CFD Solver for Two-Dimensional Cavity Flow Problem

This work aims to solve the Navier-Stokes (N-S) equations using Finite Difference Method (FDM) for a Two-Dimensional Cavity Fluid Problem.

The CFD Solver is written in C while the obtained velocity profiles are visualized with Python.

Mathematics

A. The Governing Equation

The equation of motion can be written as:

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p - \nabla \cdot \vec{\tau} + \rho \vec{g}$$

For an incompressible fluid with constant ρ and μ, the equation becomes:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla p - \mu \nabla^2 \vec{v} + \rho \vec{g}$$

The *Navier-Stokes equations* in two-dimensions are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

B. Pressure Poisson Equation

The pressure Poisson equation can be written as:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right) + \rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

C. Discretization

The 2 momentum equations and a pressure equation are discretized using central difference method.

For velocity component *u*:

$$u_{i,j}^{n+1} = u_{i,j}^n - u_{i,j}^n \Delta t \frac{(u_{i+1,j}^n - u_{i-1,j}^n)}{2\Delta x} - v_{i,j}^n \Delta t \frac{(u_{i,j+1}^n - u_{i,j-1}^n)}{2\Delta y} - \frac{\Delta t}{\rho}\frac{(p_{i+1,j}^n - p_{i-1,j}^n)}{2\Delta x} + \nu \Delta t\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right)$$

For velocity component *v*:

$$v_{i,j}^{n+1} = v_{i,j}^n - u_{i,j}^n \Delta t \frac{(v_{i+1,j}^n - v_{i-1,j}^n)}{2\Delta x} - v_{i,j}^n \Delta t \frac{(v_{i,j+1}^n - v_{i,j-1}^n)}{2\Delta y} - \frac{\Delta t}{\rho}\frac{(p_{i,j+1}^n - p_{i,j-1}^n)}{2\Delta y} + \nu \Delta t\left(\frac{v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n}{\Delta x^2} + \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{\Delta y^2}\right)$$

For pressure component *p*:

$\text{Let } b = [\text{R.H.S.}]$

$$\frac{p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n}{\Delta x^2} + \frac{p_{i,j+1}^n - 2p_{i,j}^n + p_{i,j-1}^n}{\Delta y^2} = b$$

$$p_{i,j}^n = \frac{\Delta y^2(p_{i+1,j}^n + p_{i-1,j}^n) + \Delta x^2(p_{i,j+1}^n + p_{i,j-1}^n) - b\Delta y^2 \Delta x^2}{2(\Delta x^2 + \Delta y^2)}$$

D. Initial Conditions (I.C.)

At the initial, $\text{let } u, v, p = 0$

E. Boundary Conditions (B.C.)

Within the domain $x_i \in [0,2] \text{ and } y_i \in [0,2]$

$$u = \begin{cases} 1 \text{ for} & y_i = 2 \\ 0 \text{ for} & x_i = 0, 2,\ y_i = 0 \end{cases}$$

$v = 0 \text{ for } x_i = 0, 2,\ y_i = 0, 2$

$p = 0\ at\ y = 2$

$\frac{\partial p}{\partial x} = 0\ at\ x = 0, 2$

Codes

A. Process

The CFD solver (C) does the following steps:

1. Set up parameters (mesh, time, and fluid properties)
2. Initialize velocity components and pressure arrays
3. For each timestep: 3.1 Solve N-S and pressure equations via central difference method 3.2 Apply boundary conditions
4. Write output arrays of velocity components and pressure to output files

The CFD visualizer (Python) does the following steps:

1. Set up mesh
2. Initialize velocity components and pressure arrays
3. Fill the arrays with values from input files
4. Plot cavity flow

B. Directory Structure

```
.
├── vectorutils.c
├── vectorutils.h
├── finitedifference.c
├── finitedifference.h
├── fluiddynamics.c
├── fluiddynamics.c
├── main.c
├── Makefile
├── results
│   ├── u.txt
│   ├── v.txt
│   ├── p.txt
│   └── cavityFlow.png
└── plotProfiles.py
```

C. Programs

The C program consists of the following sections:

- **finitedifference** - compute FDM approximations
- **fluiddynamics** - solve N-S and Poisson equations
- **vectorutils** - utilities
- **main** - setup, solve, and write outputs

C Program, finitedifference.c

```c
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "vectorutils.h"

float** central_ddx(float** f, float dx, int m, int n){
  float** diff = init_zero_vector(m,n);

  for (int i=1; i<m-1; ++i){
    for (int j=1; j<n-1; ++j){
      diff[i][j] = (f[i][j+1] - f[i][j-1])/(2.0*dx);
    }
```

```
  }
  return diff;
}


float** central_ddy(float** f, float dy, int m, int n){
  float** diff = init_zero_vector(m,n);

  for (int i=1; i<m-1; ++i){
    for (int j=1; j<n-1; ++j){
      diff[i][j] = (f[i+1][j] - f[i-1][j])/(2.0*dy);
    }
  }
  return diff;
}


float** laplace(float** f, float dx, float dy, int m, int n){
  float** diff = init_zero_vector(m,n);

  for (int i=1; i<m-1; ++i){
    for (int j=1; j<n-1; ++j){
      diff[i][j] = (f[i][j+1] - 2*f[i][j] + f[i][j-1])/pow(dx, 2) + (f[i+1][j] - 2*f[i][j] + f[i-1]
[j])/pow(dy, 2);
    }
  }
  return diff;
}
```

C Program, fluiddynamics.c

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "vectorutils.h"
#include "finitedifference.h"

float** compute_b(float dx,float dy,float rho,float dt, float** u, float** v, int m, int n){
  float** b = init_zero_vector(m,n);
  float** dudx = central_ddx(u, dx, m, n);
  float** dvdy = central_ddy(v, dy, m, n);

  float** dudy = central_ddy(u, dy, m, n);
  float** dvdx = central_ddx(v, dx, m, n);

  for (int i=1; i<m-1; ++i){
      for (int j=1; j<n-1; ++j){
        b[i][j] = -rho*(pow(dudx[i][j],2) + 2*dudy[i][j]*dvdx[i][j] + pow(dvdy[i][j],2)) + rho/dt*
(dudx[i][j] + dvdy[i][j]);
      }
    }

  free(dudx);
  free(dvdy);
  free(dudy);
  free(dvdx);

  return b;
}


float** pressure_poisson(float** p, float dx, float dy, float rho, float dt, float** u, float** v, int
m, int n){
  float** pn = init_zero_vector(m,n);
  int maxIteration = 50;

  float** b = compute_b(dx,dy,rho,dt,u,v,m,n);

  for (int it=0; it<maxIteration; ++it){
```

```
    copy_vector(pn, p, m, n);

    for (int i=1; i<m-1; ++i){
      for (int j=1; j<n-1; ++j){
        p[i][j] = (pow(dy,2)*(pn[i][j+1] + pn[i][j-1]) + pow(dx,2)*(pn[i+1][j] + pn[i-1][j]) - b[i]
[j]*pow(dy,2)*pow(dx,2))/(2*pow(dx,2) + 2*pow(dy,2));

      }
    }
    // B.C.
    for (int i=0; i<m; ++i){
      p[i][n-1] = p[i][n-2]; // dp/dx = 0 at x = 2 (right)
      p[i][0] = p[i][1]; // dp/dx = 0 at x = 0
    }

    for (int j=0; j<n; ++j){
      p[0][j] = p[1][j]; // dp/dy = 0 at y = 0
      p[m-1][j] = 0.0; // p = 0   at y = 2 (top)
    }
  }
  free(pn);

  return p;
}


int cavity_flow(int nt, float** u, float** v, float dt, float dx, float dy, float** p, float rho,
float nu, float ut, int m, int n){
  float** un = init_zero_vector(m,n);
  float** vn = init_zero_vector(m,n);

  for (int it=0; it<nt; ++it){
    copy_vector(un, u, m, n);
    copy_vector(vn, v, m, n);

    p = pressure_poisson(p, dx, dy, rho, dt, un, vn, m, n);

    float** dudx = central_ddx(un, dx, m, n);
    float** dudy = central_ddy(un, dy, m, n);
    float** dpdx = central_ddx(p, dx, m, n);
    float** laplaceU = laplace(un, dx, dy, m, n);

    float** dvdx = central_ddx(vn, dx, m, n);
    float** dvdy = central_ddy(vn, dy, m, n);
    float** dpdy = central_ddy(p, dy, m, n);
    float** laplaceV = laplace(vn, dx, dy, m, n);
    for (int i=1; i<m-1; ++i){
        for (int j=1; j<n-1; ++j){
          u[i][j] = un[i][j] + dt*(-un[i][j]*dudx[i][j]-vn[i][j]*dudy[i][j]-1/rho*dpdx[i]
[j]+nu*laplaceU[i][j]);
          v[i][j] = vn[i][j] + dt*(-un[i][j]*dvdx[i][j]-vn[i][j]*dvdy[i][j]-1/rho*dpdy[i]
[j]+nu*laplaceV[i][j]);
        }
    }

    free(dudx);
    free(dudy);
    free(dpdx);
    free(laplaceU);
    free(dvdx);
    free(dvdy);
    free(dpdy);
    free(laplaceV);

    // B.C.
      for (int j=0; j<n; ++j){
      u[0][j] = 0.0;   // bottom
      v[0][j] = 0.0;
      u[m-1][j] = ut;  // top
      v[m-1][j] = 0.0;
    }
```

```
    for (int i=0; i<m; ++i){
      u[i][0] = 0.0;   // left
      v[i][0] = 0.0;
      u[i][n-1] = 0.0; // right
      v[i][n-1] = 0.0;
    }
  }

  free(un);
  free(vn);

  return 0;
}
```

C Program, vectorutils.c

```
#include <stdio.h>
#include <stdlib.h>

float** init_zero_vector(int m, int n){
  float* values = calloc(m*n, sizeof(float));
  float** vec = malloc(m*sizeof(float*));

  for (int i=0; i<m; ++i){
    vec[i] = values + i*n;
  }
  return vec;
}


void copy_vector(float** newV, float** previousV, int m, int n){
  size_t size = sizeof(float)*n;
  for (int i=0; i<m; ++i){
    for (int j=0; j<n; ++j){
      newV[i][j] = previousV[i][j];
    }
  }
}


void print_vector(float** vec, int m, int n){
  for (int i = 0; i < m; i++) {
        for (int j = 0; j < n; j++)
            printf("%f,", vec[i][j]);
        printf("\n");
    }
}


int write_result(char* fileName, float** vec, int m, int n){
  FILE *outFile;
  outFile=fopen(fileName,"w+");
  if (outFile==NULL){
        printf("[Error] Failed to open output file %s", fileName);
        return -1;
    }

  for (int i = 0; i < m; i++) {
        for (int j = 0; j < n; j++)
            fprintf(outFile, "%f,", vec[i][j]);
        fprintf(outFile, "\n");
    }
  fclose(outFile);

  return 0;
}
```

C Program, main.c

```c
#include <stdio.h>
#include <stdlib.h>
#include "vectorutils.h"
#include "fluiddynamics.h"

int main(void) {
    printf("Solving Cavity Fluid Flow using Finite-Difference Approach\n");

    // Mesh and parameter set-up
    const int N_GRID_X = 41;
    const int N_GRID_Y = 41;
    const float LENGTH_X = 2.0;
    const float LENGTH_Y = 2.0;

    const float dx = LENGTH_X / (N_GRID_X - 1);
    const float dy = LENGTH_Y / (N_GRID_Y - 1);

    const int N_TIMESTEP = 100;
    const float dt = 0.001;

    const float DENSITY = 1.0;
    const float KINEMATIC_VISCOSITY = 0.1;
    const float U_TOP = 1.0;

    // I.C.
    float** u = init_zero_vector(N_GRID_X, N_GRID_Y);
    float** v = init_zero_vector(N_GRID_X, N_GRID_Y);
    float** p = init_zero_vector(N_GRID_X, N_GRID_Y);

    // Loop through timesteps
    cavity_flow(
        N_TIMESTEP, u, v, dt, dx, dy, p, DENSITY,
        KINEMATIC_VISCOSITY, U_TOP, N_GRID_X, N_GRID_Y);

    int u_txt = write_result("./results/u.txt", u, N_GRID_X, N_GRID_Y);
    int v_txt = write_result("./results/v.txt", v, N_GRID_X, N_GRID_Y);
    int p_txt = write_result("./results/p.txt", p, N_GRID_X, N_GRID_Y);

    if (u_txt!=0 || u_txt!=0 || u_txt!=0) {
        printf("[ERROR] Failed to write output files\n");
    } else {
        printf("[SUCCESS] Wrote output files\n");
        return 0;
    }

}
```

The Python program consists of the following section:

- **plotProfiles** - plot velocity profiles

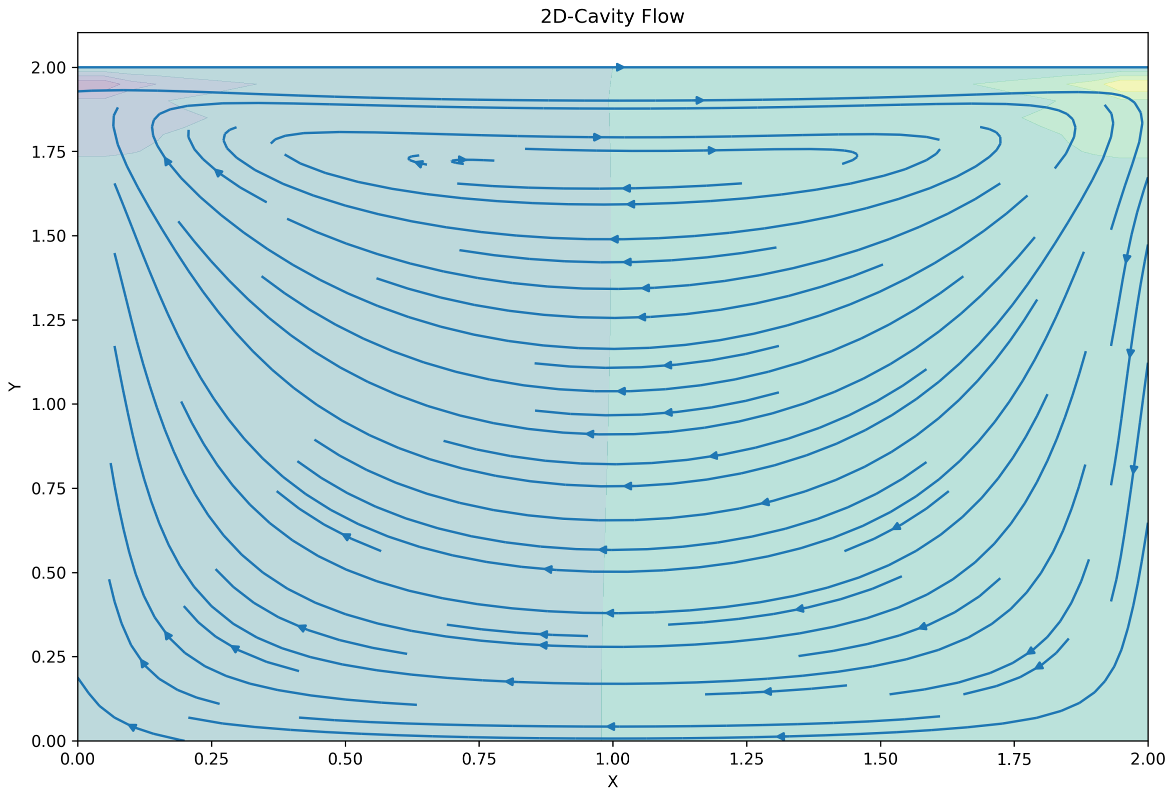
Python Program, plotProfiles.py

```python
import numpy as np
import matplotlib.pyplot as plt


def fill_profile(filename, vec):
    with open(filename) as f:
        for i, line in enumerate(f):
            data = line.split(",")[:-1]
            for j, value in enumerate(data):
                vec[i][j] = float(value)


def plot_flow(lx,ly,nx,ny,u,v,p):
    x = np.linspace(0, lx, nx)
    y = np.linspace(0, ly, ny)
```

```python
    X, Y = np.meshgrid(x, y)

    fig = plt.figure(figsize=(12,8), dpi=100)
    plt.contourf(X, Y, p, alpha=0.3)
    plt.streamplot(X, Y, u, v)
    plt.xlabel('X')
    plt.ylabel('Y')
    plt.title('2D-Cavity Flow')
    return fig

def main():
    N_GRID_X = 41
    N_GRID_Y = 41
    LENGTH_X = 2.0
    LENGTH_Y = 2.0

    u = np.zeros((N_GRID_X, N_GRID_Y))
    v = np.zeros((N_GRID_X, N_GRID_Y))
    p = np.zeros((N_GRID_X, N_GRID_Y))

    fill_profile("./results/u.txt", u)
    fill_profile("./results/v.txt", v)
    fill_profile("./results/p.txt", p)

    cavity_flow_plot = plot_flow(
        lx=LENGTH_X,
        ly=LENGTH_Y,
        nx=N_GRID_X,
        ny=N_GRID_Y,
        u=u,
        v=v,
        p=p)

    cavity_flow_plot.savefig("./results/cavityFlow.png", dpi=300)

main()
```

D. Output

After compile and execute the C program to obtain velocities and pressure profiles, run the python program to obtain the cavity flow plot as shown:



Appendix

Additional Code Sections of CFD Solver

C header files and Makefile

C Program, finitedifference.h

```
float** central_ddx(float** f, float dx, int m, int n);
float** central_ddy(float** f, float dy, int m, int n);
float** laplace(float** f, float dx, float dy, int m, int n);
```

C Program, fluiddynamics.h

```
float** compute_b(float dx,float dy,float rho,float dt, float** u, float** v, int m, int n);
float** pressure_poisson(float** p, float dx, float dy, float rho, float dt, float** u, float** v, int m, int n);
int cavity_flow(int nt, float** u, float** v, float dt, float dx, float dy, float** p, float rho, float nu, float ut, int m, int n);
```

C Program, vectorutils.h

```
float** init_zero_vector(int m, int n);
void copy_vector(float** newV, float** previousV, int m, int n);
void print_vector(float** vec, int m, int n);
int write_result(char* fileName, float** vec, int m, int n);
```

C Program, Makefile

```
```

```
CC      = gcc

all: vectorutils.o finitedifference.o fluiddynamics.o main.o
	$(CC) vectorutils.o finitedifference.o fluiddynamics.o main.o -o cavityflow

vectorutils.o: vectorutils.c
	$(CC) -c vectorutils.c -o vectorutils.o

finitedifference.o: finitedifference.c
	$(CC) -c finitedifference.c -o finitedifference.o

fluiddynamics.o: fluiddynamics.c
	$(CC) -c fluiddynamics.c -o fluiddynamics.o

main.o: main.c
	$(CC) -c main.c -o main.o

clean:
	rm *.o cavityflow
```