Second-order solutions in the sequence space

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- We argue: actually **no**
  - Perfect foresight is **gateway** to full aggregate risk solution
  - Unless shocks are huge, nonlinearities tend to be modest

Plan

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1. Aggregate risk solution in the sequence space
2. First and second order perturbations
3. Implementation in canonical HANK model

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- **Q:** How are (1) and (2) related?

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- What is the generalized IRF for a linear MA?

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- Linear in ν, not dependent of past ϵ's, and equal to z_t for size-1 shock!

Perturbation in sequence space

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- X is steady state value, and $\frac{\partial \mathcal{X}}{\partial \epsilon_{-k}}\left(\mathbf{0}\right)$ is the (deterministic) effect on X_k from a small shock at date 0, with all other shocks turned off ($\sigma = 0, \boldsymbol{\epsilon} = \mathbf{0}$)

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- Per result above, we have $\dfrac{\partial \mathcal{X}}{\partial \epsilon_{-j}} = x_j$. Let's rearrange…

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- $X + \frac{1}{2}\mathcal{X}_{\sigma\sigma}\sigma^2 + \frac{1}{2}\sum_{j=0}^{\infty}\frac{\partial^2 \mathcal{X}}{\partial \epsilon_{-j}^2}\sigma^2$: **ergodic mean**

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- For the steady state, we'll also need to get $\dfrac{1}{2}\mathcal{X}_{\sigma\sigma}\sigma^2$

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- For the IRF, we need the terms $\dfrac{\partial^2 \mathcal{X}}{\partial \epsilon_{-t}^2}$ and $\dfrac{\partial^2 \mathcal{X}}{\partial \epsilon_{-t}\partial \epsilon_{-(t+k)}}$

That's "it"!

- For the steady state, we'll also need to get $\frac{1}{2}\mathcal{X}_{\sigma\sigma}\sigma^2$

Implementation: canonical HANK model

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Nonlinear error from first-order impulse

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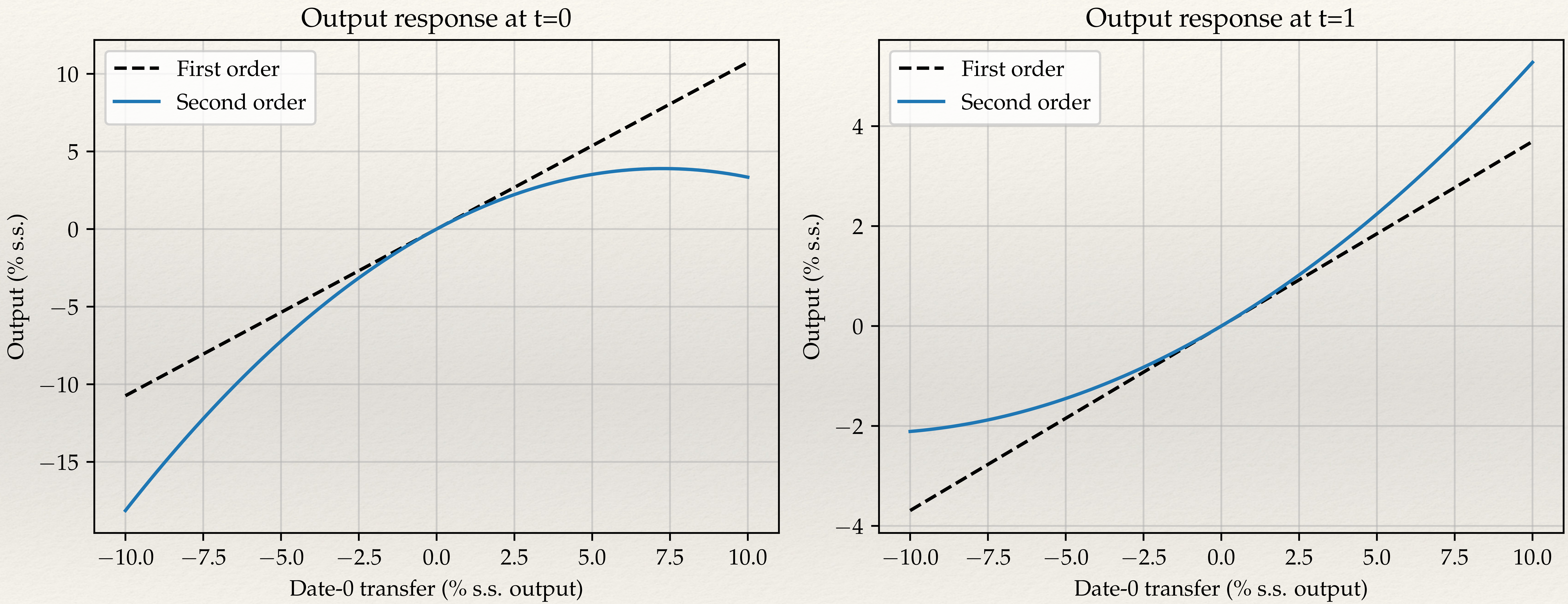
- To get size dep. impulse, compute $1/\nu^2\mathbf{A}_{\mathbf{Y}}^{-1} \left(\mathbf{A} \left(Z + \nu\mathbf{A}_{\mathbf{Y}}^{-1}\mathbf{b} \right) - B - \nu\mathbf{b} \right)$

Size dependence impulse response visualized



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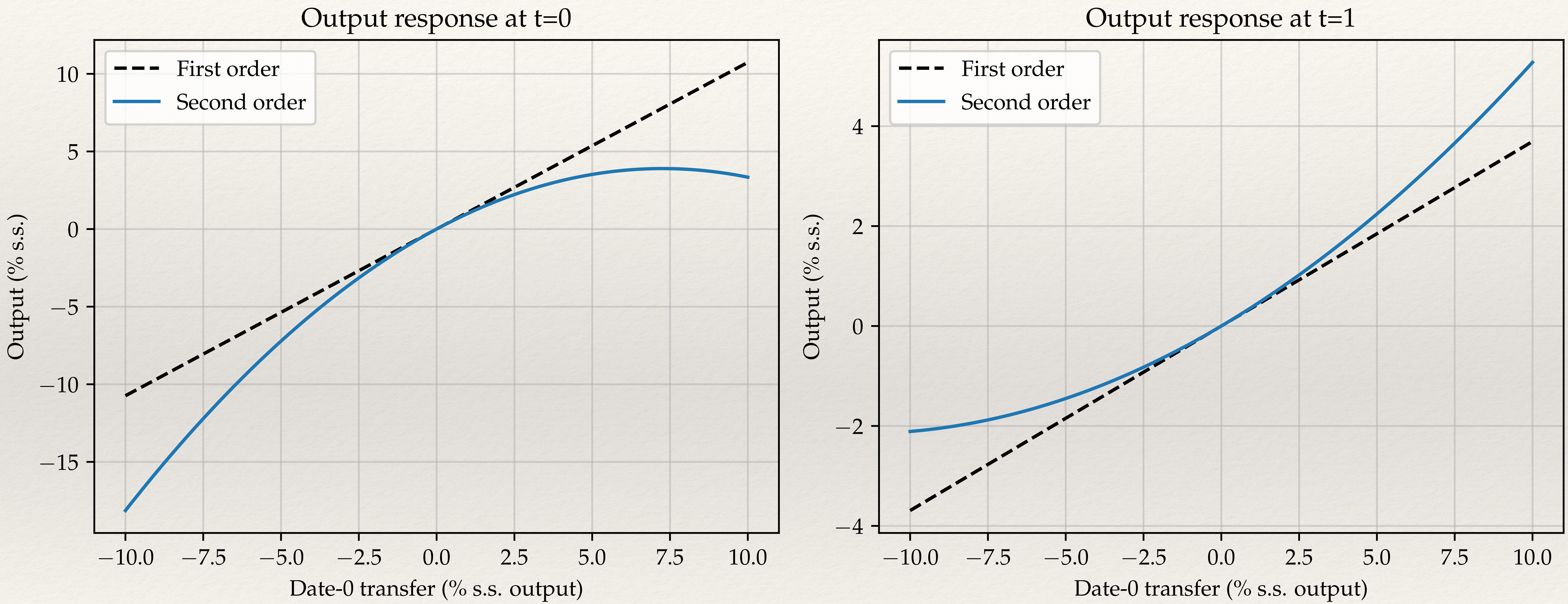

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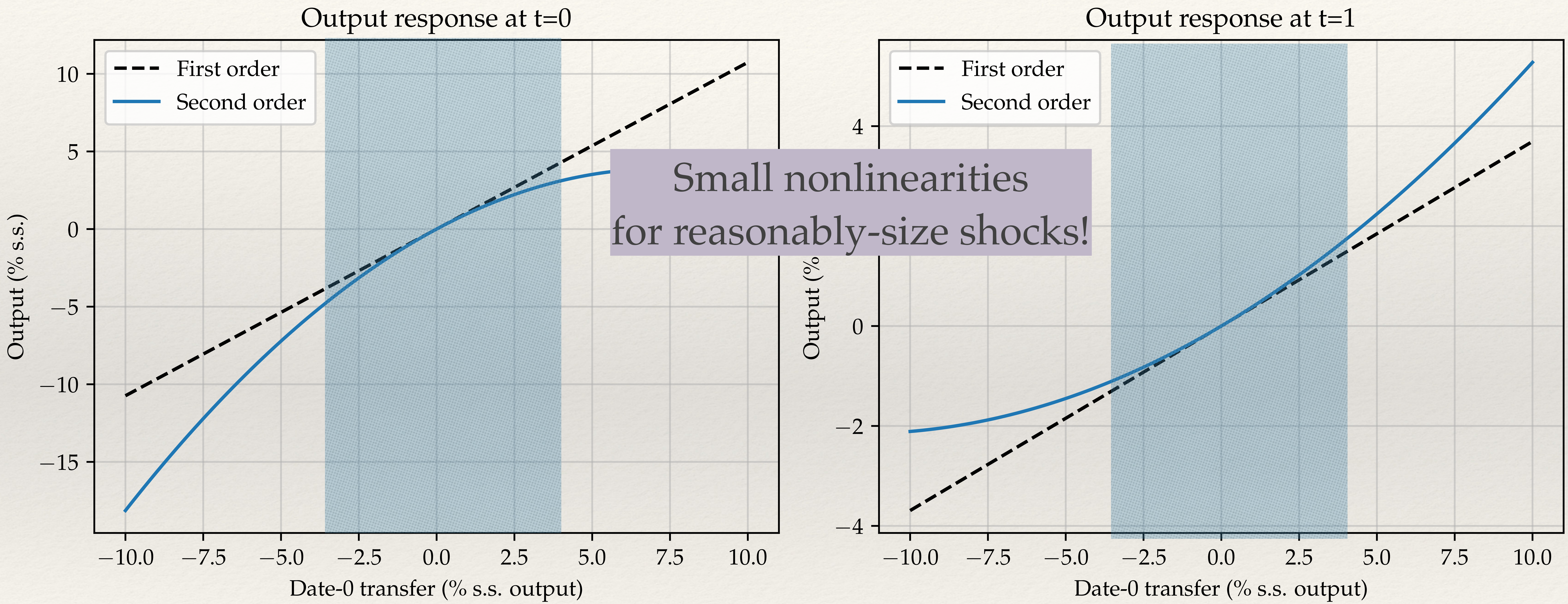

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- Note that we need 2K impulse responses for history terms up to K

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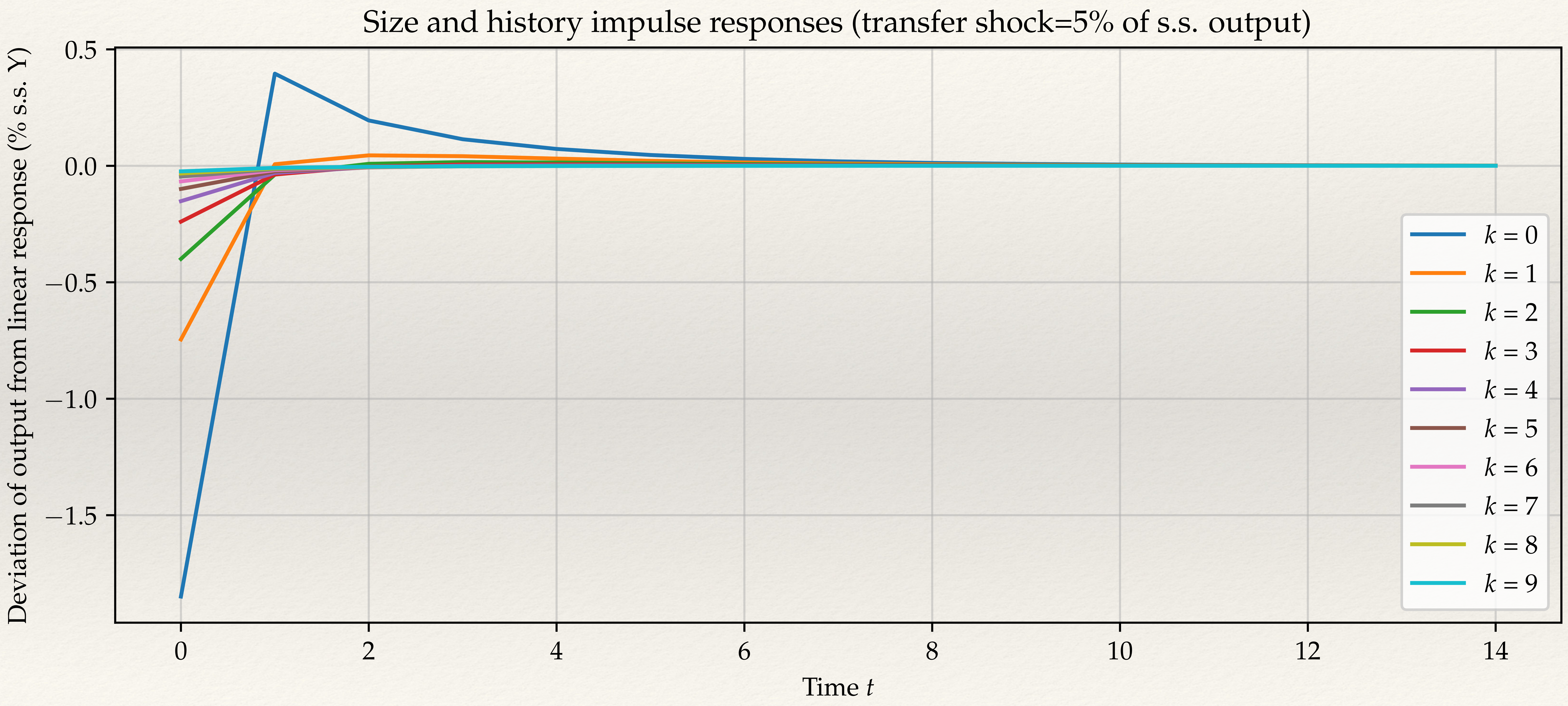

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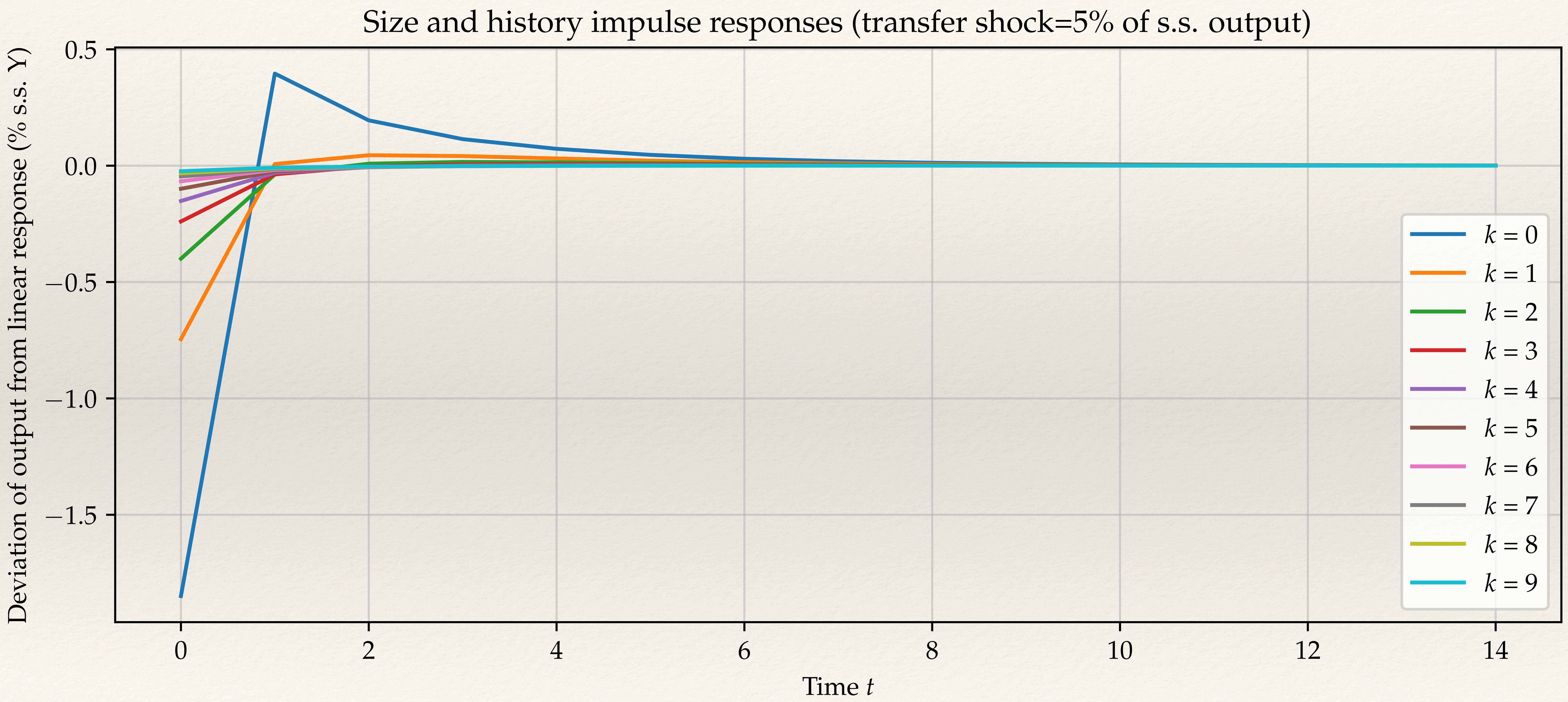

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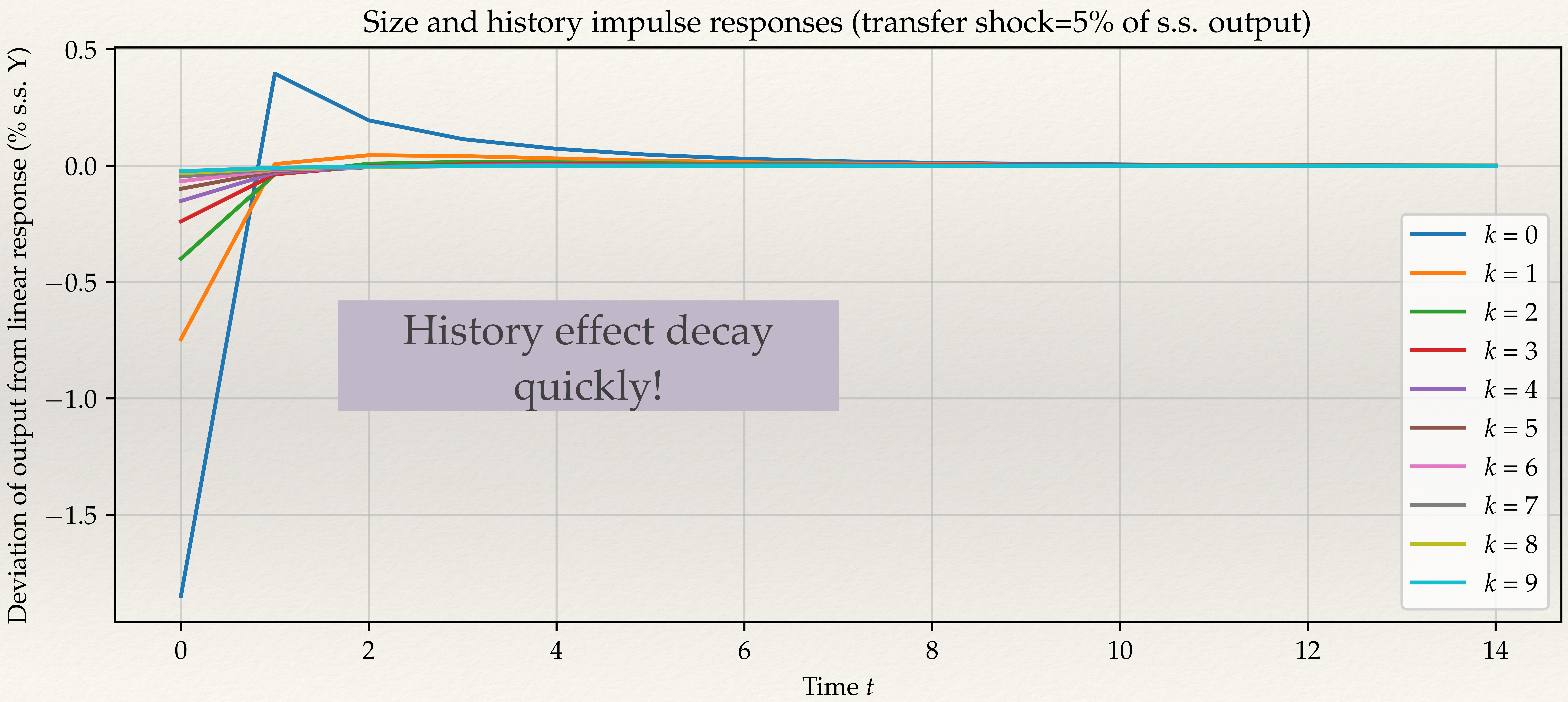

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  - use s.s. routines to get effect of this shift on equilibrating variables

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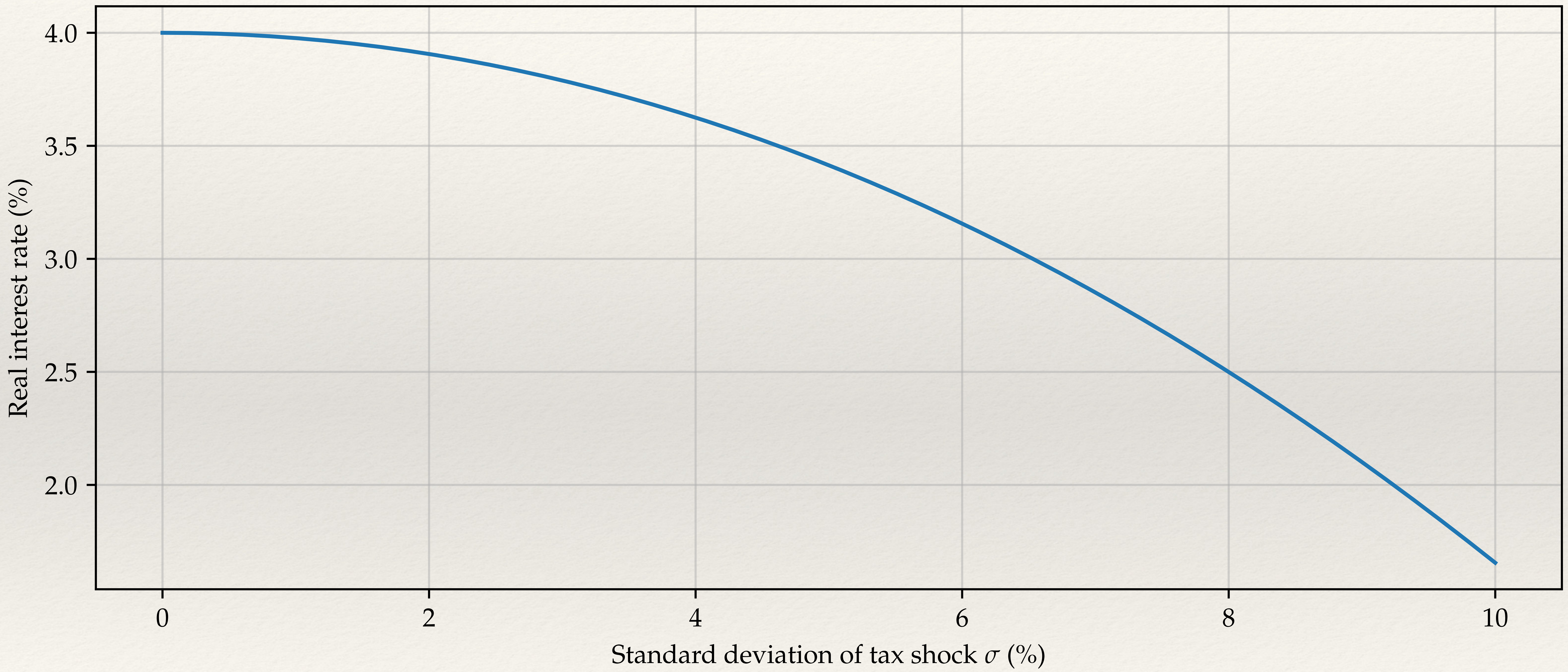

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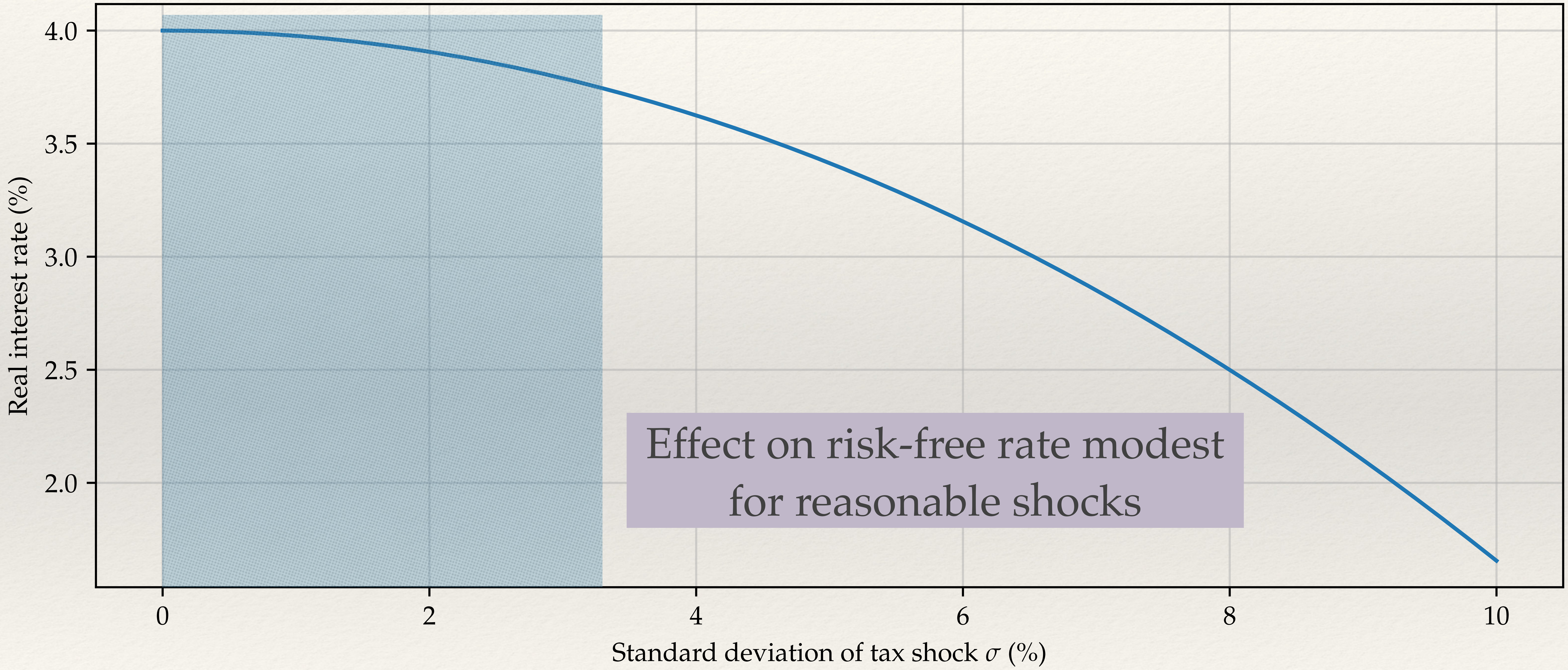

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Conclusion

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- Perfect foresight is a gateway to getting the solution with full aggregate risk
  - Size dependence
  - History dependence
  - Anticipation effect
- Implementation can be done with numerical differentiation (or fancier)
- Small nonlinearities unless shocks are huge, but interesting (nonlinear iMPCs)
- Many potential other exciting applications!