The extension principle, membership functions and fuzzy control systems

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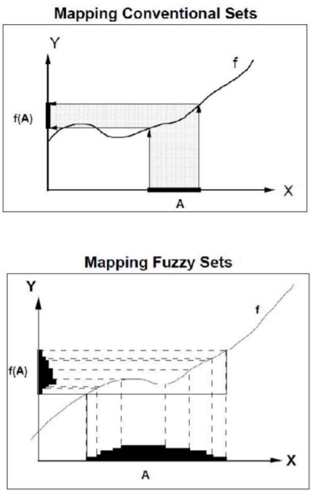
  - what happens when f acts on fuzzy subsets of $\mathcal{X}$?

Extension Principle - One to one function $f : \mathcal{X} \to \mathcal{Y}$



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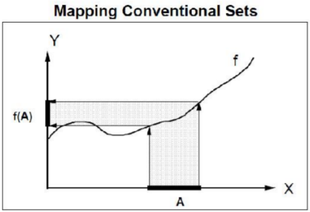

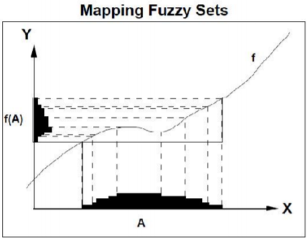

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$\mu_B(y) = \{\mu_A(x) \mid f(x) = y, x \in \mathcal{X}\}$ for all $y \in \mathcal{Y}$

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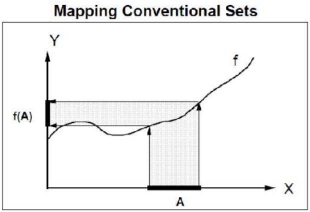

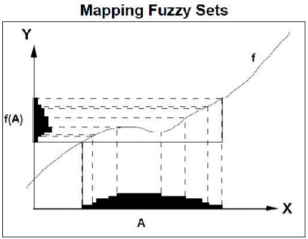

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Example: Let $A = (\mathcal{X}, \mu_A) = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n$

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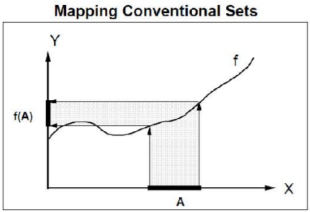

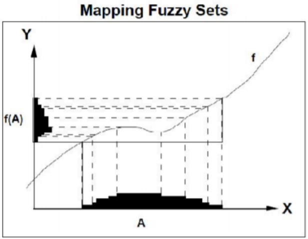

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$$\begin{aligned} B &= \hat{f}(A) \\ &= \mu_A(x_1)/y_1 + \ldots + \mu_A(x_n)/y_n \end{aligned}$$

where $y_i = f(x_i)$

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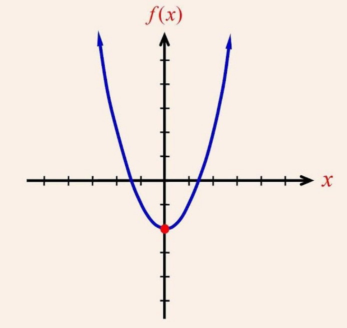

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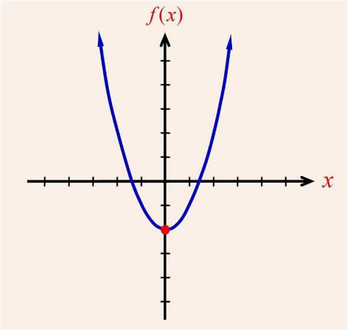

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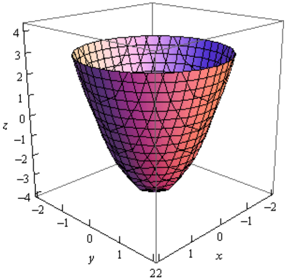

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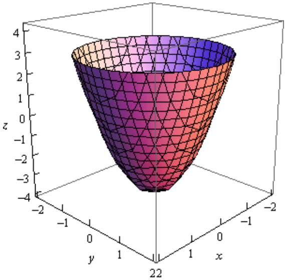

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$$\begin{aligned}\mu_C(13) &= \max\{\min\{0.5, 1\}, \min\{1, 0.6\}, \min\{0.5, 0.3\}\} \\ &= \max\{0.5, 0.6, 0.3\} = 0.6\end{aligned}$$

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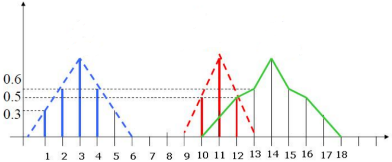

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  - triangular, trapezoidal, piecewise-linear, Gaussian, bell-shaped, etc.

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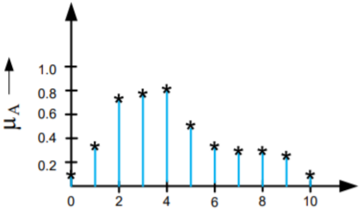
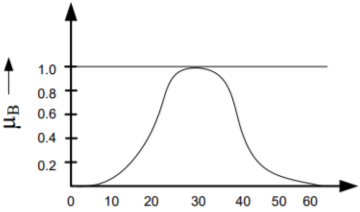
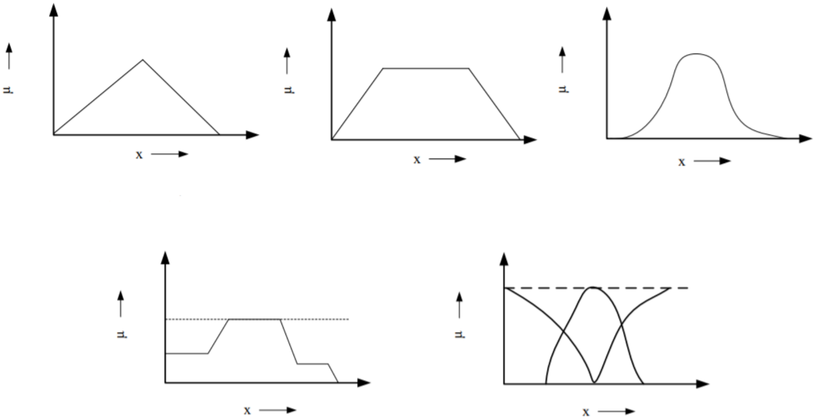




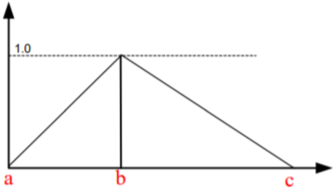
Fuzzy membership functions



Fuzzy membership functions - Triangular MF

A **triangular membership function** is specified by three parameters $\{a, b, c\}$ and can be defined as follows:

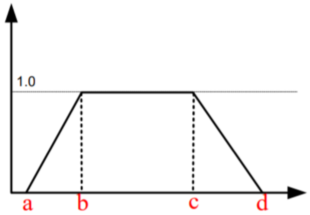
$$triangle(x; a, b, c) = \begin{cases} 0 \text{ if } x \leq a \\ \frac{x-a}{b-a} \text{ if } a \leq x \leq b \\ \frac{c-x}{c-b} \text{ if } b \leq x \leq c \\ 0 \text{ if } x \geq c \end{cases}$$



Fuzzy membership functions - Trapezoidal MF

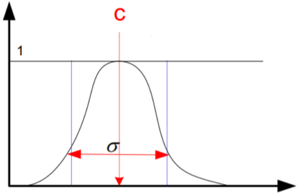
A **trapezoidal membership function** is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

$$trapezoid(x; a, b, c, d) = \begin{cases} 0 \text{ if } x \leq a \\ \frac{x-a}{b-a} \text{ if } a \leq x \leq b \\ 1 \text{ if } b \leq x \leq c \\ \frac{c-x}{c-b} \text{ if } c \leq x \leq d \\ 0 \text{ if } x \geq d \end{cases}$$



Fuzzy membership functions - Gaussian MF

A **Gaussian membership function** is specified by two parameters $\{c, \sigma\}$ and can be defined as follows:



$$gaussian(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

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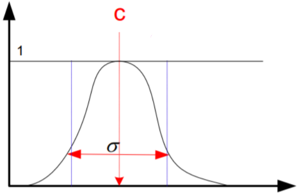

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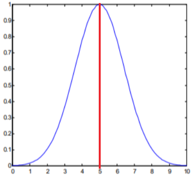
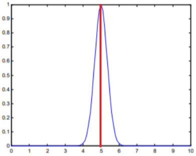


c=5, s=0.5

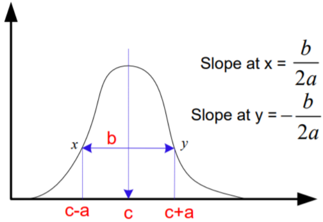


c=5, s=0.2

c=5, s=5

Fuzzy membership functions - Generalized Bell

A **Generalized Bell membership function** is specified by three parameters $\{a, b, c\}$ and is defined as follows:



$$bell(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

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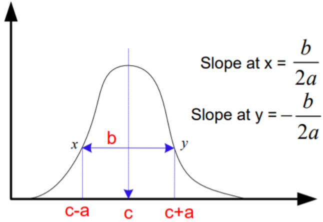

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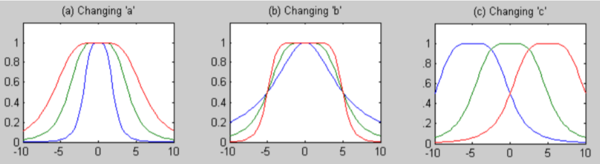

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- find applications in air conditioners, refrigerators, washing machines etc.

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- the processing stage invokes a set of logic rules and combines the results of the rules into a set of fuzzy outputs
- the output stage combines the fuzzy outputs into discrete values needed to drive the control mechanism **(defuzzification)**
- fuzzy systems have the advantage of casting the solution to a problem in terms that humans can understand

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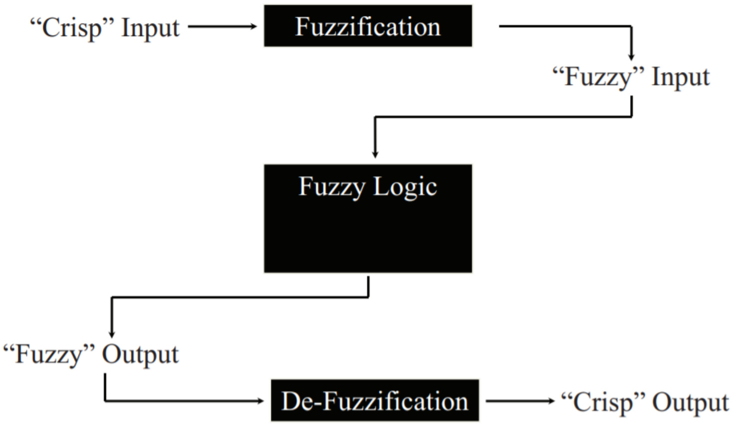
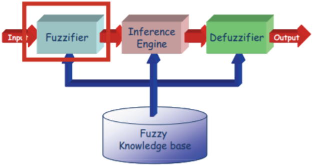
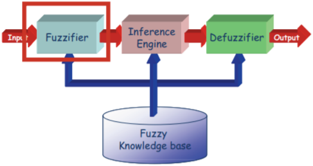


Fuzzification



- process of making a crisp quantity fuzzy using membership functions

Fuzzification



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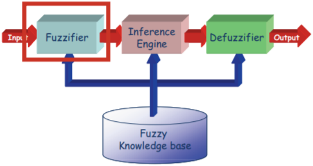

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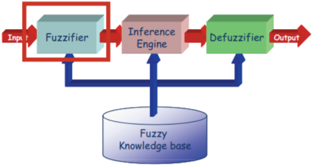
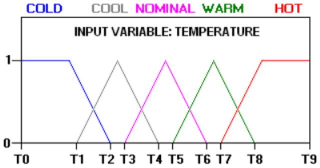
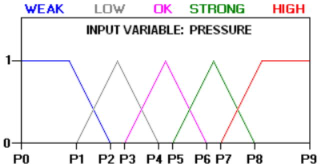


- process of making a crisp quantity fuzzy using membership functions
- hardware such as a digital voltmeter generates crisp data, but are subject to experimental error
- fuzzification helps deal with such ambiguity in data
- input variables are assigned degrees of membership to various classes

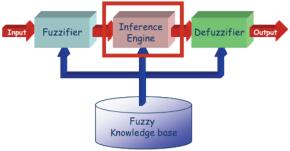
Fuzzification

For instance, the input variable temperature can be assigned membership to a number of fuzzy sets such as `cold`, `cool`, `nominal`, `warm`, `hot`, etc.



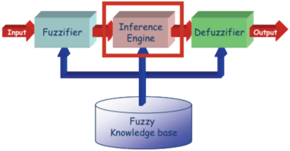


Inference Engine



- converts fuzzy input into fuzzy output based on a collection of `IF-THEN` type logic rules
  - `IF` part is called the **antecedent**
  - `THEN` part is called the **consequent**

Inference Engine



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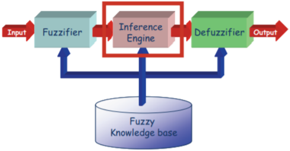

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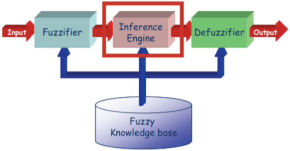

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- example: "`IF (temperature is "cold") THEN turn (heater is "high")`"
- the truth value of the rules inferred using `MAX-MIN` method
- this method sets the fuzzy value of the antecedent to the output membership value

Inference Engine

Example

Rule 1: `IF` x_1 is **Small** `AND` x_2 is **High** `THEN` y is **Medium**

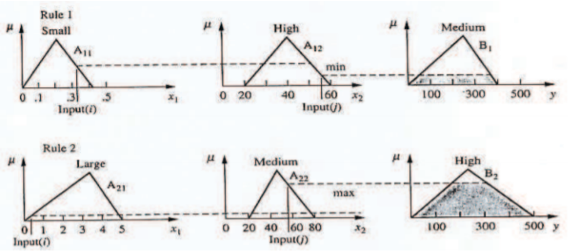
Rule 2: `IF` x_1 is **Large** `OR` x_2 is **Medium** `THEN` y is **High**

Inference Engine

Example

Rule 1: IF x_1 is **Small** AND x_2 is **High** THEN y is **Medium**

Rule 2: IF x_1 is **Large** OR x_2 is **Medium** THEN y is **High**



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