qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \mathbb{C}^2$$

. . . n qubits

$|\psi\rangle$ $|\varphi\rangle$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \otimes \begin{bmatrix}\gamma \\ \delta\end{bmatrix} = \begin{bmatrix}\alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta\end{bmatrix}$$

Visualize the state of a single qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

normalization cond $|\alpha|^2 + |\beta|^2 = 1$

What is the impact of gates on this state?

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha, \beta \in \mathbb{C}$$

$$|\psi\rangle = r_0 e^{i\gamma_0}|0\rangle + r_1 e^{i\gamma_1}|1\rangle$$

Apply normalization $|\alpha|^2 + |\beta|^2 = 1$

$$|r_0 e^{i\gamma_0}|^2 = r_0^2$$

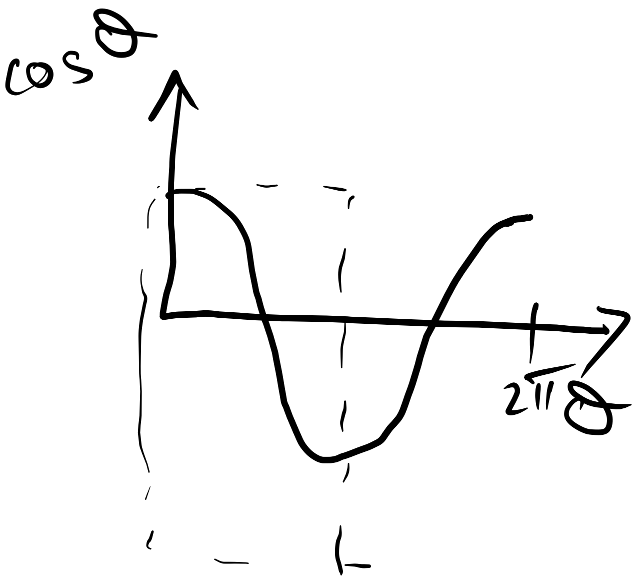
$$r_0^2 + r_1^2 = 1 \qquad \text{2 parameter.}$$

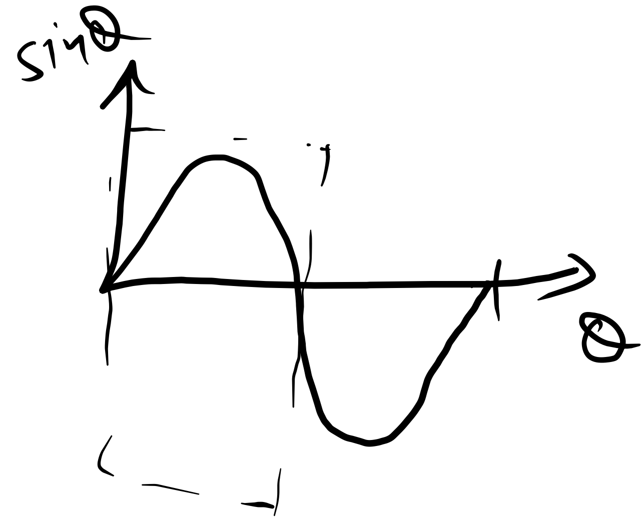
$$\cos^2\bar{\theta} + \sin^2\bar{\theta} = 1 \qquad \text{1 parameter}$$

For data $\cos^2\theta/2 + \sin^2\theta/2 = 1 \qquad 0 \leq \theta < \pi$

$$|\psi\rangle = \cos\frac{\theta}{2} e^{i\gamma_0}|0\rangle + \sin\frac{\theta}{2} e^{i\gamma_1}|1\rangle$$

$$0 \le \theta < \pi$$





$$|\psi\rangle = e^{i\gamma_0}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i(\gamma_1-\gamma_0)}|1\rangle\right)$$

$$= e^{i\delta}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle\right)$$

notebook notation

$$\mathrm{Prob}(0) = \left|e^{i\delta}\cos\frac{\theta}{2}\right|^2 = \cos^2\theta/2$$

$$\mathrm{Prob}(1) = \left|e^{i\delta}\sin\frac{\theta}{2}e^{i\varphi}\right|^2 = \sin^2\frac{\theta}{2}$$

$$|\psi\rangle \;—\!\!—\!\!\boxed{G}\!\!—\!\!—\; |\psi'\rangle \,?$$

$$|\psi\rangle = e^{i\delta}\begin{bmatrix}\cos\theta/2 \\ \sin\theta/2\, e^{i\varphi}\end{bmatrix} \xrightarrow{G} G|\psi\rangle = e^{i\delta}\begin{bmatrix}\text{changes} \\ \text{changes}\end{bmatrix}$$

Measure $G|\psi\rangle$

the $e^{i\delta}$ factor has no impact

$e^{i\delta}$ is the global phase and can be discarded.

So

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\phi}|1\rangle$$

$0 \leq \theta < \pi$

$0 \leq \phi < 2\pi$

$e^{i\phi}$ is the local phase.

θ Visualization $\quad 0 \le \frac{\theta}{2} < \frac{\pi}{2}$

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow \frac{\theta}{2} = 45^\circ$$

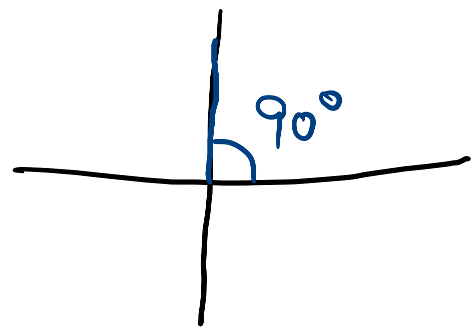
ϕ visualization

$$|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$\frac{\theta}{2} = 45^\circ$$

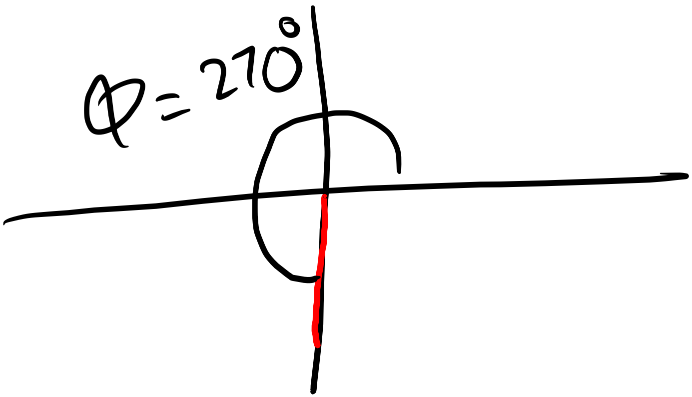
$$e^{i\pi/2} = i$$

$$\phi = 90^\circ$$



$$|\psi\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}$$

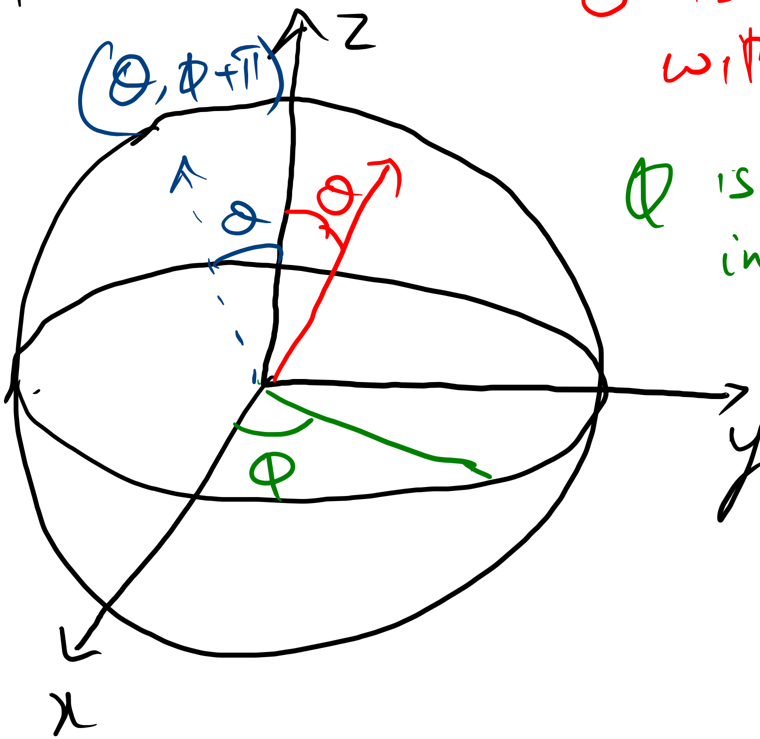
$\phi = 270°$



Bloch sphere visualization

$0 \leq \theta < \pi$

$0 < \phi < 2\pi$

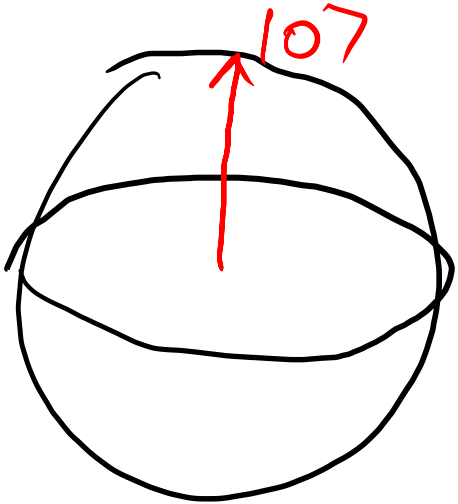


θ is the angle with the z-axis

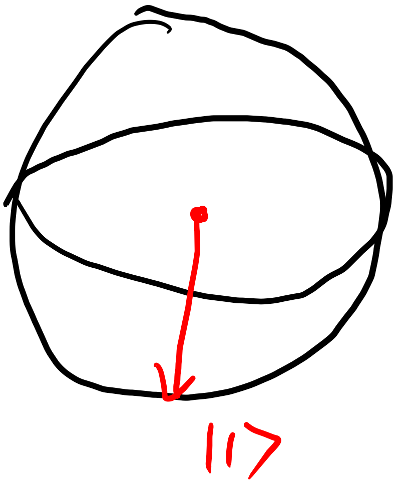
ϕ is the angle in the x-y plane wrt the x-axis

$$|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$|\psi(0,0)\rangle = |0\rangle$$



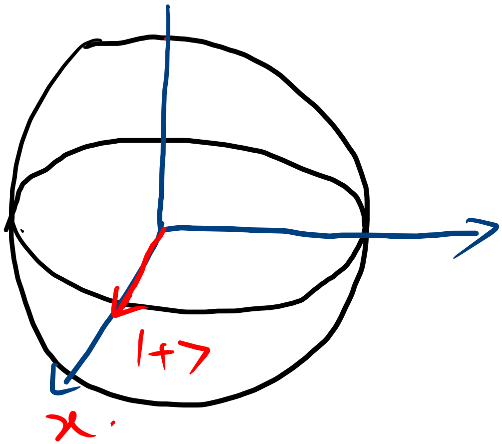
$$|\psi(\pi,0)\rangle = |1\rangle$$

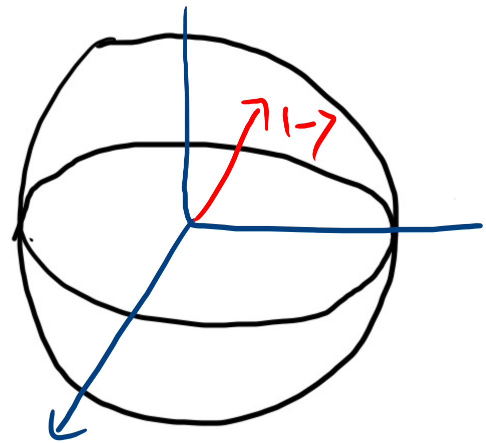


$$|\psi(\tfrac{\pi}{2}, 0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

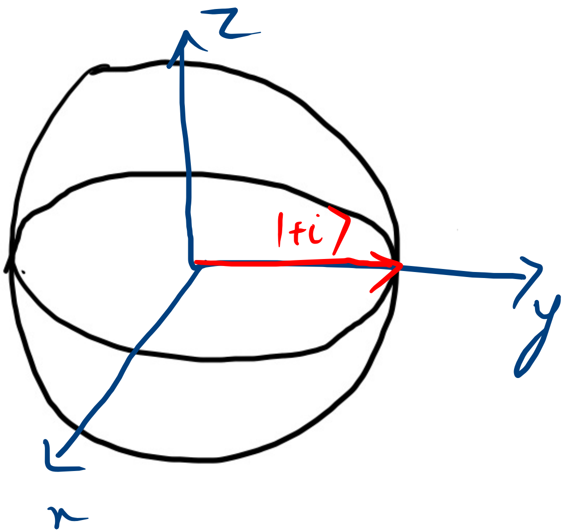
$$e^{i\pi} = -1$$

$$|\psi(\pi/2, \pi)\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

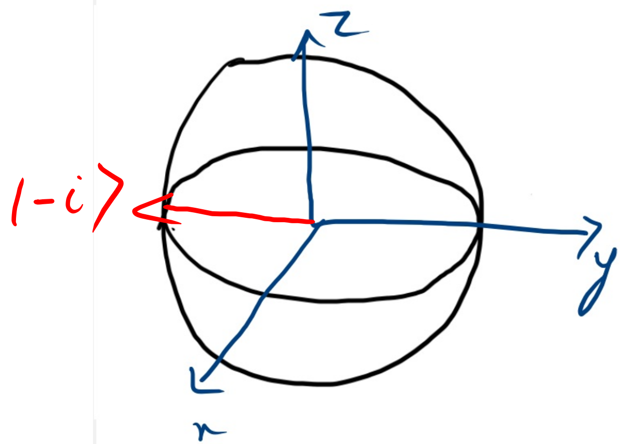




$$|\psi(\tfrac{\pi}{2}, \tfrac{\pi}{2})\rangle = \frac{|0\rangle + i|1\rangle}{2} = |i\rangle$$



$$|\psi(\pi/2, \tfrac{3\pi}{2})\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$
$$= |-i\rangle$$



Impact of gates on the visualization

Any one qubit gate is of the form

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos\theta/2 & -e^{i\lambda}\sin\theta/2 \\ e^{i\phi}\sin\theta/2 & e^{i(\lambda+\phi)}\cos\theta/2 \end{pmatrix}$$

$0 < \theta \leq \pi$ $\qquad\qquad$ $0 \leq \lambda, \phi < 2\pi$

$$U(0, 0, \pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

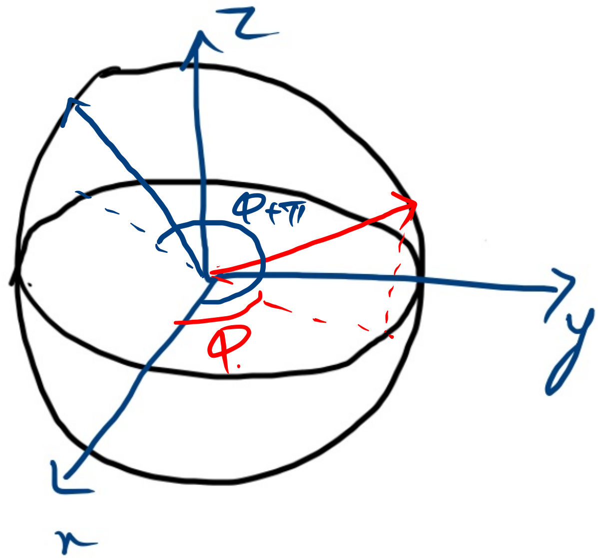
$$Z\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right)$$

$$= \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$= \cos\frac{\theta}{2}|0\rangle + e^{i\pi}\sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i(\phi+\pi)}|1\rangle$$

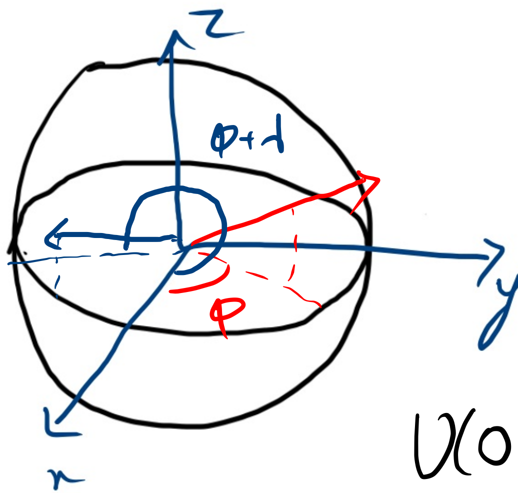
$$Z|\psi(\theta,\phi)\rangle = |\psi(\theta,\phi+\pi)\rangle$$



Z rotates the state by π degrees for the ϕ angle

$$U(0,0,\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{bmatrix}$$

$$|\psi\rangle \xrightarrow{U(0,0,\lambda)} \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i(\phi+\lambda)}|1\rangle$$



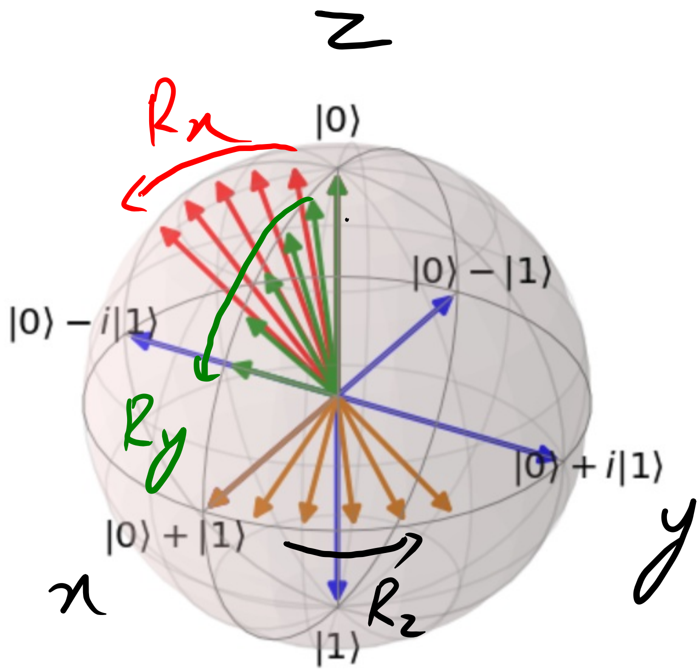
Rotating around the z-axis in anti-clockwise direction

$$U(0,0,\lambda) = R_z(\lambda)$$

Rotations around x-axis

$$R_x(\theta) = U(\theta, -\pi/2, \pi/2)$$

$$R_y(\theta) = U(\theta, 0, 0)$$


z
R_x
$|0\rangle$
$|0\rangle - |1\rangle$
$|0\rangle - i|1\rangle$
R_y
$|0\rangle + i|1\rangle$
$|0\rangle + |1\rangle$
y
x
R_z
$|1\rangle$

$$i = e^{i\pi/2}$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}.$$

sdg

"S-dagger"

$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \text{ and } T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}.$$

Swap gate =