{\lambda^2} - \cancel{3} - \cancel{\lambda^2} - \lambda^3 + 3\lambda = \cancel{\lambda} - \cancel{3}$

$\lambda^3 - 3\lambda = 0$ $\Longrightarrow$ $\underline{\lambda = 0, \sqrt{3}, -\sqrt{3}}$

eigen values

when $\lambda = 0 \Rightarrow$

$$(T - \lambda I)x = 0 \Rightarrow Tx = 0$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{-y}{1} = \frac{z}{1}$$

$\therefore$ eigen vector for $\lambda = 0$ is $k(3, -1, 1)$

when $\lambda = \sqrt{3} \Rightarrow$

$$Tx = \sqrt{3} I x$$

$$(T - \sqrt{3} I)x = 0$$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1-\sqrt{3} & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1-\sqrt{3} & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1-\sqrt{3} & 2 \\ 0 & 1-\sqrt{3} \end{vmatrix}}$$

$$\frac{x}{3-\sqrt{3}} = \frac{-y}{1-\sqrt{3}} = \frac{z}{4-2\sqrt{3}}$$

$\therefore$ eigen vector for $\lambda = \sqrt{3}$ is $k(3-\sqrt{3}, -1+\sqrt{3}, 4-2\sqrt{3})$

when $\lambda = -\sqrt{3} \Rightarrow$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1+\sqrt{3} & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1+\sqrt{3} & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1+\sqrt{3} & 2 \\ 0 & 1+\sqrt{3} \end{vmatrix}}$$

$$\frac{x}{3+\sqrt{3}} = \frac{-y}{1+\sqrt{3}} = \frac{z}{4+2\sqrt{3}}$$

$\therefore$ eigen vector for $\lambda = -\sqrt{3}$ is $k(3+\sqrt{3}, -1-\sqrt{3}, 4+2\sqrt{3})$

(iv) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$q_1 = \frac{a}{\|a\|}$ $\quad \|a\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

$q_1 = \frac{a}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$e = b - (q_1 T b) q_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left(3/\sqrt{2}\right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$

$\|e\| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{3/2}$

$q_2 = \frac{e}{\|e\|} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} & 1 & -1/2 \end{bmatrix}$

$q_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{-1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$

$E = c - (q_1 T c) q_1 - (q_2 T c) q_2$

$(q_1 T_c) q_1 \Rightarrow \left\{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}\right\} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \left(\frac{-3}{\sqrt{2}}\right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \\ -3/2 \end{bmatrix}$

$(q_2 T_c) q_2 \Rightarrow \left\{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}\right\} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = \left(\frac{3}{\sqrt{6}}\right) \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$

$$E = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} -3/2 \\ 0 \\ -3/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix}$$

$$q_1^T a = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2}$$

$$q_1^T b = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 3/\sqrt{2}$$

$$q_1^T c = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = -3/\sqrt{2}$$

$$q_2^T b = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 3/\sqrt{6}$$

$$q_3^T b = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = 3/\sqrt{6}$$

$$q_3^T c = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = 0$$

$$T = QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix}$$

Ans 4. $y = c + dx$

$+c - 4d = 4, \quad c + d = 6, \quad c + 2d = 10, \quad c + 3d = 8$

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\hat{x} = (x^T A)^{-1} . A^T . b$$

$$(A^T A)^{-1} = \frac{\begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}}{116} \Rightarrow (A^T A)^{-1} . A^T =$$

$$\frac{\begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}}{116} . \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 38 & 28 & 26 & 24 \\ -18 & 2 & 6 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} . A^T . b = \frac{\begin{bmatrix} 38 & 28 & 26 & 24 \\ -18 & 2 & 6 & 10 \end{bmatrix}}{116} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix} = \frac{\begin{bmatrix} 772 \\ 80 \end{bmatrix}}{116}$$

$\therefore$ best fit line $\Rightarrow y = \frac{193}{29} + \frac{20x}{29}$

$$\Rightarrow 29y = 193 + 20x$$

Ans 5. $x_1 + x_2 + 3x_3 + 4x_5 = 0$

$$\begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \end{bmatrix} = 0$$

$$x_1 = -x_2 - 3x_3 - 4x_5$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - 3x_3 - 4x_5 \\ x_2 \\ x_3 \\ x_5 \end{bmatrix}$$

FV → x_2, x_3, x_5

PV → x_1

$$= x_2\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = A(A^T A)^{-1} A^T$$

$$(A^T A) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 10 & 12 \\ 4 & 12 & 17 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{27}\begin{bmatrix} 26 & -3 & -4 \\ -3 & 18 & -12 \\ -4 & -12 & 11 \end{bmatrix}$$

$$A(A^T \cdot A)^{-1} = \frac{1}{27}\begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 26 & -3 & -4 \\ -3 & 18 & -12 \\ -4 & -12 & 11 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -1 & -3 & -4 \\ 26 & -3 & -4 \\ -3 & 18 & -12 \\ -4 & -12 & 11 \end{bmatrix}$$

$$A(A^T A)^{-1} A^T = \frac{1}{27}\begin{bmatrix} -1 & -3 & -4 \\ 26 & -3 & -4 \\ -3 & 18 & -12 \\ -4 & -12 & 11 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \frac{1}{27}\begin{bmatrix} 26 & -1 & -3 & -4 \\ -1 & 26 & -3 & -4 \\ -3 & -3 & 18 & -12 \\ -11 & -4 & -12 & 11 \end{bmatrix}$$

$P + Q = I$ $\quad Q = I - P \Rightarrow \begin{bmatrix} 1/27 & 1/27 & 3/27 & 4/27 \\ 1/27 & 1/27 & 3/27 & 4/27 \\ 3/27 & 3/27 & 9/27 & 12/27 \\ 11/27 & 4/27 & 12/27 & 16/27 \end{bmatrix}$

Ans 6. $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

a) $|a| = a > 0$ — (1)

$\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} = a^2 - 4 > 0$

$(a-2)(a+2) > 0 \Rightarrow a \in (-\infty, -2) \cup (2, \infty)$

— (2)

$\begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} = a(a^2-4) - 2(2a-4) - 2(4-2a) > 0$

$a^3 - 4a - 4a + 8 + 8 - 4a > 0$

$\Rightarrow a^3 - 12a + 16 > 0$

$\Rightarrow (a-2)^2 - (a+4) > 0$

$\Rightarrow (a-2)^2 > 0 \quad \therefore (a+4) > 0$ — (3)

from (1) & (2) $a \in (2, \infty)$ — (4)

from (3) & (4) $a \in (2, \infty) \Rightarrow 2 < a < \infty$

b) $f = 2x_1^2 + 2x_2^2 + 2x_3^2 + x_1x_2 - x_2x_3$ — (1)

$f = x^T A x$

$= (x_1 \quad x_2 \quad x_3) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ — (2)

$a_{11} = 2 \quad a_{22} = 2 \quad a_{33} = 2 \quad a_{12} = -1 \quad a_{23} = -1 \quad a_{21} = -1$

$a_{32} = -1$

$B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Ans 7. $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

eigen values, are

$$|A - \lambda I| = 0 \quad \begin{vmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 90\lambda = 0$$

$$\lambda_1 = 90, \quad \lambda_2 = 0$$

eigen vectors for $A \cdot A^T$ are $(A - \lambda I)(x) = 0$

$$\lambda_1 = 90 \Rightarrow \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$81x - 27y = 0$$

$$3x = y$$

$$x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$V_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}, \quad V_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$\sigma_1 = \sqrt{90}, \quad \sigma_2 = 0 \quad \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ eigen values of AA^T are $90, 0, 0$

$$v_1 = \frac{Av_2}{\sqrt{1}} = \frac{\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}{\sqrt{90}} = \frac{1}{\sqrt{90}} \begin{bmatrix} \sqrt{10} \\ -2\sqrt{10} \\ -2\sqrt{10} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$\lambda = 0 \Rightarrow (AA^T - 0I)x = 0$

$$\Rightarrow \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is rank 1 matrix $\Rightarrow N(AA^T - 0I)$ can be formed by $10x - 20y - 20z = 0$ where x is pivot variable & y, z are F.V's.

$$x = 2y + 2z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 2z \\ y \\ z \end{pmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are given vectors of AA^T w.r.t. $\lambda = 0$

$$u_2 = \frac{x_2}{\|x_2\|} \quad \therefore \quad u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

To find u_3, x_3 is $\perp$ to u_1 but not v_2

$$C = x_3 - (v_1^T x_3)u_1 - (u_2^T x_3)u_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left(\left(\frac{1}{3} - \frac{2}{3} - \frac{2}{5}\right)\right)\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -4/ \\ 2/ \end{pmatrix}$$

$$\therefore \Rightarrow A = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ -2/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$