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The connected disk covering problem

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Abstract Let P be a convex polygon with n vertices. We consider a variation of the K-center problem called *the connected disk covering problem* (*CDCP*), i.e., finding K congruent disks centered in P whose union covers P with the smallest possible radius, while a connected graph is generated by the centers of the K disks whose edge length can not exceed the radius. We give a 2.81-approximation algorithm in $O(Kn)$ time.

Keywords K-center problem · Computational geometry · Facility location problem · Unit disk graphs

1 Introduction

1.1 Background

The facility location problem has been studied for many years. It concerns the choice of the location of one or multiple facilities, in a given geographical space and is subject to some constraints, to optimally fulfill predetermined objectives. The facility location

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problem has always been considered based on some characteristics such as the number of the facilities, the construction cost of the facilities, the location of the facilities and the shape of the facilities (Rezaei and FazelZarandi 2011).

1.2 Related work

The K-center location problem is to find K centers such that the maximum distance from the stations to the nearest centers is minimized. When K is part of the input, the problem is known to be NP-complete (Megiddo and Supowit 1984). Drezener (1984) proposed an algorithm which runs in $O(n^{2K+1}\log n)$ time. By combining the result of 1-center problem proposed by Megiddo (1983), the K-center problem can be solved in $O(n^{2K-1}\log n)$ time. The best known algorithm is proposed by Hwang et al. They improved the time complexity to $O(n^{O(\sqrt{K})})$ using the slab dividing method (Hwang et al. 1993). Meanwhile, the 2-approximation algorithms have been presented (Hochbaum and Shmoys 1985; Gonzalez 1985; Feder and Greene 1988).

Various constrained versions of the K-center problem have also been studied extensively in the literature. Das et al. (2006) considered place a given number of congruent disks in a given convex polygon such that the radius is as small as possible. By using Voronoi diagram, they developed an efficient algorithm for this problem. Brass et al. (2009) studied the constrained K-center problem where the centers must be lying on a given straight line and gave an $O(n\log^2 n)$ time algorithm. Karmakar et al. (2013) proposed three algorithms for this problem with time complexities $O(nK\log n)$, $O(nK + K^2\log^3 n)$ and $O(n\log n + K\log^4 n)$. Recently, Huang et al. (2016) gave a linear time algorithm solving this problem. Considering covering a convex polygon with n vertices, Das et al. (2008) provided an $(1+\epsilon)$-approximation algorithm for the K-center problem on a convex polygon where the centers must be lying on a fixed edge of the polygon. Du and Xu (2014) studied the K-center problem for a convex polygon where the centers lie on the boundary of the polygon and presented an 1.8841-approximation algorithm which runs in $O(nK)$ time.

1.3 Problem statement

For K congruent disks on the plane, if there exists a connected graph generated by the centers of K disks whose edge length cannot exceed the radius, we call these disks connected. These K disks, as a result, form a unit disk graph. Unit disk graphs have been proven to be useful in modeling various physical real world problems. The application of unit disk graphs can be found in the field of wireless networking (Salhieh et al. 2001; wu et al. 2006; Clark et al. 1990). In this paper, we consider a variation of the K-center problem called *the connected disk covering problem* (*CDCP*). Let P be a convex polygon with n vertices. We hope to find K connected disks centered in P whose union covers P with the smallest possible radius.

1.4 Motivation

To build K transmitter stations in a city, one can view the effective broadcast range of the transmitter as a circle. Suppose each station has the same power. For any two stations, they can exchange information through the other stations effectively. As a result, choosing the location of these stations is crucial. We consider this problem as the connected disk covering problem and several applications are shown in Clark et al. (1990).

1.5 Main contribution

In this paper, we consider a variation of the K-center problem called *the connected disk covering problem*. We give a 2.81-approximation algorithm in $O(Kn)$ time. The main idea is that we first choose a proper rectangle to cover the convex polygon. Then we partition the rectangle into several small rectangles with the same length. Next we partition these small rectangles into several squares, use less connected disks covering each square and paste these squares into the original rectangle. A similar problem which proposed by Nurmela and Ostergard. They considered covering a square with up to 30 equal disks (Nurmela and Ostergard 2000). When all the K disks have covered the rectangle, we terminate the algorithm and move the centers which is out of the convex polygon into it.

1.6 Organization

The remainder of this paper is organized as follows. In Sect. 2, we consider the k ($k \geq 3$) connected disks covering a square. In Sect. 3, we consider K connected disks covering a rectangle. In Sect. 3.1, we consider the connected disks covering several squares one by one in the horizontal direction. In Sect. 3.2, we consider the connected disks covering several rectangles one by one in the vertical direction. In Sect. 3.3, we consider that how the connected disks cover a rectangle if $k = 2$. In Sect. 4, we design the algorithm solving *CDCP* and prove the approximation factor. Finally, we give a conclusion in Sect. 5.

2 Connected disk covering a square

To begin with, we list some notations we need as follows (Table 1):

In this section, we hope to find k ($k \geq 3$) connected disks covering W_h whose radius is as small as possible. The main idea is to compute k congruent disks whose union covers $D(o, R)$ and each center is in the other disk, and all the disks contain o. Furthermore, one of the intersection points of two adjacent disks must be lying on $C(o, R)$. Then shrink the radius such that the union of these disks covers W_h properly. The following lemma is needed when we consider k congruent disks covering $D(o, R)$. Let o be the original point of a rectangular coordinates.

Lemma 1 *If k is even, there exists a combination of k congruent disks such that the union of k connected disks is symmetric with respect to both x-axis and y-axis; if k is odd, there exists a combination of the k disks such that the union of k connected disks is symmetric with respect to x-axis.*

Table 1 Notations

K	The number of disks covering the convex polygon
k	The number of disks covering a disk or a square (except Sect. 3.3)
$d(p, q)$	The Euclidean distance between two points p and q
l_{pq}	The line segment connecting p and q
L_{pq}	The line through p and q
$D(o, r)$	The disk with center o and radius r
$C(o, r)$	The circle bounding $D(o, r)$
A, B, C, D	The vertices of a square or a rectangle in counterclockwise order
W_h	A square with edge length h
$W_{L,H}$	The rectangle with length L and height H
Central line L_{cl}	The line through the midpoints of l_{AD} and l_{BC}
$D(o, R)$	The circumscribing disk of W_h
$C(o, R)$	The circumscribing circle of W_h
x_s(resp. y_s)	The x-coordinate (resp. y-coordinate) of point s
$\widehat{ab}$	The minor arc with two points a and b

Proof First, for $k = 3, 4, 5$, we compute the k congruent disks covering $D(o, R)$ directly, which are shown in Fig. 1i–iii.

Next we consider the case where $k \geq 6$. If k is even, we only consider the first quadrant and second quadrant. For two adjacent disks $D(o_i, r)$ and $D(o_{i+1}, r)$, the angle of two centers and o is $\angle o_i o o_{i+1} = \frac{360°}{k}$. Thus there are $\frac{k}{2} + 1$ disks covering $D(o, R)$ in the first quadrant and second quadrant. However, k may not divide 360° exactly. If k can divide 360° exactly, there are two cases: if $\frac{k}{2} + 1$ is even, there are two centers symmetric with respect to y-axis (see Fig. 1iv); if $\frac{k}{2} + 1$ is odd, there is only one center lying on y-axis (see Fig. 1vi). Otherwise, we can modify these k disks to make sure that the union of them is symmetric with respect to y-axis. Let the angle between each two adjacent centers be $\lceil \frac{360°}{k} \rceil$.

To cover $D(o, R)$ in the first quadrant and second quadrant, there exists at least one angle of two adjacent centers which is less than $\lceil \frac{360°}{k} \rceil$. Therefore, we can properly modify these $\frac{k}{2} + 1$ disks to make sure that the union of these disks is symmetric with respect to y-axis. If $\frac{k}{2}+1$ is even, then we properly modify these $\frac{k}{2}+1$ disks to make sure that the angle of these two centers and o is $180° - (\frac{k}{2} - 1)\lceil \frac{360°}{k} \rceil$ and y-axis is the angle bisector. If $\frac{k}{2} + 1$ is odd, then we modify these $\frac{k}{2} + 1$ disks to make sure that one center is lying on y-axis and the two angles adjacent to this center are both $\frac{180° - (\frac{k}{2} - 2)\lceil \frac{360°}{k} \rceil}{2}$. Figure 2i shows an example where $\angle o_i o o_{i+1} < \angle o_{i+1} o o_{i+2} = \lceil \frac{360°}{k} \rceil$.

We only consider the first quadrant and second quadrant. Due to symmetric, we have for k disks covering $D(o, r)$, the union of these disks is symmetric with respect to x-axis.

If k is odd and k can divide 360°, we choose one disk centered at x-axis and each time set two disks on the right of the left disk. For each pair of two adjacent centers, the angle of two centers and o is $\frac{360°}{k}$. If k can not divide 360°, similar to the proof above, let

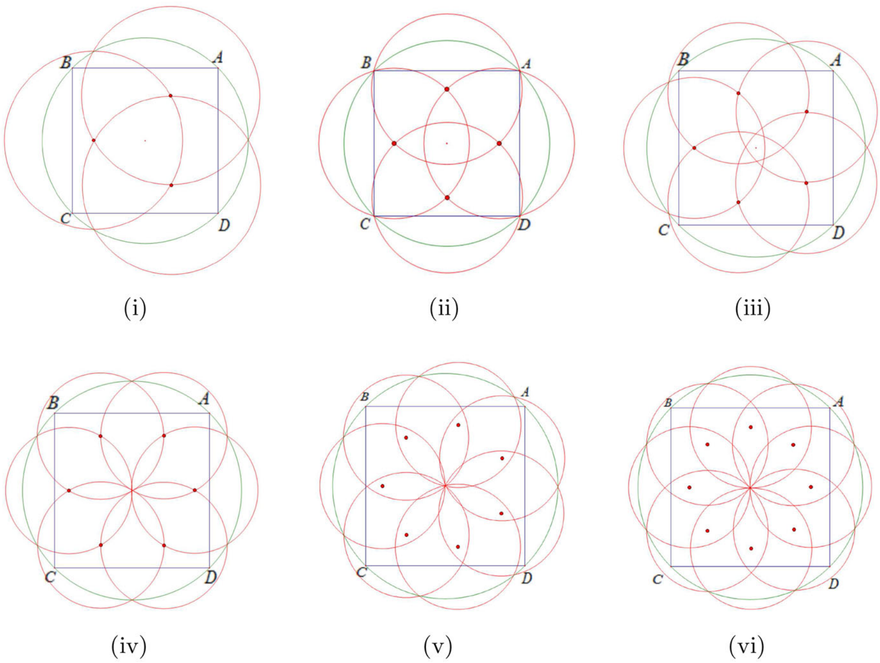


Fig. 1 The k connected disks covering $D(o, R)$. (**i**) $k = 3$, (**ii**) $k = 4$, (**iii**) $k = 5$, (**iv**) $k = 6$, (**v**) $k = 7$, (**vi**) $k = 8$

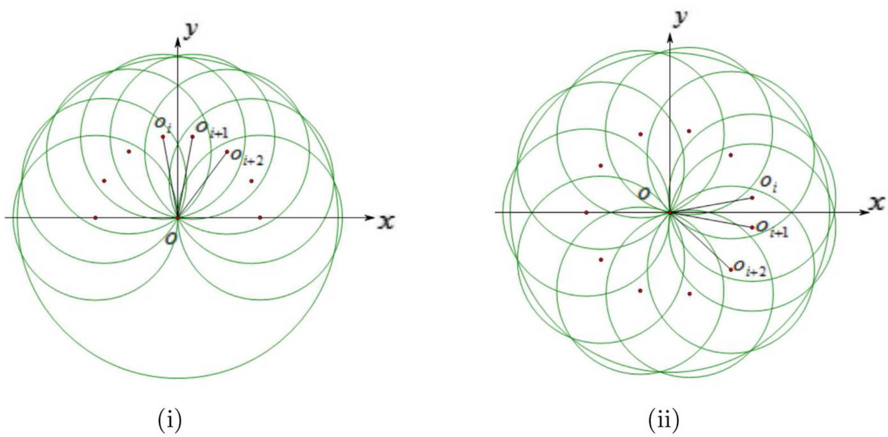


Fig. 2 Two examples for k which can not divide 360° exactly. (**i**) k is even, (**ii**) k is odd

the angle of the two centers and o be $\lceil \frac{360°}{k} \rceil$. We can modify these k disks such that the angle of the rightmost two centers and o be $\theta = 360° - (k-1)\lceil \frac{360°}{k} \rceil$ and x-axis be the angle bisector. Figure 2ii shows an example where $\angle o_i o o_{i+1} < \angle o_{i+1} o o_{i+2} = \lceil \frac{360°}{k} \rceil$.

Table 2 The radii of the disks covering a disk

$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k > 6$
$r = \frac{\sqrt{6}}{4}h$	$r = \frac{\sqrt{6}-\sqrt{2}}{2}h$	$r = \frac{\sqrt{2}\tan\frac{\pi}{5}}{\sqrt{3}\tan\frac{\pi}{5}+1}h$	$r = \frac{\sqrt{6}}{6}h$	$r = \frac{\sqrt{2}}{4\cos\frac{\pi\lceil\frac{360}{k}\rceil}{360}}h$

Due to symmetric, we have, for k disks covering $D(o, r)$, the union of these disks is symmetric with respect to x-axis.

□

Note that when $k \geq 6$, o is the intersection of all the disks. Thus the radius of the disk can be written as $r = \frac{\sqrt{2}}{4\cos\frac{\pi\lceil\frac{360}{k}\rceil}{360}}h$. When $k = 3, 4, 5$, we list the radii of k disks covering $D(o, R)$ in Table 2.

According to Lemma 1, if k is even, due to symmetry, we choose two disks centered at L_{cl}; if k is odd, we choose only one disk centered at L_{cl} and the other disks symmetric with respect to L_{cl}. We call the union of the disks the *flower disk* with boundary F_k.

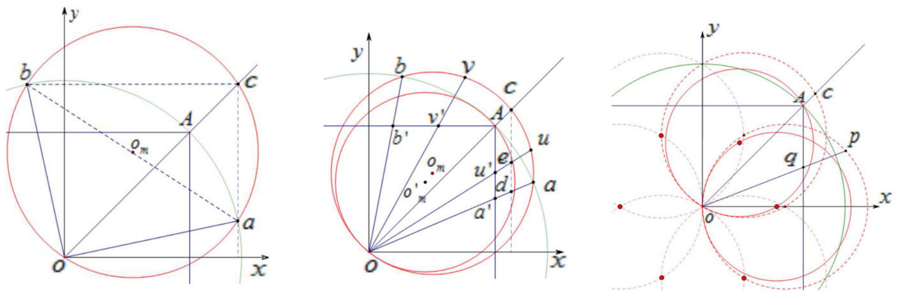
Now we shrink the radius to make sure that the union of k disks covers the square W_h properly. Let o be the original point and L_{cl} be the x-axis. We consider the case where $k \geq 5$. The following lemma is needed.

Lemma 2 *If k ($k \geq 6$) is even, $\forall p \in F_k$, and point q at which l_{po} intersects W_h, we have $\frac{d(p,o)}{d(q,o)} \geq \frac{d(c,o)}{d(A,o)}$, where c is the intersection point of F_k and L_{Ao}.*

Proof If k is even, due to symmetry, we only consider F_k in the first quadrant. When $k \geq 6$, for any two adjacent centers o_i and o_{i+1}, $\angle o_i o o_{i+1} \leq 60°$. Let m be the positive integer minimizing $|45° - m\lceil\frac{360°}{k}\rceil|$ and $\alpha = min|45° - m\lceil\frac{360°}{k}\rceil|$. Without loss of generality, suppose $45° < \alpha \leq 90°$. Let the center be o_m and the radius of the covering disks be r_m. Vertex A must be covered by $D(o_m, r_m)$. Let $C(o_m, r_m)$ intersect $C(o, R)$ at a and b counterclockwise and intersect L_{oA} at c. Then l_{oa} (resp. l_{ob}) intersects l_{AD} (resp. l_{AB}) at a' (resp. b'). We have that $y_b > y_c$ holds and $\angle o_i o o_{i+1} = \angle aob, x_a > x_c$. Otherwise, if $x_a = x_c$, then l_{ca} is vertical to the x-axis. As $\widehat{oa}$ is the common arc of $\angle abo$ and $\angle aco$, we have $\angle abo = \angle aco = 45°$. According to $d(o, a) = d(o, b)$, we have $\angle aob = 90° > 60°$ and l_{ab} is the diameter of $C(o_m, r_m)$. On the other hand, as c is on $C(o_m, r_m)$, we have $\angle acb = 90°$, thus $y_b = y_c$. If $x_a < x_c$, we have $y_b < y_c$ and $\angle aob$ will be larger than 90° which contradicts to $\angle o_i o o_{i+1} \leq 60°$. See Fig. 3i. Thus we draw a line vertical to the x-axis crossing c and intersecting l_{oa} at d. For any point u on arc $\widehat{ac}$, let e be the intersection point of l_{cd} and l_{ou}. Since l_{ou} intersects l_{AD} at u', we have $\frac{d(u,u')}{d(u',o)} \geq \frac{d(u',e)}{d(u',o)} = \frac{d(c,A)}{d(A,o)}$. In addition, as $y_b > y_c$, for any point v on cb, we have $d(v, v') > d(A, c)$ and $d(v'o) < d(A, o)$. Thus $\frac{d(v,v')}{d(v',o)} > \frac{d(c,A)}{d(A,o)}$. As shown in Fig. 3ii, we draw a circle with center o'_m lying on l_{oo_m} and crossing A. During the shrink operation, the disk $D(o_m, r_m)$ overlaps $D(o'_m, d(o, o'_m))$, and point c first intersects W_h at A.

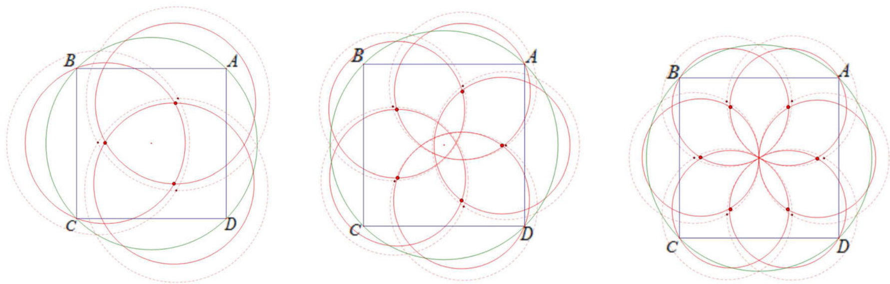
Therefore, $\forall p \in F_k$, and q where l_{op} intersects W_h, we have $\frac{d(p,o)}{d(q,o)} \geq \frac{d(c,o)}{d(A,o)}$, which is shown in Fig. 3iii. This proves the lemma.

□



(i) (ii) (iii)

Fig. 3 Illustration for the proof of Lemma 2



(i) (ii) (iii)

Fig. 4 The shrinking process of the k connected disks. (**i**) $k = 3$, (**ii**) $k = 5$, (**iii**) $k = 6$

As the flower disk is symmetric with respect to the x-axis, we consider the cases in the first quadrant and second quadrant if k is odd. Similar to the lemma above, we have the following lemma.

Lemma 3 *If k ($k \geq 5$) is odd, $\forall p \in F_k$, and point q at which l_{po} intersects W_h, we have $\frac{d(p,o)}{d(q,o)} \geq min\left\{\frac{d(p',o)}{d(A,o)}, \frac{d(p'',o)}{d(B,o)}\right\}$, where p' is the intersection point of F_k and L_{Ao}, p'' is the intersection point of F_k and L_{Bo}, respectively.*

According to Lemmas 2 and 3, we compute F_k and shrink F_k by the ratio $min\left\{\frac{d(p',o)}{d(A,o)}, \frac{d(p'',o)}{d(B,o)}\right\}$ from o. Next we compute the radius. Fortunately, when $k = 3, 4$, F_3 and F_4 also satisfy Lemmas 2 and 3. Figure 4 shows three cases when $k = 3, 5, 6$.

As k has been already fixed, when we shrink F_k, the angle between every two adjacent centers and the original point is fixed. Letting o_m be the center nearest to l_{Ao} or l_{Bo} and its coordinate be (x_m, y_m), we have $min(\angle o_m oA, \angle o_m oB) = \arctan \frac{|y_m - |x_m||}{|x_m| + y_m}$ and $\frac{\sqrt{2}}{2}h = 2r \cos \arctan \frac{|y_m - |x_m||}{|x_m| + y_m}$, i.e., $r = \frac{\sqrt{2}}{4 \cos \arctan \frac{|y_m - |x_m||}{|x_m| + y_m}} h$.

Finally, we shrink the flower disk, satisfying that the flower disk covers the square properly, i.e., at least two of $\{A, B, C, D\}$ are on the boundary of the flower disk. When $k < 5$, we list the radius in Table 3.

Table 3 The radii of the disks covering a square

$k = 3$	$k = 4$	$k = 5$	$k \geq 6$
$r_3 = \frac{3}{\sqrt{3}+\sqrt{5}} h \approx 0.5357h$	$r_4 = \frac{\sqrt{2}}{\sqrt{3}+1} h \approx 0.5176h$	$r_5 \approx 0.4216h$	$r_k = \frac{\sqrt{2}}{4 \cos \arctan \frac{\lvert y_m - \lvert x_m \rvert \rvert}{\lvert x_m \rvert + y_m}} h$

Remark Although we can not give an explicit expression of the radius covering the square with k congruent disks, for a given square W_h, we can take the coverings in which the radius is easy to formulate with h and k according to Lemmas 2 and 3.

3 Connected disk covering a rectangle

In this section, we consider the problem of covering a rectangle $W_{L,H}$ by connected disks with the smallest possible radius. We consider the problem by partitioning the rectangle into Δ $(1 \leq \Delta \leq K)$ rectangles with height h and length L. We partition $W_{L,h}$ into M squares where $L = Mh + L_{remain}$ and use less disks to cover the remaining rectangle $W_{L_{remain},h}$ where $L_{remain} < h$. Without loss of generality, we consider the case where $M = 2$ and $k \geq 5$. Let the two flower disks be F_{k_1} and F_{k_2}, respectively. The problem is considered in two parts: the horizontal connected case and the vertical connected case.

3.1 The horizontal connected case

We paste F_{k_2} to the right of F_{k_1}. We partition the problem into two cases:
Case 1 k is $(k \geq 6)$ even. According to Lemma 1, F_k is symmetric with respect to both the x-axis and the y-axis. Letting the rightmost disk be $D(o_r, r)$ of F_{k_1}, we only need to paste F_{k_2} such that the center of the leftmost disk $D(o_l, r)$ of F_{k_2} must be in $D(o_r, r)$ of F_{k_1}. Due to symmetry, we need to compute the distance δ between o_r and l_{AD}, and we have $\delta = \frac{h}{2} - r$.

If $2\delta \leq r$, we just paste F_{k_2} to the right of F_{k_1} satisfying that o_l of F_{k_2} is in $D(o_r, r)$ of F_{k_1} and vertices $\{A, D\}$ are on the boundary of F_{k_2}.
If $2\delta > r$, we move o_l of F_{k_2} towards o_r of F_{k_1} until o_l is on $C(o_r, r)$ (see Fig. 5i).
Case 2 k $(k > 5)$ is odd. According to Lemma 1, F_k is symmetric with respect to the x-axis. If the center of the rightmost disk is on the x-axis, then there are two leftmost disks symmetric with respect to the x-axis, and the intersection point u (except for o) of the two disks is also on the x-axis. Before shrinking, we have u on $C(o, R)$. Thus, u is the leftmost intersection point. We denote Δ_r as the distance between o_r and L_{AD}, Δ_l as the distance between l_{BC} and u. If k can divide 360° exactly, we have $\Delta_r = \frac{h}{2} - r$, and $\Delta_l = 2r \cos \frac{\pi}{k} - \frac{h}{2}$. Otherwise, we have $\Delta_l = 2r \cos \frac{\theta}{2} - \frac{h}{2}$ where θ is the modified angle in Lemma 1.

If $\Delta_r \leq \Delta_l$, we just paste F_{k_2} to the right of F_{k_1} satisfying that u of F_{k_2} is in $D(o_r, r)$ of F_{k_1} and vertices $\{A, D\}$ are in F_{k_2}.

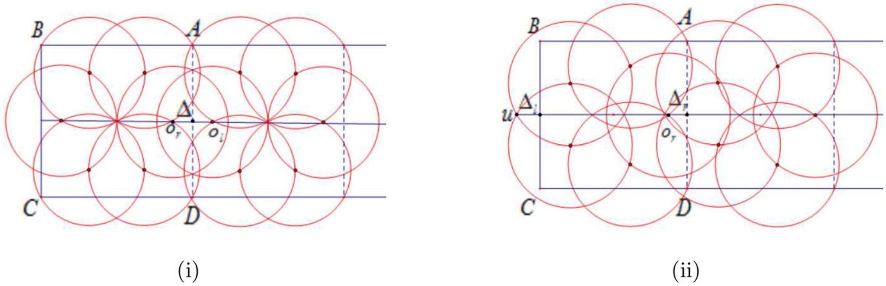


Fig. 5 Examples for the horizontal connected. (**i**) $k = 6$, (**ii**) $k = 5$

If $\Delta_r > \Delta_l$, we move u of F_{k_2} to overlap o_r of F_{k_1} (see Fig. 5ii as an example). By direct computation, when $k = 4$, we have $2\Delta \leq r$; when $k = 5$, we have $\Delta_r \leq \Delta_l$; when $k = 3$, we have $\Delta_r > \Delta_l$.

Now we consider how to cover a rectangle $W_{L,h}$. Without loss of generality, suppose each F_k satisfies that $\Delta_r \leq \Delta_l$. The length of the remaining part of the rectangle is $L_{remain} = L - Mh$. As F_k is symmetric with respect to the x-axis, for any two points u on l_{AB} and v on l_{CD}, we have the following lemma.

Lemma 4 *For a square W_h, where l_{uv} is vertical to l_{AB}, there exist at least two disks covering l_{uv} and the union of these disks is symmetric with respect to the x-axis.*

Proof As each vertex is covered by at least one disk, without loss of generality, we consider the two disks $D(o_i, r)$ and $D(o_j, r)$ cover vertices $\{A, D\}$. Since the radius is larger than $\frac{\sqrt{2}}{4}h$, the intersection point w of the two disks must be on the x-axis. We prove the lemma by discussing the following two cases:

Case 1 $x_w \geq x_u = x_v$. $\forall x_w \geq x_u = x_v$, there exists a point w^* which is the intersection point of two disks symmetric with respect to the x-axis, such that $x_u \leq x_{w^*} \leq x_w$ for a given u ($u > 0$). Therefore, there must exist two disks with intersection point w^* and their union covers l_{uv}. See Fig. 6.

Case 2 $x_w < x_u = x_v$. In this situation, we choose all the disks right to the disks $D(o_i, r)$ and $D(o_j, r)$ (including $D(o_i, r)$ and $D(o_j, r)$). Then, for all l_{uv}, the union of all the disks right to $D(o_i, r)$ and $D(o_j, r)$ can cover l_{uv}. For example in Fig. 4ii, iii, if l_{uv} is very close to l_{AD}, we need all the disks whose centers are in the first quadrant and forth quadrant to cover l_{uv}. □

Lemma 4 shows that one can compute the intersection points of the two disks symmetric with respect to the x-axis for a square W_h. Then, the interval $\left[-\frac{h}{2}, \frac{h}{2}\right]$ can be partitioned into $O(k)$ intervals. For a point u, there exists an interval to which u belongs. Consequently, we choose all the disks whose centers are on the left of l_{uv} with the union of disks covering l_{uv} to cover $W_{L_{remain},h}$. This can be done in $O(\log k)$ time.

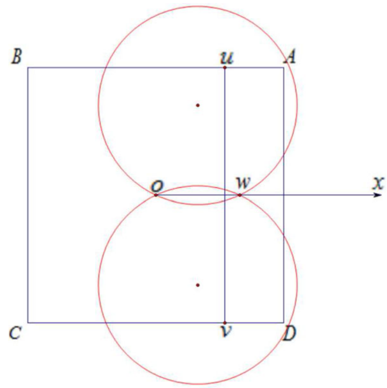


Fig. 6 Illumination for Lemma 4

3.2 The vertical connected case

As F_k is symmetric with respect to the x-axis, there exist two symmetric centers farthest to L_{cl}. Let the two centers be o_{f_1} and o_{f_2}. Let $L_{o_{f_1}o_{f_2}}$ intersect W_h at f_1, f_2, we have $d(o_{f_1}, f_1) = d(o_{f_2}, f_2)$. Without loss of generality, we choose two flower disks F_{k_1} and F_{k_2} and paste F_{k_2} on top of F_{k_1} satisfying that o_{f_2} in F_{k_2} overlaps o_{f_1} in F_{k_1} symmetrically. In particular, when we paste another $F_{k_{i+1}}$ on top of F_{k_i}, some disks may be eliminated since they are redundant. The following lemma is needed.

Lemma 5 *If l is the line parallel to L_{CD} and symmetric with respect to o_{f_2}, then at least one of the disks whose centers are in the region between l and L_{CD} is redundant.*

Proof According to $d(o_{f_1}, f_1) = d(o_{f_2}, f_2)$, we draw a line l' parallel with L_{AB} and symmetric with respect to o_{f_1}. Let S be the region between l' and L_{AB}. When F_{k_2} is pasted on top of F_{k_1} satisfying that o_{f_2} in F_{k_2} overlaps o_{f_1} in F_{k_1}, at least one center of F_{k_2} is in S. If the region covered by the disks of F_{k_2} whose centers are in S can be covered by F_{k_1}, these disks can be seen as redundant. From the assumption, the disk with center o_{f_2} in F_{k_2} must be redundant. Figure 7 shows an example. We consider the case where $k = 5$ and disks with centers o_{f_2} and o'_b are both redundant disks. □

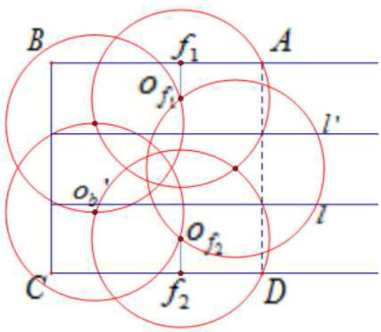


Fig. 7 Illumination for Lemma 5

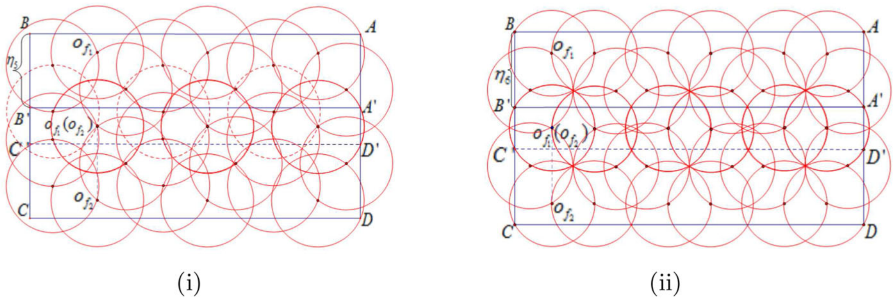


Fig. 8 Examples for the vertical connected case. (**i**) $k = 5$, (**ii**) $k = 6$

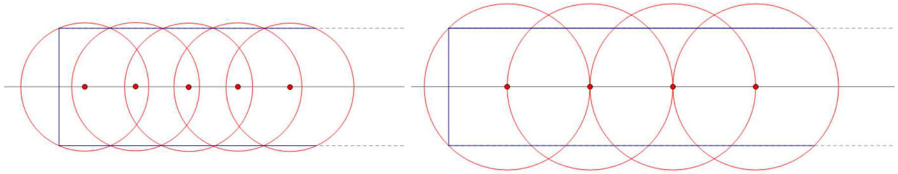


Fig. 9 Examples for the connected disks whose centers are on L_{cl}

Before we paste another $F_{k_{i+1}}$ on top of F_{k_i}, we need to check the redundant disks. For a fixed k, this can be done in $O(k)$ time as a preprocessing. Then we do the vertical connected case. Note that when we paste another $F_{k_{i+1}}$ on top of F_{k_i}, the height has been increased by $\eta_k = d(o_{f_1}, o_{f_2})$. For each $k \geq 3$, η_k can be computed directly. Thus, $W_{L,H}$ has been divided into Δ rectangles with length L where $\Delta = 1 + \frac{H-h}{\eta_k}$. Figure 8 shows two examples for $k = 5$ and $k = 6$. The dash circles are the boundary of redundant disks. The rectangle has been partitioned into two rectangles with vertices $\{A, B, C', D'\}$ and $\{A', B', C, D\}$.

Hence we can compute the radius r_k of the connected disks. For a fixed k ($3 \leq k < K$) and rectangle $W_{L,H}$, go through Δ from 1 to K. By direct computation, we can obtain η_k, $r_{k,\Delta}$ and K_Δ where $r_{k,\Delta}$ is the radius of the connected disks covering square W_h and K_Δ is the number of the disks covering the whole rectangle $W_{L,H}$. Then, one can obtain an increasing order as $K_1 < K_2 < K_3 < \cdots < K_K$ with respect to Δ. By binary search, one can obtain an interval $K \in [K_i, K_{i+1})$. Therefore, we can obtain the radius r_k corresponding to the covering number K_i for the fixed k in $O(\log K)$ time.

3.3 The case for $k = 2$

In the previous subsections, we consider the case where $k \geq 3$. In this section, we consider the case where $k = 2$. Different from the cases with $k \geq 3$, we consider the case that the centers are on the central line L_{cl} for each rectangle $W_{L,h}$ (see Fig. 9). For a fixed rectangle $W_{L,h}$, we need the following lemma. (In the following text, k denotes the number of connected disks covering a rectangle $W_{L,h}$.)

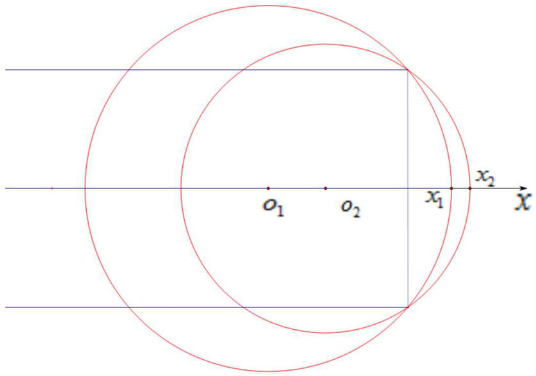


Fig. 10 Illumination for Lemma 6

Lemma 6 *If the center of each disk is on the boundary of the other disk, letting $k_1 < k_2$ and the corresponding radii be r_1, r_2, we have $(k_1 + 1)r_1 < (k_2 + 1)r_2$.*

Proof It is clear that $r_1 > r_2$. From the assumption, as $W_{L,h}$ is fixed and k_1 (resp. k_2) disks cover the rectangle, letting the rightmost disk be $D(o_1, r_1)$ (resp. $D(o_2, r_2)$), L_{cl} of the rectangle be the x-axis, the coordinate of the intersection point of the x-axis and $C(o_1, r_1)$ (resp. $C(o_2, r_2)$) be $(x_1, 0)$ (resp. $(x_2, 0)$). Since o_1 and o_2 are both on the x-axis and $r_1 > r_2$, we have $x_1 < x_2$, as shown in Fig. 10. Thus, due to symmetry, we have $(k_1 + 1)r_1 < (k_2 + 1)r_2$. □

Hence we know that for k disks with radius r covering $W_{L,h}$ whose centers are on L_{cl} and each center is on the boundary of the other disk, we have $r \geq \frac{h}{\sqrt{3}}$ and $(k + 1)r \leq L + (\frac{h}{\sqrt{3}})$. Similarly, if each center is in the interior of the other disk, we have $\frac{h}{2} < r < \frac{h}{\sqrt{3}}$ and $(k + 1)r > L + (\frac{h}{\sqrt{3}})$. One can thus compute the upper bound κ of the disks whose centers are on L_{cl} and each center is on the boundary of the other disk. If $K > \kappa$, the center of each disk is in the interior of the other disk. Then we have the following lemma.

Lemma 7 *For a rectangle $W_{L,h}$, if k disks cover $W_{L,h}$ whose centers are on L_{cl} and the radius r is larger than or equal to $\frac{h}{\sqrt{3}}$, then $\forall h' < h$ and $k' < k$, the radius r' (of k' disks covering $W_{L,h'}$ with centers on L_{cl}) is still larger than $\frac{h'}{\sqrt{3}}$; if the radius r is less than $\frac{h}{\sqrt{3}}$, then $\forall h' > h$ and $k' > k$, the radius r' (of k' disks covering $W_{L,h'}$ with centers on L_{cl}) is still less than $\frac{h'}{\sqrt{3}}$.*

Proof We consider the case with $r \geq \frac{h}{\sqrt{3}}$. The proof for the case $r < \frac{h}{\sqrt{3}}$ is symmetric. Assume $r' \leq \frac{h'}{\sqrt{3}}$, we have $L + \frac{h'}{\sqrt{3}} \leq (k' + 1)r'$, i.e., $\frac{\sqrt{3}L}{h'} \leq k'$. On the other hand, as $r \geq \frac{h}{\sqrt{3}}$, $L + \frac{h}{\sqrt{3}} \geq (k + 1)r$, i.e., $\frac{\sqrt{3}L}{h} \geq k$. Hence $\forall h' < h$ and $k' < k$, we have $\frac{\sqrt{3}L}{h'} > \frac{\sqrt{3}L}{h} \geq k > k'$, which comes to a contradiction. □

Suppose the rectangle $W_{L,H}$ is partitioned into Δ $(1 \leq \Delta \leq K)$ rectangles with height $h = \frac{2H}{\Delta+1}$, there must be $\lfloor \frac{K}{\Delta} \rfloor$ disks covering each rectangle. From Lemma 7, there must exist a Δ^* whose corresponding radius r^* satisfies that either $r^* \leq \frac{2H}{\sqrt{3}(\Delta^*+1)}$ and a radius $r^{*'}$ satisfies that $r^{*'} > \frac{2H}{\sqrt{3}(\Delta^*+2)}$ for $\Delta^{*'} = \Delta^* + 1$, or for $\Delta^* = 1$, $r^* > \frac{2H}{\sqrt{3}(\Delta^*+1)} = \frac{H}{\sqrt{3}}$. Then we choose r^* as the radius r_2.

4 The algorithm of solving *CDCP*

4.1 The algorithm

Now we design the algorithm for *CDCP*. To begin with, we compute the rectangle covering P (Du and Xu 2014). From Lemma 3 in Du and Xu (2014), the author considered two ways to compute the proper rectangle covering a convex polygon. By using their method, one can obtain a proper rectangle covering P in $O(n)$ time. For a fixed k ($3 \leq k < K$) and $1 \leq \Delta \leq K$, we partition the rectangle $W_{L,H}$ into Δ rectangles and obtain $W_{L,h}$. For fixed k, Δ and H, h and η_k can be computed directly. Then we partition the rectangle $W_{L,h}$ into M squares and remainders. For each square, we compute k congruent disks covering the square W_h and consider horizontal connected case. Next, we consider vertical connected case and obtain the increasing order $\{K_1, K_2, K_3, \ldots, K_K\}$ with respect to Δ. Therefore, we obtain the interval $K \in [K_i, K_{i+1})$ and the radius r_k corresponding to K_i. By iterating k from 3 to K, we choose the minimum radius $r_{min} = min\{r_3, r_4, \ldots, r_K\}$. On the other hand, we consider the case with $k = 2$. According to Sect. 3.3, we compute r_2 and then choose the minimum one between r_2 and r_{min} as the output radius of the algorithm. Finally, we move the centers out of the convex polygon into it. From the fourth step of the algorithm in Du and Xu (2014), this can be done in $O(Kn)$ time. We simply show the algorithm below.

Algorithm 1 The Algorithm Solving $CDCP$

1: Compute the rectangle $W_{L,H}$ covering P.
2: **for** $k = 3$ to K **do**
3:     **for** $\Delta = 1$ to K **do**
4:         Compute the number of the covering disks K_Δ and the radius r_Δ.
5:     **end for** Δ
6:     $r_k \leftarrow r_\Delta$ where $K \in [K_\Delta, K_{\Delta+1})$.
7: **end for** k
8: $r_{min} \leftarrow min\{r_k\}$.
9: **for** $k = 2$ **do**
10:     Compute r_2.
11: **end for**
12: $r \leftarrow min\{r_2, r_{min}\}$.
13: Move the centers into the polygon.

4.2 Time complexity

Step 1 takes $O(n)$ time. By binary search, Step 3 to Step 5 take $O(\log K)$ time. Step 2 to Step 8 take $O(K \log K)$ time. From Lemma 7, Step 9 to Step 11 take $O(\log K)$ time.

Step 12 takes $O(1)$ time. Step 13 takes $O(Kn)$ time. Thus, the total time complexity is $O(Kn)$.

4.3 The approximation factor

In this section, we analyze the approximation factor in the worst case with $k = 2$. In the following, let $k = \lfloor \frac{K}{\Delta} \rfloor$ and Δ instead of Δ^* ($\Delta \geq 1$), which provides the following:

$$(k+1)r \geq L + \frac{h}{\sqrt{3}}, r \leq \frac{h}{\sqrt{3}}.$$

When $\Delta' = \Delta + 1$, $h' = \frac{2H}{\Delta+2}$, $k' = \lfloor \frac{K}{\Delta+1} \rfloor$, we have:

$$(k'+1)r' < L + \frac{h'}{\sqrt{3}}, r' > \frac{h'}{\sqrt{3}}.$$

Then,

$$\frac{L + \frac{h}{\sqrt{3}}}{(k+1)} \leq \frac{h}{\sqrt{3}}, \qquad \frac{h'}{\sqrt{3}} < \frac{L + \frac{h'}{\sqrt{3}}}{(k'+1)}.$$

According to $\frac{K-\Delta}{\Delta+1} \leq \lfloor \frac{K}{\Delta+1} \rfloor \leq \frac{K}{\Delta+1}$, we have:

$$\lfloor \frac{K}{\Delta} \rfloor \geq \frac{\sqrt{3}L}{2H}(\Delta+1), \quad \frac{\sqrt{3}L}{2H}\Delta(\Delta+1) \leq K < \frac{\sqrt{3}L}{2H}(\Delta+1)(\Delta+2) + \Delta.$$

If each center is in the interior of the other disk, then the radius can be written as the following:

$$r = \sqrt{\frac{L^2}{4k^2} + \frac{h^2}{4}} = \sqrt{\frac{L^2}{4\lfloor \frac{K}{\Delta} \rfloor^2} + \frac{H^2}{(\Delta+1)^2}}.$$

Moreover, we can compute the lower bound of the radius of the optimal solution of *CDCP*. From Du and Xu (2014), the area of the union of the optimal disks is larger than $\frac{LH}{2}$. Let the optimal radius be r_{opt}, then $\frac{2}{3}(K-1)\pi r_{opt}^2 + \pi r_{opt}^2 > \frac{LH}{2}$, i.e., $r_{opt} > \sqrt{\frac{3HL}{2\pi(2K+1)}}$. Since $r_{opt} \geq \frac{L}{K+1}$, a lower bound of r_{opt} can be obtained as $max\left\{\sqrt{\frac{3HL}{2\pi(2K+1)}}, \frac{L}{K+1}\right\}$. Now we compute the approximation ratio of the algorithm.

Case 1 $\sqrt{\frac{3HL}{2\pi(2K+1)}} \geq \frac{L}{K+1}$. As $\Delta \geq 1$, $K \geq 2$ and $\frac{L+\frac{h}{\sqrt{3}}}{(k+1)} \leq \frac{h}{\sqrt{3}}, \forall x > \sqrt{3}$, it satisfies that either $\frac{h}{x} < \frac{L+\frac{h}{\sqrt{3}}}{(k+1)} \leq \frac{h}{\sqrt{3}}$, or $\frac{L+\frac{h}{\sqrt{3}}}{(k+1)} < \frac{h}{x}$ and $\frac{h'}{\sqrt{3}} < \frac{L+\frac{h'}{\sqrt{3}}}{(k'+1)}$.

If $\frac{h}{x} < \frac{L+\frac{h}{\sqrt{3}}}{(k+1)}$, we have:

$$\frac{\sqrt{3}L}{2H}\Delta(\Delta+1) \leq K < \frac{xL}{2H}\Delta(\Delta+1) + \frac{x\Delta}{\sqrt{3}} - 1.$$

Then,

$$\begin{aligned}\frac{r}{r_{opt}} &< \sqrt{\frac{L^2}{4\lfloor\frac{K}{\Delta}\rfloor^2}+\frac{H^2}{(\Delta+1)^2}}\Big/\sqrt{\frac{3HL}{2\pi(2K+1)}} \le \sqrt{\frac{H^2}{3(\Delta+1)^2}+\frac{H^2}{(\Delta+1)^2}}\sqrt{\frac{2\pi(2K+1)}{3HL}}\\ &< \frac{2\sqrt{2\pi}}{3}\sqrt{\frac{H}{L}}\sqrt{\frac{xL}{H}\frac{\Delta(\Delta+1)}{(\Delta+1)^2}+\frac{2x}{\sqrt{3}}\frac{\Delta}{(\Delta+1)^2}-\frac{1}{(\Delta+1)^2}}\\ &< \frac{2\sqrt{2\pi}}{3}\sqrt{\frac{x\Delta}{\Delta+1}+\frac{2x}{\sqrt{3}}\frac{\Delta}{(\Delta+1)^2}-\frac{1}{(\Delta+1)^2}} = f_1(x,\Delta).\end{aligned}$$

If $\frac{L+\frac{h}{\sqrt{3}}}{(k+1)} \le \frac{h}{x}$, we have:

$$\left\lfloor\frac{K}{\Delta}\right\rfloor \ge \frac{xL(\Delta+1)}{2H}+\frac{x}{\sqrt{3}}-1 > \frac{xL(\Delta+1)}{2H}.$$

Then,

$$\begin{aligned}\frac{r}{r_{opt}} &< \sqrt{\frac{L^2}{4\lfloor\frac{K}{\Delta}\rfloor^2}+\frac{H^2}{(\Delta+1)^2}}\Big/\sqrt{\frac{3HL}{2\pi(2K+1)}} < \sqrt{\frac{H^2}{x^2(\Delta+1)^2}+\frac{H^2}{(\Delta+1)^2}}\sqrt{\frac{2\pi(2K+1)}{3HL}}\\ &< \sqrt{\frac{2\pi}{3}}\frac{\sqrt{x^2+1}}{x}\sqrt{\frac{H}{L}}\sqrt{\frac{\frac{\sqrt{3}L}{H}(\Delta+1)(\Delta+2)+2\Delta+1}{(\Delta+1)^2}}\\ &< \sqrt{\frac{2\pi}{3}}\frac{\sqrt{x^2+1}}{x}\sqrt{\frac{\sqrt{3}(\Delta+2)}{\Delta+1}+\frac{2\Delta+1}{(\Delta+1)^2}} = f_2(x,\Delta).\end{aligned}$$

For the fixed x, one can obtain $max\{sup(f_1(x,\Delta)), sup(f_2(x,\Delta))\}$ for $\Delta \ge 1$. Therefore, we just need to minimize $max\{sup(f_1(x,\Delta)), sup(f_2(x,\Delta))\}$ for $x \ge \sqrt{3}$. By numerical calculations, we have $minmax\{sup(f_1(x,\Delta)), sup(f_2(x,\Delta))\} = 2.8100$ when $x = 2.8160$ (see Fig. 11).

Case 2 $\sqrt{\frac{3HL}{2\pi(2K+1)}} < \frac{L}{K+1}$, i.e., $K+1 < \frac{4\pi L}{3H}$, then we have

$$\frac{\sqrt{3}L}{2H}\Delta(\Delta+1) \le K < \frac{4\pi L}{3H}-1.$$

By direct computation, we have $\Delta = 1$. Thus,

$$\begin{aligned}\frac{r}{r_{opt}} &< \sqrt{\frac{L^2}{4\left\lfloor\frac{K}{\Delta}\right\rfloor^2}+\frac{H^2}{(\Delta+1)^2}}\Big/\frac{L}{K+1} = \frac{1}{2}\sqrt{\frac{(K+1)^2}{K^2}+(K+1)^2\frac{H^2}{L^2}}\\ &< \frac{1}{2}\sqrt{\frac{9}{4}+\frac{16\pi^2}{9}} < 2.2246.\end{aligned}$$

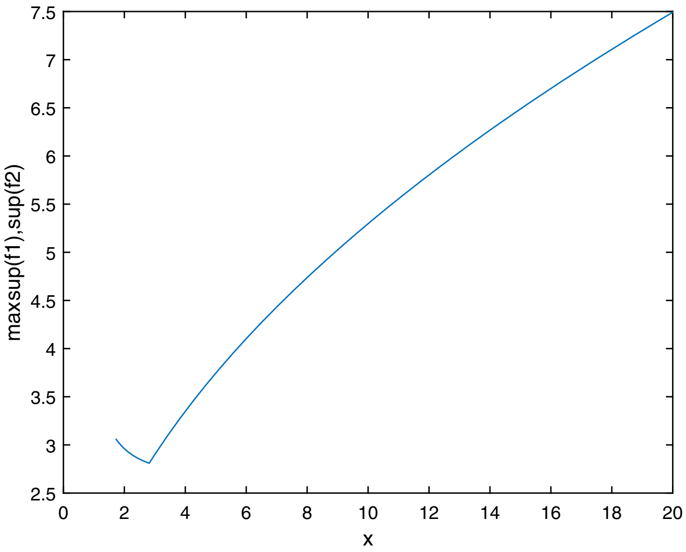


Fig. 11 Numerical result

However, for K disks and $W_{L,H}$, if each center is on the boundary of the other disk, then the radius satisfies that:

$$r \leq \frac{L + \frac{H}{\sqrt{3}}}{K+1}.$$

If $\sqrt{\frac{3HL}{2\pi(2K+1)}} \leq \frac{L}{K+1}$, we have:

$$\frac{r}{r_{opt}} \leq \frac{\frac{L+\frac{H}{\sqrt{3}}}{K+1}}{\frac{L}{K+1}} = 1 + \frac{\sqrt{3}H}{3L} < 1 + \frac{\sqrt{3}}{3} < 1.5774.$$

If $\frac{L}{K+1} < \sqrt{\frac{3HL}{2\pi(2K+1)}}$, we have:

$$\frac{2\pi L}{3H} < \frac{(K+1)^2}{2K+1} < \frac{(K+1)^2}{2K} = \frac{1}{2}(K+1)(1+\frac{1}{K}) < \frac{3}{4}(K+1).$$

i.e., $K+1 > \frac{8\pi L}{9H}$. Hence we obtain:

$$\frac{r}{r_{opt}} \leq \frac{\frac{L+\frac{H}{\sqrt{3}}}{K+1}}{\sqrt{\frac{3HL}{2\pi(2K+1)}}} = \sqrt{\frac{2\pi}{3}}\frac{(L+\frac{H}{\sqrt{3}})}{\sqrt{HL}}\frac{\sqrt{2K+1}}{K+1} < 2\sqrt{\frac{\pi}{3}}\frac{1}{\sqrt{K+1}}\left(\sqrt{\frac{L}{H}}+\sqrt{\frac{H}{3L}}\right)$$
$$< 2\sqrt{\frac{\pi}{3}}\left(\frac{3}{2\sqrt{2\pi}} + \frac{\sqrt{3}}{2\sqrt{2\pi}}\right) < 1.9320.$$

Theorem 1 *Algorithm* 1 *finds* K *connected disks covering* P *with factor* 2.81-*approximation in* $O(Kn)$ *time.*

5 Conclusion

In this paper, we consider a variation of the K-center problem, called *the connected disk covering problem*, which is to find K congruent disks centered in P covering P with the smallest possible radius, while a connected graph is generated by the centers of the K disks whose edge length cannot exceed the radius. We give a 2.81-approximation algorithm in $O(Kn)$ time. In future work, we hope to develop better approximation algorithms.

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