Determinacy and Large-Scale Models in the Sequence Space

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Roadmap for determinacy and existence

- Want a sequence-space criterion for determinacy and existence
- To get there: obtain a **structure theorem** for sequence-space Jacobians
- When het-agent model is *stationary*, Jacobians are *quasi-Toeplitz*:

$$\mathbf{J} = T(\mathbf{j}) + \mathbf{E}$$

  i.e. sum of Toeplitz operator $T(\mathbf{j})$ and compact operator $\mathbf{E}$ on ℓ^2
- Will exploit this structure in many ways, but start with:
  - **"Winding number" criterion** on $\mathbf{j}$ for **determinacy** & existence

Toeplitz and quasi-Toeplitz operators

What is a Toeplitz operator?

- We'll work with **semi-infinite** Toeplitz matrices $T(\mathbf{j})$ with constant diagonals $\{j_s\}_{s=-\infty}^{\infty}$
- Assuming $\sum_s |j_s| < \infty$, these induce bounded operators on ℓ^2
- Can define series $j(z) \equiv \sum_{s=-\infty}^{\infty} j_s z^s$, sometimes called "**symbol**" of $T(\mathbf{j})$

$$T(\mathbf{j}) = \begin{pmatrix} j_0 & j_{-1} & j_{-2} & \\ j_1 & j_0 & j_{-1} & \ddots \\ j_2 & j_1 & j_0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

Examples: lag and lead operators

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \ddots \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Lag operator $\mathbf{L} : (x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, \ldots)$

[Injective but not surjective (range missing 1 dimension)]

$$\begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Lead operator $\mathbf{F} : (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots)$

[Surjective but not injective (null-space missing 1 dimension)]

[= taking one-period-ahead expectations given MIT shock]

More complex example: reset prices given marginal cost

- Log-linearizing standard Calvo model, get:

$$p_t^* = (1-\beta\theta)\cdot\sum_{s=0}^{\infty}(\beta\theta)^s\cdot\mathbb{E}_t MC_{t+s}$$

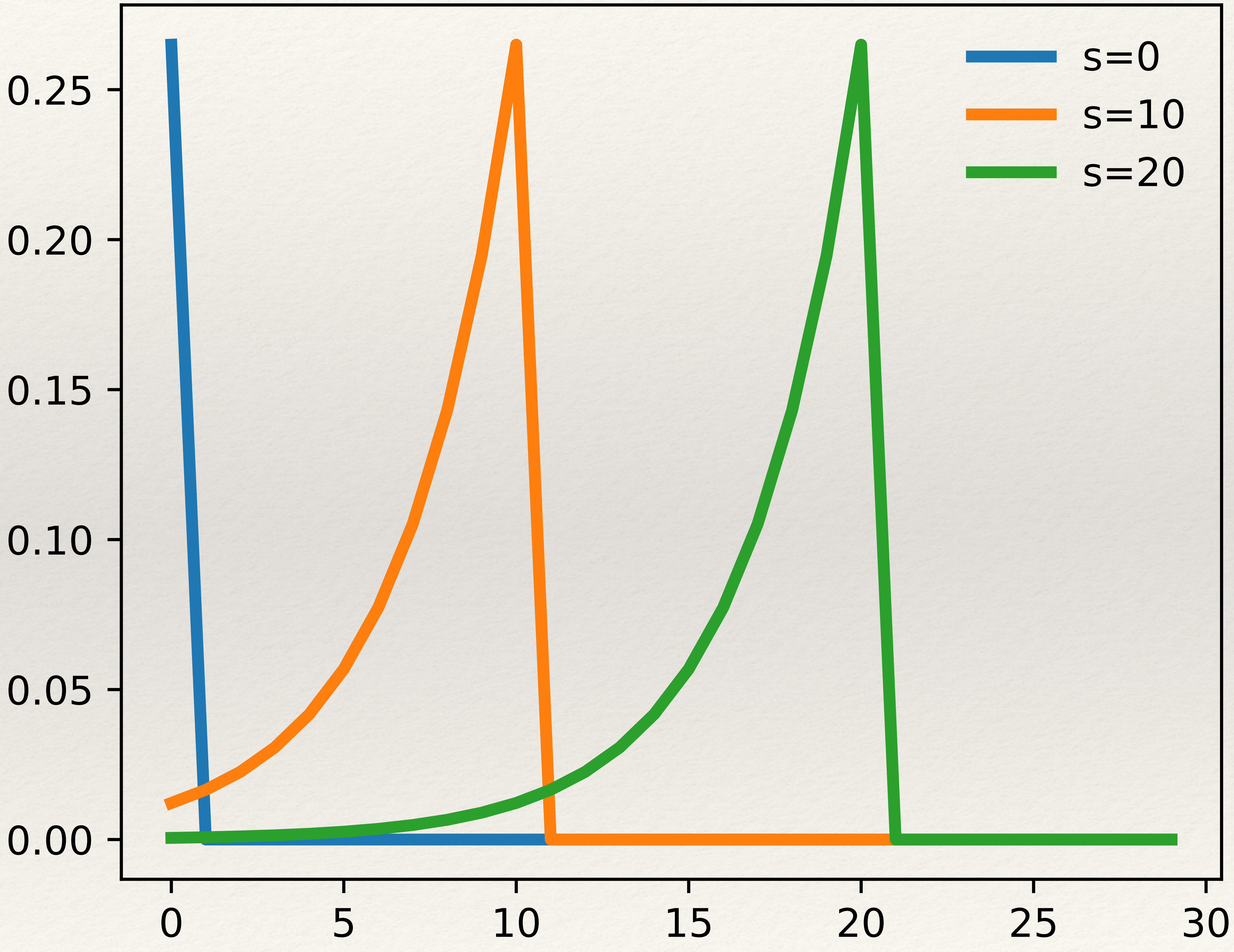
- For MIT shock, mapping from $\{MC_t\}_{t=0}^{\infty}$ to $\{p_t^*\}_{t=0}^{\infty}$ is Toeplitz, equal to $(1-\beta\theta)$ times matrix on right
  [forward-looking → upper triangular]

$$\begin{pmatrix} 1 & \beta\theta & (\beta\theta)^2 & \\ 0 & 1 & \beta\theta & \ddots \\ 0 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

Columns of Toeplitz matrix: costs to reset prices

$$(1-\beta\theta)\times\begin{pmatrix} 1 & \beta\theta & (\beta\theta)^2 & \\ 0 & 1 & \beta\theta & \ddots \\ 0 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

$(\beta = 0.98,\ \theta = 0.75)$



Aggregate prices given reset prices

- Again log-linearizing standard Calvo model, we get:

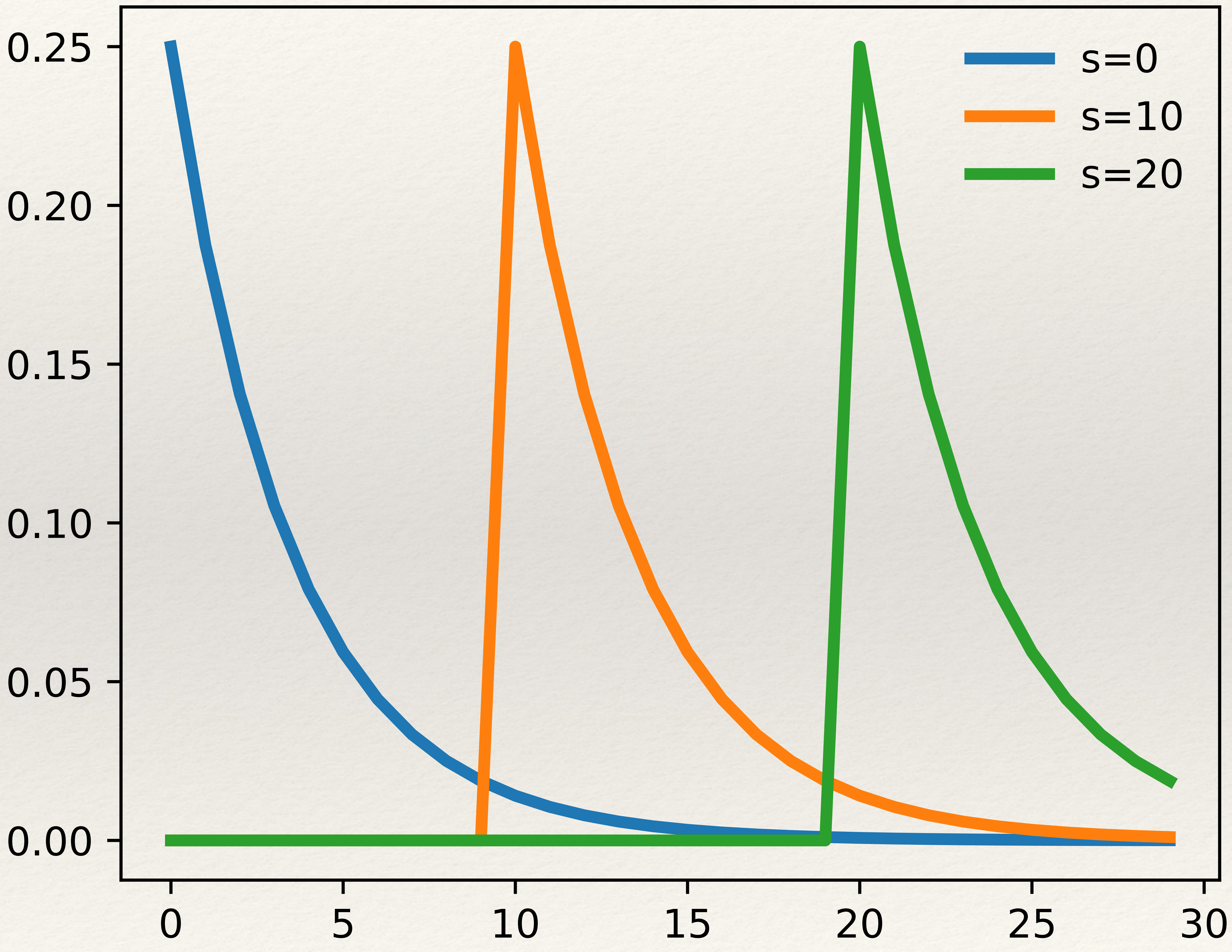
$$p_t = (1-\theta)\cdot \sum_{s=0}^{\infty} \theta^s \cdot p^*_{t-s}$$

- Like before, mapping from $\{p^*_t\}_{t=0}^{\infty}$ to $\{p_t\}_{t=0}^{\infty}$ is Toeplitz, equal to $(1-\theta)$ times matrix on right
  [backward-looking → lower triangular]

$$\begin{pmatrix} 1 & 0 & 0 & \\ \theta & 1 & 0 & \ddots \\ \theta^2 & \theta & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

Columns of Toeplitz matrix: reset prices to agg prices

$$(1-\theta)\times\begin{pmatrix} 1 & 0 & 0 & \\ \theta & 1 & 0 & \ddots \\ \theta^2 & \theta & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$



What if we want to compose Toeplitz operators?

- Would be nice to deal only with Toeplitz operators
- **Problem:** class of Toeplitz operators not closed under composition / multiplication
- Simple example: $\mathbf{FL} = \mathbf{I}$ (lead of lag is identity), but $\mathbf{LF} \neq \mathbf{I}$, and instead:

$$\mathbf{LF} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \ddots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

- So $(\mathbf{LF}) \cdot (x_0, x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$
- Interpretation for MIT shocks? **"Missing anticipation."**
    - $\mathbb{E}_{t-1} x_t = x_t$ for $t > 0$, but $\mathbb{E}_{-1} x_0 = 0 \neq x_0$

But we **do** stay in larger quasi-Toeplitz class

- Define **quasi-Toeplitz** operators as Toeplitz operator plus compact operator $\mathbf{E}$
  [compact in ℓ^2 = can be uniformly well-approximated by finite rank]

$$\mathbf{J} = T(\mathbf{j}) + \mathbf{E}$$

- **Result:** product of any two Toeplitz operators $T(\mathbf{j}_1)$ and $T(\mathbf{j}_2)$ is quasi-Toeplitz like above, with $\mathbf{j} = \mathbf{j}_1 \cdot \mathbf{j}_2$ given by convolution [so $j(z) = j_1(z)j_2(z)$]

- Intuitively, why do we need $\mathbf{E}$? "Missing anticipation": taking lags of leads!

- Quasi-Toeplitz operators closed under multiplication!
  [multiplying Toeplitz parts → quasi-Toeplitz, multiplying any bounded by compact → compact]

Compose to get map $\mathbf{J}$ from costs to aggregate prices

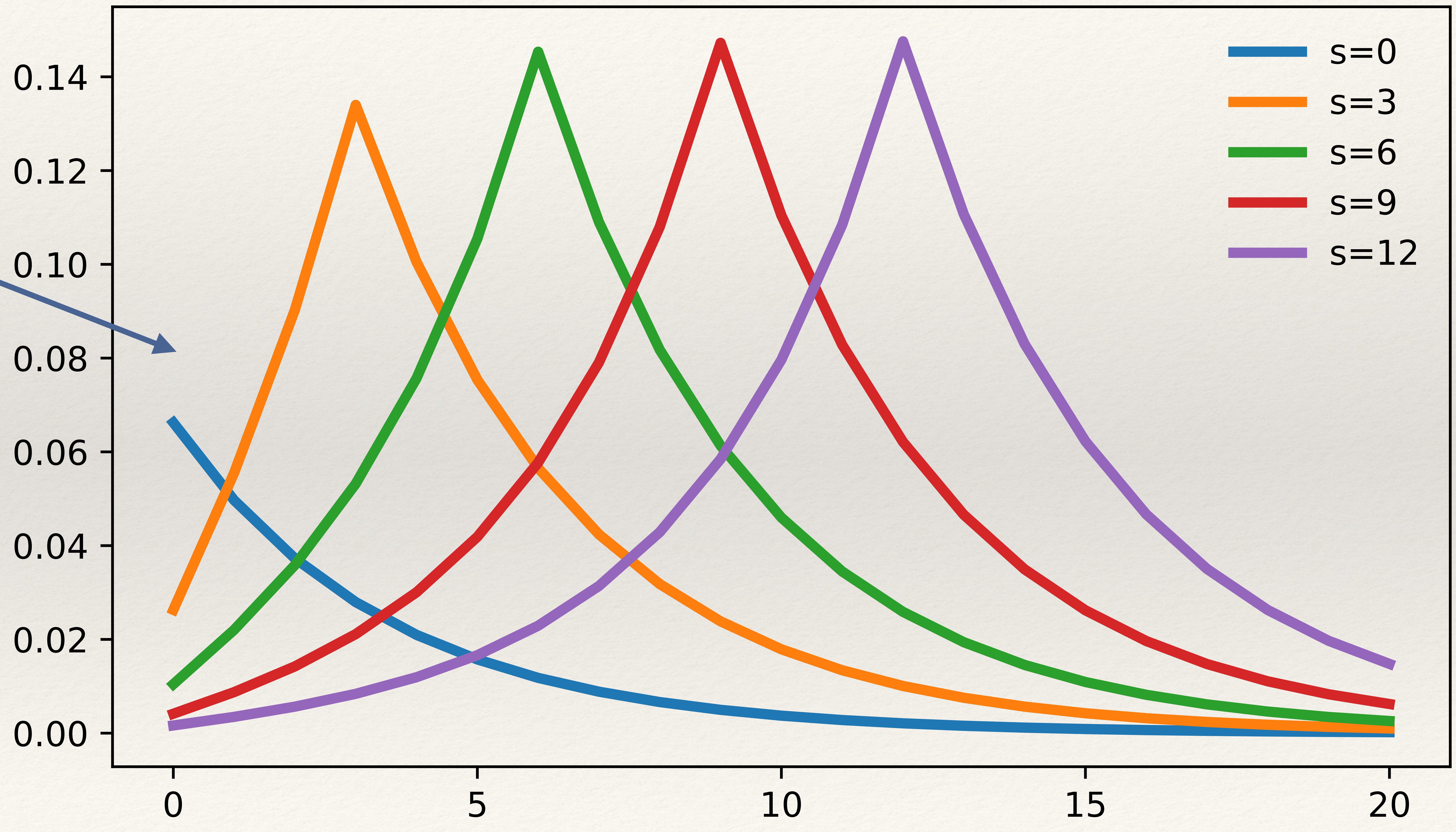
$$(1-\theta)\times \begin{pmatrix} 1 & 0 & 0 & \\ \theta & 1 & 0 & \ddots \\ \theta^2 & \theta & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \times (1-\beta\theta)\times \begin{pmatrix} 1 & \beta\theta & (\beta\theta)^2 & \\ 0 & 1 & \beta\theta & \ddots \\ 0 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

(Lag of lead: we should expect a compact "correction" $\mathbf{E}$!)

Visualizing columns of this **J**

Eventually converge to Toeplitz pattern, but "missing anticipation" leads to a limited price response near date 0.

Deviation **E** from Toeplitz is **compact** (and, here, negative)



Structure theorem
for heterogeneous-agent models

Structure theorem for heterogeneous-agent models

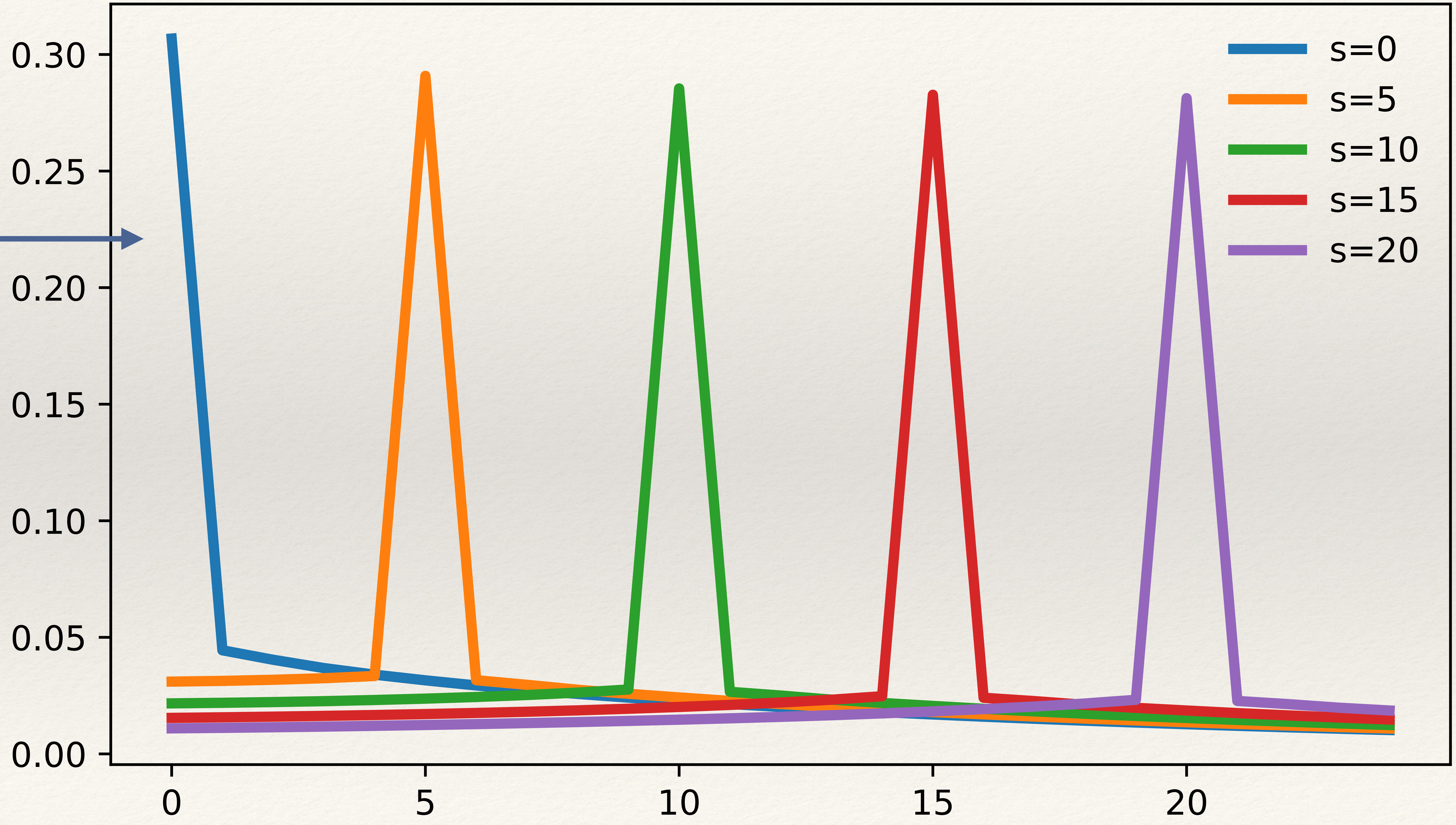
- Such models have some steady-state transition matrix Λ_{ss}, some backward mapping $\mathbf{v}_{t+1} \to \mathbf{v}_t$ on value function, with steady-state derivative $\mathbf{v}_{\mathbf{v}}$
- Suppose we have "**stationarity**":
  - Λ_{ss} has all eigenvalues but one strictly inside unit circle
  - $\mathbf{v}_{\mathbf{v}}$ has eigenvalues strictly inside unit circle
- **Structure theorem:** if model is stationary, all sequence-space Jacobians are **quasi-Toeplitz** with

$$|E_{ts}| \le K\Delta^{t+s}$$

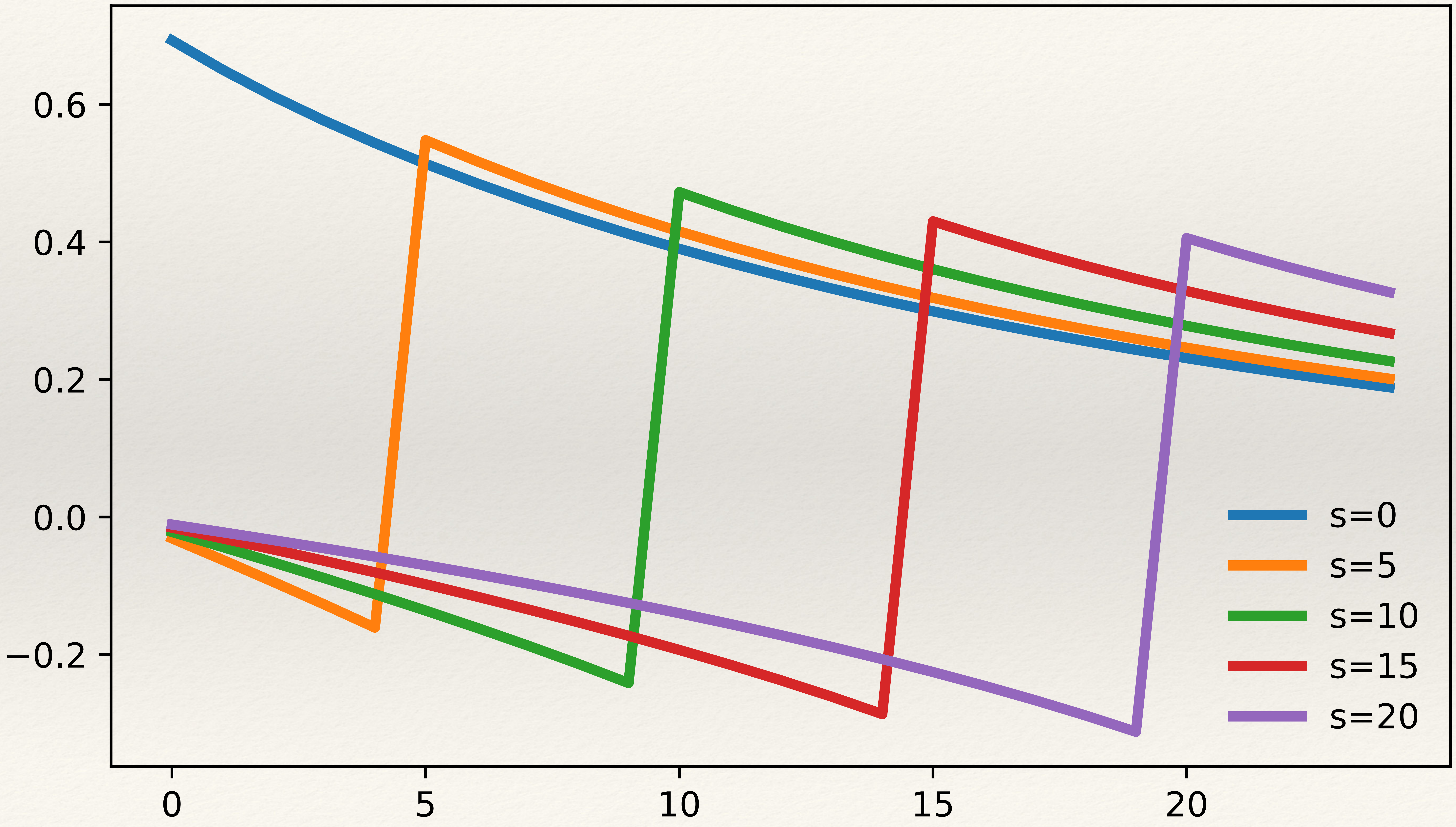
  for bound Δ on eigenvalues and some constant K
- Intuition: if effect of far future on value function eventually dies off…
  - … and effect of distribution on future eventually dies off, "missing anticipation" dies off too

Example: intertemporal MPCs in a SIM model

E is positive, because not having anticipated income means you didn't spend ahead of time, leaving more to spend now



Same, but Jacobian of assets vs. income



Taking stock

- We've shown that **quasi-Toeplitz** Jacobians naturally emerge:
  - as closure under multiplication of Toeplitz Jacobians (which themselves emerge from simple aggregate equations)
  - as Jacobians of stationary heterogeneous-agent problems
- Now will discuss **applications**:
  - **now:** for testing **determinacy** and **existence** of solutions
  - **later:** directly getting sequence-space solutions in "**truncation-free**" way, and solving **huge sequence-space systems**

Determinacy, existence, and inversion

Winding number of a Toeplitz operator

- **Winding number** wind(j) is # of times symbol $j(z)$ rotates counterclockwise around 0 as z goes counterclockwise around the unit circle [see Onatski 2006]
- **Result:** can invert Toeplitz $T(\mathbf{j})$, resulting in quasi-Toeplitz, iff winding number 0
- For quasi-Toeplitz $T(\mathbf{j}) + \mathbf{E}$, this result is "generic" on open & dense set of $\mathbf{E}$:
  - If wind(j) $= 0$, $\mathbf{J}$ is generically **invertible**
  - If wind(j) < 0, $\mathbf{J}$ is **not injective (indeterminacy)**, but generically surjective
  - If wind(j) > 0, $\mathbf{J}$ is **not surjective (nonexistence)**, but generically injective
- [Example: lag operator has $j(z) = z$, winding number of 1, so not surjective.]

Application: uniqueness in Intertemporal Keynesian Cross model

- In Intertemporal Keynesian Cross model (Auclert Rognlie Straub 2024), equilibrium output $d\mathbf{Y}$ given taxes $d\mathbf{T}$ and bonds $d\mathbf{B}$ given by

$$\mathbf{A}(d\mathbf{Y} - d\mathbf{T}) = d\mathbf{B}$$

  where $\mathbf{A}$ is Jacobian of household assets to post-tax income

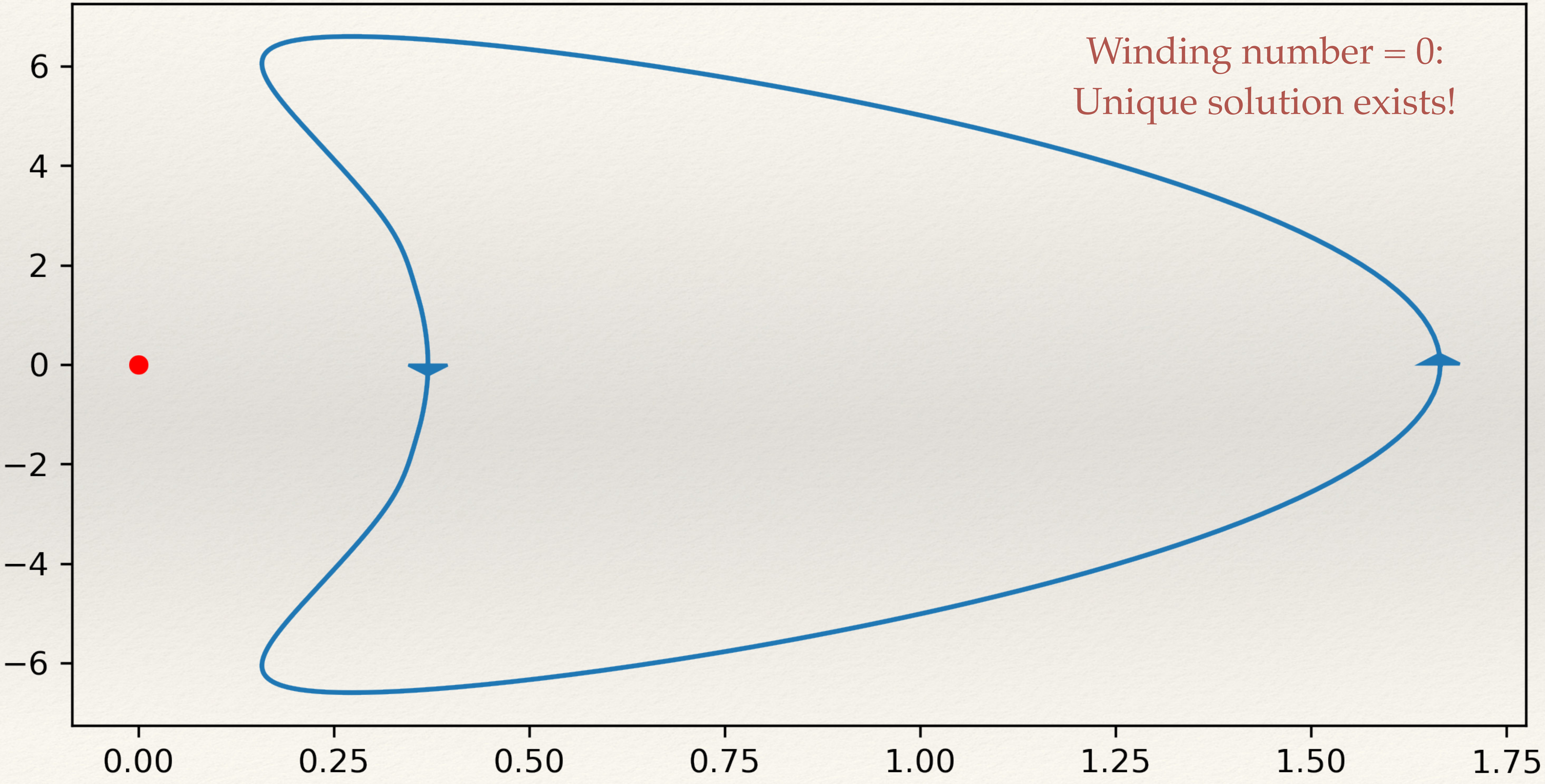
- $\mathbf{A}$ is quasi-Toeplitz, will generically be invertible with solution

$$d\mathbf{Y} = \mathbf{A}^{-1} d\mathbf{B} + d\mathbf{T}$$

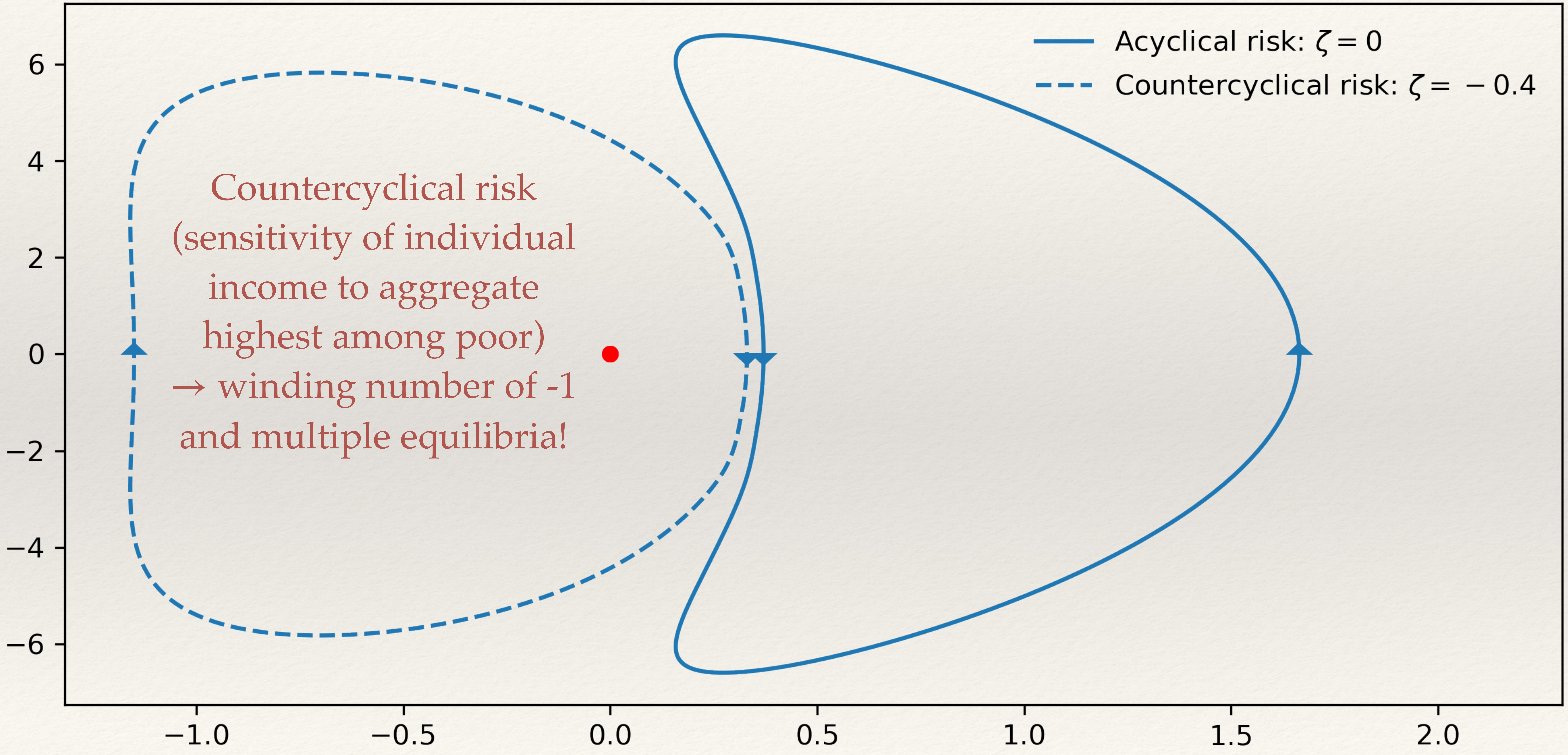
  if winding number of $a(z)$ is zero

- Multiple equilibria $d\mathbf{Y}$ if winding number is negative
  - (Self-fulfilling booms or busts in output!)

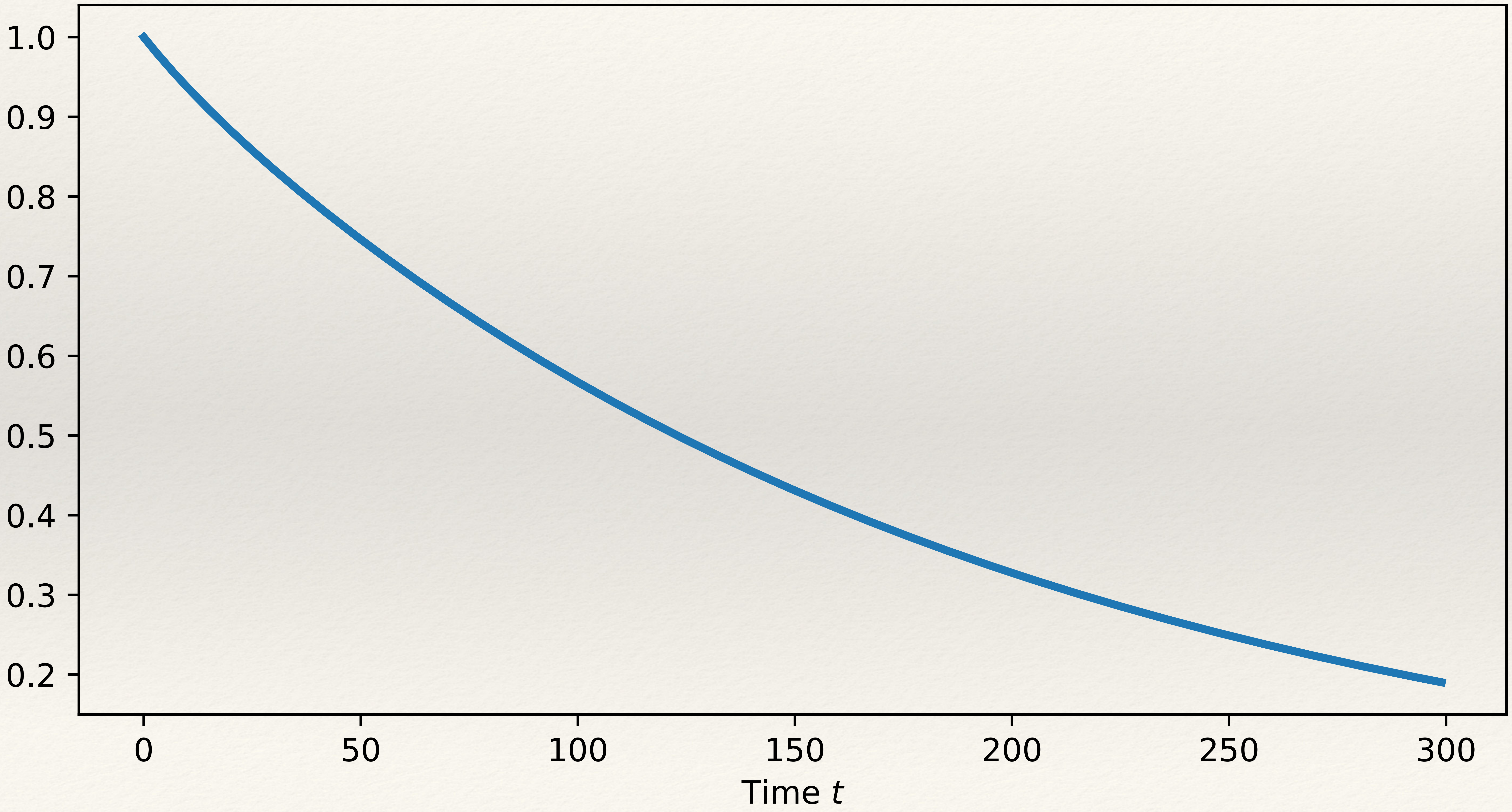
Winding number plot for **A** with a standard calibration



Case with countercyclical risk: **self-fulfilling boom**

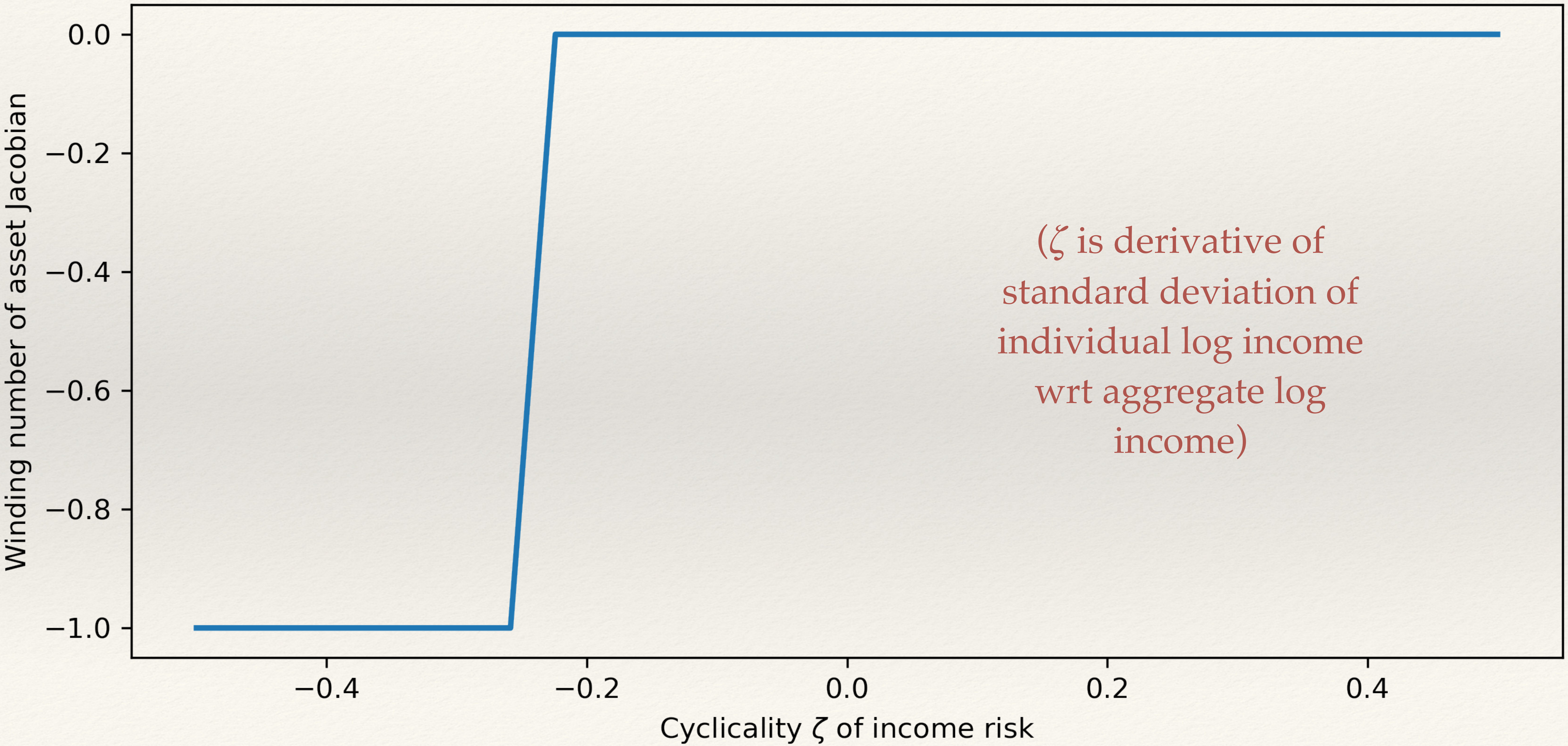


Get shape of indeterminacy from approx SVD null vector



Self-fulfilling boom that is **very** long-lasting: boom increases income and decreases future risk, leading to self-fulfilling boom in consumption

How does winding number vary with cyclicality ζ of income risk?



Under the hood: winding number code

```python
def sample_values(j, N=8192):
    """Evaluate Laurent polynomial j(z) (with equally many positive
    and negative powers) counterclockwise at N evenly spaced roots of
    unity z, wrapping back around to z=1, using FFT"""
    assert N % 2 == 0 and len(j) % 2 == 1
    Tau = len(j) // 2 + 1 # Tau-1 is the maximum pos or neg power in j(z)

    # center j(z) at N/2
    jj = np.zeros(N)
    jj[N//2-Tau+1:N//2+Tau] = j

    # take FFT to evaluate j(z) * z^(N/2) at roots of unity (could exploit
    conjugate symmetry to halve work)
    e = np.fft.fft(jj)

    # divide by z^(N/2) at same roots, which is alternating 1 and -1, to
    get j(z)
    alt = np.tile([1, -1], N//2)
    e = e * alt

    # return wrapped back to z=1, reversed to make counterclockwise
    return np.concatenate((e, [e[0]]))[::-1]
```

Once you have *j*(*z*), it's easy to evaluate at sample points around unit circle very efficiently with the FFT—see left.

(**Much** better than a naive implementation.)

Then we just need to count the number of times the path wraps around the origin—tedious function to write, header below, full function in **winding_number.py**.

```python
@njit
def winding_number_of_path(x, y):
    """Compute winding number around origin of (x,y) coordinates that make
    closed path by
    counting number of counterclockwise crossings of ray from (0,0) ->
    (infty,0) on x axis"""
```

Block quasi-Toeplitz case

- Say we have N^2 quasi-Toeplitz matrices from N unknowns to one of N targets
- Can think of this as being one **block quasi-Toeplitz operator**, like a quasi-Toeplitz but where entries are each $N \times N$ blocks
- Then $\{j_k\}_{k=-\infty}^{\infty}$ is two-sided sequence of $N \times N$ matrices, so *matrix-valued* $j(z)$:

$$j(z) \equiv \sum_{k=-\infty}^{\infty} j_k z^k$$

- Winding number test still holds generically, now for $\mathrm{wind}(\det j)$
- Important case in practice

Beyond determinacy:
quasi-Toeplitz as a computational tool

Solving het-agent models to first order

- Two key considerations:
  - Size of idiosyncratic state space **S**
  - Number of endogenous aggregate variables **N**
- **State-space approach:** costly when S large. Has determinacy criterion.

  [Reiter, Ahn-Kaplan-Moll-Winberry-Wolf, Bayer-Luetticke, …]
- **Sequence-space approach:** fast when S large, costly when N large.

  [Boppart-Krusell-Mitman, Auclert-Bardoczy-Rognlie-Straub, ...]

- How do we solve models <u>when both S and N are large?</u>

Further exploiting quasi-Toeplitz structure

- **Three** ways of exploiting quasi-Toeplitz structure:
  - **Winding number criterion** on $\mathbf{j}$ for **determinacy** & existence [did this!]
  - **"Truncation-free" solution** working directly with $\mathbf{j}$ and $\mathbf{E}$ **[next!]**
  - Using $T(\mathbf{j}^{-1})$ as guess for $\mathbf{J}^{-1}$ gives **rapid iterative solution**, even when N large **[next!]**

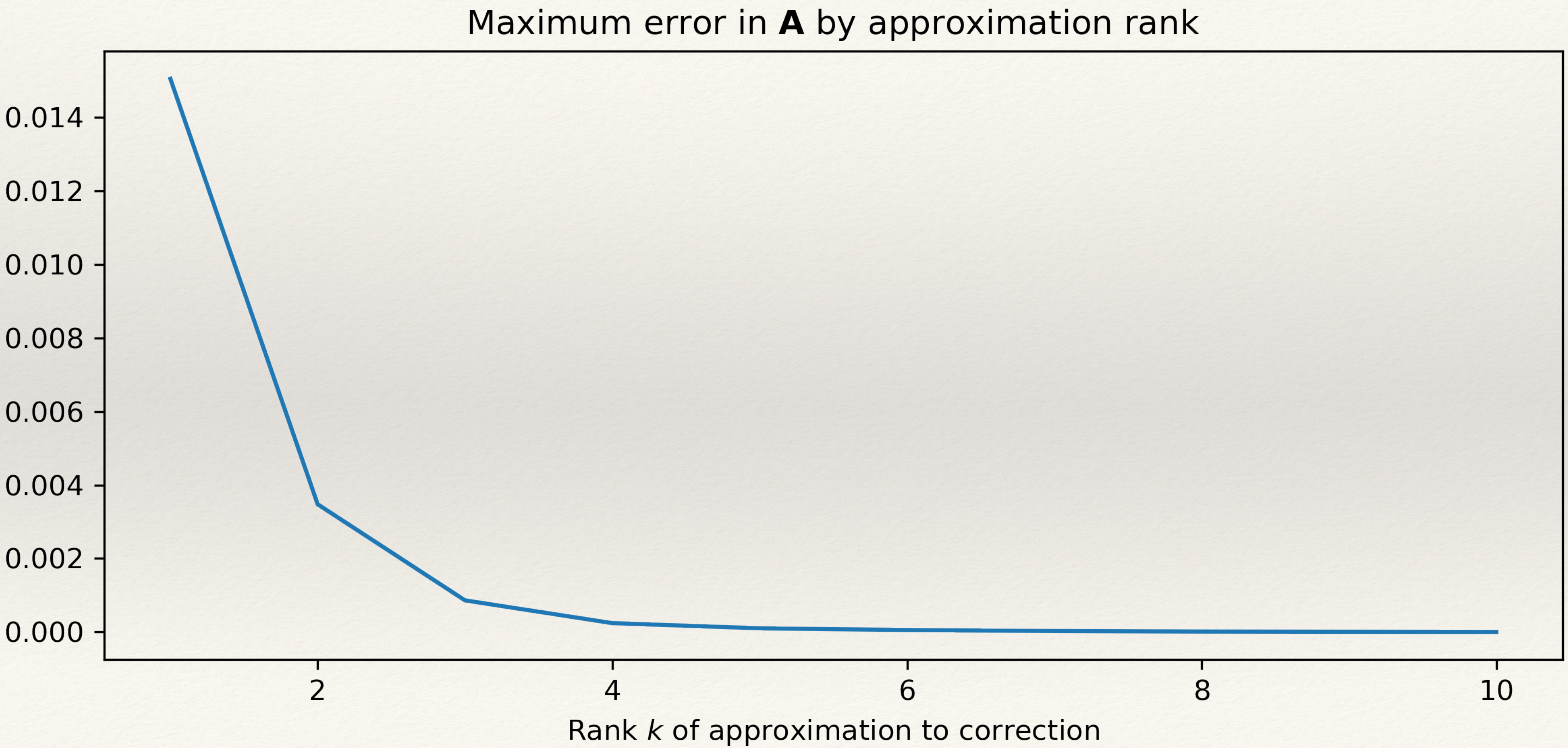
Operations with quasi-Toeplitz operators:
No more truncation!

Directly use quasi-Toeplitz form

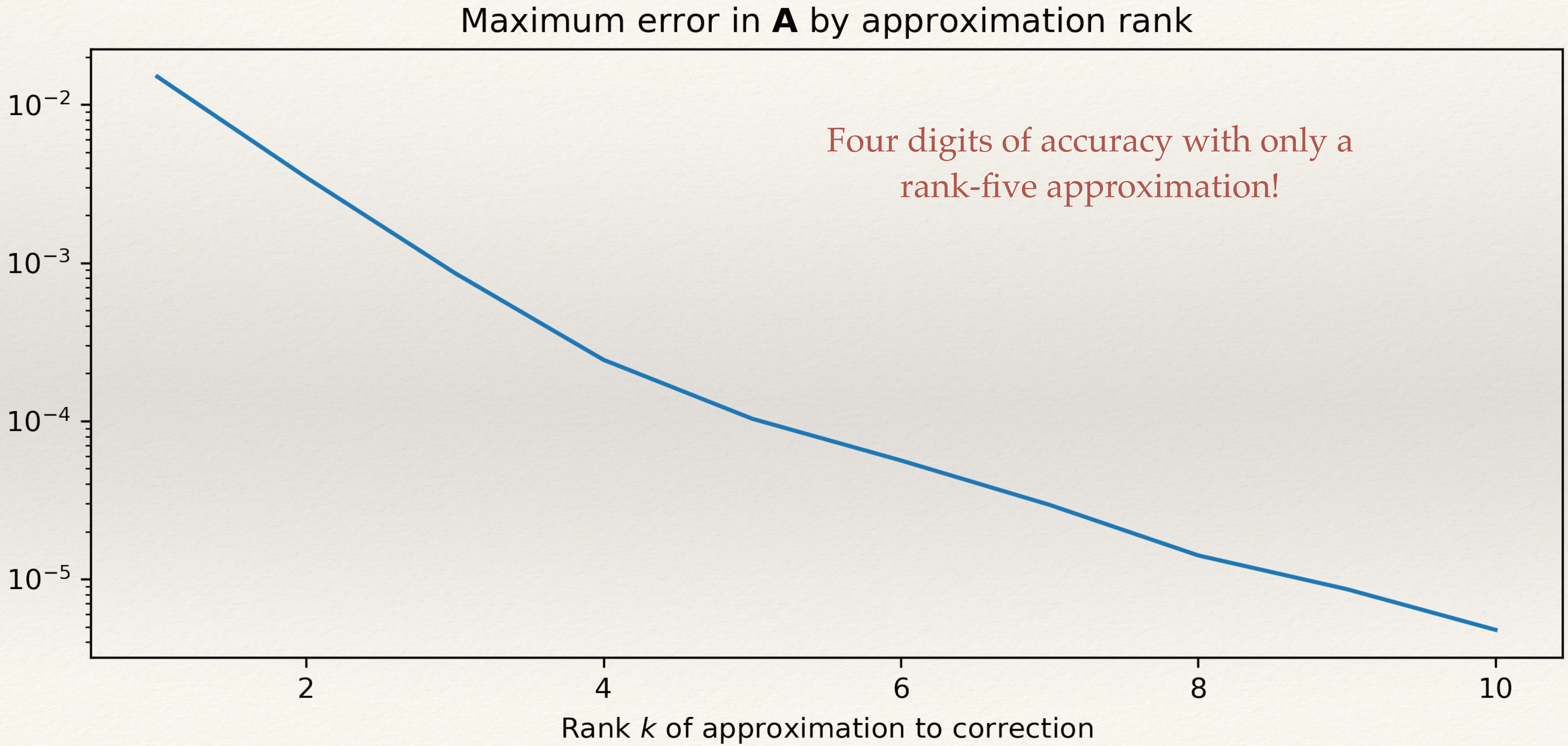
- We have quasi-Toeplitz representation $\mathbf{J} = T(\mathbf{j}) + \mathbf{E}$ of Jacobians
- So far, we've used the winding number of $\mathbf{j}$ to assess determinacy
- Going beyond this: directly do computations with this representation!
  - Benefit: $\mathbf{E}$ decays quickly to zero, and often close to low-rank $\mathbf{E} \approx \mathbf{U}\mathbf{V}'$
  - Cheap to get $T(\mathbf{j}_1\mathbf{j}_2)$, multiply $T(\mathbf{j}_1)\mathrm{U}_2'$, etc., to construct $\mathbf{J}_1\mathbf{J}_2$ (using FFT)
  - Similar, though a bit more complex, for inversion
- Working directly with quasi-Toeplitz (infinite!), not truncated matrices, avoids errors from truncation

[Bini, Massei, Robol 2019]

How well can we approximate **A**?



Big easier to visualize with a log scale…



Alternative: use structure for iterative solutions

(and solve giant models in the process!)

First point: easy to get Toeplitz part of inverse

- Suppose we want to solve $\mathbf{A}d\mathbf{Z} = d\mathbf{B}$
- $\mathbf{A}^{-1}$ is quasi-Toeplitz of form $T(\mathbf{a}^{-1}) + \mathbf{E}$, with $\mathbf{E}$ low-rank like we saw
- Key point: $\mathbf{a}^{-1}$ is **really** easy to calculate!
  - Get $a(z)$ at many z using FFT, then go from $a(z)^{-1}$ to $\mathbf{a}^{-1}$ with inverse FFT
  - Cost is only $O(T \log T)$, way cheaper than $O(T^3)$ matrix inversion
  - What can we do with just $T(\mathbf{a}^{-1})$?
  - [conceptually, $\mathbf{a}^{-1}$ is inverse for infinitely-well-anticipated shocks]

What can we do with $\mathbf{a}^{-1}$?

- Start with $(T(\mathbf{a}) + \mathbf{E})d\mathbf{Z} = d\mathbf{B}$, multiply both sides by $T(\mathbf{a}^{-1})$:

$$T(\mathbf{a}^{-1})(T(\mathbf{a}) + \mathbf{E})d\mathbf{Z} = T(\mathbf{a}^{-1})d\mathbf{B}$$

- Both $T(\mathbf{a}^{-1})T(\mathbf{a}) - \mathbf{I}$ and $T(\mathbf{a}^{-1})\mathbf{E}$ compact, well-approximated by low rank, so can be written in form

$$(\mathbf{I} + \mathbf{C})d\mathbf{Z} = d\mathbf{y}$$

- Iterative method (GMRES) **very** good at solving $(\mathbf{I} + \mathbf{C})^{-1}d\mathbf{y}$ if $\mathbf{C}$ low-rank

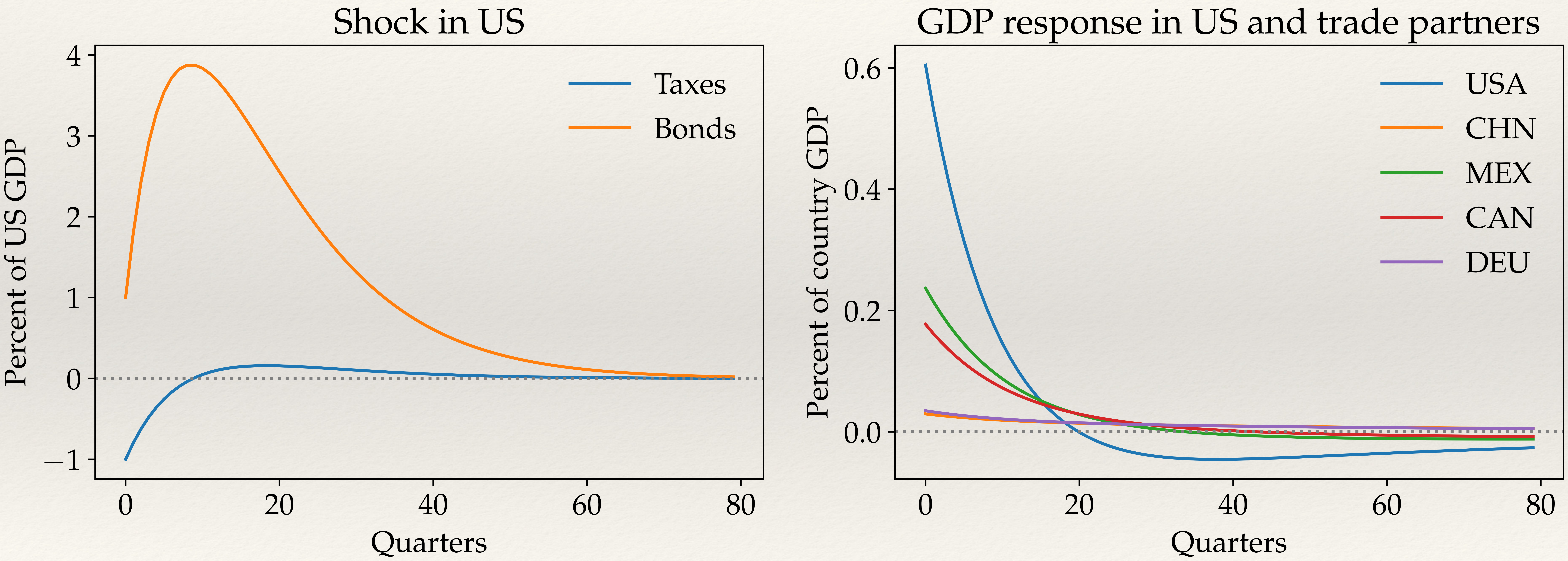
  [Multiplying by $T(\mathbf{a}^{-1})$ is called "preconditioning".]

- Cheap, doesn't require explicitly forming new matrices like $\mathbf{C}$

We expand this to HUGE model

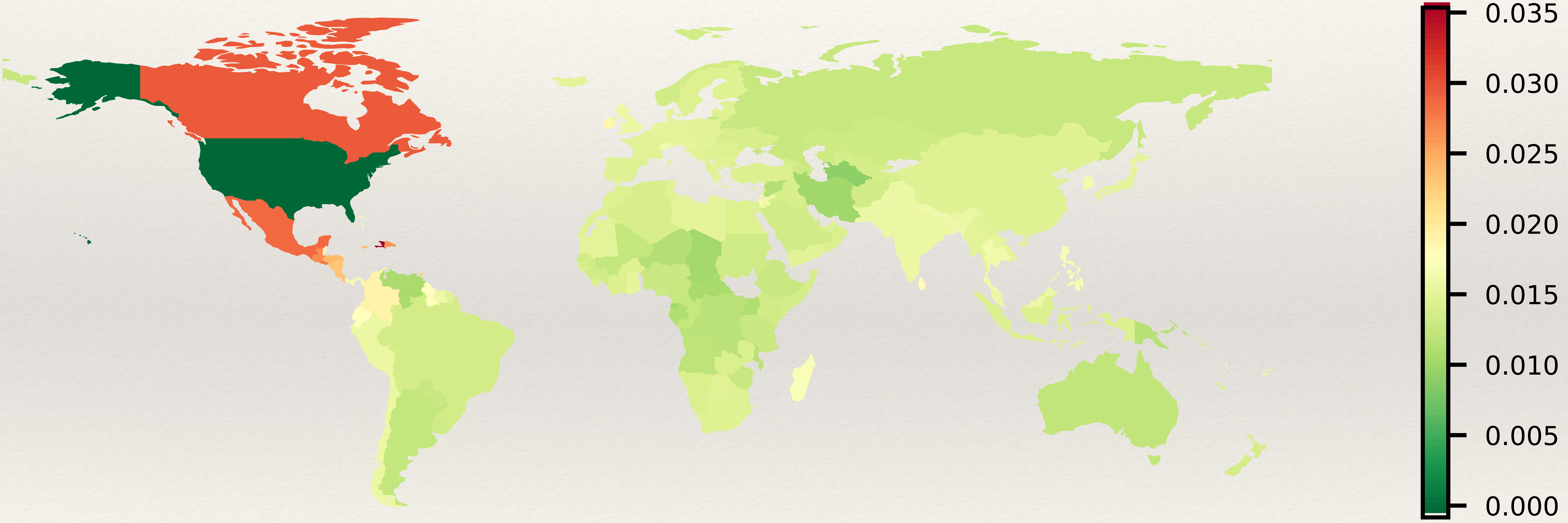
- N-country extension of IKC model, constant r in each country n
- Fiscal policy in n chooses $\{B_t^n, T_t^n\}$ consistent with budget constraint
- n spends share $\Pi_{n,n'}$ on output from others n', take from data for 177 countries
- Assume same HA model in each n, for simplicity assume all share $\mathbf{A}, \mathbf{M}$
- Solve for GDP $\{Y_{nt}\}$ in all N countries, in response to US deficit-financed tax cut, need long horizon $T = 1000$
- Usual sequence-space approach: Jacobian size $(177{,}000)^2$: can't even store!
- With iterative approach, solves in a few seconds on laptop!

Peek at solution: selected countries over time

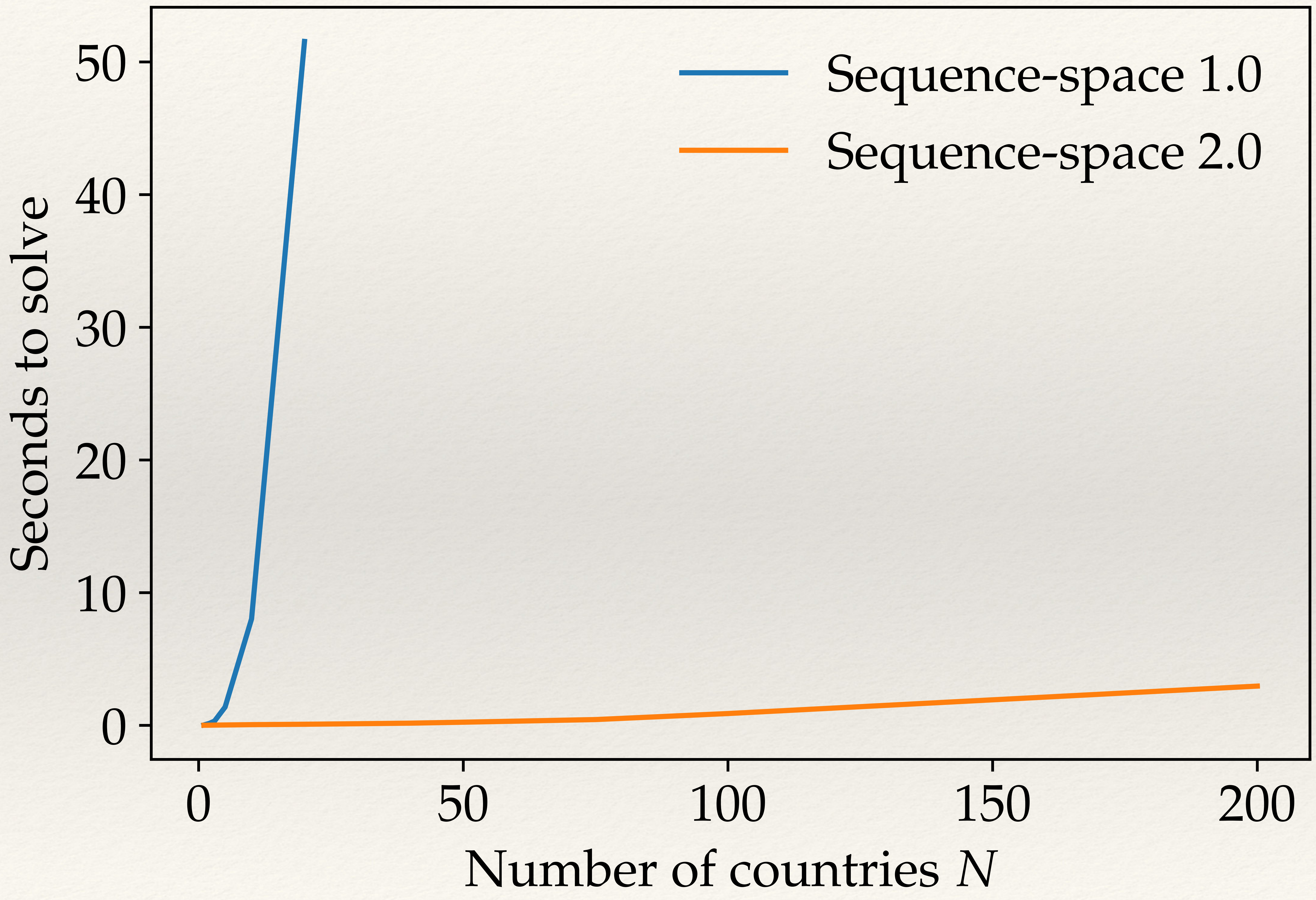


Peek at solution: on impact across countries

Peek at solution: after 20 quarters across countries

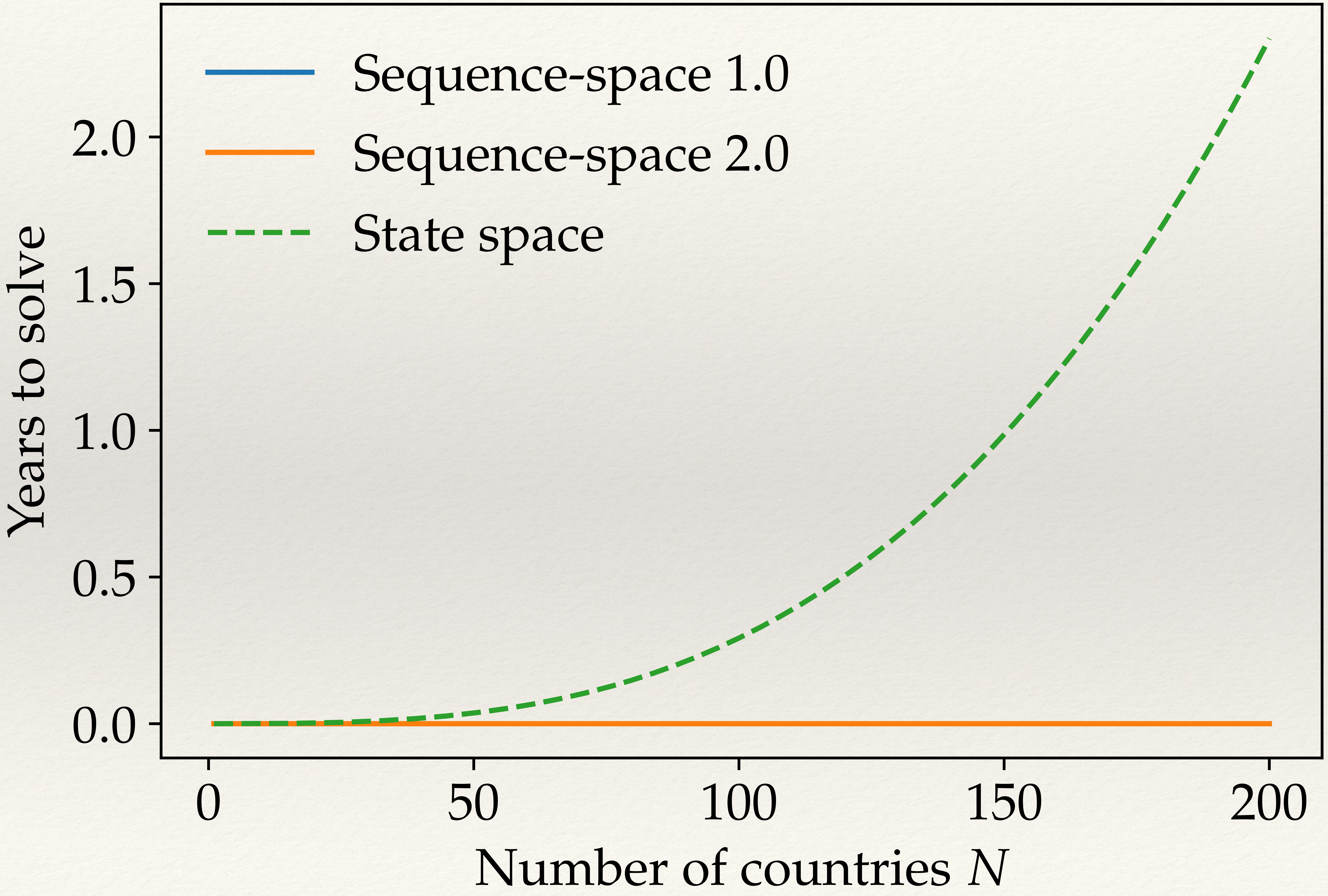


How fast is this?



How fast is this? Compare to state space

(Extrapolation using cubic cost of solving state-space system!)



Conclusion

- Quasi-Toeplitz structure of Jacobians delivers:
  - **winding number test** for determinacy
  - **truncation-free computations** exploiting the structure
  - extremely fast **iterative computations, even in huge models**
    - solves 177-country HANK in 3 seconds!!