Soliton: preliminaries

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Abstract

This document has an outline of a report with figures, with the sections that your report should contain. The abstract should have a paragraph encapsulation of the motivation, methods, and major results of the report.

1 Introduction

Relevant equation:

$$\partial_t u + \epsilon u \partial_x u + \mu \partial_x^3 u = 0, \tag{1}$$

2 Prep work

[Shadi: I can write up some solutions to the prep work section today.]

We first consider the simpler problem, namely the advection equation for $u(x,t)$ given by

$$\partial_t u + \epsilon \partial_x u = 0, \tag{2}$$

were we have suppressed functional dependence and ϵ is a constant.

It is straightforward to verify that *any* function of the form

$$u(x,t) = f(x - \epsilon t), \tag{3}$$

is a solution of Eq. (2). Plugging in this ansatz we have

$$\begin{aligned} \partial_t u + \epsilon \partial_x u &= \partial_t f(x-\epsilon t) + \epsilon \partial_x f(x - \epsilon t), \\ &= \frac{\partial f(x-\epsilon t)}{\partial (x - \epsilon t)} \frac{\partial (x-\epsilon t)}{\partial t} + \epsilon \frac{\partial f(x-\epsilon t)}{\partial (x-\epsilon t)} \frac{\partial (x - \epsilon t)}{\partial x}, \\ &= -\epsilon f' + \epsilon f' = 0. \end{aligned} \tag{4}$$

Here, the prime denotes differentiation relative to the variable $\xi := x - \epsilon t$.

The solution denotes an arbitrarily shaped wave-form for the height of the water that is being carried, without changing its shape, by a constant velocity ϵ. Suppose that we turn off dissipation in the Kortweg-de-Vries (KdV) equation (1) by setting $\mu = 0$, then the simplified equation takes a similar form to the advection equation

$$\partial_t u + \epsilon u \partial_x u = 0, \tag{5}$$

with the only difference being the dependence on the shape of u in the second term. So, we expect similar solutions that are carried with velocity ϵ in the non-dissipative regime, with the caveat that the velocity becomes dependent on the height of the wave itself precisely due to the non-linear dependence on u in the second term of the above equation.

Indeed, we may be more quantitative by characterizing these solutions. Let us plug in the ansatz in Eq. (3) in the KdW equation, with $\epsilon \to c$. By similar reasoning to the solution verification of the advection equation, we obtain

$$\begin{aligned}\partial_t u + \epsilon u \partial_x u + \mu \partial_x^3 u &= \partial_t f + \epsilon f \partial_x f + \mu \partial_x^3 u, \\ &= f' \frac{\partial(x-ct)}{\partial t} + \epsilon f f' \frac{\partial(x-ct)}{\partial x} + \mu f''' \frac{\partial(x-ct)}{\partial x}, \\ &= -cf' + \epsilon f f' + \mu f''' = 0. \end{aligned} \tag{6}$$

The above equation characterizes the waveforms that are solutions to the KdW equation in the non-dissipative regime, and that are of the form $u = f(x - ct)$ for some constant c.

To get a feel for this equation, we may integrate this once relative to ξ to obtain

$$-cf + \frac{\epsilon}{2} f^2 + \mu f'' = C_1,$$

where C_1 is an integration constant. Rearranging gives

$$\mu f'' = C_1 + cf - \frac{\epsilon}{2} f^2.$$

As physicists, we will attempt to solve the simplest possible case, so we set both $C_1 = 0$, multiply throughout by f', and integrate once more relative to ξ to get

$$\frac{1}{2}(f')^2 = \frac{c}{2\mu} f^2 - \frac{\epsilon}{6\mu} f^3 = \frac{f^2}{\mu}\left(c - \frac{\epsilon}{3} f\right).$$

Taking the square root and integrating on both sides, we obtain

$$\int \frac{\sqrt{\mu} df}{f\sqrt{c - \frac{\epsilon}{3} f}} = \int d\xi.$$

We perform the integral on the left via a substitution

$$f = \frac{3c}{\epsilon} \operatorname{sech}^2(z),$$

under which the integration measure becomes

$$df = -\frac{6c}{\epsilon} \frac{\sinh(z)}{\cosh(z)^3} dz.$$

The square root in the denominator simply evaluates to a $\tanh z$, and there is a massive simplification yielding

$$-\frac{2}{\sqrt{c}} \sqrt{\mu} z = \xi - \xi_0,$$

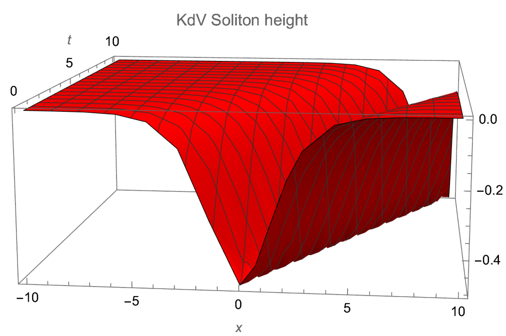
where ξ_0 is an integration constant. Solving for z and plugging into the expression for f, we obtain the general solution to the KdW equation as

$$f = \frac{3c}{\epsilon}\text{sech}^2\left[\frac{\sqrt{c\mu}}{2}(\xi - \xi_0)\right].$$

Plugging this back into the equation, we find that this will be a solution for any c, ϵ as long as $\mu = \pm 1$. For $\mu = 1$ and $\epsilon = -6$, we obtain

$$f = -\frac{c}{2}\text{sech}^2\left[\frac{\sqrt{c}}{2}(\xi - \xi_0)\right]. \tag{7}$$

We plot this in Fig. (2) for $c = 1$ and $\xi_0 = 0$.



3 Introduction

4 Methods

4.1 Formulation of the problem

4.2 Computational methods

5 Results

6 Discussion