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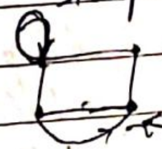
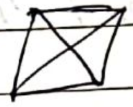
Date __/__/____ **Graph Theory:**

Graph (G) is structure with an order pair (V, E) where V is a set of vertices (nodes, points) and E is set of edges (unordered pair of vertices.)



Simple | Multigraph → undirected graph

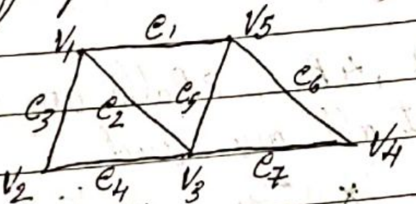


→ Containing loops & multiple edges

no multiple edges on 2 vertices | multiple edges on 2 vertices

 → directed graph (digraph)

Theorem 8 :- State Handshaking Lemma & verify it for foll. graph i.e. $\Sigma_i d(V_i) = 2e$.



$\therefore$ Degree of vertices :- $d(V_1) = 3$, $d(V_2) = 2$, $d(V_3) = 4$, $d(V_4) = 2$, $d(V_5) = 3$

$\therefore$ LHS $\Rightarrow$ $\Sigma d(V_i) = d(V_1) + \ldots + d(V_5)$

$= 3 + 2 + 4 + 2 + 3 = 14$

$= 2(7)$

$\therefore$ No. of edges $= 7$

1. No. of odd degree vertices in a graph will be even in number.
2. Degree of vertices (d) = no. of edges incident on vertices.

If $d(V) = 0$ → isolate vertex
$d(V) = 1$ → Pendant vertex / leaf

Outward degree (d^+), Inward degree (d^-)

eg. 

$\therefore$ $d^+(A) = 2$, $d^-(A) = 1$

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1. A graph has 12 edges then 2 vertices of degree 3, 2 vertices of degree 4 other remaining vertices of degree 5. Find total no. of vertices in a graph.

Sol. By handshake lemma, $\Sigma d(V_i) = 2e$

Let x be no. of vertices of degree 5

$\therefore (2\times3) + (2\times4) + x.5 = 2(12)$

$\therefore x = 2$

$\therefore$ Total vertices $= 2+2+2 = \underline{6}$

2. ~~Q~~ How many vertices does the follo. graph has.

1. 16 edges and all vertices of degree 2, $x = 2$,
2. Graph has 21 edges, 3 vert. of deg. 4 & others of deg. 3

$12 + 3x = 42$

$3x = 30$, $x = 10$,

$\therefore$ Total vertices $= 10 + 3 = \underline{13}$

3. Suppose the graph has vertices of degree 0, 2, 2, 3 and 9 how many edges does the graph have, i.e. $e = ?$

Sol. $0 + 2 + 2 + 3 + 9 = 2e$

$\therefore e = 8$

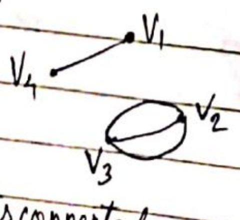
Q. Draw the graph having given properties or explain why ~~no~~ such graph does not exist.

~~Sol.~~ 1. A graph with 4 vertices 1, 1, 2, 3

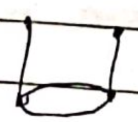
Sol. Such graph will not exist as ~~odd~~ no. of odd degree vertices is odd,

2. A graph with 4 vertices 1, 1, 3, 3

→



Disconnected graph



Connected graph

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3. Simple graph with 1, 1, 3, 3

Sol No simple graph is possible as the graph drawn will contain loops & multiple edges

4. 6 vertices each having degree 3

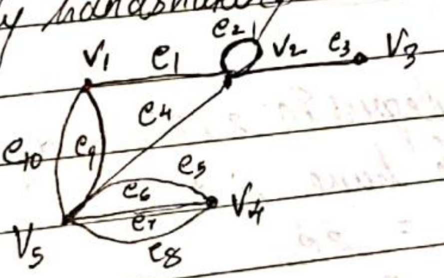
→

5. graph with 6 vertices & 4 edges, $n = 6$, $e = 4$

→

disconnected graph

H.W. Verify handshaking lemma for given graph



☆ Note:- The degree of vertex in a simple graph (G) on n vertices cannot exceed $(n-1)$.

☆ $\delta = \min^m \{ d(v_i) \mid v_i \in V(G) \}$

$\Delta = \max^m \{ d(v_i) \mid v_i \in V(G) \}$

6. δ

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4. Show that the max^m no. of edges in a simple graph with n vertices is $n(n-1)/2$

Sol. By handshake lemma, $\sum d(V_i) = 2e$

$$d(V_1) + d(V_2) + \dots d(V_n) = 2e$$

As in simple graph max^m degree of any vertex is $(n-1)$

$$\therefore (n-1) + (n-1) + \dots (n-1) = 2e$$

$$\therefore n(n-1) = 2e$$

$$\therefore \boxed{e = \frac{n(n-1)}{2}}$$ → Max^m no. of edges in any graph.

★ Types of Graph :-

1. Null graph - A finite graph with no edges is called null graph denoted by N_n → Null graph on n vertices

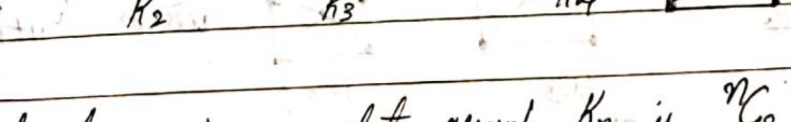
eg. N_4 N_1 → Null graph with 1 vertex

⇒ Every vertex of null graph is isolate vertex. (degree 0)

2. A graph is said to be complete graph if there is an edge between every pair of distinct vertices

Denoted by K_n → Complete graph on n vertices

eg.



K_1 K_2 K_3 K_4

⇒ No. of edges in complete graph K_n is $^nC_2 = \frac{n(n-1)}{2}$

In complete graph K_n, degree of each vertex $= (n-1)$

3. A graph (G) (V, E) is said to be regular graph if degree of each vertex is said to be same

→ If degree of each vertex is r then that graph is called as r-regular graph edge → { }, Gaurini's puzzle

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eg. $d(v_i) = 2$, 2-regular graph → 
$d(V_i) = 3$, 3 " " or [cubic graph] → 

1. What is the size of r regular graph in (p,q) graph?

Sol. $\sum d(V_i) = 2e$ $|V| = p$ = order
$|E| = q$ → size

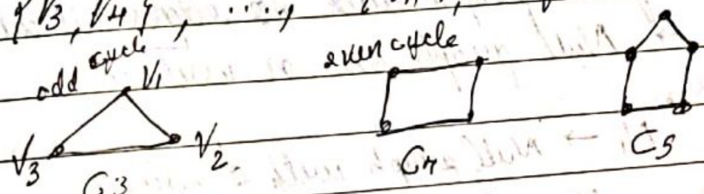
$r + r + \dots + r = 2q$

$pr = 2q$

$q = \frac{pr}{2}$

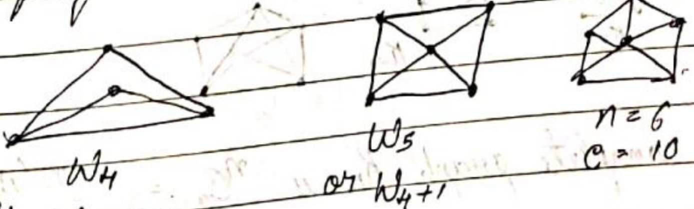
4. The graph $G(V,E)$ is called cycle if set of vertices is a set of n vertices and has $\{V_1 V_2\}$ edge $\{V_2, V_3\}$, $\{V_3, V_4\}, \dots, \{V_{n-1}, V_n\}$

C_3 → odd cycle
C_4 → even cycle

eg. 

C_n → vertices n, edges n

5. Wheel graph can be obtained by cycle C_n and adding one vertex to it and connecting that every vertex to every vertex of cycle. → Wheel graph



~~verte~~ W_{n+1} = vertices $= n+1$
$e = 2n$

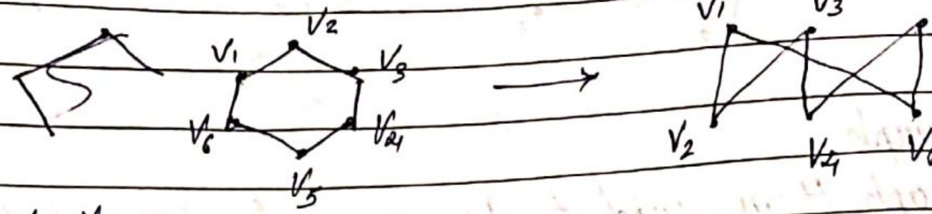
6. Bipartite graph: A graph $G(V,E)$ is said to be bipartite graph if vertex set of G can be partitioned in to two disjoint set V_1 & V_2 if $|V_1| = m$, $|V_2| = n$
Where each edge in E in G has one end point in V_1 & another end point is $|n| =$

e.g. 1. $|V| = m + n$

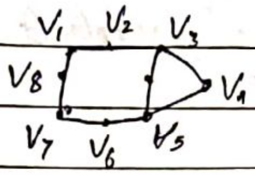
$m = 4$
$n = 3$

The vertices in m, n should be independent

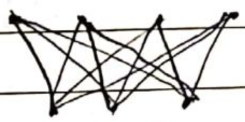
2.



✓ It's a bipartite



7. Complete bipartite graph - $(K_{m,n})$:- A graph $G(V,E)$ is said to be complete bipartite graph if vertex set of G can be partitioned in to two disjoint sets V_1 & V_2



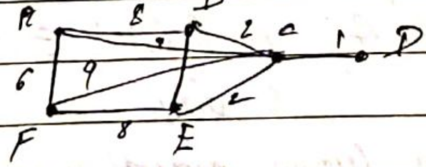
$|V| = m + n$
$|E| = mn$

Each vertex of m should be connected with n.

8. Weighted graph :- The graph $G(V,E)$ is said to be weighted graph if each edge have been assigned a positive real no. (weight)

weight → cost, dist,

eg.



9. graph $G(V,E)$ is called a cube if $|V| = 2^n$ of bit length n.

10.

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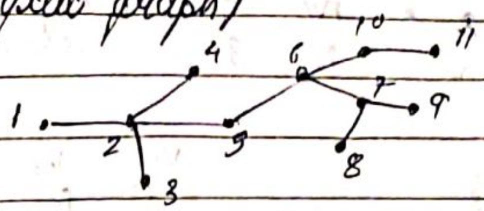
eg. Tree (acyclic graph)



$|V| = n$

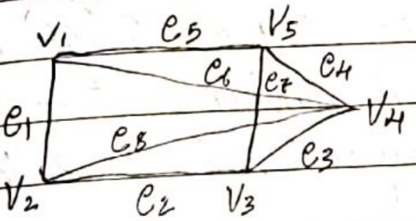
$|E| = (n-1)$

* Subgraph

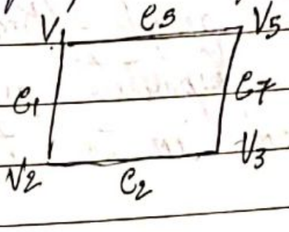
1. A graph H is said to be subgraph of graph G if $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$

2. & if $V(H) \subset V(G)$ & $E(H) \subset E(G)$ then H is called as proper subgraph

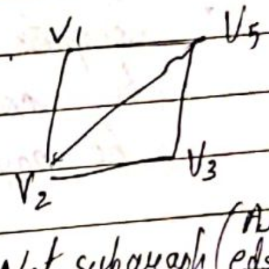
eg.



graph (G)



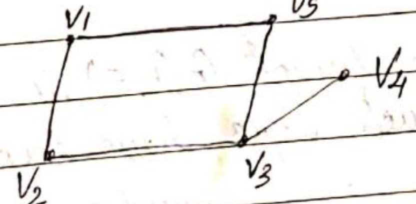
Subgraph



Not subgraph (As edge (V_2-V_5) is not in main graph)

3. If $V(H) = V(G)$ then it is called as spanning subgraph

eg

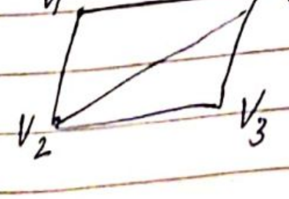


→ spanning subgraph of main graph.

4. Induced subgraph :- A graph H is called a induced subgrap of G if $(u,v) \in E(H)$, $u,v \in H$ (edge)

All edges amongst the vertices from main graph should be present

eg.



→ Induced

If Graph is self complementary no. of vertices = $2K$ or $(2K+1)$

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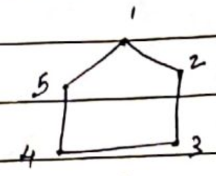
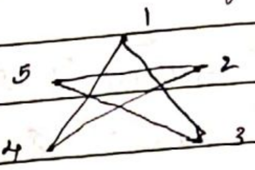
* Complement of graph (G)
A Graph $\bar{G}$ is said to be complement of G if vertex set of $\bar{G}$ is equal to vertex set of G
i.e. $V(\bar{G}) = V(G)$ and edges which are not in G i.e. $(u, v) \in E(\bar{G})$ if $(u, v) \notin E(G)$

eg. 1.

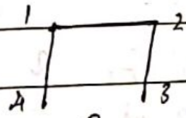


C_5

$\xrightarrow{\text{Complement}}$

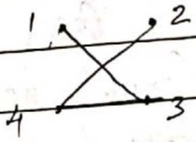


2.



G

$\bar{G} \rightarrow$



* Isomorphism graph :- graphs G_1 and G_2 are said to be isomorphic if there exist a one one correspondence if both $V_1 = V(G_1)$ & $V_2 = V(G_2)$
then f is called as isomorphic & $G_1 \cong G_2$

* Note
1. Isomorphic graphs has $|V_1| = |V_2|$, $|E_1| = |E_2|$ and same degree sequence
2. Isomorphic graphs has same cycles of same length
3. If edge (v_1, v_2) is a loop in G then $(f(v_1), f(v_2))$ is a loop
4. If $(v_1, v_2), (v_3, v_4)$ are multiple edges in G then $(f(v_1), f(v_2))$ & $(f(v_3), f(v_4))$ — " " —

* By seeing how to identify isomorphism → (Pull or push the points and if we are getting another ~~or~~ the other graph then both isomorphic.)

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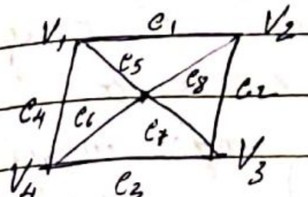
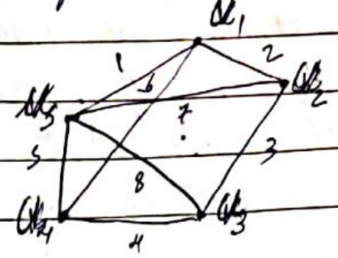
Check whether foll. graph are isomorphic ~~are~~ or not. Justify

1.





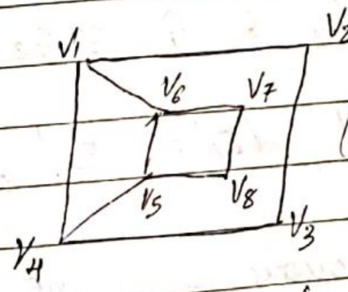
G1 (V, E)

G2 (V_2, E_2)

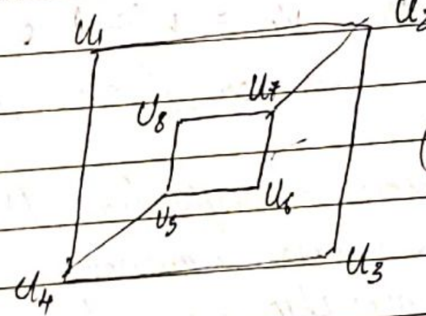
1. G1 has 5 vertices & 8 edges
G2 has 5 ver 8 edges
2. Degree seq. in G1 = 3, 3, 3, 3, 4
G2 = 3 3 3 3 4
3. Total cycles (count no. of cycles in G_1 & G_2)
4. Check adjacency such as $V_1 \rightarrow U_1$
$V_2 \rightarrow U_2$
⋮

∴ Adjacency ~~proved~~ preserved

2.



(G1)



(G2)

Sol. 1. $|V_1| = 8$, $|V_2| = 8$. ∴ $|V_1| = |V_2|$
$|E_1| = 10$ $|E_2| = 10$ $|E_1| = |E_2|$

2. Degree seq. 3, 2, 2, 3, 3, 3, 2, 2 → G1
G2 :

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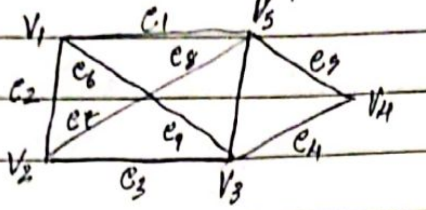
1. Walk :- It is an alternating sequence of vertices and edges starting and ending in vertices such that each edge btw vertices V_i & V_j has end pts. at V_i and V_j

eg.



∴ Walk: $V_1\ e_1\ V_5\ e_7\ V_6\ e_6\ V_1\ e_1\ V_5\ e_5\ V_4\ e_6\ V_3$ [edges & vertices

2. Trail :- It is an open walk with no repitition of edge [vertices may repeat]

Trail - $V_1\ e_1\ V_5\ e_7\ V_6\ e_6\ V_1\ e_2\ V_2\ e_3\ V_3$

3. ~~Open~~ Closed walk - A walk in which start and end vertex are same

4. Open walk - " start & end are not same.

5. Path - It is an open trail in which no vertex is repeated

Path - $V_1\ e_1\ V_5\ V_4\ e_4\ V_3$ [No edge & vertex is repeated

6. Cycle :- Closed path

Cycle - $V_1\ e_1\ V_5\ e_{10}\ V_3\ e_3\ V_2\ e_2\ V_1$

Eulerian graph

1. Eulerian path (trail)

A trail in a connected graph G is called Euler trail if it includes every edge exactly once then it is called as Eulerian path / trail [vertex may repeat] [It should not be closed walk]

2. Eulerian path circuit :-

A circuit in a connect graph G is called Eulerian circuit if it is closed Euler trail. [All edges should be covered]

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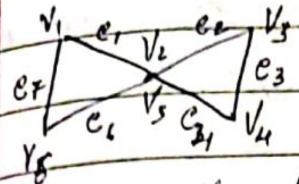
3. Eulerian graph :- A graph which having Eulerian circuit



Euler path not exists
Euler circuit exists
∴ It is an euler graph

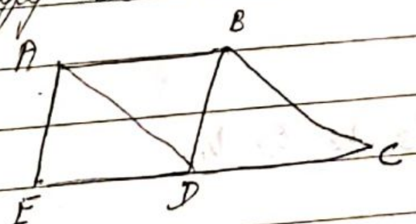
*1. Theorem :- A connected graph is Eulerian if and only if all vertices are of even degree.

2. If G is a connected graph having exactly two vertices u and v of odd degree then there is Eulerian path.

Trick:- (To find Euler path start with odd vertex and end with odd vertex)

Find an Euler path and an Euler circuit if it exist, Justify the ans

→ 1.

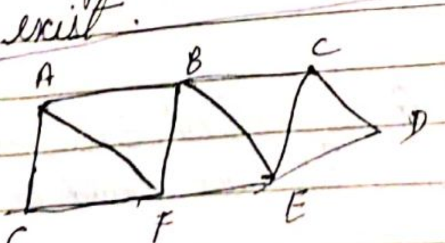


∴ degree seq :- 3, 3, 2, 4, 2

∴ This graph G has exactly two odd vertices.
∴ Euler path exist.
∴ Path :- A → B → C → D → E → A → D → B

As degree of each vertex is not even, Euler circuit does not exist.

2.



degree sequence :- 3, 4, 3, 2, 4, 4, 2
Euler path exists
Euler circuit not exists
Not an Eulerian graph

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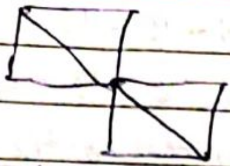
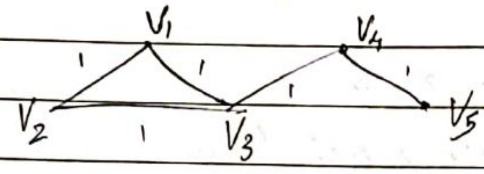
3.  Euler path exists
No euler circuit & graph

Extra (Out of Syllabus)

* Distance btw two (pts) vertices = shortest path btw the two vertices

Eccentricity :- $E(v) = \text{Max}\{d(v,u),\ u \in V\}$
i.e. take farthest distance shortest path from every vertex and considers the max. no.

eg. 

$E(V_1) = 3$
$\therefore d(V_1, V_5) = 3$

$E(\cancel{V_1}) = \{1, 1, 2, 3\}$

$V_3, V_4 \rightarrow$ Centre
Radius = 2 (minimum eccentricity)

$E(V_2) = 3$ $E(V_4) = 2$
$E(V_3) = 2$ $E(V_5) = 3$

Diameter → (maximum eccentricity) → 3

Hamiltonian :-

1. Hamiltonian Trail (Path) :- A trail which contains every vertex exactly once is called as Hamiltonian trail [open]
2. Hamiltonian circuit (Hami.. cycle) :- A circuit which contains every vertex exactly once is called Hamiltonian circuit [closed]

3. Hamiltonian Graph :- A graph which contains Hamiltonian circuit is called Hamiltonian graph

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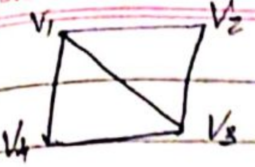
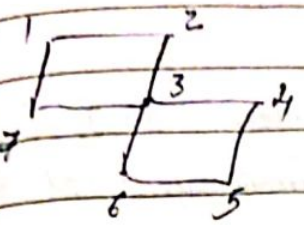
eg. 1. 

Hamiltonian Path $V_1 - V_2 - V_3 - V_4$

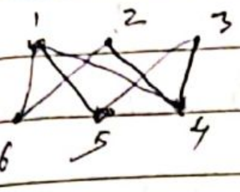
Circuit $V_1 - V_2 - V_3 - V_4 - V_1$

$\therefore$ It is a Hamiltonian graph

eg. 2 

$\therefore$ Path :- 2 - 1 - 7 - 3 - 4 - 5 - 6

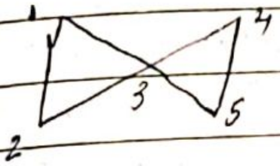
$\therefore$ Ha Cycle - no, $\because$ 3 will repeat

3. 

path :- 1 - 6 - 2 - 5 - 3 - 4

Cycle :- 1 - 6 - 2 - 5 - 3 - 4 - 1

$\therefore$ It's ham- path, circuit

4. 

1 - 2 - 5 - 3 - 4

No cycle, not hami

* **Dirac's Theorem**

A simple connected graph G with $n \geq 3$ vertices is Hamiltonian if $d(V) \geq n/2$ for every vertex V in G

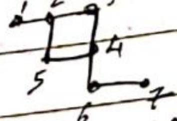
i.e. if $d(V) \geq \frac{n}{2} \Rightarrow$ Hamiltonian

* if $d(V) \leq \frac{n}{2} \Rightarrow$ may or may not be

eg. 1. 

here $d(V) \leq \frac{n}{2}$

But it is hamiltonian

2. 

here $d(V)$ =

here $d(V) = 2, 3 \leq \frac{7}{2}$

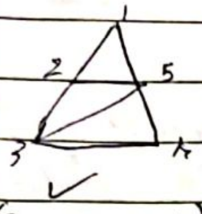
It is not hamiltonian

Date ___/___/___

Ores theorem

Let G be a simple connected graph with $n \geq 3$ &
If $d(v) + d(w) >= n$, for each v & w not connected by an edge
Then G is an Hamiltonian.

eg.

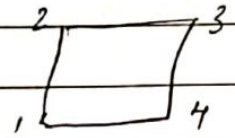


Hamilt(Graph, Path)

qs.1. Given an example of graph which contains Eulerian

1. Eulerian circuit which is same as Hamil. circuit

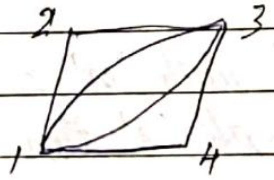
Sol.



Eulerian ckt :- 1-2-3-4-1
Hamil. 1-2-3-4-1

2. Eulerian and Hamiltonian ckt are different

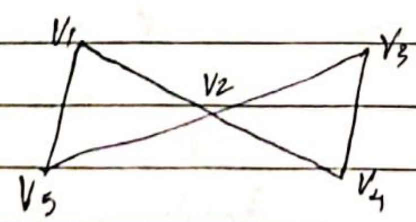
Sol.



Eulerian ckt :- 1-2-3-4-1-3-1
+c

Hamil - 1-2-3-4-1

3. Eulerian but not Hamiltonian



Saathi

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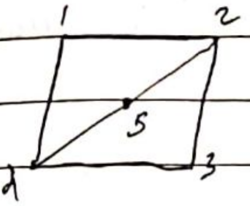
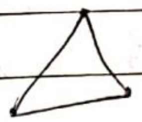
13 Graph which does not contains Eulerian ckt but not contains Hamiltonian



Planar Graph :- A graph is said to be planar if there exist some geometric representation such that no
17 two edges intersect each other, otherwise called non planar.



Planar

If G is a simple connected planar graph with n vertices, where and E edges with $n \geq 3$ and 'e' edges then $e \leq 3n - 6$.

If G is a simple connected planar graph with 'n' ≥ 3 vertices and 'e' edges and no circuit of length '3' then ~~e~~ e is $e \leq 2n - 2$

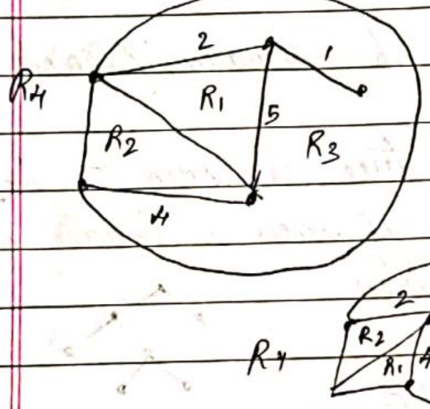
Date ___ /___ /______

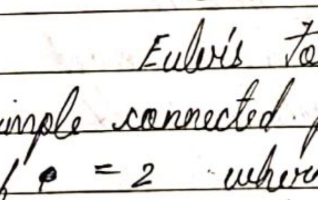
1. A connected plane graph has 10 vertices each of degree $d(V) = 3$ in how many regions does a representation of this planar graph split the plane.

Region (face) and its degree - Planar representation of graph divides the plane into several regions (faces, windows, meshes). A region is characterized by set of edges forming its boundary. And region lying outside the graph is called infinite or unbounded region. The degree of region R, the length of closed walk which that bounds the region.

eg

$d(R_1) = 3 \quad d(R_2) = 3 \quad d(R_3) =$





Euler's formula

If G is simple connected planar graph then $V - e + r = 2$ or $V - e + f = 2$ where

where V are number of vertices e = no. edges

f or r = no. of regions

Sol. $n = 10 \quad d(V) = 3 \quad , f = ?$

Euler formula

$$V - e + r = 2$$

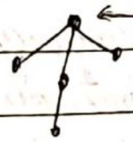
∴ By handshake lemma.

* **Tree** → A connected simple graph without cycle (circuit) is called tree. Its another name is acyclic graph.

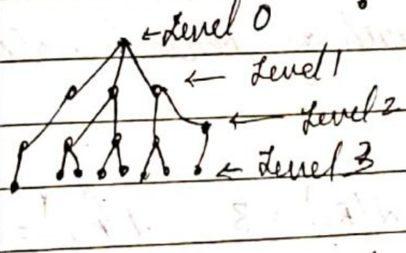
Note → A tree with n vertice has (n-1) edges

Types of Tree

1. Rooted tree → A tree is called as rooted tree if one vertex is designated as the root of the tree and it is distinguishable from other vertices.



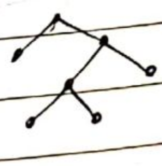
Family tree



2. Binary tree - A tree is said to be binary, if each vertex has atmost two children

eg.

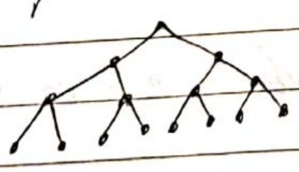
- A binary tree is said to be full binary tree if each internal vertex has 2 children



eg.

- A binary tree is called complete binary tree if every level except possibly the last is completely filled.

eg.



3. Spanning tree :- A subgraph T of graph G is said to be a spanning tree if

1. T is a tree 2. T has every vertex of graph G.

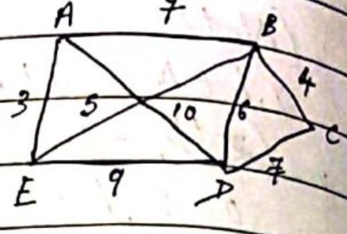
30/3/22

e.g. Give spanning tree for graph(G) → isomorphic tree

(G)

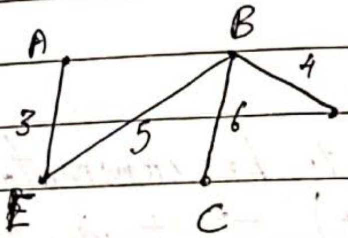
Kruskal's Algorithm.

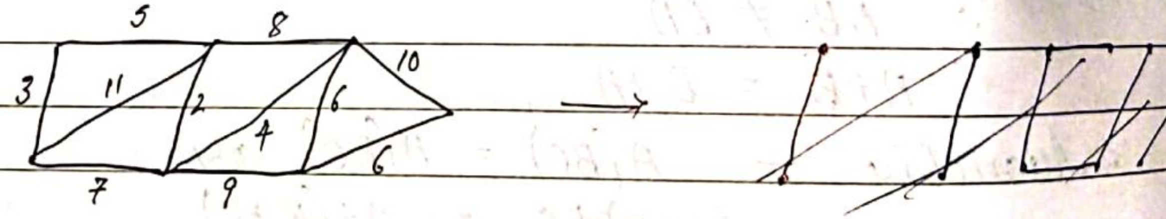
To find minimal spanning tree :-

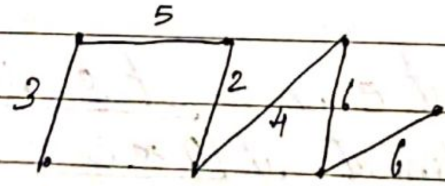


Sol. Edges in ascending order of their weight of edges.

Edges	Weight
(A, E)	3
(B, C)	4
(B, E)	5
(B, D)	6
(A, B)	7
(E, D)	7
(D, C)	9
(A, D)	10







$\boxed{MST = 26}$