Liouville's Theorem
and the application to
propagation of Cosmic Rays

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Liouville's Theorem

derived 1838 by Joseph Liouville (1809 - 1882) 

Theorem

The phase-space distribution function $\rho(q_i, p_i; t)$ is constant along trajectory of a conservative system

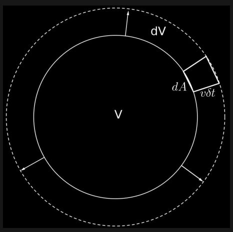
$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_i \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0$$

i. e. the density of system points is constant in the vicinity of a given system point traveling through phase-space.

Liouville's Theorem

Theorem

The Volume enclosed by a closed surface in phase-space is constant as surface moves through phase-space



Proof for Hamiltonian systems:

$$dV = \int_S dA\hat{n}\vec{v}\delta t \quad \Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}t} = \int_S dA\hat{n}\vec{v} = \int_V \nabla\vec{v}dV$$

$$\vec{v} = (\dot{q}, \dot{p}) \quad \Rightarrow \quad \nabla\vec{v} = \partial_q\dot{q} + \partial_p\dot{p} = \partial_q(\partial_p H) + \partial_p(-\partial_q H) = 0 = \frac{\mathrm{d}V}{\mathrm{d}t}$$

Examples

Harmonic Oscillator

Pendulum

Liouville's Theorem

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Orbits in phase-space are unique.

i. e. trajectories in phase-space cannot cross each other.

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application to Cosmic Rays?

Liouville's Theorem for Cosmic Rays

from Parker 1967: Dynamics of the Magnetosphere:
Consider a static magnetic field B. Density of particles at point s on a field line of B with pitch angle θ at time t: $\Psi(s, \theta, t)$.
"Liouville said that [...] the particle density in phase-space is preserved along the particle trajectory. That is to say the total derivative of the density function Ψ is zero in moving with the particle along the line of force."

$$0 = \frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\partial\Psi}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

with velocity along B-line: $\frac{\mathrm{d}s}{\mathrm{d}t} = v\sin\theta$ and invariance of magnetic moment $\frac{mv_\perp^2}{2B} = const$. this differential equation can be written as

$$0 = \frac{\partial\Psi}{\partial t} + v\cos\theta\frac{\partial\Psi}{\partial s} + \frac{v\sin\theta}{2B}\frac{\mathrm{d}B}{\mathrm{d}s}\frac{\partial\Psi}{\partial\theta}$$

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assuming small variation in time, $\frac{\partial \Psi}{\partial t} \simeq 0$, the general solution has the from $\Psi(s, \theta) = f\left(\frac{\sin^2\theta}{B}\right)$ with arbitrary function f. More conveniently one can write

$$\Psi = A \left(\frac{\sin^2\theta}{B}\right)^{\alpha}$$

Liouville's Theorem for Cosmic Rays

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Assume: particle density Ψ is isotropic at any point along the trajectory. This demands $\alpha = 0$ and therefore $\Psi = A = const$ along the trajectory. Also Ψ is no longer a function of B.

Theorem

In presence of a static or slowly changing magnetic field, if the particle density is isotropic at one point in space it is isotropic everywhere.

This implies that **magnetic fields cannot change the flux of cosmic rays measured at earth**. Especially, no anisotropy can be intorduced purely by deflection in B

q.e.d

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