Unit 11
Principal Component Analysis

EL-GY 6143/CS-GY 6923: INTRODUCTION TO MACHINE LEARNING

PROF. PEI LIU

Learning Objectives

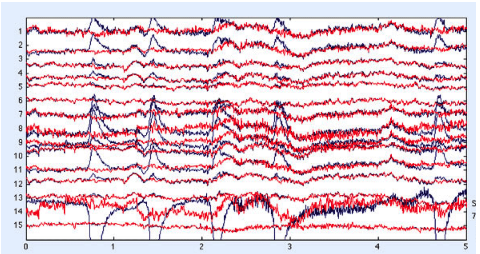
- Identify cases to use dimensionality reduction
- Mathematically describe principal components representations of data
- Compute principal components via SVDs
- Compute PC components in python
- Add PCA transforms as a pre-processing step to classification and regression
- Implement low-rank transforms for recommender systems

Outline

- Why dimensionality reduction?
- Principal components and directions of variance
- Approximation with PCs
- Computing PCs via the SVD
- Face example in python
- Training models from PCs
- Low rank approximations and recommender systems

High-Dimensional Data

- Many data sets have very high dimension
- Training can be difficult
  - Especially when number of samples is small
  - Classifier needs many parameters

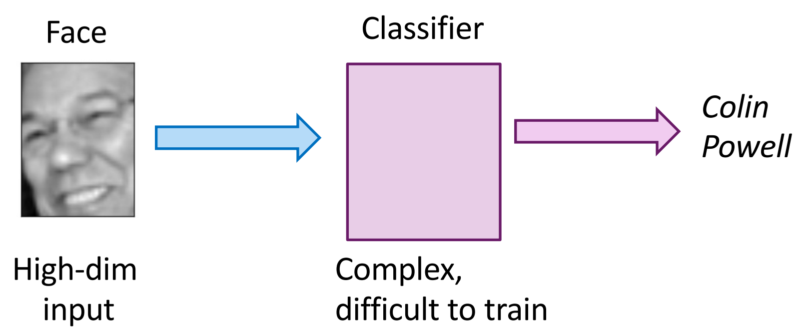


EEG
Ex: 32 channels x 1 kHz x 10s

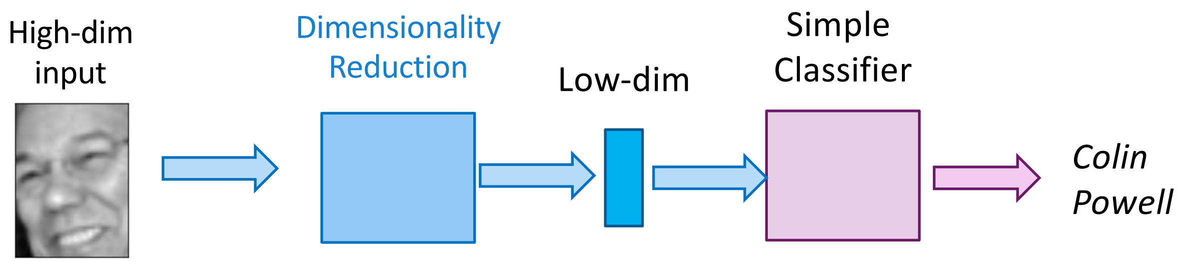
Face recognition with high-resolution images

Problems with High-Dimensions

- Consider face recognition
- Input is high-dimensional
  - Esp. for high resolution image
- Resulting classifier:
  - Requires many parameters
  - Difficult to train
  - Needs many samples
  - Computationally complex



Dimensionality Reduction



- Dimensionality reduction:
  - Reduce the input dimension to lower dimensional representation
- Can build simpler classifier
- Low-dimensional representational also good for:
  - Visualizing data
  - Clustering and other unsupervised tasks
  - Finding underlying structure of the data

Outline

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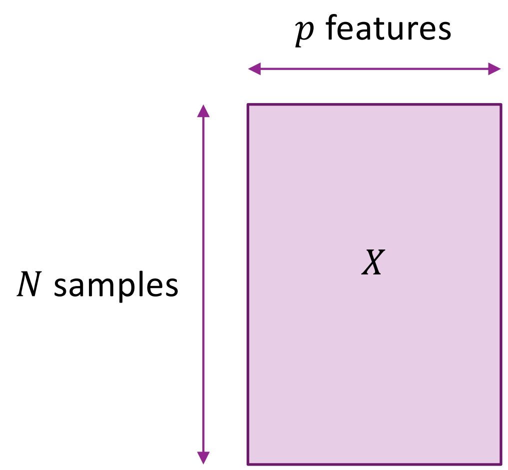
Data Definitions

- Given data: $\boldsymbol{x}_i, i = 1, \dots, N$
  - Each sample has p features: $\boldsymbol{x}_i = (x_{i1}, \dots, x_{ip})$
  - Represent as an $N \times p$ matrix



- Unsupervised learning
  - Samples do not have a label
  - Or we choose to ignore the label for now

- Dimension p is large

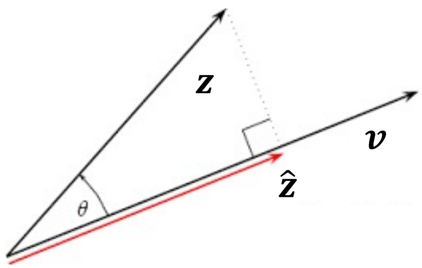
- How do we reduce the dimension?

Projections

- PCA reduces dimensionality by "projecting" data to a lower dim subspace
- Projection: Given vectors $\boldsymbol{z}$ and $\boldsymbol{v}$, the projection of $\boldsymbol{z}$ onto $\boldsymbol{v}$ is:

$$\hat{\boldsymbol{z}} = \mathrm{Proj}_{\boldsymbol{v}}(\boldsymbol{z}) = \alpha \boldsymbol{v}, \qquad \alpha = \frac{\boldsymbol{v}^T \boldsymbol{z}}{\boldsymbol{v}^T \boldsymbol{v}} = \frac{||\boldsymbol{z}||}{||\boldsymbol{v}||} \cos\theta$$

  - α = coefficient of the projection



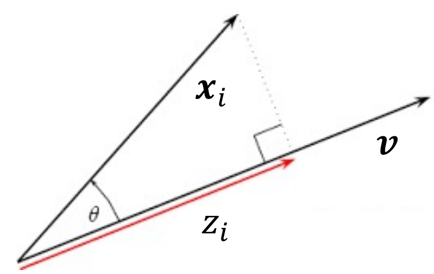
- Theorem: $\mathrm{Proj}_{\boldsymbol{v}}(\boldsymbol{z})$ is closest point in V to $\boldsymbol{z}$:

$$\hat{\boldsymbol{z}} = \arg\min_{\boldsymbol{w} \in V} ||\boldsymbol{z} - \boldsymbol{w}||^2$$

  - $V = \{\alpha \boldsymbol{v} | \alpha \in R\}$ = vectors on the line spanned by $\boldsymbol{v}$

Maximal Directional Variance

- Given data: $\boldsymbol{x}_i, i = 1, \dots, N$ and direction $\boldsymbol{v}$ with $\|\boldsymbol{v}\| = 1$
- Let $z_i = \boldsymbol{v}^T \boldsymbol{x}_i$ = coefficient of the projection of $\boldsymbol{x}_i$ onto $\boldsymbol{v}$
- Sample mean and variance in direction $\boldsymbol{v}$ is :
  - Sample mean $\bar{z} = \frac{1}{N} \sum_{i=1}^{N} z_i$
  - Sample variance $s_z^2 = \frac{1}{N} \sum_{i=1}^{N} (z_i - \bar{z})^2$



> Problem: Find the direction $\boldsymbol{v}$ that maximizes the variance s_z^2

- Why?
  - Captures the most variation of the data
  - Provides the best vector for dimensionality reduction

Sample Covariance Matrix

- Sample mean of the data: $\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i$, and matrix $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1^T \\ \vdots \\ \boldsymbol{x}_N^T \end{bmatrix}$
- Sample covariance matrix: Matrix $\boldsymbol{Q}$ with components:

$$Q_{k\ell} = \frac{1}{N}\sum_{i=1}^{N}(x_{ik} - \bar{x}_k)(x_{i\ell} - \bar{x}_\ell)$$

  - Covariance between feature k and ℓ in the dataset
  - Matrix is $p\times p$
- Define $\widetilde{\boldsymbol{x}}_i = \boldsymbol{x}_i - \overline{\boldsymbol{x}}$, data with sample mean removed
- Sample covariance is given by

$$\boldsymbol{Q} = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^T = \frac{1}{N}\begin{bmatrix}\widetilde{\boldsymbol{x}}_1 & \widetilde{\boldsymbol{x}}_2 & \cdots & \widetilde{\boldsymbol{x}}_N\end{bmatrix}\begin{bmatrix} \widetilde{\boldsymbol{x}}_1^T \\ \widetilde{\boldsymbol{x}}_2^T \\ \vdots \\ \widetilde{\boldsymbol{x}}_N^T \end{bmatrix} = \frac{1}{N}\widetilde{\boldsymbol{X}}^T\widetilde{\boldsymbol{X}}$$

Sample Covariance and Directional Variance

- Let $z_i = \boldsymbol{v}^T \boldsymbol{x}_i$ = coefficient of the projection of $\boldsymbol{x}_i$ onto $\boldsymbol{v}$
- Can compute the sample mean and variance of z_i from $\overline{\boldsymbol{x}}$ and $\boldsymbol{Q}$
- Sample mean of the coefficients:

$$\bar{z} = \frac{1}{N}\sum_{i=1}^{N} z_i = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{v}^T \boldsymbol{x}_i = \boldsymbol{v}^T\left[\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i\right] = \boldsymbol{v}^T \overline{\boldsymbol{x}}$$

- Sample variance of the coefficients:

$$s_z^2 = \frac{1}{N}\sum_{i=1}^{N} (z_i - \bar{z})^2 = \frac{1}{N}\sum_{i=1}^{N} (\boldsymbol{v}^T(\boldsymbol{x}_i - \overline{\boldsymbol{x}}))^2$$

$$= \frac{1}{N}[\boldsymbol{v}^T\widetilde{\boldsymbol{x}}_1 \quad \boldsymbol{v}^T\widetilde{\boldsymbol{x}}_2 \quad \cdots \quad \boldsymbol{v}^T\widetilde{\boldsymbol{x}}_N]\begin{bmatrix}\widetilde{\boldsymbol{x}}_1^T\boldsymbol{v} \\ \widetilde{\boldsymbol{x}}_2^T\boldsymbol{v} \\ \vdots \\ \widetilde{\boldsymbol{x}}_N^T\boldsymbol{v}\end{bmatrix} = \frac{1}{N}\boldsymbol{v}^T[\widetilde{\boldsymbol{x}}_1 \quad \widetilde{\boldsymbol{x}}_2 \quad \cdots \quad \widetilde{\boldsymbol{x}}_N]\begin{bmatrix}\widetilde{\boldsymbol{x}}_1^T \\ \widetilde{\boldsymbol{x}}_2^T \\ \vdots \\ \widetilde{\boldsymbol{x}}_N^T\end{bmatrix}\boldsymbol{v}$$

$$= \boldsymbol{v}^T \boldsymbol{Q} \boldsymbol{v}$$

Maximizing Directional Variance

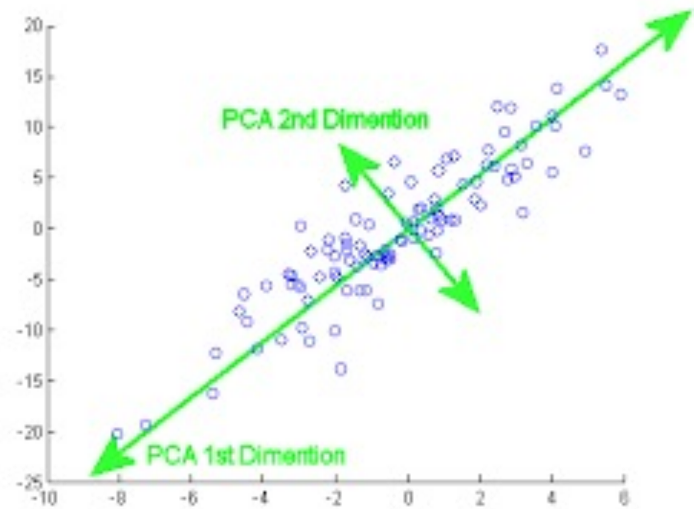
- From previous slide: Directional variance $s_z^2 = \frac{1}{N}\sum_{i=1}^{N}(z_i - \bar{z})^2 = \boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{Q}\boldsymbol{v}$
- Maximizing directional variance can be formulated as an optimization problem:

$$\max_{\boldsymbol{v}} \boldsymbol{v}^T\boldsymbol{Q}\boldsymbol{v} \quad \text{s.t.} \quad \|\boldsymbol{v}\| = 1$$

- Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ be the eigenvectors of $\boldsymbol{Q}$: $\boldsymbol{Q}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$
- Sort eigenvalues in descending order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$
  - Can show that eigenvalues are real and non-negative
- Theorem: Any local maxima of the variance directional is an eigenvector
  - $\boldsymbol{v} = \boldsymbol{v}_j$ for some j and $\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{Q}\boldsymbol{v} = \lambda_j$
  - Proof below

Visualizing Principal Components

- Principal components: The eigenvectors of $\boldsymbol{Q}$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$
  - Always normalized $\|\boldsymbol{v}_j\| = 1$
  - Sorted by eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$
  - Each vector is of dimension p
- Key property: Vectors are orthogonal
  - $\boldsymbol{v}_j^T \boldsymbol{v}_k = 0$ if $j \neq k$
- Represents directions of decreasing variance:
  - $\boldsymbol{v}_1$: PC 1 = Direction of max variance
  - $\boldsymbol{v}_2$: PC 2 = Direction of second most variance
  - $\boldsymbol{v}_3$: PC 3 = Direction of third most variance
  - …



Proof PCs = Eigenvectors of $\boldsymbol{Q}$

- PC constrained optimization problem:

$$\max_{\boldsymbol{v}} \boldsymbol{v}^{\boldsymbol{T}} \boldsymbol{Q} \boldsymbol{v} \quad \text{s.t.} \ \|\boldsymbol{v}\| = 1$$

- Define Lagrangian: $L(\boldsymbol{v}, \lambda) = \boldsymbol{v}^{\boldsymbol{T}} \boldsymbol{Q} \boldsymbol{v} - \lambda[\|\boldsymbol{v}\|^2 - 1\,]$
- At any local maxima:

$$\frac{\partial L}{\partial v} = 0 \Rightarrow \boldsymbol{Q}\boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{0}$$

- This shows that $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$

In-Class Exercise

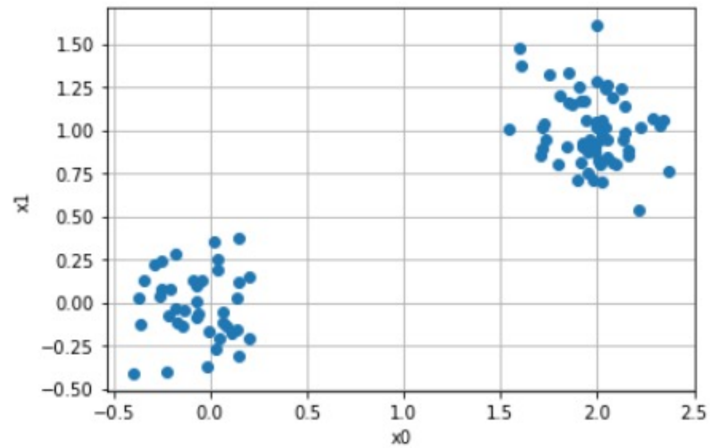
Exercise 1

We begin by showing how to compute and visualize PCs manually on a simple 2-dim synthentic data set. First, run the following code to generate 100 samples of synthetic data. Each data point has `d=2` dimensions.

```
p = 0.5
std = 0.2
s = np.array([2,1])
d = 2
ns = 100

U = np.random.normal(0,std,(ns,d))
v = np.random.uniform(0,1,ns)
X = U + (v < p)[:,None]*s[None,:]
```

Create a scatter plot of the data of the two features of the data, `X[:,0]` and `X[:,1]`.



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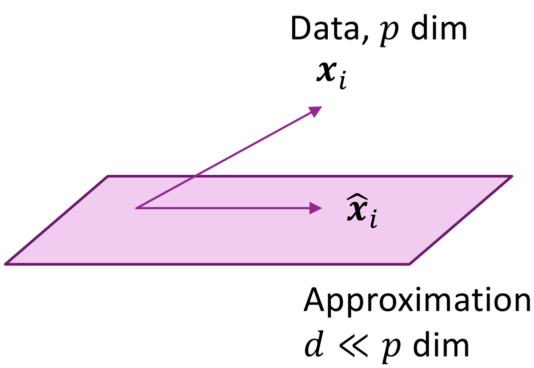
Low-Dimensional Representations

- Given data $\boldsymbol{x}_i, i = 1, \ldots, N$. Each $\boldsymbol{x}_i \in \mathbb{R}^p$
- Problem: Find basis vectors $\boldsymbol{v}_j, j = 1, \ldots, d$ such that:

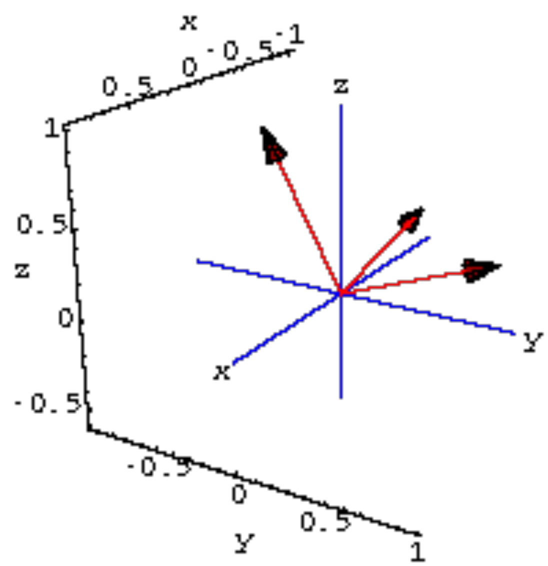
$$\boldsymbol{x}_i \approx \widehat{\boldsymbol{x}}_i = \overline{\boldsymbol{x}} + \sum_{j=1}^{d} \alpha_{ij} \boldsymbol{v}_j$$

  - Sample mean + linear combination of basis vectors
  - $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{id})$ is an approximate coordinates of $\boldsymbol{x}_i$ in basis $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$
- Dimensionality reduction:
  - If $d \ll p$ we have represented $\boldsymbol{v}_i$ with a smaller number of coefficients.



Orthonormal Sets and Bases

- Definition: A set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ are an orthonormal set if:
  - $\|\boldsymbol{v}_j\| = 1$ for all j (unit length)
  - $\boldsymbol{v}_j^T \boldsymbol{v}_k = 0$ if $j \neq k$ (perpendicular to one another)
- Matrix form: If $\boldsymbol{V} = [\boldsymbol{v}_1 \ \ldots \boldsymbol{v}_d]$, then $\boldsymbol{V}^T\boldsymbol{V} = I_d$
- If $d = p$ then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ is called an orthonormal basis
  - $\boldsymbol{V}$ is an orthogonal matrix
- Key property: the PCs form an orthonormal basis



Coefficients in an Orthonormal Basis

- Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ is an orthonormal basis
- Given a vector $\boldsymbol{z}$, can write

$$\boldsymbol{z} = \sum_{j=1}^{p} \alpha_j \boldsymbol{v}_j\,, \qquad \alpha_j = \boldsymbol{v}_j^T \boldsymbol{z}$$

  - Simple expression for computing coefficients in an orthonormal basis
- Matrix form:

$$\boldsymbol{\alpha} = \boldsymbol{V}^T \boldsymbol{z}, \qquad \boldsymbol{z} = \boldsymbol{V}\alpha$$

Approximating the Data Matrix

- Given data $\boldsymbol{x}_i, i = 1, \dots, N$
- Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ be the PCs
- Find coefficient expansion of each data sample:

$$\boldsymbol{x}_i = \overline{\boldsymbol{x}} + \sum_{j=1}^{p} \alpha_{ij} \boldsymbol{v}_j \,, \qquad \alpha_{ij} = \boldsymbol{v}_j^T (\boldsymbol{x}_i - \overline{\boldsymbol{x}})$$

- Approximation with d coefficients:

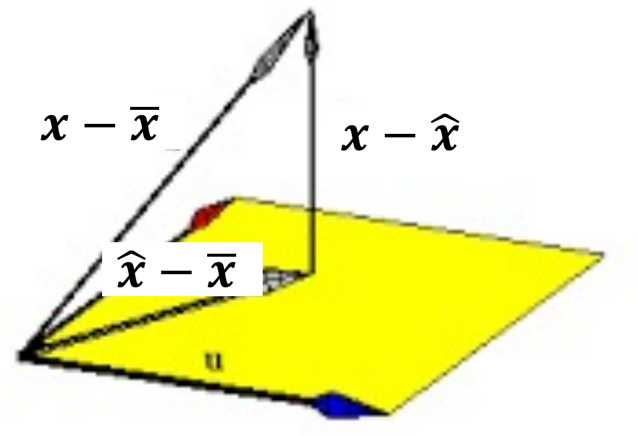
$$\widehat{\boldsymbol{x}}_i = \overline{\boldsymbol{x}} + \sum_{j=1}^{d} \alpha_{ij} \boldsymbol{v}_j$$

Geometry of Approximations

- Approximation can be interpreted geometrically
- Let V be set of all linear combinations

$$\sum_{j=1}^{d} \alpha_j \boldsymbol{v}_j$$

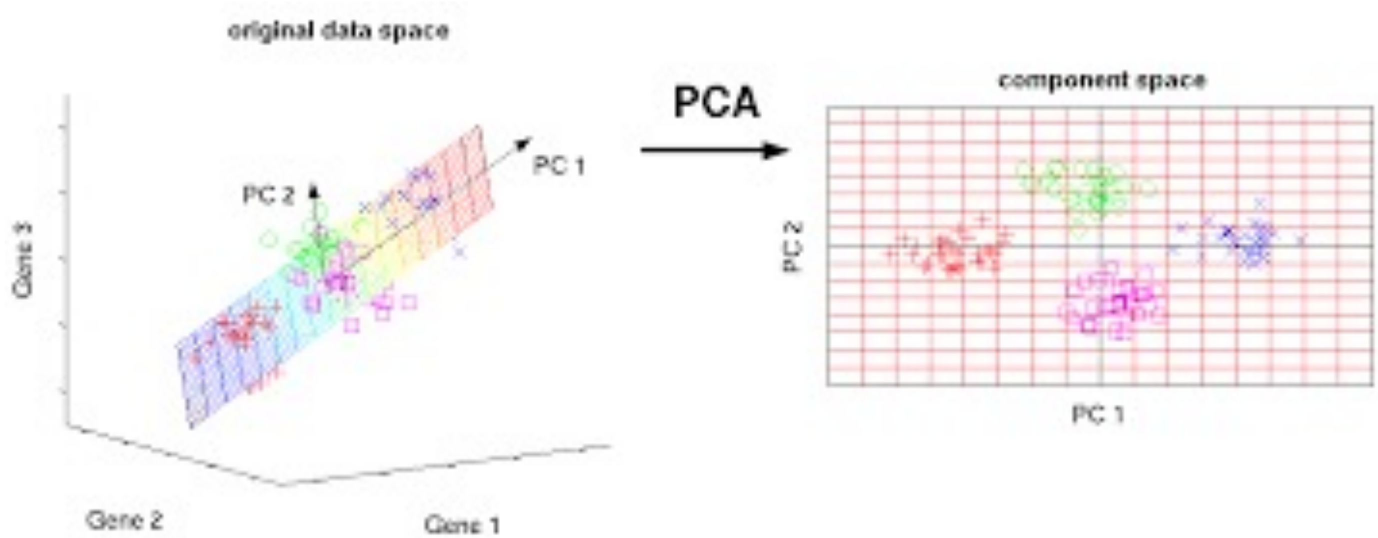
  - V is a vector space
  - Called the span of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$
- $\hat{\boldsymbol{x}} - \overline{\boldsymbol{x}}$ is the closest vector in V to $\boldsymbol{x} - \overline{\boldsymbol{x}}$
  - Note the subtraction of the mean



Space spanned by $v_1, \ldots, v_d$

Visualizing the Representation

- Finds a low-dimensional representation



Example Calculation

- Problem:
  - Let $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}[1,1,0]$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{6}}[1,-1,2]$
  - Show v_1 and v_2 are orthonormal
- Solution:
  - $\boldsymbol{v}_1^T\boldsymbol{v}_1 = \frac{1}{2}(1^2 + 1^2 + 0^2)$
  - $\boldsymbol{v}_2^T\boldsymbol{v}_2 = \frac{1}{6}(1^2 + (-1)^2 + 2^2) = 1$
  - $\boldsymbol{v}_1^T\boldsymbol{v}_2 = \frac{1}{\sqrt{2(3)}}(1(1) + 1(-1) + 0(2)) = 0$

Example Calculation Continued

❑ Problem:

- Let $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}[1,1,0]$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{6}}[1,-1,2]$ be two PCs
- Let $\overline{\boldsymbol{x}} = [0,1,2]$ be the mean of the data
- Find the approximation of $\boldsymbol{x} = [2,4,4]$ with the two PCs

❑ Solution:

- Subtract mean: $\boldsymbol{x} - \overline{\boldsymbol{x}} = [2,3,2]$
- Coeff on PC1: $\alpha_1 = \boldsymbol{v}_1^T(\boldsymbol{x} - \overline{\boldsymbol{x}}) = \frac{1}{\sqrt{2}}[2 + 3 + 0] = \frac{5}{\sqrt{2}}$
- Coeff on PC2: $\alpha_2 = \boldsymbol{v}_2^T(\boldsymbol{x} - \overline{\boldsymbol{x}}) = \frac{1}{\sqrt{6}}[2 - 3 + 4] = \frac{3}{\sqrt{6}}$
- Approximation: $\widehat{\boldsymbol{x}} = \overline{\boldsymbol{x}} + \sum_{j=1}^{d} \alpha_j \boldsymbol{v}_j = [0,1,2] + \frac{5}{2}[1,1,0] + \frac{3}{6}[1,-1,2] \approx [3,3,3]$

Average Approximation Error

- Change base of $\boldsymbol{x}_i - \overline{\boldsymbol{x}}$ to eigen vectors of sample variances:

$$\boldsymbol{x}_i - \overline{\boldsymbol{x}} = \sum_{j=1}^{p} \alpha_{ij} \boldsymbol{v}_j$$

Eigen vectors sorted by decreasing order of eigenvalues

- Let $\widehat{\boldsymbol{x}}_i$ = approximation with d PCs

$$\widehat{\boldsymbol{x}}_i - \overline{\boldsymbol{x}} = \sum_{j=1}^{d} \alpha_{ij} \boldsymbol{v}_j$$

- Error in sample i:

$$\boldsymbol{x}_i - \widehat{\boldsymbol{x}}_i = \sum_{j=d+1}^{p} \alpha_{ij} \boldsymbol{v}_j$$

- Theorem: Average error with a d PC approximation is:

$$\frac{1}{\mathrm{N}} \sum_{i=1}^{N} \|\boldsymbol{x}_i - \widehat{\boldsymbol{x}}_i\|^2 = \sum_{j=d+1}^{p} \lambda_j$$

Sum of the smallest $p - d$ eigenvalues

Proportion of Variance (PoV)

- Total variance of data set: Trace of covariance matrix

$$\frac{1}{\mathrm{N}}\sum_{i=1}^{N}\|\boldsymbol{x}_i-\overline{\boldsymbol{x}}\|^2=\sum_{j=1}^{p}\lambda_j$$

- Average approximation error:

$$\frac{1}{\mathrm{N}}\sum_{i=1}^{N}\|\boldsymbol{x}_i-\widehat{\boldsymbol{x}}_i\|^2=\sum_{j=d+1}^{p}\lambda_j$$

- The proportion of variance explained by d PCs is:

$$PoV(d)=\frac{\sum_{j=1}^{d}\lambda_j}{\sum_{j=1}^{p}\lambda_j}$$

  - Measure of approximation error in using d PCs

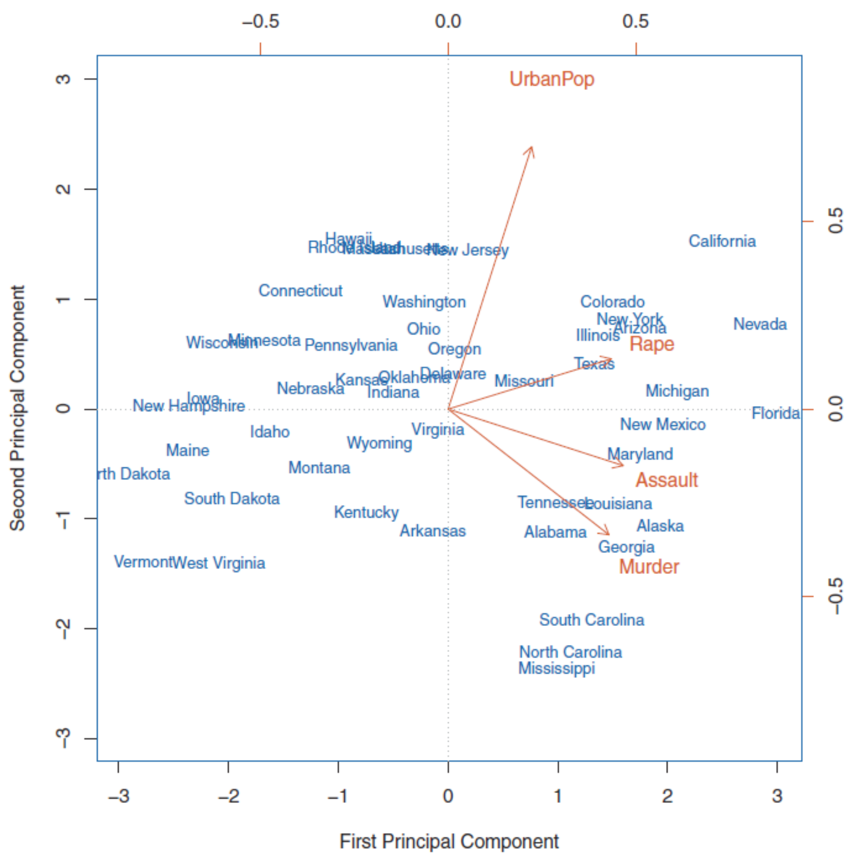
Example

PC index	λ_i	POV(i)
1	10	10/14.3≈ 0.70
2	4	14/14.3≈ 0.98
3	0.2	14.2/14.3≈ 0.99
4	0.1	14.3/14.3= 1

Latent Representations

- Each record is of the form: $\boldsymbol{x}_i \approx \overline{\boldsymbol{x}} + \sum_{j=1}^{d} \alpha_{ij} \boldsymbol{v}_j$
- Variance in $\boldsymbol{x}_i$ explained by small number of "latent components"
  - Coefficients α_{ij} are the latent representations of $\boldsymbol{x}_i$
- Example:
  - $\boldsymbol{x}_i$ = list of movie preferences for customer i
  - Movie preferences are highly correlated.
  - Could be explained by small number of components (action, romance, presence of stars, …)
  - PCA can be used to find these out

Example: USArrests



- ❑ Arrests per capita in four categories
  - One record per US state
- ❑ Visualize PCA in a biplot
  - See the scores (i.e. coefficients of each state)
  - Loading (PC vectors)
- ❑ Fig from ISL 10.1

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In-Class Exercise

Exercise 2: Computing the Approximation Error

We now verify the approximation errors. For each `k` :

- Compute `Xhat` the PC approximation of `X` using `k` PC coefficients
- Compute the approximation error, `err[k]` the average error `1/ns sum_{ij} (X[i,j]-Xhat[i,j])**2)`
- Compute the expected approximation error, `err_pred[k]` the expected approximation error based on the eigenvalues `lam` .

Remember you will need to sort the eigenvalues in descending order. You can use the command

```
I = np.argsort(lam)[::-1]
```

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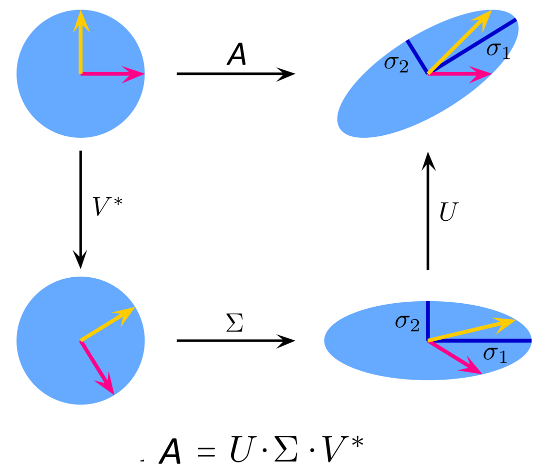
Singular Value Decomposition

- SVD: Powerful method in linear algebra

- Given a matrix A:
  - Decomposes the matrix into a product: $A = USV^T$
  - Provides orthonormal bases of the input and output spaces
  - Multiplication of A is equivalent to scaling in that basis

- For PCA:
  - Identifies low rank subspaces for data
  - Computes coefficients in that subspace



Singular Value Decomposition Defined

- Given matrix $\boldsymbol{A} \in \mathbb{F}^{n \times d}$
  - For PCA, this will be a scaled version of the data matrix

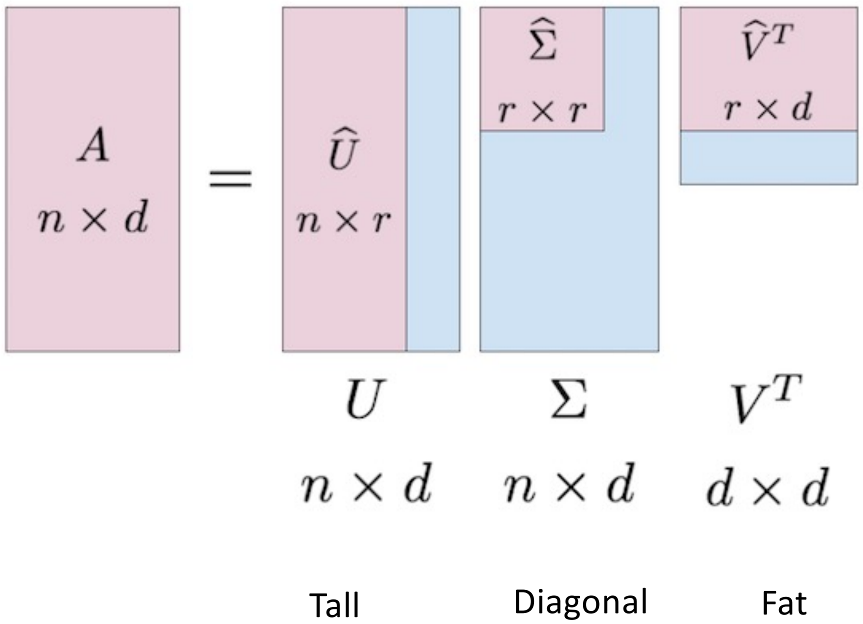
- SVD is $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, where
  - $\boldsymbol{U} \in \mathbb{F}^{n \times r}$, columns are orthonormal
  - $\boldsymbol{V} \in \mathbb{F}^{d \times r}$, columns are orthonormal
  - $\boldsymbol{\Sigma} = \mathrm{diag}(s_1, \ldots, s_r)$, sorted $s_1 \geq s_2 \geq \cdots \geq s_r \geq 0$.
  - Called the singular values
- All matrices have an SVD
  - Matrices do not have to be square.
- Number of singular values $r \leq \min(n, d)$

Economy vs. Full SVD

- Suppose $\boldsymbol{A} \in \mathbb{R}^{n \times d}$ with rank $r \leq \min\{n, d\}$
- Two types of SVDs
- Economy SVD: $A = USV^*$
  - $U \in \mathbb{F}^{n \times r}$, columns are orthonormal
  - $V \in \mathbb{F}^{d \times r}$, columns are orthonormal
  - $\Sigma \in \mathbb{F}^{r \times r}$ diagonal $\Sigma = diag(s_1, \ldots, s_r)$,
- Full SVD: $A = USV^*$
  - $U \in \mathbb{F}^{n \times n}$, columns are an orthonormal basis of $\mathbb{R}^n$
  - $V \in \mathbb{F}^{d \times d}$, columns are an orthonormal basis of $\mathbb{R}^d$
  - $\Sigma \in \mathbb{F}^{n \times d}$ with diagonal upper left $\Sigma = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$

SVD Visualized

- ❑Pink matrices represent “economy” SVD
- ❑Blue represent “full SVD”



Example

- Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}$
- Then can check that $A = U\Sigma V^*$

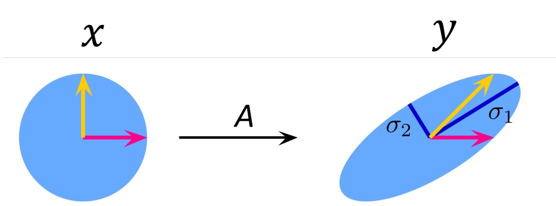
$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

  - Also verify that $UU^* = I_5$ and $VV^* = I_5$
  - This can be found by (cleverly) permute the rows of A
  - But, in general, use a computer to compute SVD

Geometric Interpretation

- Matrix can be viewed as a linear transformation/map

$$y = Ax$$

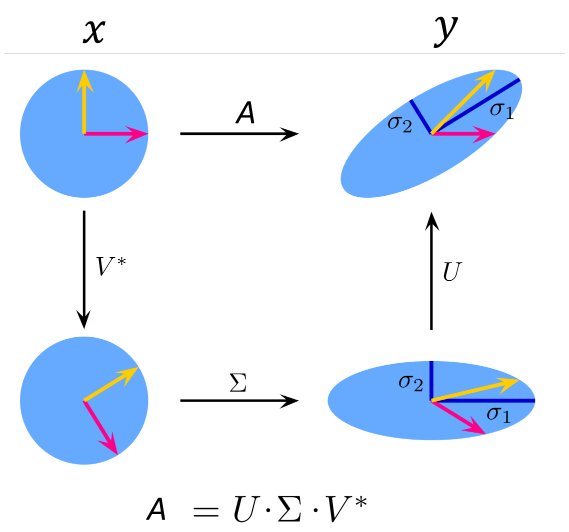


- Transform a vector x to a vector y
  - $T: R^p \rightarrow R^n$
  - The columns of A are the coordinates of the transformed basis vectors

- *Check Essence of linear algebra on Youtube by 3Blue1Brown*
  - *https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab*

Geometric Interpretation

- Let $A = U\Sigma V^*$ and $y = Ax$
- Consider a transformed space
  - $\boldsymbol{w} = \boldsymbol{V}^*\boldsymbol{x} = [w_1, \dots, w_N]$ coefficients in input basis $V = [v_1, \dots, v_N]$
  - $\boldsymbol{z} = \boldsymbol{U}^*\boldsymbol{y} = [z_1, \dots, z_M]$: coefficients in output basis $U = [u_1, \dots, u_M]$
- Then: $\boldsymbol{z} = \boldsymbol{\Sigma}\boldsymbol{w}$ so $z_i = \sigma_i w_i$
- Each input direction $\boldsymbol{v}_i$ is mapped to $\sigma_i \boldsymbol{u}_i$
- Consequence:
  - SVD finds orthonormal bases U, V such that matrix A is a linear scaling in each basis vector



Example Problem

- Suppose that $A = U\Sigma V^* \in \mathbb{R}^{3\times 4}$ with $\Sigma = diag(3,0.2,0,0)$
- If $\boldsymbol{x} = 2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 + 4\boldsymbol{v}_3 + 5\boldsymbol{v}_4$ find $y = Ax$ in terms of basis u_1, u_2, u_3
- Solution:
  - $\boldsymbol{A}\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ for all i
  - Therefore,

$$
\begin{aligned}
\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} &= 2\boldsymbol{A}\boldsymbol{v}_1 + 3\boldsymbol{A}\boldsymbol{v}_2 + 4\boldsymbol{A}\boldsymbol{v}_3 + 5\boldsymbol{A}\boldsymbol{v}_4 \\
&= 2(3)\boldsymbol{u}_1 + 3(0.2)\boldsymbol{u}_2 + 4(0)\boldsymbol{u}_3 \\
&= 6\boldsymbol{u}_1 + 0.6\,\boldsymbol{u}_2
\end{aligned}
$$

Computing the SVD in Python

- Random matrix

```
# Create some random matrix
A = np.random.normal(0,1,(100,10))
```

- Full SVD

```
# Full SVD
U,s,Vtr = np.linalg.svd(A)
```

```
A.shape   = (100, 10)
U.shape   = (100, 100)
s.shape   = (10,)
Vtr.shape = (10, 10)
```

- Economy SVD

```
# Economy SVD
U,s,Vtr = np.linalg.svd(A, full_matrices=False)
```

```
U.shape   = (100, 10)
s.shape   = (10,)
Vtr.shape = (10, 10)
```

- Reconstruction:

```
# Recovers back A
Ahat = (U*s[None,:]).dot(Vtr)
```

Computing the PCA via SVD

- Let $\boldsymbol{A} = \frac{1}{\sqrt{N}}\widetilde{\boldsymbol{X}}$ = scaled data matrix with sample mean removed.
- Take SVD: $\boldsymbol{A} = \mathbf{U\Sigma V}^T$
- Properties:
  - Sample covariance matrix is $\boldsymbol{Q} = \frac{1}{N}\widetilde{\boldsymbol{X}}^{\boldsymbol{T}}\widetilde{\boldsymbol{X}} = \boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A} = \boldsymbol{V\Sigma}\mathbf{U}^T\boldsymbol{U\Sigma V}^T = \boldsymbol{V\Sigma}^{\mathbf{2}}\boldsymbol{V}^T$
  - Eigenvalues of $\mathbf{Q}$ = squared singular values of $\boldsymbol{A}$
  - PCs are $\boldsymbol{v}_j$, columns of $\boldsymbol{V}$
  - Coefficients are $\boldsymbol{Z} = \widetilde{\boldsymbol{X}}\boldsymbol{V} = \sqrt{N}\boldsymbol{AV} = \sqrt{N}\boldsymbol{U\Sigma}$
- Hence, SVD provides PCs, eigenvalues coefficients $\boldsymbol{Z}$ in the PCA representation.

In-Class Exercise

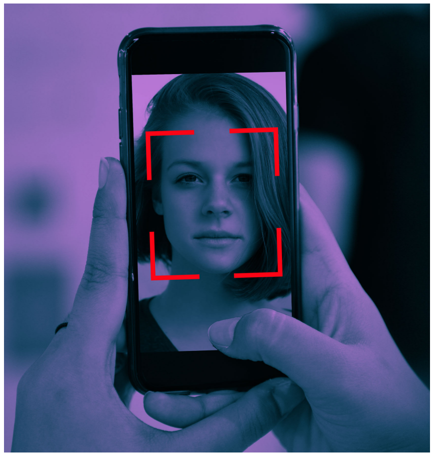
Exercise 3: Computing the PCA via the SVD ¶

- Compute the matrix `A = 1/np.sqrt(ns)*Xm`
- Compute the economy SVD of `A` with the `np.linalg.svd()` command. Use `full_matrices=False` option. Print the dimensions of the components of the SVD

Outline

- Dimensionality reduction
- Principal components and directions of variance
- Approximation with PCs
- Computing PCs via the SVD
- Face recognition using PCA in python
- Training models from PCs
- Low rank approximations and recommender systems

Example: Face Recognition



Labeled Faces in the Wild Home

- Face recognition challenges:
  - Face images can be high-dimensional
  - We will use 50 x 37 = 1850 pixels
- Applying PCA:
  - Should be few degrees of freedom
  - Can transform to lower dimensional representations
- Data Labelled Faces in the Wild project
  - http://vis-www.cs.umass.edu/lfw
  - Large collection of faces (13000 images)
  - Taken from web articles about 20 years ago

Loading the Data

- Built-in routines to load data is sciket-learn
- Can take several minutes the first time (Be patier

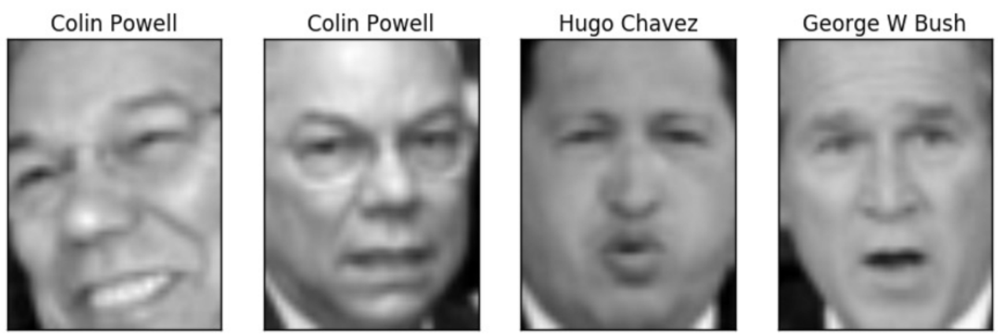
```
Image size     = 50 x 37 = 1850 pixels
Number faces   = 1288
Number classes = 7
```

```
from sklearn.datasets import fetch_lfw_people
lfw_people = fetch_lfw_people(min_faces_per_person=70, resize=0.4)
```

```
2016-11-14 14:15:30,862 Downloading LFW metadata: http://vis-www.cs.umass.edu/lfw/pairsDevTrain.txt
2016-11-14 14:15:30,958 Downloading LFW metadata: http://vis-www.cs.umass.edu/lfw/pairsDevTest.txt
2016-11-14 14:15:31,028 Downloading LFW metadata: http://vis-www.cs.umass.edu/lfw/pairs.txt
2016-11-14 14:15:31,294 Downloading LFW data (~200MB): http://vis-www.cs.umass.edu/lfw/lfw-funneled.tgz
2016-11-14 14:20:10,056 Decompressing the data archive to C:\Users\Sundeep\scikit_learn_data\lfw_home\lfw_funneled
2016-11-14 14:22:08,605 Loading LFW people faces from C:\Users\Sundeep\scikit_learn_data\lfw_home
2016-11-14 14:22:09,735 Loading face #00001 / 01288
2016-11-14 14:22:13,640 Loading face #01001 / 01288
```

Plotting the Data

- Some example faces
- You may be too young to remember them all




```python
def plt_face(x):
    h = 50
    w = 37
    plt.imshow(x.reshape((h, w)), cmap=plt.cm.gray)
    plt.xticks([])
    plt.yticks([])

I = np.random.permutation(n_samples)
plt.figure(figsize=(10,20))
nplt = 4;
for i in range(nplt):
    ind = I[i]
    plt.subplot(1,nplt,i+1)
    plt_face(X[ind])
    plt.title(target_names[y[ind]])
```

Computing the PCA

```
npix = h*w
Xmean = np.mean(X,0)
Xs = X - Xmean[None,:]
```

```
U,S,Vtr = np.linalg.svd(Xs, full_matrices=False)
```

- Manually compute the PCs with SVD
  - Remove the mean
  - Use broadcasting
  - Compute the SVD

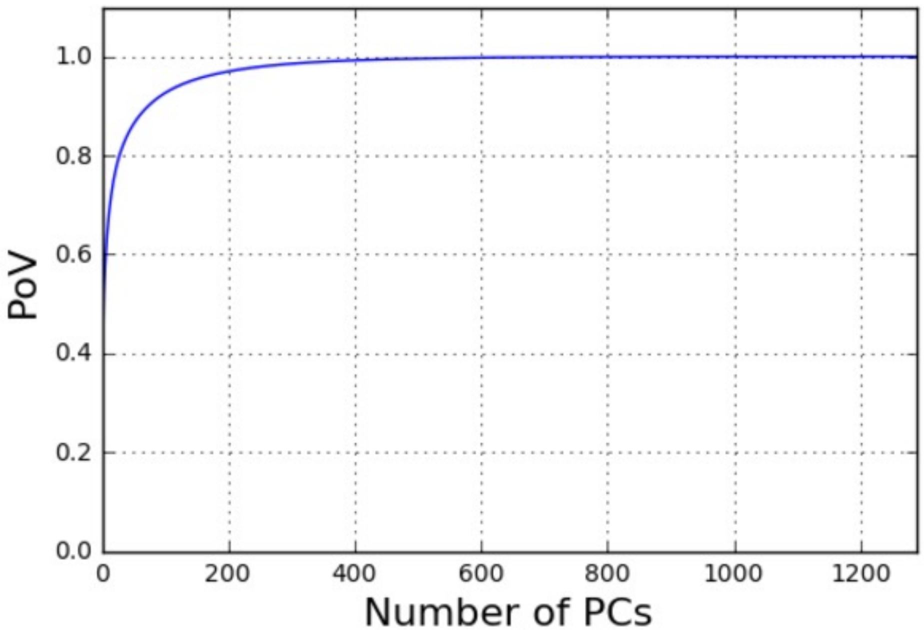
```
from sklearn.decomposition import PCA

# Construct the PCA object
pca = PCA(n_components=ncomp,
          svd_solver='randomized', whiten=True)

# Fit the PCA components on the entire dataset
pca.fit(X)
```

- Use sklearn builtin PCA function
  - Construct a PCA object
  - Call fit: Computes mean and PC components
  - Stores values internally in the pca class

Finding the PoV



- Most variance explained in about 400 components
- Some reduction

```
lam = S**2
PoV = np.cumsum(lam)/np.sum(lam)

plt.plot(PoV)
plt.grid()
plt.axis([1,n_samples,0, 1.1])
plt.xlabel('Number of PCs', fontsize=16)
plt.ylabel('PoV', fontsize=16)
```

Plotting Approximations

```
nplt = 2                # number of faces to plot
ds = [0,5,10,20,100]    # number of SVD approximations
use_pca = True          # True=Use sklearn reconstruction, else use SVD
```

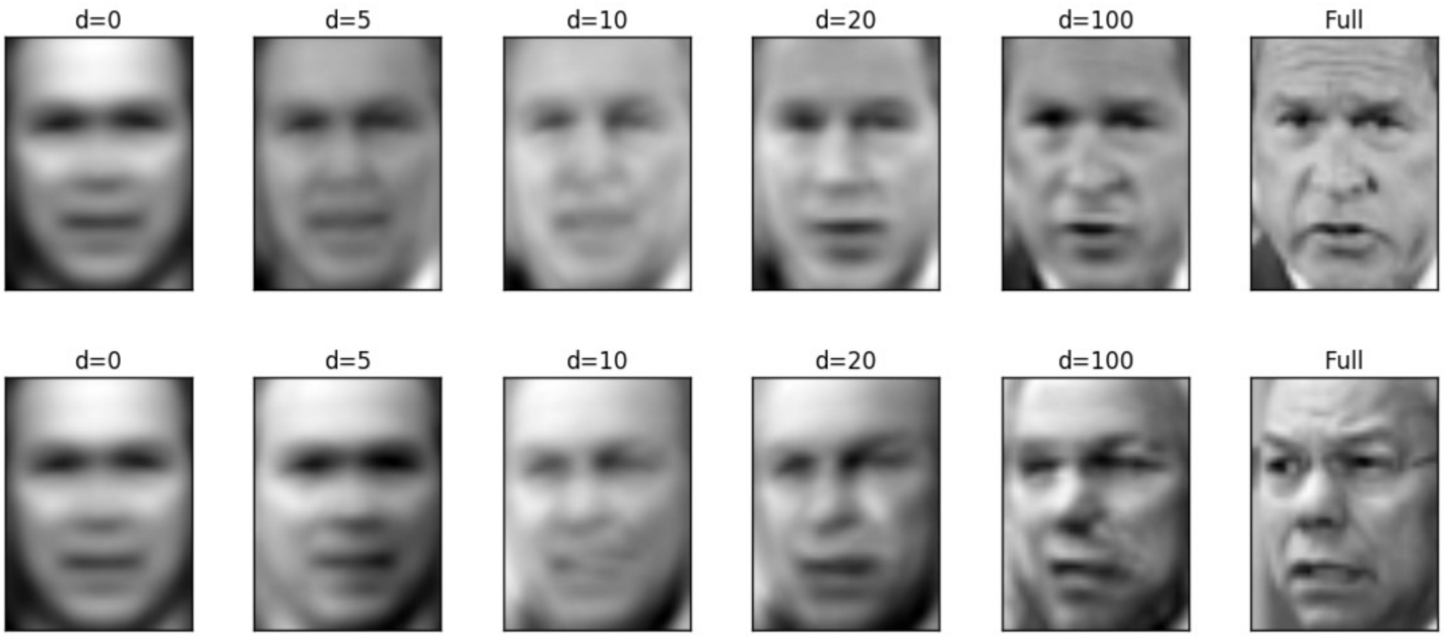
```
# Loop over figures
iplt = 0
for ind in inds:
    for d in ds:
        plt.subplot(nplt,nd+1,iplt+1)
        if use_pca:
            # Zero out coefficients after d.
            # Note, we need to copy to not overwrite the coefficients
            Zd = np.copy(Z[ind,:])
            Zd[d:] = 0
            Xhati = pca.inverse_transform(Zd)
        else:
            # Reconstruct with SVD
            Xhati = (U[ind,:d]*S[None,:d]).dot(Vtr[:d,:]) + Xmean

        plt_face(Xhati)
        plt.title('d={0:d}'.format(d))
        iplt += 1

    # Plot the true face
    plt.subplot(nplt,nd+1,iplt+1)
    plt_face(X[ind,:])
    plt.title('Full')
    iplt += 1
```

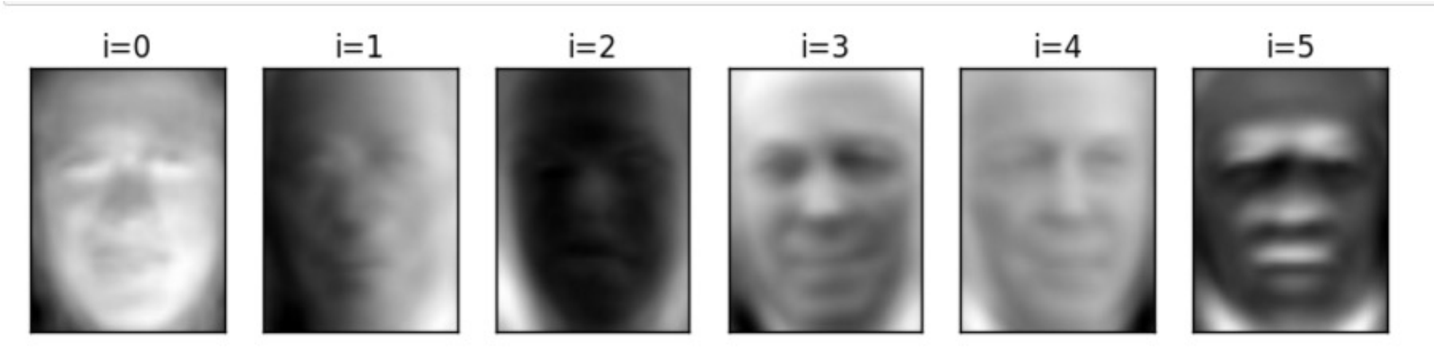
- Reconstruction using sklearn method
  - Uses the inverse_transform method
- Reconstruction using SVD
  - Note use of broadcasting

Plotting the Approximations



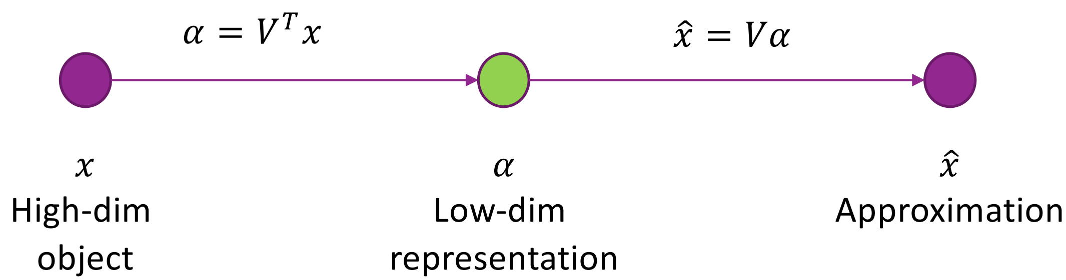
Plotting the PCs

- The PCs can be plotted as well



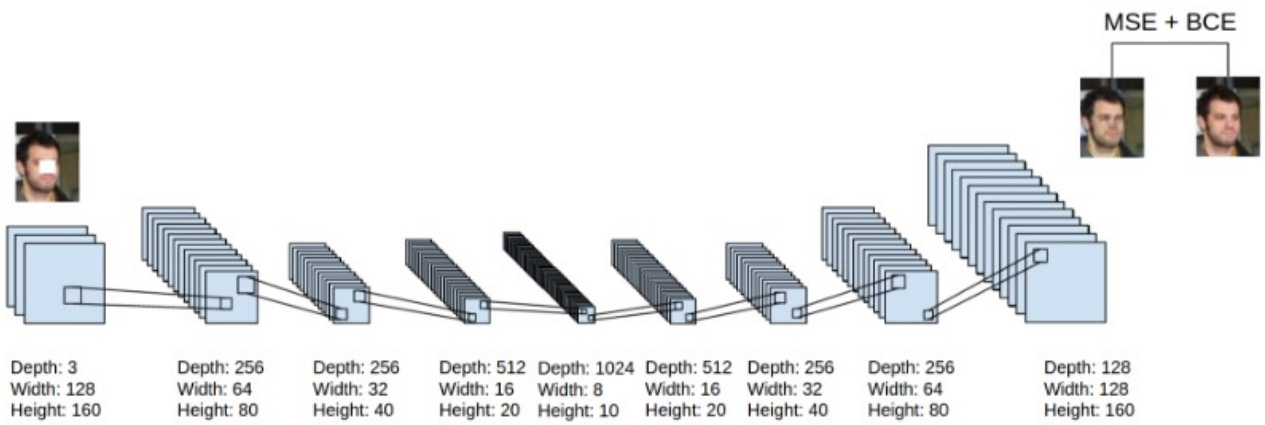
State-of-the-Art: Auto-Encoders

- PCA is a simple example of an autoencoder
- Tries to find low-dim representation
- Restricted to linear transforms
- Not very good for images and complex data



Deep Auto-Encoders

- Can use deep networks for learning complex latent representations and their inverses
  - http://www.cc.gatech.edu/~hays/7476/projects/Avery_Wenchen/
  - https://swarbrickjones.wordpress.com/2016/01/13/enhancing-images-using-deep-convolutional-generative-adversarial-networks-dcgans/ (Code in Theano not tensorflow)



Outline

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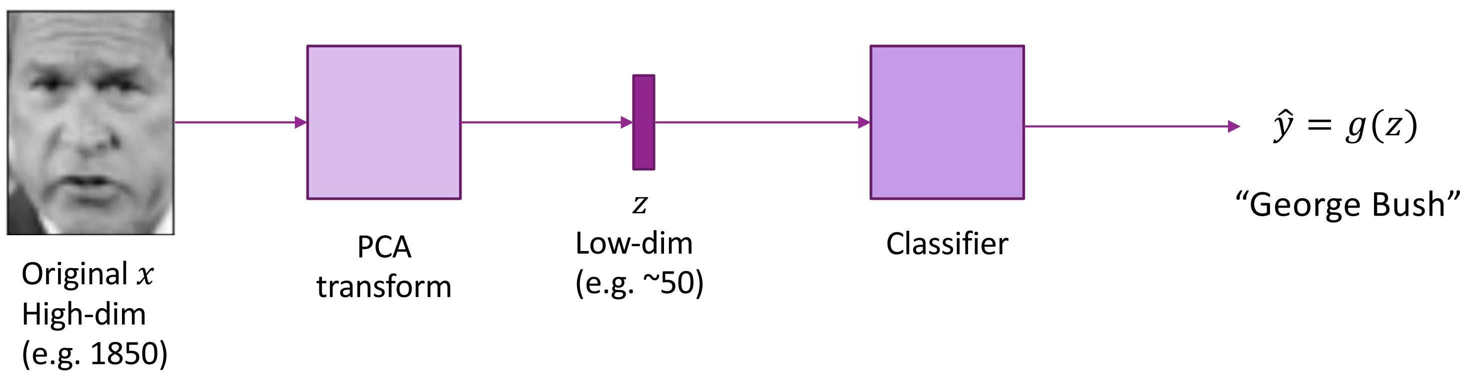
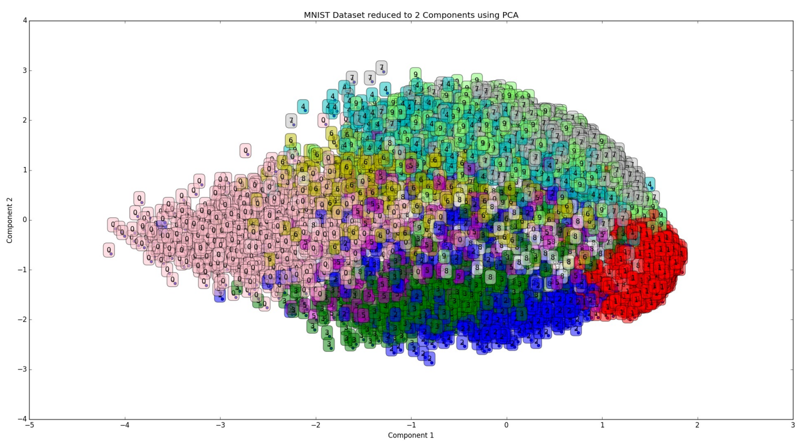
Classification Using PCs



- Many problems: Dimensionality of data x is too large
  - Classifier in original space will have too many parameters
- Key idea:
  - Learn a dimension reducing transform via PCA: $z = f(x)$
  - Train classifier on low-dim transform $\hat{y} = g(z)$

Why This Would Work?



- PCA works if:
  classes are separable in transformed domain

- Example to right:
  - MNIST digits plotted in two PCs
  - Can mostly separate the classses

Training and Testing

- Split data in training and test: $X_{tr}, y_{tr}, X_{ts}, y_{ts}$
- Fit PCA transform on $Z = g(X)$ on training data X_{tr}
  - Do not include test data in PCA fit!
  - Many students make this mistake.
- Transform training and test:
  - $Z_{tr} = g(X_{tr}),\ Z_{ts} = g(X_{ts})$
- Fit classifier $\hat{y} = f(z)$ on transformed training data (Z_{tr}, y_{tr})
- Predict classifier on transformed test data: $\hat{y}_{ts} = f(Z_{ts})$
- Score error rate / MSE on test data: $\epsilon = \frac{1}{N} \#\{\hat{y}_{ts}^{i} \neq y_{ts}^{i}\}$

Cross-Validation

- To find number of PCs and other parameters use cross-validation
- Split data in training and test: $X_{tr}, y_{tr}, X_{ts}, y_{ts}$
- For each set of parameters:
  - Fit PCA transform on $Z = g(X, \mathrm{numPCs})$ on training data X_{tr}
  - Transform training and test: $Z_{tr} = g(X_{tr}),\ Z_{ts} = g(X_{ts})$
  - Fit classifier $\hat{y} = f(z)$ on transformed training data (Z_{tr}, y_{tr})
  - Predict classifier on transformed test data: $\hat{y}_{ts} = f(Z_{ts})$
  - Score (e.g. error rate / MSE) on test data: $\epsilon = \frac{1}{N} \#\{\hat{y}_{ts}^{i} \neq y_{ts}^{i}\}$
- Select the parameters with lowest score

Example: SVM classification with PCAs

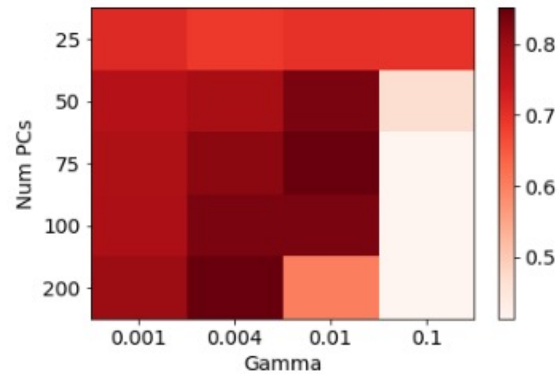
```
npc_test = [25,50,75,100,200]
gam_test = [1e-3,4e-3,1e-2,1e-1]
C = 100
n0 = len(npc_test)
n1 = len(gam_test)
acc = np.zeros((n0,n1))
acc_max = 0

for i0, npc in enumerate(npc_test):

    # Fit PCA on the training data
    pca = PCA(n_components=npc, svd_solver='randomized', whiten=True)
    pca.fit(Xtr)

    # Transform the training and test
    Ztr = pca.transform(Xtr)
    Zts = pca.transform(Xts)

    for i1, gam in enumerate(gam_test):

        # Fiting on the transformed training data
        svc = SVC(C=c, kernel='rbf', gamma = gam)
        svc.fit(Ztr, ytr)

        # Predict on the test data
        yhat = svc.predict(Zts)

        # Compute the accuracy
        acc[i0,i1] = np.mean(yhat == yts)
        print('npc=%d gam=%12.4e acc=%12.4e' % (npc,gam,acc[i0,i1]))
```

- ❑ Parameters to search
  - Number of PCs and gamma
- ❑ Fit on the training data.
  - This is in the loop!
- ❑ Transform the data
- ❑ Fit classifier on transformed training data
- ❑ Test on the transformed test data
- ❑ Score on test data

Example: Parameter Search

- Search over:
  - Number of PCs $\in \{25,50,75,100,200\}$
  - $\gamma \in \{0.001, 0.004, 0.01, 0.1\}$
- Plotted is the test accuracy
- Best test accuracy $\approx 85\%$

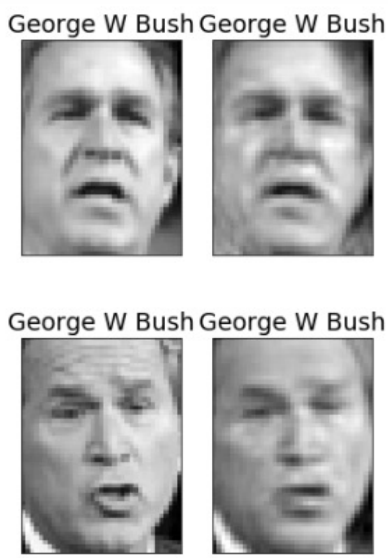
Test Accuracy




```
Optimal num PCs = 75
Optimal gamma   = 0.010000
```

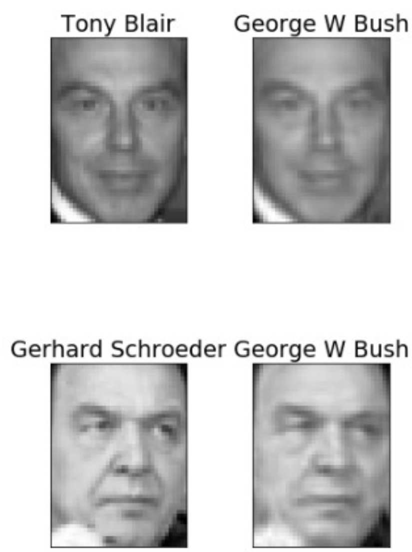
Examples

- Correct images



Original Reduced

- Error images



Original Reduced

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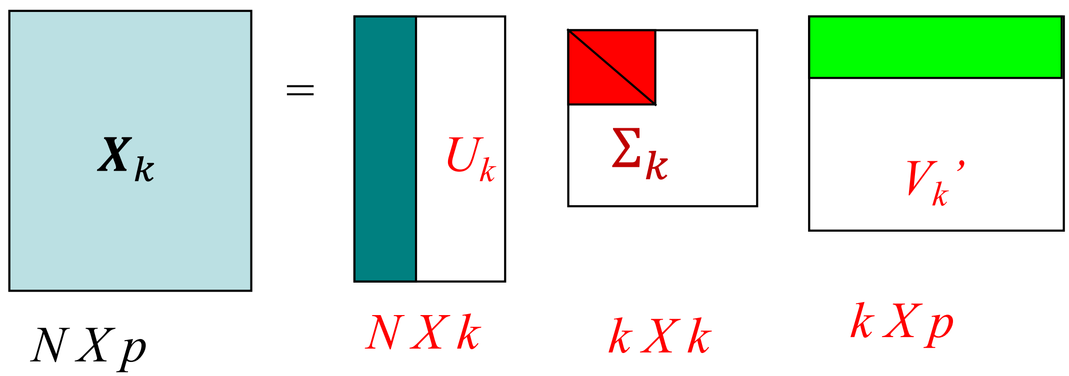
Low-Rank Approximations

- SVD can be used for a low-rank approximation
- SVD can be written: $\boldsymbol{X} = \mathbf{U\Sigma V}^T = \sum_{j=1}^{r} \alpha_j \boldsymbol{u}_j \boldsymbol{v}_j^T$
- Consider k −term approximation: $\boldsymbol{X_k} = \sum_{j=1}^{k} \alpha_j \boldsymbol{u}_j \boldsymbol{v}_j^T$
- Properties:
  - $\boldsymbol{X}_k$ is rank k
  - $\boldsymbol{X}_k = \boldsymbol{U_k \Sigma_k V_k^T}$
  - Error is $\|X - X_k\|_F^2 = \sum_i \sum_j \left(X_{ij} - X_{k,ij}\right)^2 = \sum_{j=k+1}^{r} \alpha_j^2$
  - If $s_{k+1}, \ldots, s_r$ is small then matrix is well approximated by rank k matrix

Low-Rank Approximation Visualized



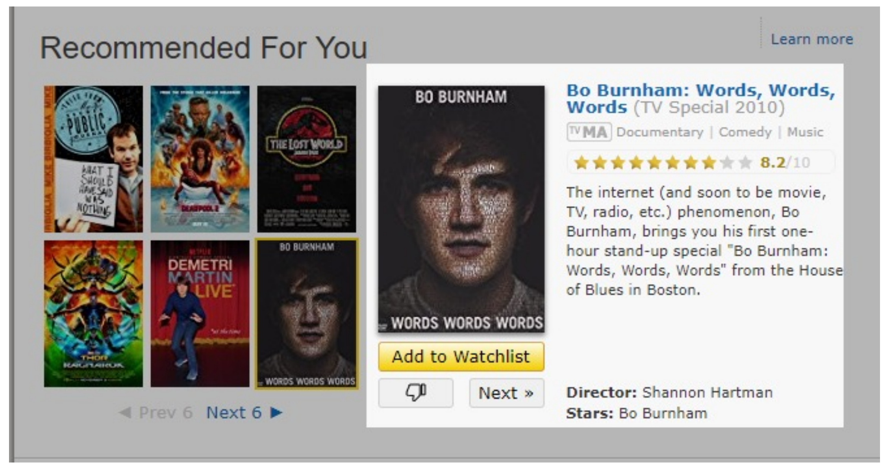
- Can show: Reconstructed matrix X_k is optimal rank k approximation

Recommender Systems

- How do you recommend a movie to a user?
- MovieLens dataset:
  - Get past ratings from users
  - Make recommendations for future

t[3]:

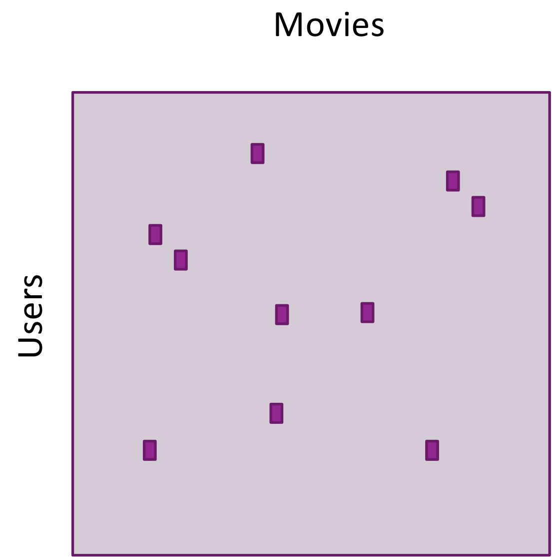
	movieId	title	genres
0	1	Toy Story (1995)	Adventure|Animation|Children|Comedy|Fantasy
1	2	Jumanji (1995)	Adventure|Children|Fantasy
2	3	Grumpier Old Men (1995)	Comedy|Romance
3	4	Waiting to Exhale (1995)	Comedy|Drama|Romance
4	5	Father of the Bride Part II (1995)	Comedy



Ratings Matrix

- Data can be represented as ratings matrix
  - Users x movies
- Problem: Most users have only rated a small fraction
- Need to estimate unseen entries
  - Very sparse
- How can we do complete this matrix

Name	Dates	Users	Movies	Ratings	Density
ML Latest	'95 – '16	247,753	34,208	22,884,377	0.003%
ML Latest Small	'96 – '16	668	10,329	105,339	0.015%



Latent Factor Model for Ratings

- Idea: Ratings for movies dependent on small number of latent factors
  - E.g. Action, famous actors, genre, …
- Mathematically model as:

$$R_{ij} \approx \hat{R}_{ij} = b_i^u + b_j^m + \sum_{k=1}^{K} A_{ik} B_{jk}$$

  - R_{ij} =Rating of movie j by user i
  - b_i^u =Bias of user i
  - b_j^m =Bias of movie j
  - K = number of latent factors. Typically small $K \ll N_{user}, N_{movies}$
  - A_{ik} = Preference of user i to factor k
  - B_{jk} = Component of factor k in movie j

More to be added

- These slides are still under construction.
- More will be added on low rank approximations and embedding layers.