Fuzzy logic, fuzzy sets and fuzzy relations

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Fuzzy logic vs Classical logic

Classical logic

- based on two values - `True` and `False`
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- cannot handle propositions with variable answers

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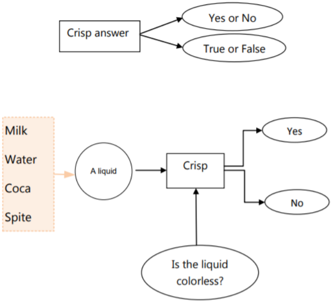
Fuzzy logic

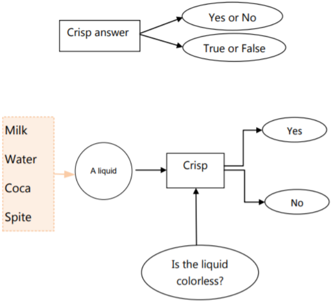
- the truth value of variables may be any real number between 0 and 1, both inclusive
- handles the concept of partial truth
- represents vagueness and imprecise information

Fuzzy logic vs. Crisp logic



(a) Crisp logic

Fuzzy logic vs. Crisp logic



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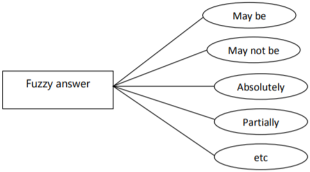
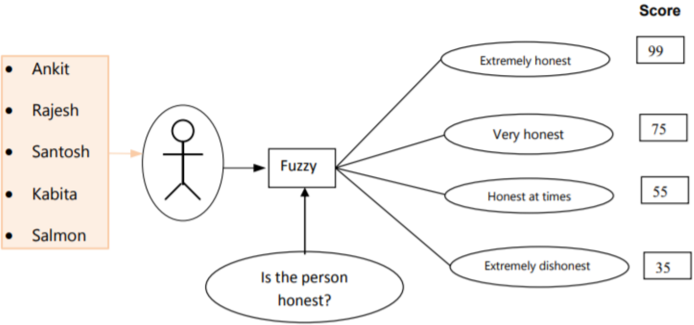


(b) Fuzzy logic

Fuzzy logic vs. Crisp logic



Fuzzy sets vs. Crisp sets

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Fuzzy Sets

- allow elements to be partially in a set
- each element given a degree of membership
- membership value ranges from 0 (not an element) to 1 (a member)
- example - consider the set of `young` people - a 100 year old person will not be a member, but people at the age of 20, 30, or 40 years can have varying degrees of membership

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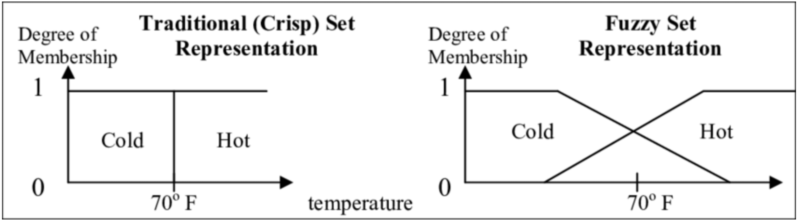

Fuzzy sets

- A fuzzy set A is characterized by a function $\mu_A : \mathcal{X} \to [0, 1]$
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  - defined over the `universe of discourse` $\mathcal{X}$
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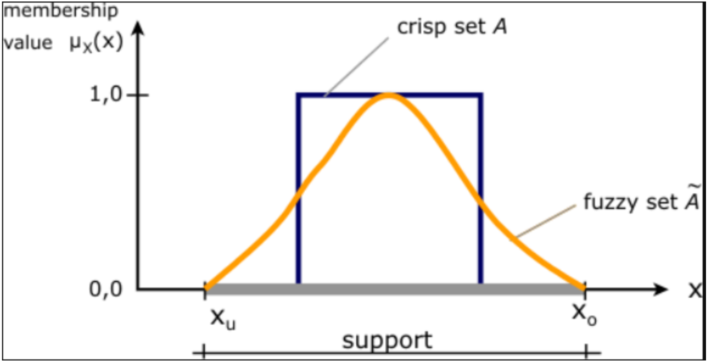
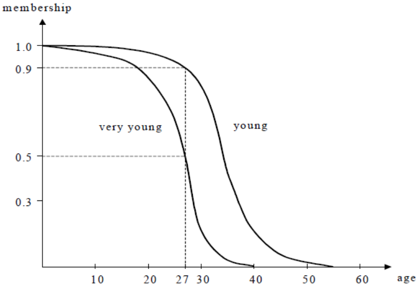
- The set of elements with a non-zero membership is called the `Support` of the fuzzy set

Membership function for Fuzzy sets[1]



[1] Image source: Google

Fuzzy set representation



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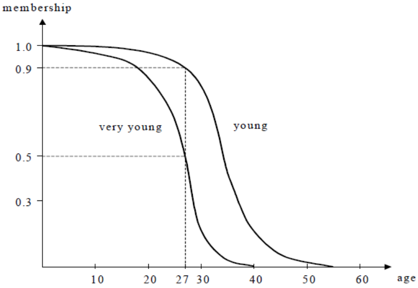

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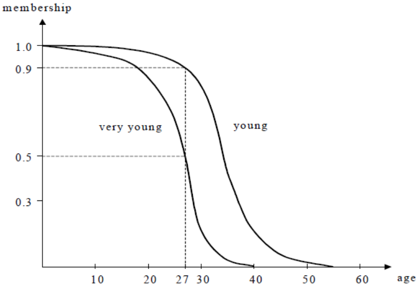

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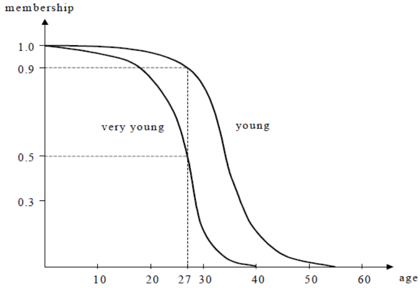

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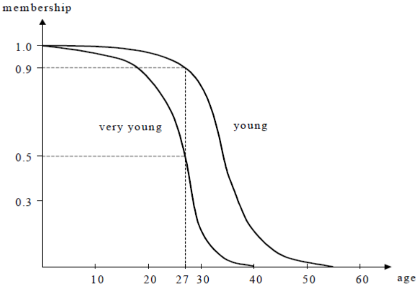

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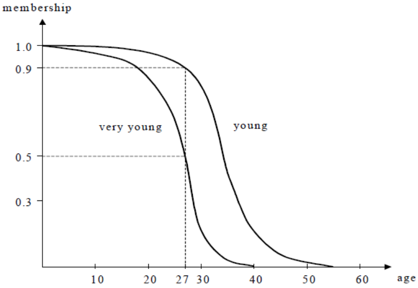

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- In these representations, $\sum, \int$ and $/$ have only symbolic meaning

Fuzzy sets - Examples

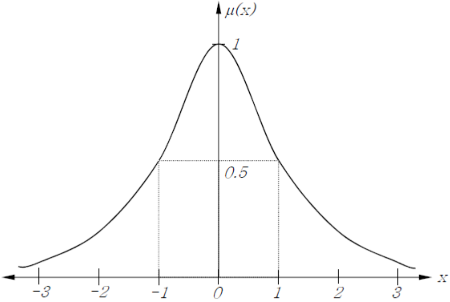

Figure: Consider the fuzzy set $A = \{\texttt{real numbers near 0}\}$ represented as $A = \int_{x \in \mathcal{X}} \mu(x)/x$ where $\mu(x) = \frac{1}{1+x^2}$

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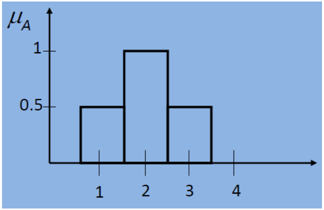

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- **Support:** $supp(A) = \{x \in \mathcal{X} : \mu_A(x) > 0\}$

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  - The fuzzy sets other than $\emptyset$ with height less than 1 are said to be **subnormal**
  - A is subnormal $\Leftrightarrow 0 < hgt(A) < 1$
  - A non-empty subnormal fuzzy set A can be normalized into the set A^* by dividing the membership function of A by $hgt(A)$ $\forall x \in \mathcal{X} : \ \mu_{A^*}(x) = \mu_A(x)/hgt(A)$

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- For a fuzzy set A any element $x \in \mathcal{X}$ that satisfies $\mu_A(x) = 0.5$ is called a **crossover point**

Crisp sets related to fuzzy sets

For a fuzzy set $A = (\mathcal{X}, \mu)$ and $\alpha \in [0, 1]$

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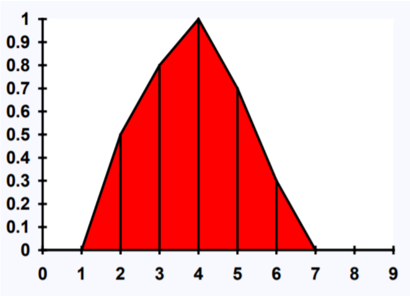
- The **level** of a fuzzy set A is defined as $A^{=\alpha} = \{x \in \mathcal{X} : \ \mu_A(x) = \alpha\}$

Crisp sets related to fuzzy sets - Example



- This set can be represented as $\boldsymbol{A} = 0.0/0 + 0.0/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6 + 0.0/7 + 0.0/8 + 0.0/9$
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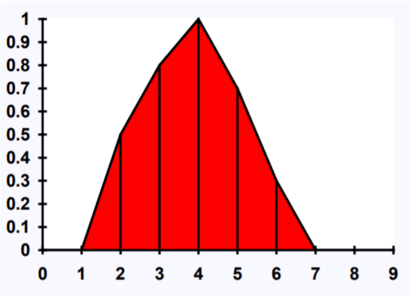

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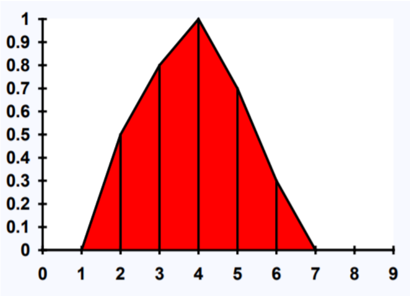

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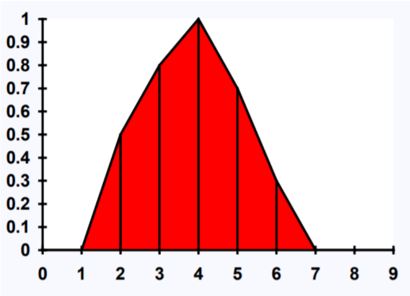

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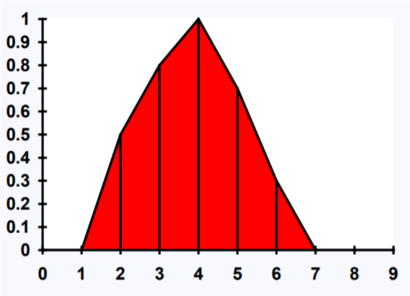

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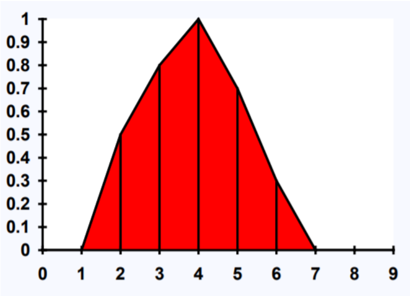

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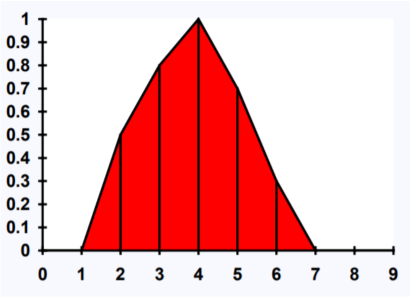

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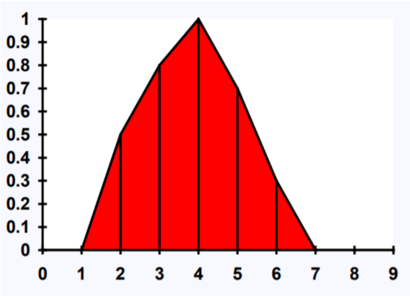

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Fuzzy set operations

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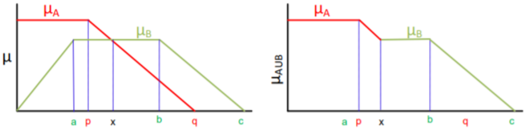

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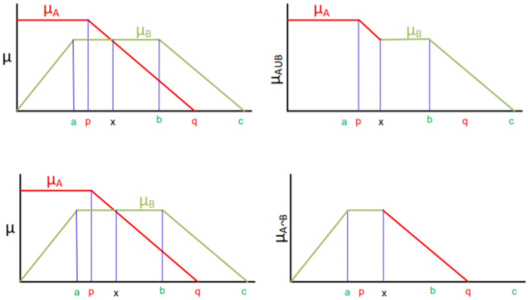

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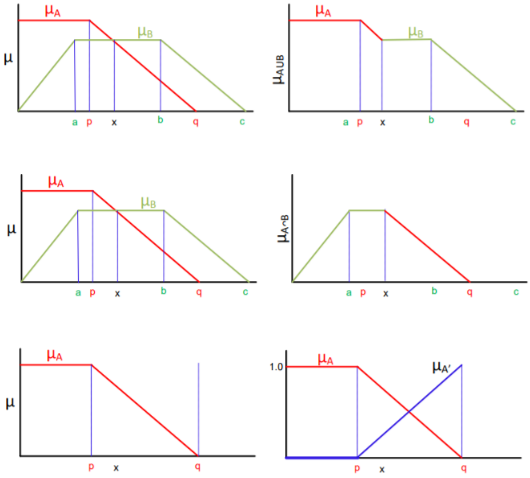

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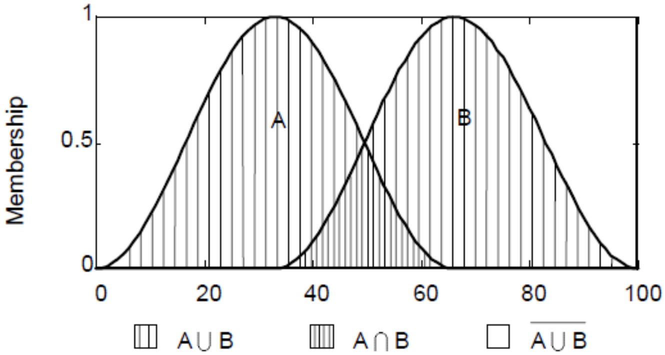

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$$\text{If } A \subseteq B,\ B \subseteq C, \text{ then } A \subseteq C$$

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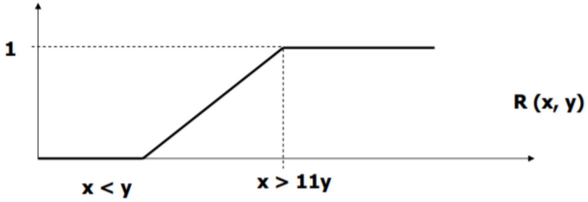
Consider the relation ***R***: "x is considerably larger than y", where $x, y \in \mathbb{R}$

$$R(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ \{x - y)/(10\, y), & \text{for } y < x \leq 11y \\ 1 & \text{for } x > 11y \end{cases}$$



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Composition: For fuzzy relations $R \in \mathbb{F}(\mathcal{X}_1 x \mathcal{X}_2)$ and $S \in \mathbb{F}(\mathcal{X}_2 x \mathcal{X}_3)$, their `composition` or `relational product` $R \circ S$ is defined as:

$$\mu_{R \circ S}(x, y) = \max_{z \in \mathcal{X}_2} \min\{\mu_R(x, z), \mu_S(z, y)\} \text{ for all } (x, y) \in \mathcal{X}_1 \mathrm{x} \mathcal{X}_3$$

Composition of fuzzy relations: Example

$X = (x_1, x_2, x_3),\ Y = (y_1, y_2),\ Z = (z_1, z_2, z_3)$

$$R = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{pmatrix} \end{array}$$

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Image Sources

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