

50. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.
51. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.
52. As mentioned in the text, the notation $\exists!x P(x)$ denotes “There exists a unique x such that $P(x)$ is true.” If the domain consists of all integers, what are the truth values of these statements?
- $\exists!x(x > 1)$
 - $\exists!x(x^2 = 1)$
 - $\exists!x(x + 3 = 2x)$
 - $\exists!x(x = x + 1)$
53. What are the truth values of these statements?
- $\exists!x P(x) \rightarrow \exists x P(x)$
 - $\forall x P(x) \rightarrow \exists!x P(x)$
 - $\exists!x \neg P(x) \rightarrow \neg \forall x P(x)$
54. Write out $\exists!x P(x)$, where the domain consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.
55. Given the Prolog facts in Example 28, what would Prolog return given these queries?
- ?instructor(chan, math273)
 - ?instructor(patel, cs301)
 - ?enrolled(X, cs301)
 - ?enrolled(kiko, Y)
 - ?teaches(grossman, Y)
56. Given the Prolog facts in Example 28, what would Prolog return when given these queries?
- ?enrolled(kevin, ee222)
 - ?enrolled(kiko, math273)
 - ?instructor(grossman, X)
 - ?instructor(X, cs301)
 - ?teaches(X, kevin)
57. Suppose that Prolog facts are used to define the predicates *mother(M, Y)* and *father(F, X)*, which represent that *M* is the mother of *Y* and *F* is the father of *X*, respectively. Give a Prolog rule to define the predicate *sibling(X, Y)*, which represents that *X* and *Y* are siblings (that is, have the same mother and the same father).
58. Suppose that Prolog facts are used to define the predicates *mother(M, Y)* and *father(F, X)*, which represent that *M* is the mother of *Y* and *F* is the father of *X*, respectively. Give a Prolog rule to define the predicate *grandfather(X, Y)*, which represents that *X* is the grandfather of *Y*. [Hint: You can write a disjunction in Prolog

either by using a semicolon to separate predicates or by putting these predicates on separate lines.]

Exercises 59–62 are based on questions found in the book *Symbolic Logic* by Lewis Carroll.

59. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements “ x is a professor,” “ x is ignorant,” and “ x is vain,” respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.
- No professors are ignorant.
 - All ignorant people are vain.
 - No professors are vain.
 - Does (c) follow from (a) and (b)?
60. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements “ x is a clear explanation,” “ x is satisfactory,” and “ x is an excuse,” respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.
- All clear explanations are satisfactory.
 - Some excuses are unsatisfactory.
 - Some excuses are not clear explanations.
 - Does (c) follow from (a) and (b)?
61. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “ x is a baby,” “ x is logical,” “ x is able to manage a crocodile,” and “ x is despised,” respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.
- Babies are illogical.
 - Nobody is despised who can manage a crocodile.
 - Illogical persons are despised.
 - Babies cannot manage crocodiles.
 - Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?
62. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “ x is a duck,” “ x is one of my poultry,” “ x is an officer,” and “ x is willing to waltz,” respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.
- No ducks are willing to waltz.
 - No officers ever decline to waltz.
 - All my poultry are ducks.
 - My poultry are not officers.
 - Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

1.4 Nested Quantifiers

Introduction

In Section 1.3 we defined the existential and universal quantifiers and showed how they can be used to represent mathematical statements. We also explained how they can be used to translate

English sentences into logical expressions. In this section we will study **nested quantifiers**. Two quantifiers are nested if one is within the scope of the other, such as

$$\forall x \exists y (x + y = 0).$$

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example,

$$\forall x \exists y (x + y = 0)$$

is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$. Nested quantifiers commonly occur in mathematics and computer science. Although nested quantifiers can sometimes be difficult to understand, the rules we have already studied in Section 1.3 can help us use them.

To understand these statements involving many quantifiers, we need to unravel what the quantifiers and predicates that appear mean. This is illustrated in Examples 1 and 2.

EXAMPLE 1 Assume that the domain for the variables x and y consists of all real numbers. The statement

$$\forall x \forall y (x + y = y + x)$$



says that $x + y = y + x$ for all real numbers x and y . This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers.

EXAMPLE 2 Translate into English the statement

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.

Solution: This statement says that for every real number x and for every real number y , if $x > 0$ and $y < 0$, then $xy < 0$. That is, this statement says that for real numbers x and y , if x is positive and y is negative, then xy is negative. This can be stated more succinctly as “The product of a positive real number and a negative real number is always a negative real number.”

THINKING OF QUANTIFICATION AS LOOPS In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. (Of course, if there are infinitely many elements in the domain of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.) For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for x , and for each x we loop through the values for y . If we find that $P(x, y)$ is true for all values for x and y , we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value x for which we hit a value y for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for x . For each x we loop through the values for y until we find a y for which $P(x, y)$ is true. If for

every x we hit such a y , then $\forall x \exists y P(x, y)$ is true; if for some x we never hit such a y , then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y . Once we find such an x , we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x , then we know that $\exists x \forall y P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x , where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.

The Order of Quantifiers

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

These remarks are illustrated by Examples 3–5.

EXAMPLE 3 Let $P(x, y)$ be the statement “ $x + y = y + x$.” What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\forall x \forall y P(x, y)$$



denotes the proposition

“For all real numbers x , for all real numbers y , $x + y = y + x$.”

Because $P(x, y)$ is true for all real numbers x and y (it is the commutative law for addition, which is an axiom for the real numbers—see Appendix 1), the proposition $\forall x \forall y P(x, y)$ is true. Note that the statement $\forall y \forall x P(x, y)$ says “For all real numbers y , for all real numbers x , $x + y = y + x$.” This has the same meaning as the statement as “For all real numbers x , for all real numbers y , $x + y = y + x$.” That is, $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning, and both are true. This illustrates the principle that the order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

EXAMPLE 4 Let $Q(x, y)$ denote “ $x + y = 0$.” What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\exists y \forall x Q(x, y)$$

denotes the proposition

“There is a real number y such that for every real number x , $Q(x, y)$.”

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x , the statement $\exists y \forall x Q(x, y)$ is false.

TABLE 1 Quantifications of Two Variables.

<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y .	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y .
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y .	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y .

The quantification

$$\forall x \exists y Q(x, y)$$

denotes the proposition

“For every real number x there is a real number y such that $Q(x, y)$.”

Given a real number x , there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true. 

Example 4 illustrates that the order in which quantifiers appear makes a difference. The statements $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent. The statement $\exists y \forall x P(x, y)$ is true if and only if there is a y that makes $P(x, y)$ true for every x . So, for this statement to be true, there must be a particular value of y for which $P(x, y)$ is true regardless of the choice of x . On the other hand, $\forall x \exists y P(x, y)$ is true if and only if for every value of x there is a value of y for which $P(x, y)$ is true. So, for this statement to be true, no matter which x you choose, there must be a value of y (possibly depending on the x you choose) for which $P(x, y)$ is true. In other words, in the second case, y can depend on x , whereas in the first case, y is a constant independent of x .

From these observations, it follows that if $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ must also be true. However, if $\forall x \exists y P(x, y)$ is true, it is not necessary for $\exists y \forall x P(x, y)$ to be true. (See Supplementary Exercises 24 and 25 at the end of this chapter.)

Table 1 summarizes the meanings of the different possible quantifications involving two variables.

Quantifications of more than two variables are also common, as Example 5 illustrates.

EXAMPLE 5 Let $Q(x, y, z)$ be the statement “ $x + y = z$.” What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$, where the domain of all variables consists of all real numbers?

Solution: Suppose that x and y are assigned values. Then, there exists a real number z such that $x + y = z$. Consequently, the quantification

$$\forall x \forall y \exists z Q(x, y, z),$$

which is the statement

“For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$,”

is true. The order of the quantification here is important, because the quantification

$$\exists z \forall x \forall y Q(x, y, z),$$

which is the statement

“There is a real number z such that for all real numbers x and for all real numbers y it is true that $x + y = z$,”

is false, because there is no value of z that satisfies the equation $x + y = z$ for all values of x and y . 

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Mathematical statements expressed in English can be translated into logical expressions, as Examples 6–8 show.

EXAMPLE 6 Translate the statement “The sum of two positive integers is always positive” into a logical expression.

Solution: To translate this statement into a logical expression, we first rewrite it so that the implied quantifiers and a domain are shown: “For every two integers, if these integers are both positive, then the sum of these integers is positive.” Next, we introduce the variables x and y to obtain “For all positive integers x and y , $x + y$ is positive.” Consequently, we can express this statement as



$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)),$$

where the domain for both variables consists of all integers. Note that we could also translate this using the positive integers as the domain. Then the statement “The sum of two positive integers is always positive” becomes “For every two positive integers, the sum of these integers is positive. We can express this as

$$\forall x \forall y (x + y > 0),$$

where the domain for both variables consists of all positive integers. 

EXAMPLE 7 Translate the statement “Every real number except zero has a multiplicative inverse.” (A **multiplicative inverse** of a real number x is a real number y such that $xy = 1$.)

Solution: We first rewrite this as “For every real number x except zero, x has a multiplicative inverse.” We can rewrite this as “For every real number x , if $x \neq 0$, then there exists a real number y such that $xy = 1$.” This can be rewritten as

$$\forall x ((x \neq 0) \rightarrow \exists y (xy = 1)).$$

One example that you may be familiar with is the concept of limit, which is important in calculus.

EXAMPLE 8 (*Requires calculus*) Express the definition of a limit using quantifiers.

Solution: Recall that the definition of the statement

$$\lim_{x \rightarrow a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\forall \epsilon \exists \delta \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon),$$

where the domain for the variables δ and ϵ consists of all positive real numbers and for x consists of all real numbers.

This definition can also be expressed as

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

when the domain for the variables ϵ and δ consists of all real numbers, rather than just the positive real numbers. [Here, restricted quantifiers have been used. Recall that $\forall x > 0 P(x)$ means that for all x with $x > 0$, $P(x)$ is true.] 

Translating from Nested Quantifiers into English

Expressions with nested quantifiers expressing statements in English can be quite complicated. The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean. The next step is to express this meaning in a simpler sentence. This process is illustrated in Examples 9 and 10.

EXAMPLE 9 Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

into English, where $C(x)$ is “ x has a computer,” $F(x, y)$ is “ x and y are friends,” and the domain for both x and y consists of all students in your school.

Solution: The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in your school has a computer or has a friend who has a computer. 

EXAMPLE 10 Translate the statement

$$\exists x \forall y \forall z ((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z))$$

into English, where $F(a, b)$ means a and b are friends and the domain for x , y , and z consists of all students in your school.

Solution: We first examine the expression $(F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z)$. This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends. It follows that the original statement, which is triply quantified, says that there is a student x such that for all students y and all students z other than y , if x and y are friends and x and z are friends, then y and z are not friends. In other words, there is a student none of whose friends are also friends with each other. 

Translating English Sentences into Logical Expressions

In Section 1.3 we showed how quantifiers can be used to translate sentences into logical expressions. However, we avoided sentences whose translation into logical expressions required the use of nested quantifiers. We now address the translation of such sentences.

EXAMPLE 11

Express the statement “If a person is female and is a parent, then this person is someone’s mother” as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution: The statement “If a person is female and is a parent, then this person is someone’s mother” can be expressed as “For every person x , if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y .” We introduce the propositional functions $F(x)$ to represent “ x is female,” $P(x)$ to represent “ x is a parent,” and $M(x, y)$ to represent “ x is the mother of y .” The original statement can be represented as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y)).$$

Using the null quantification rule in part (b) of Exercise 47 in Section 1.3, we can move $\exists y$ to the left so that it appears just after $\forall x$, because y does not appear in $F(x) \wedge P(x)$. We obtain the logically equivalent expression

$$\forall x \exists y ((F(x) \wedge P(x)) \rightarrow M(x, y)).$$

EXAMPLE 12

Express the statement “Everyone has exactly one best friend” as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution: The statement “Everyone has exactly one best friend” can be expressed as “For every person x , person x has exactly one best friend.” Introducing the universal quantifier, we see that this statement is the same as “ $\forall x$ (person x has exactly one best friend),” where the domain consists of all people.

To say that x has exactly one best friend means that there is a person y who is the best friend of x , and furthermore, that for every person z , if person z is not person y , then z is not the best friend of x . When we introduce the predicate $B(x, y)$ to be the statement “ y is the best friend of x ,” the statement that x has exactly one best friend can be represented as

$$\exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

Consequently, our original statement can be expressed as

$$\forall x \exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

[Note that we can write this statement as $\forall x \exists! y B(x, y)$, where $\exists!$ is the “uniqueness quantifier” defined on page 37.]

EXAMPLE 13

Use quantifiers to express the statement “There is a woman who has taken a flight on every airline in the world.”

Solution: Let $P(w, f)$ be “ w has taken f ” and $Q(f, a)$ be “ f is a flight on a .” We can express the statement as

$$\exists w \forall a \exists f (P(w, f) \wedge Q(f, a)),$$

where the domains of discourse for w , f , and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

The statement could also be expressed as

$$\exists w \forall a \exists f R(w, f, a),$$

where $R(w, f, a)$ is “ w has taken f on a .” Although this is more compact, it somewhat obscures the relationships among the variables. Consequently, the first solution is usually preferable. ◀

Negating Nested Quantifiers



Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier. This is illustrated in Examples 14–16.



EXAMPLE 14 Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

Solution: By successively applying De Morgan’s laws for quantifiers in Table 2 of Section 1.3, we can move the negation in $\neg \forall x \exists y (xy = 1)$ inside all the quantifiers. We find that $\neg \forall x \exists y (xy = 1)$ is equivalent to $\exists x \neg \exists y (xy = 1)$, which is equivalent to $\exists x \forall y \neg (xy = 1)$. Because $\neg (xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y (xy \neq 1)$. ◀

EXAMPLE 15 Use quantifiers to express the statement that “There does not exist a woman who has taken a flight on every airline in the world.”

Solution: This statement is the negation of the statement “There is a woman who has taken a flight on every airline in the world” from Example 13. By Example 13, our statement can be expressed as $\neg \exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$, where $P(w, f)$ is “ w has taken f ” and $Q(f, a)$ is “ f is a flight on a .” By successively applying De Morgan’s laws for quantifiers in Table 2 of Section 1.3 to move the negation inside successive quantifiers and by applying De Morgan’s law for negating a conjunction in the last step, we find that our statement is equivalent to each of this sequence of statements:

$$\begin{aligned} \forall w \neg \forall a \exists f (P(w, f) \wedge Q(f, a)) &\equiv \forall w \exists a \neg \exists f (P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a)). \end{aligned}$$

This last statement states “For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.” ◀

EXAMPLE 16 (*Requires calculus*) Use quantifiers and predicates to express the fact that $\lim_{x \rightarrow a} f(x)$ does not exist.

Solution: To say that $\lim_{x \rightarrow a} f(x)$ does not exist means that for all real numbers L , $\lim_{x \rightarrow a} f(x) \neq L$. By using Example 8, the statement $\lim_{x \rightarrow a} f(x) \neq L$ can be expressed as

$$\neg \forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon).$$

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\begin{aligned}
 & \neg \forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
 & \equiv \exists \epsilon > 0 \neg \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
 & \equiv \exists \epsilon > 0 \forall \delta > 0 \neg \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
 & \equiv \exists \epsilon > 0 \forall \delta > 0 \exists x \neg (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
 & \equiv \exists \epsilon > 0 \forall \delta > 0 \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).
 \end{aligned}$$

In the last step we used the equivalence $\neg(p \rightarrow q) \equiv p \wedge \neg q$, which follows from the fifth equivalence in Table 7 of Section 1.2.

Because the statement “ $\lim_{x \rightarrow a} f(x)$ does not exist” means for all real numbers L , $\lim_{x \rightarrow a} f(x) \neq L$, this can be expressed as

$$\forall L \exists \epsilon > 0 \forall \delta > 0 \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. 

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.

- a) $\forall x \exists y (x < y)$
- b) $\forall x \forall y ((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0)$
- c) $\forall x \forall y \exists z (xy = z)$

2. Translate these statements into English, where the domain for each variable consists of all real numbers.

- a) $\exists x \forall y (xy = y)$
- b) $\forall x \forall y ((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0)$
- c) $\forall x \forall y \exists z (x = y + z)$

3. Let $Q(x, y)$ be the statement “ x has sent an e-mail message to y ,” where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

- | | |
|----------------------------------|----------------------------------|
| a) $\exists x \exists y Q(x, y)$ | b) $\exists x \forall y Q(x, y)$ |
| c) $\forall x \exists y Q(x, y)$ | d) $\exists y \forall x Q(x, y)$ |
| e) $\forall y \exists x Q(x, y)$ | f) $\forall x \forall y Q(x, y)$ |

4. Let $P(x, y)$ be the statement “student x has taken class y ,” where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

- | | |
|----------------------------------|----------------------------------|
| a) $\exists x \exists y P(x, y)$ | b) $\exists x \forall y P(x, y)$ |
| c) $\forall x \exists y P(x, y)$ | d) $\exists y \forall x P(x, y)$ |
| e) $\forall y \exists x P(x, y)$ | f) $\forall x \forall y P(x, y)$ |

5. Let $W(x, y)$ mean that student x has visited website y , where the domain for x consists of all students in your

school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

- a) $W(\text{Sarah Smith, www.att.com})$
- b) $\exists x W(x, \text{www.imdb.org})$
- c) $\exists y W(\text{Jose Orez, } y)$
- d) $\exists y (W(\text{Ashok Puri, } y) \wedge W(\text{Cindy Yoon, } y))$
- e) $\exists y \forall z (y \neq (\text{David Belcher}) \wedge (W(\text{David Belcher, } z) \rightarrow W(y, z)))$
- f) $\exists x \exists y \forall z ((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

6. Let $C(x, y)$ mean that student x is enrolled in class y , where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

- a) $C(\text{Randy Goldberg, CS 252})$
- b) $\exists x C(x, \text{Math 695})$
- c) $\exists y C(\text{Carol Sitea, } y)$
- d) $\exists x (C(x, \text{Math 222}) \wedge C(x, \text{CS 252}))$
- e) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$
- f) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

7. Let $T(x, y)$ mean that student x likes cuisine y , where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

- a) $\neg T(\text{Abdallah Hussein, Japanese})$
- b) $\exists x T(x, \text{Korean}) \wedge \forall x T(x, \text{Mexican})$

- c) $\exists y(T(\text{Monique Arsenault}, y) \vee T(\text{Jay Johnson}, y))$
- d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$
- e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$
- f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$
8. Let $Q(x, y)$ be the statement “student x has been a contestant on quiz show y .” Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.
- There is a student at your school who has been a contestant on a television quiz show.
 - No student at your school has ever been a contestant on a television quiz show.
 - There is a student at your school who has been a contestant on *Jeopardy* and on *Wheel of Fortune*.
 - Every television quiz show has had a student from your school as a contestant.
 - At least two students from your school have been contestants on *Jeopardy*.
9. Let $L(x, y)$ be the statement “ x loves y ,” where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.
- Everybody loves Jerry.
 - Everybody loves somebody.
 - There is somebody whom everybody loves.
 - Nobody loves everybody.
 - There is somebody whom Lydia does not love.
 - There is somebody whom no one loves.
 - There is exactly one person whom everybody loves.
 - There are exactly two people whom Lynn loves.
 - Everyone loves himself or herself.
 - There is someone who loves no one besides himself or herself.
10. Let $F(x, y)$ be the statement “ x can fool y ,” where the domain consists of all people in the world. Use quantifiers to express each of these statements.
- Everybody can fool Fred.
 - Evelyn can fool everybody.
 - Everybody can fool somebody.
 - There is no one who can fool everybody.
 - Everyone can be fooled by somebody.
 - No one can fool both Fred and Jerry.
 - Nancy can fool exactly two people.
 - There is exactly one person whom everybody can fool.
 - No one can fool himself or herself.
 - There is someone who can fool exactly one person besides himself or herself.
11. Let $S(x)$ be the predicate “ x is a student,” $F(x)$ the predicate “ x is a faculty member,” and $A(x, y)$ the predicate “ x has asked y a question,” where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.
- Lois has asked Professor Michaels a question.
 - Every student has asked Professor Gross a question.
- c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
- d) Some student has not asked any faculty member a question.
- e) There is a faculty member who has never been asked a question by a student.
- f) Some student has asked every faculty member a question.
- g) There is a faculty member who has asked every other faculty member a question.
- h) Some student has never been asked a question by a faculty member.
12. Let $I(x)$ be the statement “ x has an Internet connection” and $C(x, y)$ be the statement “ x and y have chatted over the Internet,” where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.
- Jerry does not have an Internet connection.
 - Rachel has not chatted over the Internet with Chelsea.
 - Jan and Sharon have never chatted over the Internet.
 - No one in the class has chatted with Bob.
 - Sanjay has chatted with everyone except Joseph.
 - Someone in your class does not have an Internet connection.
 - Not everyone in your class has an Internet connection.
 - Exactly one student in your class has an Internet connection.
 - Everyone except one student in your class has an Internet connection.
 - Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
 - Someone in your class has an Internet connection but has not chatted with anyone else in your class.
 - There are two students in your class who have not chatted with each other over the Internet.
 - There is a student in your class who has chatted with everyone in your class over the Internet.
 - There are at least two students in your class who have not chatted with the same person in your class.
 - There are two students in the class who between them have chatted with everyone else in the class.
13. Let $M(x, y)$ be “ x has sent y an e-mail message” and $T(x, y)$ be “ x has telephoned y ,” where the domain consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)
- Chou has never sent an e-mail message to Koko.
 - Arlene has never sent an e-mail message to or telephoned Sarah.
 - Jose has never received an e-mail message from Deborah.
 - Every student in your class has sent an e-mail message to Ken.

- e) No one in your class has telephoned Nina.
- f) Everyone in your class has either telephoned Avi or sent him an e-mail message.
- g) There is a student in your class who has sent everyone else in your class an e-mail message.
- h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.
- i) There are two different students in your class who have sent each other e-mail messages.
- j) There is a student who has sent himself or herself an e-mail message.
- k) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.
- l) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.
- m) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.
- n) There are two different students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.
14. Use quantifiers and predicates with more than one variable to express these statements.
- There is a student in this class who can speak Hindi.
 - Every student in this class plays some sport.
 - Some student in this class has visited Alaska but has not visited Hawaii.
 - All students in this class have learned at least one programming language.
 - There is a student in this class who has taken every course offered by one of the departments in this school.
 - Some student in this class grew up in the same town as exactly one other student in this class.
 - Every student in this class has chatted with at least one other student in at least one chat group.
15. Use quantifiers and predicates with more than one variable to express these statements.
- Every computer science student needs a course in discrete mathematics.
 - There is a student in this class who owns a personal computer.
 - Every student in this class has taken at least one computer science course.
 - There is a student in this class who has taken at least one course in computer science.
 - Every student in this class has been in every building on campus.
 - There is a student in this class who has been in every room of at least one building on campus.
 - Every student in this class has been in at least one room of every building on campus.
16. A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.
- There is a student in the class who is a junior.
 - Every student in the class is a computer science major.
 - There is a student in the class who is neither a mathematics major nor a junior.
 - Every student in the class is either a sophomore or a computer science major.
 - There is a major such that there is a student in the class in every year of study with that major.
17. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.
- Every user has access to exactly one mailbox.
 - There is a process that continues to run during all error conditions only if the kernel is working correctly.
 - All users on the campus network can access all websites whose url has a .edu extension.
 - *d) There are exactly two systems that monitor every remote server.
18. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.
- At least one console must be accessible during every fault condition.
 - The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.
 - For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.
 - There are at least two paths connecting every two distinct endpoints on the network.
 - No one knows the password of every user on the system except for the system administrator, who knows all passwords.
19. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.
- The sum of two negative integers is negative.
 - The difference of two positive integers is not necessarily positive.
 - The sum of the squares of two integers is greater than or equal to the square of their sum.
 - The absolute value of the product of two integers is the product of their absolute values.
20. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.
- The product of two negative integers is positive.
 - The average of two positive integers is positive.

- c) The difference of two negative integers is not necessarily negative.
d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.
21. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.
22. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.
23. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.
- The product of two negative real numbers is positive.
 - The difference of a real number and itself is zero.
 - Every positive real number has exactly two square roots.
 - A negative real number does not have a square root that is a real number.
24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
- $\exists x \forall y (x + y = y)$
 - $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
 - $\exists x \exists y (((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$
 - $\forall x \forall y ((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$
25. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
- $\exists x \forall y (xy = y)$
 - $\forall x \forall y (((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$
 - $\exists x \exists y ((x^2 > y) \wedge (x < y))$
 - $\forall x \forall y \exists z (x + y = z)$
26. Let $Q(x, y)$ be the statement " $x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?
- $Q(1, 1)$
 - $Q(2, 0)$
 - $\forall y Q(1, y)$
 - $\exists x Q(x, 2)$
 - $\exists x \exists y Q(x, y)$
 - $\forall x \exists y Q(x, y)$
 - $\exists y \forall x Q(x, y)$
 - $\forall y \exists x Q(x, y)$
 - $\forall x \forall y Q(x, y)$
27. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
- $\forall n \exists m (n^2 < m)$
 - $\exists n \forall m (n < m^2)$
 - $\forall n \exists m (n + m = 0)$
 - $\exists n \forall m (nm = m)$
 - $\exists n \exists m (n^2 + m^2 = 5)$
 - $\exists n \exists m (n^2 + m^2 = 6)$
 - $\exists n \exists m (n + m = 4 \wedge n - m = 1)$
 - $\exists n \exists m (n + m = 4 \wedge n - m = 2)$
 - $\forall n \forall m \exists p (p = (m + n)/2)$
28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.
- $\forall x \exists y (x^2 = y)$
 - $\forall x \exists y (x = y^2)$
- c) $\exists x \forall y (xy = 0)$
d) $\exists x \exists y (x + y \neq y + x)$
e) $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$
f) $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$
g) $\forall x \exists y (x + y = 1)$
h) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$
i) $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$
j) $\forall x \forall y \exists z (z = (x + y)/2)$
29. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y , where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.
- $\forall x \forall y P(x, y)$
 - $\exists x \exists y P(x, y)$
 - $\exists x \forall y P(x, y)$
 - $\forall y \exists x P(x, y)$
30. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
- $\neg \exists y \exists x P(x, y)$
 - $\neg \forall x \exists y P(x, y)$
 - $\neg \exists y (Q(y) \wedge \forall x \neg R(x, y))$
 - $\neg \exists y (\exists x R(x, y) \vee \forall x S(x, y))$
 - $\neg \exists y (\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$
31. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
- $\forall x \exists y \forall z T(x, y, z)$
 - $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$
 - $\forall x \exists y (P(x, y) \wedge \exists z R(x, y, z))$
 - $\forall x \exists y (P(x, y) \rightarrow Q(x, y))$
32. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
- $\exists z \forall y \forall x T(x, y, z)$
 - $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$
 - $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$
 - $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$
33. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
- $\neg \forall x \forall y P(x, y)$
 - $\neg \forall y \exists x P(x, y)$
 - $\neg \forall y \forall x (P(x, y) \vee Q(x, y))$
 - $\neg (\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$
 - $\neg \forall x (\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$
34. Find a common domain for the variables x , y , and z for which the statement $\forall x \forall y ((x \neq y) \rightarrow \forall z ((z = x) \vee (z = y)))$ is true and another domain for which it is false.
35. Find a common domain for the variables x , y , z , and w for which the statement $\forall x \forall y \forall z \exists w ((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.
36. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in

- simple English. (Do not simply use the words “It is not the case that.”)
- No one has lost more than one thousand dollars playing the lottery.
 - There is a student in this class who has chatted with exactly one other student.
 - No student in this class has sent e-mail to exactly two other students in this class.
 - Some student has solved every exercise in this book.
 - No student has solved at least one exercise in every section of this book.
37. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words “It is not the case that.”)
- Every student in this class has taken exactly two mathematics classes at this school.
 - Someone has visited every country in the world except Libya.
 - No one has climbed every mountain in the Himalayas.
 - Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.
38. Express the negations of these propositions using quantifiers, and in English.
- Every student in this class likes mathematics.
 - There is a student in this class who has never seen a computer.
 - There is a student in this class who has taken every mathematics course offered at this school.
 - There is a student in this class who has been in at least one room of every building on campus.
39. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
- $\forall x \forall y (x^2 = y^2 \rightarrow x = y)$
 - $\forall x \exists y (y^2 = x)$
 - $\forall x \forall y (xy \geq x)$
40. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
- $\forall x \exists y (x = 1/y)$
 - $\forall x \exists y (y^2 - x < 100)$
 - $\forall x \forall y (x^2 \neq y^3)$
41. Use quantifiers to express the associative law for multiplication of real numbers.
42. Use quantifiers to express the distributive laws of multiplication over addition for real numbers.
43. Use quantifiers and logical connectives to express the fact that every linear polynomial (that is, polynomial of degree 1) with real coefficients and where the coefficient of x is nonzero, has exactly one real root.
44. Use quantifiers and logical connectives to express the fact

that a quadratic polynomial with real number coefficients has at most two real roots.

45. Determine the truth value of the statement $\forall x \exists y (xy = 1)$ if the domain for the variables consists of
- the nonzero real numbers.
 - the nonzero integers.
 - the positive real numbers.
46. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain for the variables consists of
- the positive real numbers.
 - the integers.
 - the nonzero real numbers.
47. Show that the two statements $\neg \exists x \forall y P(x, y)$ and $\forall x \exists y \neg P(x, y)$, where both quantifiers over the first variable in $P(x, y)$ have the same domain, and both quantifiers over the second variable in $P(x, y)$ have the same domain, are logically equivalent.
- *48. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain, are logically equivalent. (The new variable y is used to combine the quantifications correctly.)
- *49. a) Show that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to $\forall x \exists y (P(x) \wedge Q(y))$, where all quantifiers have the same nonempty domain.
 b) Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x \exists y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain.
- A statement is in **prenex normal form (PNF)** if and only if it is of the form
- $$Q_1 x_1 Q_2 x_2 \cdots Q_k x_k P(x_1, x_2, \dots, x_k),$$
- where each Q_i , $i = 1, 2, \dots, k$, is either the existential quantifier or the universal quantifier, and $P(x_1, \dots, x_k)$ is a predicate involving no quantifiers. For example, $\exists x \forall y (P(x, y) \wedge Q(y))$ is in prenex normal form, whereas $\exists x P(x) \vee \forall x Q(x)$ is not (because the quantifiers do not all occur first).
- Every statement formed from propositional variables, predicates, T, and F using logical connectives and quantifiers is equivalent to a statement in prenex normal form. Exercise 51 asks for a proof of this fact.
- *50. Put these statements in prenex normal form. [Hint: Use logical equivalence from Tables 6 and 7 in Section 1.2, Table 2 in Section 1.3, Example 19 in Section 1.3, Exercises 45 and 46 in Section 1.3, and Exercises 48 and 49 in this section.]
- $\exists x P(x) \vee \exists x Q(x) \vee A$, where A is a proposition not involving any quantifiers.
 - $\neg(\forall x P(x) \vee \forall x Q(x))$
 - $\exists x P(x) \rightarrow \exists x Q(x)$
- **51. Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement.
- *52. Express the quantification $\exists! x P(x)$, introduced on page 37, using universal quantifications, existential quantifications, and logical operators.

1.5 Rules of Inference

Introduction

Later in this chapter we will study proofs. Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an **argument**, we mean a sequence of statements that end with a conclusion. By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument. That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements.

Before we study mathematical proofs, we will look at arguments that involve only compound propositions. We will define what it means for an argument involving compound propositions to be valid. Then we will introduce a collection of rules of inference in propositional logic. These rules of inference are among the most important ingredients in producing valid arguments. After we illustrate how rules of inference are used to produce valid arguments, we will describe some common forms of incorrect reasoning, called **fallacies**, which lead to invalid arguments.

After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements. We will describe how these rules of inference can be used to produce valid arguments. These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.

Finally, we will show how rules of inference for propositions and for quantified statements can be combined. These combinations of rule of inference are often used together in complicated arguments.

Valid Arguments in Propositional Logic

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

“If you have a current password, then you can log onto the network.”

“You have a current password.”

Therefore,

“You can log onto the network.”

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion “You can log onto the network” must be true when the premises “If you have a current password, then you can log onto the network” and “You have a current password” are both true.

Before we discuss the validity of this particular argument, we will look at its form. Use p to represent “You have a current password” and q to represent “You can log onto the network.” Then, the argument has the form

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

where \therefore is the symbol that denotes “therefore.”

We know that when p and q are propositional variables, the statement $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology (see Exercise 10(c) in Section 1.2). In particular, when both $p \rightarrow q$ and p are true, we know that q must also be true. We say this form of argument is **valid** because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true. Now suppose that both “If you have a current password, then you can log onto the network” and “You have a current password” are true statements. When we replace p by “You have a current password” and q by “You can log onto the network,” it necessarily follows that the conclusion “You can log onto the network” is true. This argument is **valid** because its form is valid. Note that whenever we replace p and q by propositions where $p \rightarrow q$ and p are both true, then q must also be true.

What happens when we replace p and q in this argument form by propositions where not both p and $p \rightarrow q$ are true? For example, suppose that p represents “You have access to the network” and q represents “You can change your grade” and that p is true, but $p \rightarrow q$ is false. The argument we obtain by substituting these values of p and q into the argument form is

“If you have access to the network, then you can change your grade.”
 “You have access to the network.”

 ∴ “You can change your grade.”

The argument we obtained is a valid argument, but because one of the premises, namely the first premise, is false, we cannot conclude that the conclusion is true. (Most likely, this conclusion is false.)

In our discussion, to analyze an argument, we replaced propositions by propositional variables. This changed an argument to an **argument form**. We saw that the validity of an argument follows from the validity of the form of the argument. We summarize the terminology used to discuss the validity of arguments with our definition of the key notions.

DEFINITION 1

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

From the definition of a valid argument form we see that the argument form with premises p_1, p_2, \dots, p_n and conclusion q is valid, when $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.

The key to showing that an argument in propositional logic is valid is to show that its argument form is valid. Consequently, we would like techniques to show that argument forms are valid. We will now develop methods for accomplishing this task.

Rules of Inference for Propositional Logic

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different

rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic.

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (Modus ponens is Latin for *mode that affirms*.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol \therefore denotes “therefore.”):

$$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion. In particular, modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true. Example 1 illustrates the use of modus ponens.

EXAMPLE 1 Suppose that the conditional statement “If it snows today, then we will go skiing” and its hypothesis, “It is snowing today,” are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, “We will go skiing,” is true. 

As we mentioned earlier, a valid argument can lead to an incorrect conclusion if one or more of its premises is false. We illustrate this again in Example 2.

EXAMPLE 2 Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.

“If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$.”

Solution: Let p be the proposition “ $\sqrt{2} > \frac{3}{2}$ ” and q the proposition “ $2 > (\frac{3}{2})^2$.” The premises of the argument are $p \rightarrow q$ and p , and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$. 

Table 1 lists the most important rules of inference for propositional logic. Exercises 9, 10, 15, and 30 in Section 1.2 ask for the verifications that these rules of inference are valid argument forms. We now give examples of arguments that use these rules of inference. In each argument, we first use propositional variables to express the propositions in the argument. We then show that the resulting argument form is a rule of inference from Table 1.

EXAMPLE 3 State which rule of inference is the basis of the following argument: “It is below freezing now. Therefore, it is either below freezing or raining now.”

TABLE 1 Rules of Inference.

<i>Rule of Inference</i>	<i>Tautology</i>	<i>Name</i>
$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\begin{array}{c} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$	$p \rightarrow (p \vee q)$	Addition
$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification
$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Solution: Let p be the proposition “It is below freezing now” and q the proposition “It is raining now.” Then this argument is of the form

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$$

This is an argument that uses the addition rule. 

EXAMPLE 4 State which rule of inference is the basis of the following argument: “It is below freezing and raining now. Therefore, it is below freezing now.”

Solution: Let p be the proposition “It is below freezing now,” and let q be the proposition “It is raining now.” This argument is of the form

$$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$$

This argument uses the simplification rule. 

EXAMPLE 5 State which rule of inference is used in the argument:

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition “It is raining today,” let q be the proposition “We will not have a barbecue today,” and let r be the proposition “We will have a barbecue tomorrow.” Then this argument is of the form

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism. 

Using Rules of Inference to Build Arguments

When there are many premises, several rules of inference are often needed to show that an argument is valid. This is illustrated by Examples 6 and 7, where the steps of arguments are displayed on separate lines, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

EXAMPLE 6 Show that the hypotheses “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”

Solution: Let p be the proposition “It is sunny this afternoon,” q the proposition “It is colder than yesterday,” r the proposition “We will go swimming,” s the proposition “We will take a canoe trip,” and t the proposition “We will be home by sunset.” Then the hypotheses become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t . We need to give a valid argument with hypotheses $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t .

We construct an argument to show that our hypotheses lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Hypothesis
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Hypothesis
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Hypothesis
8. t	Modus ponens using (6) and (7)

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p , q , r , s , and t , such a truth table would have 32 rows. 

EXAMPLE 7 Show that the hypotheses “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go

to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.”

Solution: Let p be the proposition “You send me an e-mail message,” q the proposition “I will finish writing the program,” r the proposition “I will go to sleep early,” and s the proposition “I will wake up feeling refreshed.” Then the hypotheses are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$. We need to give a valid argument with hypotheses $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$ and conclusion $\neg q \rightarrow s$.

This argument form shows that the hypotheses p lead to the desired conclusion.

Step	Reason
1. $p \rightarrow q$	Hypothesis
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Hypothesis
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Hypothesis
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)



Resolution

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution**. This rule of inference is based on the tautology



$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r).$$

(The verification that this is a tautology was addressed in Exercise 30 in Section 1.2.) The final disjunction in the resolution rule, $q \vee r$, is called the **resolvent**. When we let $q = r$ in this tautology, we obtain $(p \vee q) \wedge (\neg p \vee q) \rightarrow q$. Furthermore, when we let $r = F$, we obtain $(p \vee q) \wedge (\neg p) \rightarrow q$ (because $q \vee F \equiv q$), which is the tautology on which the rule of disjunctive syllogism is based.

EXAMPLE 8

Use resolution to show that the hypotheses “Jasmine is skiing or it is not snowing” and “It is snowing or Bart is playing hockey” imply that “Jasmine is skiing or Bart is playing hockey.”



Solution: Let p be the proposition “It is snowing,” q the proposition “Jasmine is skiing,” and r the proposition “Bart is playing hockey.” We can represent the hypotheses as $\neg p \vee q$ and $p \vee r$, respectively. Using resolution, the proposition $q \vee r$, “Jasmine is skiing or Bart is playing hockey,” follows.



Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied). Furthermore, it can be used to build automatic theorem proving systems. To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses**, where a clause is a disjunction of variables or negations of these variables. We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses. For example, suppose we have a statement of the form $p \vee (q \wedge r)$. Because $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, we can replace the single statement $p \vee (q \wedge r)$ by two statements $p \vee q$ and $p \vee r$, each of which is a clause. We can replace a statement of

the form $\neg(p \vee q)$ by the two statements $\neg p$ and $\neg q$ because De Morgan's law tells us that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. We can also replace a conditional statement $p \rightarrow q$ with the equivalent disjunction $\neg p \vee q$.

EXAMPLE 9 Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Solution: We can rewrite the hypothesis $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$. Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conclude $p \vee s$. 

Fallacies

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference but are based on contingencies rather than tautologies. These are discussed here to show the distinction between correct and incorrect reasoning.



The proposition $[(p \rightarrow q) \wedge q] \rightarrow p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. In other words, they treat the argument with premises $p \rightarrow q$ and q and conclusion p as a valid argument form, which it is not. This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.

EXAMPLE 10 Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.

Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \rightarrow q$ and q , then p . This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.) 

The proposition $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis**.

EXAMPLE 11 Let p and q be as in Example 10. If the conditional statement $p \rightarrow q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \rightarrow q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. 

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$. Universal instantiation is used when we conclude from the statement “All women are wise” that “Lisa is wise,” where Lisa is a member of the domain of all women.

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

We summarize these rules of inference in Table 2. We will illustrate how one of these rules of inference for quantified statements is used in Example 12.

EXAMPLE 12 Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	Name
$\frac{\forall x P(x)}{P(c)}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{P(c) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\exists x P(x)}$	Existential generalization

Solution: Let $D(x)$ denote “ x is in this discrete mathematics class,” and let $C(x)$ denote “ x has taken a course in computer science.” Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.



The following steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)



EXAMPLE 13 Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

Solution: Let $C(x)$ be “ x is in this class,” $B(x)$ be “ x has read the book,” and $P(x)$ be “ x passed the first exam.” The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)



Combining Rules of Inference for Propositions and Quantified Statements

We have developed rules of inference both for propositions and for quantified statements. Note that in our arguments in Examples 12 and 13 we used both universal instantiation, a rule of inference for quantified statements, and modus ponens, a rule of inference for propositional logic. We will often need to use this combination of rules of inference. Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**. This rule tells us that if $\forall x(P(x) \rightarrow Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ is true. Then, by modus ponens, $Q(a)$ must also be true. We can describe universal modus ponens as follows:

$$\begin{array}{c} \forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore Q(a) \end{array}$$

Universal modus ponens is commonly used in mathematical arguments. This is illustrated in Example 14.

EXAMPLE 14 Assume that “For all positive integers n , if n is greater than 4, then n^2 is less than 2^n ” is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

Solution: Let $P(n)$ denote “ $n > 4$ ” and $Q(n)$ denote “ $n^2 < 2^n$. The statement “For all positive integers n , if n is greater than 4, then n^2 is less than 2^n ” can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(n)$ is true, namely that $100^2 < 2^{100}$.

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **universal modus tollens**. Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\frac{\begin{array}{c} \forall x(P(x) \rightarrow Q(x)) \\ \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \end{array}}{\therefore \neg P(a)}$$

We leave the verification of universal modus tollens to the reader (see Exercise 25). Exercise 26 develops additional combinations of rules of inference in propositional logic and quantified statements.

Exercises

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.

Socrates is human.

\therefore Socrates is mortal.

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not an insect.

George is an insect.

\therefore George has eight legs.

3. What rule of inference is used in each of these arguments?

- a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
- b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
- c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
- d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
- e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

4. What rule of inference is used in each of these arguments?

- a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
- b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.
- c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.
- d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.
- e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.
- 5. Use rules of inference to show that the hypotheses “Randy works hard,” “If Randy works hard, then he is a dull boy,” and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job.”
- 6. Use rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,” and “The trophy was not awarded” imply the conclusion “It rained.”
- 7. What rules of inference are used in this famous argument? “All men are mortal. Socrates is a man. Therefore, Socrates is mortal.”
- 8. What rules of inference are used in this argument? “No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.”

9. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
- "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."
 - "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."
 - "I am either clever or lucky." "I am not lucky." "If I am lucky, then I will win the lottery."
 - "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."
 - "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."
 - "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."
10. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
- "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."
 - "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."
 - "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."
 - "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."
 - "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."
 - "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."
11. Show that the argument form with premises p_1, p_2, \dots, p_n and conclusion $q \rightarrow r$ is valid if the argument form with premises p_1, p_2, \dots, p_n, q , and conclusion r is valid.
12. Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, and $\neg s$ and conclusion $q \rightarrow r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.
13. For each of these arguments, explain which rules of inference are used for each step.
- "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."
 - "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."
 - "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."
 - "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."
14. For each of these arguments, explain which rules of inference are used for each step.
- "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."
 - "Each of five roommates, Melissa, Aaron, Ralph, Venesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."
 - "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."
 - "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."
15. For each of these arguments determine whether the argument is correct or incorrect and explain why.
- All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.
 - Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
 - All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
 - Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.
16. For each of these arguments determine whether the argument is correct or incorrect and explain why.
- Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.
 - A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
 - Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

- d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.
17. What is wrong with this argument? Let $H(x)$ be “ x is happy.” Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.
18. What is wrong with this argument? Let $S(x, y)$ be “ x is shorter than y .” Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.
19. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?
- If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.
 - If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.
 - If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.
20. Determine whether these are valid arguments.
- If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.
 - If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.
21. Which rules of inference are used to establish the conclusion of Lewis Carroll’s argument described in Example 26 of Section 1.3?
22. Which rules of inference are used to establish the conclusion of Lewis Carroll’s argument described in Example 27 of Section 1.3?
23. Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \wedge \exists x Q(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.
- | | |
|-----------------------------------------|------------------------------------|
| 1. $\exists x P(x) \vee \exists x Q(x)$ | Premise |
| 2. $\exists x P(x)$ | Simplification from (1) |
| 3. $P(c)$ | Existential instantiation from (2) |
| 4. $\exists x Q(x)$ | Simplification from (1) |
| 5. $Q(c)$ | Existential instantiation from (4) |
| 6. $P(c) \wedge Q(c)$ | Conjunction from (3) and (5) |
| 7. $\exists x(P(x) \wedge Q(x))$ | Existential generalization |
24. Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.
- | | |
|------------------------------------------|-----------------------------------|
| 1. $\forall x(P(x) \vee Q(x))$ | Premise |
| 2. $P(c) \vee Q(c)$ | Universal instantiation from (1) |
| 3. $P(c)$ | Simplification from (2) |
| 4. $\forall x P(x)$ | Universal generalization from (3) |
| 5. $Q(c)$ | Simplification from (2) |
| 6. $\forall x Q(x)$ | Universal generalization from (5) |
| 7. $\forall x(P(x) \vee \forall x Q(x))$ | Conjunction from (4) and (6) |
25. Justify the rule of universal modus tollens by showing that the premises $\forall x(P(x) \rightarrow Q(x))$ and $\neg Q(a)$ for a particular element a in the domain, imply $\neg P(a)$.
26. Justify the rule of **universal transitivity**, which states that if $\forall x(P(x) \rightarrow Q(x))$ and $\forall x(Q(x) \rightarrow R(x))$ are true, then $\forall x(P(x) \rightarrow R(x))$ is true, where the domains of all quantifiers are the same.
27. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.
28. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.
29. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.
30. Use resolution to show the hypotheses “Allen is a bad boy or Hillary is a good girl” and “Allen is a good boy or David is happy” imply the conclusion “Hillary is a good girl or David is happy.”
31. Use resolution to show that the hypotheses “It is not raining or Yvette has her umbrella,” “Yvette does not have her umbrella or she does not get wet,” and “It is raining or Yvette does not get wet” imply that “Yvette does not get wet.”
32. Show that the equivalence $p \wedge \neg p \equiv \mathbf{F}$ can be derived using resolution together with the fact that a conditional statement with a false hypothesis is true. [Hint: Let $q = r = \mathbf{F}$ in resolution.]
33. Use resolution to show that the compound proposition $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ is not satisfiable.
- *34. The Logic Problem, taken from *WFF’N PROOF, The Game of Logic*, has these two assumptions:
- “Logic is difficult or not many students like logic.”
 - “If mathematics is easy, then logic is not difficult.”
- By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:
- That mathematics is not easy, if many students like logic.
 - That not many students like logic, if mathematics is not easy.
 - That mathematics is not easy or logic is difficult.
 - That logic is not difficult or mathematics is not easy.
 - That if not many students like logic, then either mathematics is not easy or logic is not difficult.
- *35. Determine whether this argument, taken from Kalish and Montague [KaMo64], is valid.
- If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

2

Basic Structures: Sets, Functions, Sequences, and Sums

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
- 2.4 Sequences and Summations

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines. These are some of the topics we will study in later chapters.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a set exactly one element of a set. Functions play important roles throughout discrete mathematics. They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways. Useful structures such as sequences and strings are special types of functions. In this chapter, we will introduce the notion of a sequence, which represents ordered lists of elements. We will introduce some important types of sequences, and we will address the problem of identifying a pattern for the terms of a sequence from its first few terms. Using the notion of a sequence, we will define what it means for a set to be countable, namely, that we can list all the elements of the set in a sequence.

In our study of discrete mathematics, we will often add consecutive terms of a sequence of numbers. Because adding terms from a sequence, as well as other indexed sets of numbers, is such a common occurrence, a special notation has been developed for adding such terms. In this section, we will introduce the notation used to express summations. We will develop formulae for certain types of summations. Such summations appear throughout the study of discrete mathematics, as, for instance, when we analyze the number of steps a procedure uses to sort a list of numbers into increasing order.

2.1 Sets

Introduction

In this section, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set. Sets are used to group objects together. Often, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion. We now provide a definition of a set. This definition is an intuitive definition, which is not part of a formal theory of sets.

DEFINITION 1

A *set* is an unordered collection of objects.



Note that the term *object* has been used without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated by the German mathematician Georg Cantor in 1895. The theory that results from this intuitive

Assessment

definition of a set, and the use of the intuitive notion that any property whatever there is a set consisting of exactly the objects with this property, leads to **paradoxes**, or logical inconsistencies. This was shown by the English philosopher Bertrand Russell in 1902 (see Exercise 38 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory beginning with axioms. We will use Cantor's original version of set theory, known as **naive set theory**, without developing an axiomatic version of set theory, because all sets considered in this book can be treated consistently using Cantor's original theory.

DEFINITION 2

The objects in a set are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

We will now introduce notation used to describe membership in sets. We write $a \in A$ to denote that a is an element of the set A . The notation $a \notin A$ denotes that a is not an element of the set A . Note that lowercase letters are usually used to denote elements of sets.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c , and d .

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

EXAMPLE 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a, 2, \text{Fred}, \text{New Jersey}\}$ is the set containing the four elements $a, 2, \text{Fred}$, and New Jersey .

Sometimes the brace notation is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (...) are used when the general pattern of the elements is obvious.

EXAMPLE 4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \dots, 99\}$.

Extra Examples

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set \mathbf{Q}^+ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p \text{ and } q\}.$$

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \dots\}$, the set of **natural numbers**

$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of **integers**

$\mathbf{Z}^+ = \{1, 2, 3, \dots\}$, the set of **positive integers**

$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**

\mathbf{R} , the set of **real numbers**

(Note that some people do not consider 0 a natural number, so be careful to check how the term *natural numbers* is used when you read other books.)

Sets can have other sets as members, as Example 5 illustrates.

EXAMPLE 5 The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are \mathbf{N} , the set of natural numbers; \mathbf{Z} , the set of integers; \mathbf{Q} , the set of rational numbers; and \mathbf{R} , the set of real numbers. 

Remark: Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *boolean* is the name of the set $\{0, 1\}$ together with operators on one or more elements of this set, such as AND, OR, and NOT.

Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

DEFINITION 3 Two sets are *equal* if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

EXAMPLE 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. 

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the **universal set** U , which contains all the objects under consideration, is represented by a rectangle. (Note that the universal set varies depending on which objects are of interest.) Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets. We show how a Venn diagram can be used in Example 7.



GEORG CANTOR (1845–1918) Georg Cantor was born in St. Petersburg, Russia, where his father was a successful merchant. Cantor developed his interest in mathematics in his teens. He began his university studies in Zurich in 1862, but when his father died he left Zurich. He continued his university studies at the University of Berlin in 1863, where he studied under the eminent mathematicians Weierstrass, Kummer, and Kronecker. He received his doctor's degree in 1867, after having written a dissertation on number theory. Cantor assumed a position at the University of Halle in 1869, where he continued working until his death.

Cantor is considered the founder of set theory. His contributions in this area include the discovery that the set of real numbers is uncountable. He is also noted for his many important contributions to analysis. Cantor also was interested in philosophy and wrote papers relating his theory of sets with metaphysics.

Cantor married in 1874 and had five children. His melancholy temperament was balanced by his wife's happy disposition. Although he received a large inheritance from his father, he was poorly paid as a professor. To mitigate this, he tried to obtain a better-paying position at the University of Berlin. His appointment there was blocked by Kronecker, who did not agree with Cantor's views on set theory. Cantor suffered from mental illness throughout the later years of his life. He died in 1918 in a psychiatric clinic.

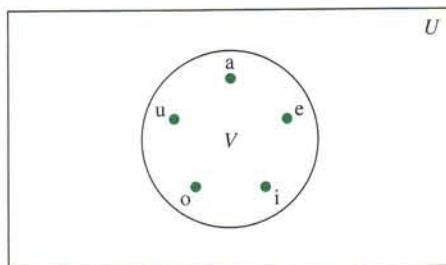


FIGURE 1 Venn Diagram for the Set of Vowels.

EXAMPLE 7 Draw a Venn diagram that represents V , the set of vowels in the English alphabet.

Solution: We draw a rectangle to indicate the universal set U , which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V . Inside this circle we indicate the elements of V with points (see Figure 1). ◀

There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by \emptyset . The empty set can also be denoted by $\{ \}$ (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set. A set with one element is called a **singleton set**.

A common error is to confuse the empty set \emptyset with the set $\{\emptyset\}$, which is a singleton set. The single element of the set $\{\emptyset\}$ is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

DEFINITION 4

The set A is said to be a *subset* of B if and only if every element of A is also an element of B . We use the notation $A \subseteq B$ to indicate that A is a subset of the set B .

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \rightarrow x \in B)$$

is true.



BERTRAND RUSSELL (1872–1970) Bertrand Russell was born into a prominent English family active in the progressive movement and having a strong commitment to liberty. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science. He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica*, written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for literature in 1950.

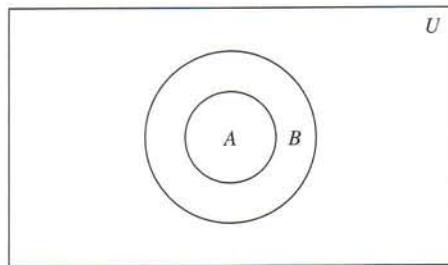


FIGURE 2 Venn Diagram Showing that A Is a Subset of B .

EXAMPLE 8

The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself). \blacktriangleleft

Theorem 1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

THEOREM 1

For every set S ,

- (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Proof: We will prove (i) and leave the proof of (ii) as an exercise.

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. That is, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. This completes the proof of (i). Note that this is an example of a vacuous proof. \blacktriangleleft

When we wish to emphasize that a set A is a subset of the set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B . For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A . That is, A is a proper subset of B if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true. Venn diagrams can be used to illustrate that a set A is a subset of a set B . We draw the universal set U as a rectangle. Within this rectangle we draw a circle for B . Because A is a subset of B , we draw the circle for A within the circle for B . This relationship is shown in Figure 2.



JOHN VENN (1834–1923) John Venn was born into a London suburban family noted for its philanthropy. He attended London schools and got his mathematics degree from Caius College, Cambridge, in 1857. He was elected a fellow of this college and held his fellowship there until his death. He took holy orders in 1859 and, after a brief stint of religious work, returned to Cambridge, where he developed programs in the moral sciences. Besides his mathematical work, Venn had an interest in history and wrote extensively about his college and family.

Venn's book *Symbolic Logic* clarifies ideas originally presented by Boole. In this book, Venn presents a systematic development of a method that uses geometric figures, known now as *Venn diagrams*. Today these diagrams are primarily used to analyze logical arguments and to illustrate relationships between sets. In addition to his work on symbolic logic, Venn made contributions to probability theory described in his widely used textbook on that subject.

One way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. This turns out to be a useful way to show that two sets are equal. That is, $A = B$, where A and B are sets, if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$, or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

Sets may have other sets as members. For instance, we have the sets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \quad \text{and} \quad B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}.$$

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

Sets are used extensively in counting problems, and for such applications we need to discuss the size of sets.

DEFINITION 5 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S . The cardinality of S is denoted by $|S|$.

EXAMPLE 9 Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

EXAMPLE 10 Let S be the set of letters in the English alphabet. Then $|S| = 26$.

EXAMPLE 11 Because the null set has no elements, it follows that $|\emptyset| = 0$.

We will also be interested in sets that are not finite.

DEFINITION 6 A set is said to be *infinite* if it is not finite.

EXAMPLE 12 The set of positive integers is infinite.



The cardinality of infinite sets will be discussed in Section 2.4.

The Power Set

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set S , we build a new set that has as its members all the subsets of S .

DEFINITION 7 Given a set S , the *power set* of S is the set of all subsets of the set S . The power set of S is denoted by $P(S)$.

EXAMPLE 13 What is the power set of the set $\{0, 1, 2\}$?



Solution: The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets.

EXAMPLE 14 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$P(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, \emptyset and the set $\{\emptyset\}$ itself. Therefore,

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

Cartesian Products

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered n -tuples**.

DEFINITION 8

The *ordered n -tuple* (a_1, a_2, \dots, a_n) is the ordered collection that has a_1 as its first element, a_2 as its second element, \dots , and a_n as its n th element.

We say that two ordered n -tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \dots, n$. In particular, 2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

Many of the discrete structures we will study in later chapters are based on the notion of the *Cartesian product* of sets (named after René Descartes). We first define the Cartesian product of two sets.



RENÉ DESCARTES (1596–1650) René Descartes was born into a noble family near Tours, France, about 200 miles southwest of Paris. He was the third child of his father's first wife; she died several days after his birth. Because of René's poor health, his father, a provincial judge, let his son's formal lessons slide until, at the age of 8, René entered the Jesuit college at La Flèche. The rector of the school took a liking to him and permitted him to stay in bed until late in the morning because of his frail health. From then on, Descartes spent his mornings in bed; he considered these times his most productive hours for thinking.

Descartes left school in 1612, moving to Paris, where he spent 2 years studying mathematics. He earned a law degree in 1616 from the University of Poitiers. At 18 Descartes became disgusted with studying and decided to see the world. He moved to Paris and became a successful gambler. However, he grew tired of bawdy living and moved to the suburb of Saint-Germain, where he devoted himself to mathematical study. When his gambling friends found him, he decided to leave France and undertake a military career. However, he never did any fighting. One day, while escaping the cold in an overheated room at a military encampment, he had several feverish dreams, which revealed his future career as a mathematician and philosopher.

After ending his military career, he traveled throughout Europe. He then spent several years in Paris, where he studied mathematics and philosophy and constructed optical instruments. Descartes decided to move to Holland, where he spent 20 years wandering around the country, accomplishing his most important work. During this time he wrote several books, including the *Discours*, which contains his contributions to analytic geometry, for which he is best known. He also made fundamental contributions to philosophy.

In 1649 Descartes was invited by Queen Christina to visit her court in Sweden to tutor her in philosophy. Although he was reluctant to live in what he called "the land of bears amongst rocks and ice," he finally accepted the invitation and moved to Sweden. Unfortunately, the winter of 1649–1650 was extremely bitter. Descartes caught pneumonia and died in mid-February.

DEFINITION 9 Let A and B be sets. The *Cartesian product* of A and B , denoted by $A \times B$, is the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE 15 Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$?



Solution: The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b) , where a is a student at the university and b is a course offered at the university. The set $A \times B$ can be used to represent all possible enrollments of students in courses at the university.

EXAMPLE 16 What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}. \quad \blacktriangleleft$$

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B . The elements of R are ordered pairs, where the first element belongs to A and the second to B . For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$. We will study relations at length in Chapter 8.

The Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$ (see Exercises 26 and 30, at the end of this section). This is illustrated in Example 17.

EXAMPLE 17 Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are as in Example 16.

Solution: The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 16.

The Cartesian product of more than two sets can also be defined.

DEFINITION 10 The *Cartesian product* of the sets A_1, A_2, \dots, A_n , denoted by $A_1 \times A_2 \times \dots \times A_n$, is the set of ordered n -tuples (a_1, a_2, \dots, a_n) , where a_i belongs to A_i for $i = 1, 2, \dots, n$. In other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}.$$

EXAMPLE 18 What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c) , where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \quad \blacktriangleleft$$

Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S . In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S . That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

EXAMPLE 19 What do the statements $\forall x \in \mathbf{R}(x^2 \geq 0)$ and $\exists x \in \mathbf{Z}(x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x , $x^2 \geq 0$. This statement can be expressed as “The square of every real number is nonnegative.” This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as “There is an integer whose square is 1.” This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

Truth Sets of Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P , and a domain D , we define the **truth set** of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

EXAMPLE 20 What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is “ $|x| = 1$,” $Q(x)$ is “ $x^2 = 2$,” and $R(x)$ is “ $|x| = x$.”

Solution: The truth set of P , $\{x \in \mathbf{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x , we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q , $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

The truth set of R , $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is \mathbf{N} , the set of nonnegative integers. ◀

Note that $\forall x P(x)$ is true over the domain U if and only if the truth set of P is the set U . Likewise, $\exists x P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Exercises

- List the members of these sets.
 - $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
 - $\{x \mid x \text{ is a positive integer less than } 12\}$
 - $\{x \mid x \text{ is the square of an integer and } x < 100\}$
 - $\{x \mid x \text{ is an integer such that } x^2 = 2\}$
- Use set builder notation to give a description of each of these sets.
 - $\{0, 3, 6, 9, 12\}$
 - $\{-3, -2, -1, 0, 1, 2, 3\}$
 - $\{m, n, o, p\}$
- Determine whether each of these pairs of sets are equal.
 - $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}, \{5, 3, 1\}$
 - $\{\{1\}\}, \{1, \{1\}\}$
 - $\emptyset, \{\emptyset\}$
- Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.
- For each of the following sets, determine whether 2 is an element of that set.
 - $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$
 - $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$
 - $\{2, \{2\}\}$
 - $\{\{2\}, \{\{2\}\}\}$
 - $\{\{2\}, \{2, \{2\}\}\}$
 - $\{\{\{2\}\}\}$

6. For each of the sets in Exercise 5, determine whether $\{2\}$ is an element of that set.
7. Determine whether each of these statements is true or false.
- a) $0 \in \emptyset$ b) $\emptyset \in \{0\}$
 c) $\{0\} \subset \emptyset$ d) $\emptyset \subset \{0\}$
 e) $\{0\} \in \{0\}$ f) $\{0\} \subset \{0\}$
 g) $\{\emptyset\} \subseteq \{\emptyset\}$
8. Determine whether these statements are true or false.
- a) $\emptyset \in \{\emptyset\}$ b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
 c) $\{\emptyset\} \in \{\emptyset\}$ d) $\{\emptyset\} \in \{\{\emptyset\}\}$
 e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
 g) $\{\{\emptyset\}\} \subset \{\{\{\emptyset\}\}, \{\emptyset\}\}$
9. Determine whether each of these statements is true or false.
- a) $x \in \{x\}$ b) $\{x\} \subseteq \{x\}$ c) $\{x\} \in \{x\}$
 d) $\{x\} \in \{\{x\}\}$ e) $\emptyset \subseteq \{x\}$ f) $\emptyset \in \{x\}$
10. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.
11. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.
12. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.
13. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.
14. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.
15. Suppose that A , B , and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.
16. Find two sets A and B such that $A \in B$ and $A \subseteq B$.
17. What is the cardinality of each of these sets?
- a) $\{a\}$ b) $\{\{a\}\}$
 c) $\{a, \{a\}\}$ d) $\{a, \{a\}, \{a, \{a\}\}\}$
18. What is the cardinality of each of these sets?
- a) \emptyset b) $\{\emptyset\}$
 c) $\{\emptyset, \{\emptyset\}\}$ d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
19. Find the power set of each of these sets, where a and b are distinct elements.
- a) $\{a\}$ b) $\{a, b\}$ c) $\{\emptyset, \{\emptyset\}\}$
20. Can you conclude that $A = B$ if A and B are two sets with the same power set?
21. How many elements does each of these sets have where a and b are distinct elements?
- a) $P(\{a, b, \{a, b\}\})$
 b) $P(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
 c) $P(P(\emptyset))$
22. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.
- a) \emptyset b) $\{\emptyset, \{a\}\}$
 c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
23. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find
- a) $A \times B$ b) $B \times A$
24. What is the Cartesian product $A \times B$, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university?
25. What is the Cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States?
26. Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?
27. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.
28. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find
- a) $A \times B \times C$ b) $C \times B \times A$
 c) $C \times A \times B$ d) $B \times B \times B$
29. How many different elements does $A \times B$ have if A has m elements and B has n elements?
30. Show that $A \times B \neq B \times A$, when A and B are nonempty, unless $A = B$.
31. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.
32. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.
33. Translate each of these quantifications into English and determine its truth value.
- a) $\forall x \in \mathbf{R} (x^2 \neq -1)$ b) $\exists x \in \mathbf{Z} (x^2 = 2)$
 c) $\forall x \in \mathbf{Z} (x^2 > 0)$ d) $\exists x \in \mathbf{R} (x^2 = x)$
34. Translate each of these quantifications into English and determine its truth value.
- a) $\exists x \in \mathbf{R} (x^3 = -1)$ b) $\exists x \in \mathbf{Z} (x + 1 > x)$
 c) $\forall x \in \mathbf{Z} (x - 1 \in \mathbf{Z})$ d) $\forall x \in \mathbf{Z} (x^2 \in \mathbf{Z})$
35. Find the truth set of each of these predicates where the domain is the set of integers.
- a) $P(x): "x^2 < 3"$ b) $Q(x): "x^2 > x"$
 c) $R(x): "2x + 1 = 0"$
36. Find the truth set of each of these predicates where the domain is the set of integers.
- a) $P(x): "x^3 \geq 1"$
 b) $Q(x): "x^2 = 2"$
 c) $R(x): "x < x^2"$
- *37. The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only

- if $a = c$ and $b = d$. [Hint: First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]
- *38. This exercise presents **Russell's paradox**. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.
- a) Show the assumption that S is a member of S leads to a contradiction.

- b) Show the assumption that S is not a member of S leads to a contradiction.

By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

- *39. Describe a procedure for listing all the subsets of a finite set.



2.2 Set Operations

Introduction

Two sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.



DEFINITION 1

Let A and B be sets. The *union* of the sets A and B , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B , or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B . This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

The Venn diagram shown in Figure 1 represents the union of two sets A and B . The area that represents $A \cup B$ is the shaded area within either the circle representing A or the circle representing B .

We will give some examples of the union of sets.

EXAMPLE 1

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

EXAMPLE 2

The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both).

DEFINITION 2

Let A and B be sets. The *intersection* of the sets A and B , denoted by $A \cap B$, is the set containing those elements in both A and B .

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B . This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$