Digital Logic

Lecture 5

2nd Stage

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Topics covered

- ✧ Boolean algebra theorems
- ✧ De Morgan’s theorem
- ✧ Reduction of logic expressions using Boolean algebra
- ✧ Canonical Forms
- ✧ Sum Of Products (SOP)
- ✧ Product Of Sums (POS)
- ✧ Non-Canonical to Canonical Conversion
- ✧ How to convert SOP to POS & POS to SOP

Boolean Algebra:

Allows us to apply provable mathematical principles to help us design logical circuits.

The algebra which deals with binary variables and the logic operators AND (•), OR (+) and NOT ($\bar{x}$) is called Boolean algebra. Boolean algebra also follows the Commutative, Associative, Absorption, Distributive laws and De Morgan's laws of the common algebra.

Basic Identities of Boolean Algebra

Identity Name	AND Form	OR Form
Identity Law	$1x = x$	$0+x = x$
Null (or Dominance) Law	$0x = 0$	$1+x = 1$
Idempotent Law	$xx = x$	$x+x = x$
Inverse Law	$x\overline{x} = 0$	$x+\overline{x} = 1$
Commutative Law	$xy = yx$	$x+y = y+x$
Associative Law	$(xy)z = x(yz)$	$(x+y)+z = x+(y+z)$
Distributive Law	$x+yz = (x+y)(x+z)$	$x(y+z) = xy+xz$
Absorption Law	$x(x+y) = x$	$x+xy = x$
DeMorgan's Law	$(\overline{xy}) = \overline{x}+\overline{y}$	$(\overline{x+y}) = \overline{x}\overline{y}$
Double Complement Law	$\overline{\overline{x}} = x$	

Basic Identities of Boolean Algebra

Simplify the following Boolean function to a minimum number of literals.

1) $\boldsymbol{A.(\bar{A}+B)=?}$

***Solution*:**

$A.(\bar{A}+B)=A.\bar{A}+A.B = 0+A.B=AB$

2) $\boldsymbol{A+(\bar{A}.B)=?}$

***Solution*:**

$A+(\bar{A}.B)=(A+\bar{A}).(A+B) = 1.(A+B)=A+B$

Basic Identities of Boolean Algebra

$\mathbf{3)}$ $(\boldsymbol{A} + \boldsymbol{B}).(\boldsymbol{A} + \overline{\boldsymbol{B}}) = ?$

Note: List the identities used at each step.

Solution:

$$(A + B).(A + \bar{B}) = A.A + A.B + A.\bar{B} + B.\bar{B}$$

Inverse Law

Idempotent Law

$$= A + A.B + A.\bar{B} + 0$$

Distributive Law

$$= A + A(B + \bar{B})$$

Inverse Law

$$= A + A.1$$

Identity Law

$$= A + A$$

Idempotent Law

$$= A$$

Basic Identities of Boolean Algebra

Find the complement of the following function by applying DeMorgan's theorems as many times as necessary.

Question: $F = (\bar{x} \,.\, y \,.\, \bar{z} + \bar{x} \,.\, \bar{y} \,.\, z)$

Solu.:

$$\bar{F} = \overline{(\bar{x} \,.\, y \,.\, \bar{z} + \bar{x} \,.\, \bar{y} \,.\, z)}$$

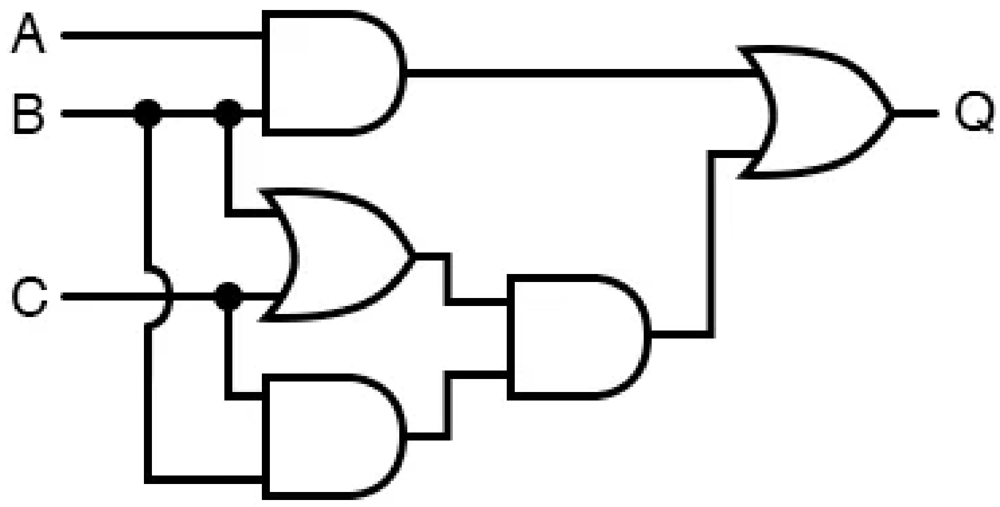
$$= (\, \overline{\bar{x}.y \,.\bar{z}} \,).(\, \overline{\bar{x}.\bar{y}.z} \,)$$

$$= (\, \bar{\bar{x}} + \bar{y} + \bar{\bar{z}} \,).(\, \bar{\bar{x}} + \bar{\bar{y}} + \bar{z} \,)$$

$$= (\, x + \bar{y} + z \,).(\, x + y + \bar{z} \,)$$

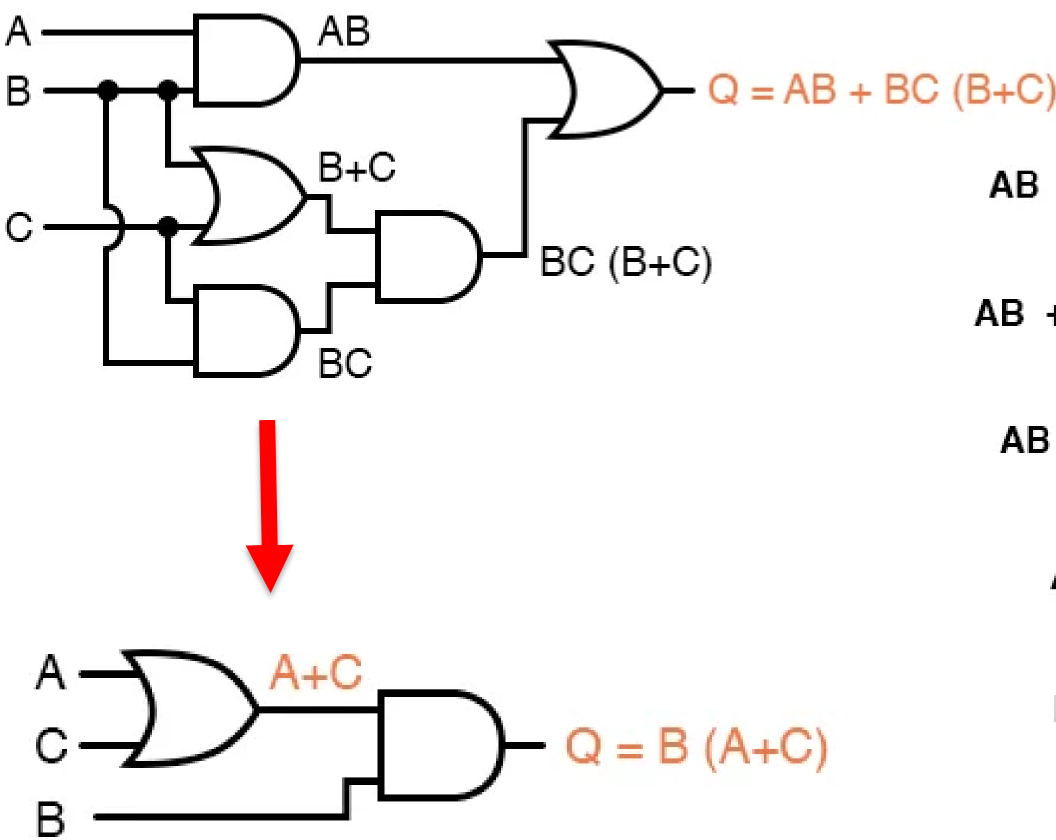
Basic Identities of Boolean Algebra

Use Boolean algebra to simplify the following logic diagram:



Basic Identities of Boolean Algebra

Use Boolean algebra to simplify the following logic diagram:



$AB + BC(B + C)$

↓ Distributing terms

$AB + BBC + BCC$

↓ Applying identity AA = A to 2nd and 3rd terms

$AB + BC + BC$

↓ Applying identity A + A = A to 2nd and 3rd terms

$AB + BC$

↓ Factoring B out of terms

$B(A + C)$

Canonical Forms

- Any Boolean function F( ) can be expressed as a *unique* **sum** of **min**terms and a unique **product** of **max**terms (under a fixed variable ordering).

- In other words, every function F() has two canonical forms:
  - Canonical Sum-Of-Products (sum of minterms)
  - Canonical Product-Of-Sums (product of maxterms)

Canonical Forms (cont.)

- Canonical Sum-Of-Products:
  The minterms included are those m_j such that F( ) = 1 in row *j* of the truth table for F( ).

- Canonical Product-Of-Sums:
  The maxterms included are those M_j such that F( ) = 0 in row *j* of the truth table for F( ).

Sum-of-products (SOP)

Minterm

- Represents exactly one combination in the truth table.
- Denoted by m_j, where j is the decimal equivalent of the minterm's corresponding binary combination (b_j).
- A variable in m_j is complemented if its value in b_j is 0, otherwise is uncomplemented.
- Example: Assume 3 variables (A,B,C), and j=3. Then, b_j = 011 and its corresponding minterm is denoted by m_j = A'BC

Sum-of-products (SOP)

Shorthand: ∑

- F(A,B,C) = ∑ m(1,2,4,6), where ∑ indicates that this is a sum-of-products form, and m(1,2,4,6) indicates that the minterms to be included are m_1, m_2, m_4, and m_6.

Sum-of-products (SOP)

✧ Also called *disjunctive normal form* (DNF) or *minterm expansion*

minterm

$$F = \overset{001}{A'B'C} + \overset{011}{A'BC} + \overset{101}{AB'C} + \overset{110}{ABC'} + \overset{111}{ABC}$$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Sum-of-products (SOP)

✧ Variables appear exactly once in each minterm in true or inverted form (but not both)

A	B	C	F	minterms	
0	0	0	0		
0	0	1	1	A'B'C	m1
0	1	0	0		
0	1	1	1	A'BC	m3
1	0	0	0		
1	0	1	1	AB'C	m5
1	1	0	1	ABC'	m6
1	1	1	1	ABC	m7

short-hand notation

F in canonical form:

$$F(A,B,C) = \Sigma m(1,3,5,6,7)$$
$$= m1 + m3 + m5 + m6 + m7$$
$$= A'B'C+A'BC+AB'C+ABC'+ABC$$

Sum-of-products (SOP)

✧ Develop a truth table for the canonical SOP expression

$$\overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + ABC$$

Inputs			Output	minterm
A	B	C	X	
0	0	0	0	
0	0	1	1	$\overline{A}\,\overline{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\overline{B}\,\overline{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

Product-of-sums (POS)

Maxterm

- Represents exactly one combination in the truth table.
- Denoted by M_j, where j is the decimal equivalent of the maxterm's corresponding binary combination (b_j).
- A variable in M_j is complemented if its value in b_j is 1, otherwise is uncomplemented.
- Example: Assume 3 variables (A,B,C), and j=3. Then, b_j = 011 and its corresponding maxterm is denoted by M_j = A+B'+C'

Product-of-sums (POS)

Shorthand: ∏

- F(A,B,C) = ∏ M(0,3,5,7), where ∏ indicates that this is a product-of-sums form, and M(0,3,5,7) indicates that the maxterms to be included are M_0, M_3, M_5, and M_7.

- Since $m_j = M_j'$ for any *j*,
  $\sum m(1,2,4,6) = \prod M(0,3,5,7) = F(a,b,c)$

Product-of-sums (POS)

✧ Also called *conjunctive normal form* (CNF) or *maxterm expansion*

$$F = \overset{000}{(A + B + C)}\ \overset{010}{(A + B' + C)}\ \overset{100}{(A' + B + C)}$$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Maxterm

Product-of-sums (POS)

✧ Variables appear exactly once in each maxterm in true or inverted form (but not both)

A	B	C	F	maxterms	
0	0	0	0	A+B+C	M0
0	0	1	1		
0	1	0	0	A+B'+C	M2
0	1	1	1		
1	0	0	0	A'+B+C	M4
1	0	1	1		
1	1	0	1		
1	1	1	1		

F in canonical form:

$$
\begin{aligned}
F(A,B,C) &= \Pi M(0,2,4) \\
&= M0 \bullet M2 \bullet M4 \\
&= (A+B+C)(A+B'+C)(A'+B+C)
\end{aligned}
$$

short-hand notation

Example

- Consider the truth table for F(a,b,c) at right.
- The canonical sum-of-products form for F is:
  $F(a,b,c) = m_1 + m_2 + m_4 + m_6$
  $= a'b'c + a'bc' + ab'c' + abc'$
- The canonical product-of-sums form for F is:
  $F(a,b,c) = M_0 \bullet M_3 \bullet M_5 \bullet M_7$
  $= (a+b+c) \bullet (a+b'+c') \bullet (a'+b+c') \bullet (a'+b'+c')$.
- Observe that: $m_j = M_j'$

#	a	b	c		F
0	0	0	0		0
1	0	0	1		1
2	0	1	0		1
3	0	1	1		0
4	1	0	0		1
5	1	0	1		0
6	1	1	0		1
7	1	1	1		0

Example

- Develop a truth table for the canonical POS expression

$$(A+B+C)(A+\overline{B}+C)(A+\overline{B}+\overline{C})$$
$$(\overline{A}+B+\overline{C})(\overline{A}+\overline{B}+C)$$

Inputs			Output	Maxterm
A	B	C	X	
0	0	0	0	$(A+B+C)$
0	0	1	1	
0	1	0	0	$(A+\overline{B}+C)$
0	1	1	0	$(A+\overline{B}+\overline{C})$
1	0	0	1	
1	0	1	0	$(\overline{A}+B+\overline{C})$
1	1	0	0	$(\overline{A}+\overline{B}+C)$
1	1	1	1	

Non-canonical Forms

- non-canonical forms are "*like*" canonical forms, except that not all variables need appear in the individual product (SOP) or sum (POS) terms.
- Example:
  $F(a,b,c) = a'b'c + bc' + ac'$
  is a non-canonical sum-of-products form
- $F(a,b,c) = (a+b+c).(b'+c').(a'+c')$
  is a non-canonical product-of-sums form.

Conversion of SOP from non-canonical to canonical form

- Expand *non-canonical* terms by inserting equivalent of 1 in each missing variable x:
  (x + x’) = 1
- Remove duplicate minterms
- F(a,b,c) = a’b’c + bc’ + ac’
  = a’b’c + (a+a’)bc’ + a(b+b’)c’
  = a’b’c + abc’ + a’bc’ + abc’ + ab’c’
  = a’b’c + abc’ + a’bc ’ + ab’c’

Conversion of SOP from non-canonical to canonical form

Convert the following Boolean expression into canonical SOP form:

$$F(A,B,C) = AB + AC$$

Sol. $AB + AC$

$= AB(C+C') + AC(B+B')$

$= ABC+ABC'+ABC+AB'C$

$= ABC+ABC'+AB'C$

Conversion of SOP from non-canonical to canonical form

✧ Convert the following Boolean expression into canonical SOP form:

$$F(A,B,C,D) = A\overline{B}C + \overline{A}\,\overline{B} + AB\overline{C}D$$

Solution:

$$A\overline{B}C = A\overline{B}C(D+\overline{D}) = A\overline{B}CD + A\overline{B}C\overline{D}$$

$$\overline{A}\,\overline{B} = \overline{A}\,\overline{B}(C+\overline{C}) = \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C}$$

$$\overline{A}\,\overline{B}C(D+\overline{D}) + \overline{A}\,\overline{B}\,\overline{C}(D+\overline{D}) = \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$$

$$A\overline{B}C + \overline{A}\,\overline{B} + AB\overline{C}D = A\overline{B}CD + A\overline{B}C\overline{D} + \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + AB\overline{C}D$$

Conversion of POS from non-canonical to canonical form

- Expand non-canonical terms by adding 0 in terms of missing variables (*e.g.*, $xx' = 0$) and using the distributive law
- Remove duplicate maxterms
- $F(a,b,c) = (a+b+c).(b'+c').(a'+c')$

$$
\begin{aligned}
&= (a+b+c).(aa'+b'+c').(a'+bb'+c') \\
&= (a+b+c).(a+b'+c').(a'+b'+c').\ (a'+b+c').(a'+b'+c') \\
&= (a+b+c).(a+b'+c').(a'+b'+c').(a'+b+c')
\end{aligned}
$$

Conversion of POS from non-canonical to canonical form

Convert the following Boolean expression into canonical POS form:

$$F(A,B,C)=(A+B)(A+C)$$

Sol. $F = (A+B).(A+C)$

$=(A+B)+(C.C') \,.\, (A+C)+(B.B')$

$=(A+B+C).(A+B+C').(A+B+C)(A+B'+C)$ **Distributive law**

$=(A+B+C).(A+B+C')(A+B'+C)$ **Remove duplicates**

Conversion of POS from non-canonical to canonical form

✧ Convert the following Boolean expression into canonical POS form:

$$F(A,B,C,D) = (A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

Solution:

$$A+\bar{B}+C = A+\bar{B}+C+D\bar{D} = (A+\bar{B}+C+D)(A+\bar{B}+C+\bar{D})$$
$$\bar{B}+C+\bar{D} = \bar{B}+C+\bar{D}+A\bar{A} = (A+\bar{B}+C+\bar{D})(\bar{A}+\bar{B}+C+\bar{D})$$

$$(A+\bar{B}+C)(\bar{B}+C+\bar{D})(A+\bar{B}+\bar{C}+D) =$$
$$(A+\bar{B}+C+D)(A+\bar{B}+C+\bar{D})(A+\bar{B}+C+\bar{D})(\bar{A}+\bar{B}+C+\bar{D})(A+\bar{B}+\bar{C}+D)$$
$$=(A+\bar{B}+C+D)(A+\bar{B}+C+\bar{D})(\bar{A}+\bar{B}+C+\bar{D})(A+\bar{B}+\bar{C}+D)$$

Conversion Between Canonical Forms

- Replace $\sum$ with $\prod$ (or *vice versa*) and replace those *j*'s that appeared in the original form with those that do not.
- Example:
  $F(a,b,c) = a'b'c + a'bc' + ab'c' + abc'$
  $= m_1 + m_2 + m_4 + m_6$
  $= \sum m(1,2,4,6)$

  $= \prod M(0,3,5,7)$
  $= (a+b+c).(a+b'+c').(a'+b+c').(a'+b'+c')$

Conversion Between Canonical Forms

SOP

$$F(x_1, x_2, x_3) = \Sigma\, m(\underbrace{0, 2, 4, 5, 6, 7})$$

Missed numbers: 1, 3

POS

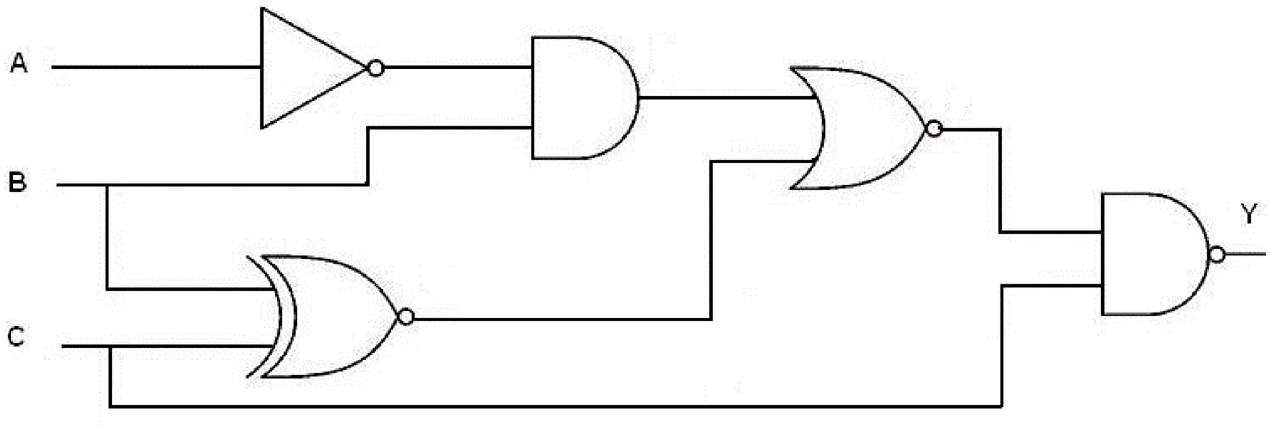
$$F(x_1, x_2, x_3) = \Pi\, M(\underbrace{1, 3})$$

$$= M_1 \bullet M_3$$

$$= (x_1 + x_2 + \overline{x_3}) \bullet (x_1 + \overline{x_2} + \overline{x_3})$$

Exercises-4

1- Express the following logic diagram as a sum of minterms.



2) Find the complement of the following expression by using DeMorgan's theorems.

$$\boldsymbol{F = (XYZ) + (\overline{XYZ}) + \overline{(Y.\overline{Z}) + \overline{X}}}$$

Deadline: October 29, 2022 @ 11:59 PM

Homework 5

1) Simplify the following Boolean function to a minimum number of literals. List the identities used at each step.

A) $F(W,X,Y,Z)= \overline{W}X(\overline{Z}+\overline{Y}Z)+X(W+\overline{W}YZ)$

B) $F(X,Y,Z)= \overline{X}YZ+X\overline{Y}Z+XY\overline{Z}+XYZ$

2) Express the following function as a sum of minterms.

$$F(A,B)= (\overline{A}\overline{B})\otimes\overline{(A\oplus B)} + (A + B)$$

Deadline: November 4, 2022 @ 11:59 PM