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Winter 2020 Assignment 1

COMP 233 Assignment 1 Solution

Question 1 Roll two fair dice. Then the sample space S is the following.

$$S = \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}$$

Let E be the event that the sum of the dice is odd, let F be the event that the first die lands on 1, and let G be the event that the sum is 5.
Describe the following events:

Solution:

$F = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$ $G = \{(1,4), (2,3), (3,2), (4,1)\}$

$E = \{(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)\}$

a) $E \cap F = \{(1,1), (1,3), (1,5)\}$
b) $E \cup F = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6),$
$(4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5), \}$
c) $F \cap G = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (3,2), (4,1), \}$
d) $E \cap F^c = \{(2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2),$
$(5,4), (5,6), (6,1), (6,3), (6,5)\}$
e) $EFG = \{(1,4)\}$

Question2

(a) Prove that $P(EF^c) = P(E) - P(EF)$
(b) Prove that $P(E^cF^c) = 1 - P(E) - P(F) + P(EF)$
(c) Show that the probability that exactly one of the events E or F occurs is equal to $P(E)+P(F)-2P(EF)$

Solution:

a) $P(E \cap F^c) = P(E) - P(E \cap F)$
$\frac{N(E \cap F^c)}{N(S)} = \frac{N(E)}{N(S)} - \frac{N(E \cap F)}{N(S)}$
$N(EF^c) = N(E) - N(E \cap F)$
$EF^c = E - EF$
$E \cdot (1 - F) = E - EF$
$E - EF = E - EF$

b) $P(E^c \cap F^c) = P((E \cup F)^c) = 1 - P(E \cup F) = 1 - \big(P(E) + P(F) - P(EF)\big)$
   $P(E^c \cap F^c) = 1 - P(E) - P(F) + P(EF)$

c) The probability that exactly 1 of the events E or F occurs is equal to the union of both events minus their intersection. Which translates to:
   $P(E \cup F) - P(E \cap F) = P(E) + P(F) - P(E \cap F) - P(E \cap F)$
   $= P(E) + P(F) - 2P(EF)$

Question 3 Consider a team of eleven (11) soccer players, all of whom are equally good players and can play any position.

(a) Suppose that the team has just finished regulation time for a play-off game and the score is tied with the other team. The coach has to select five players for penalty kicks to decide which team wins the game. Since each player takes penalty kicks differently, the order in which the players are arranged for the penalty kicks is important and can affect the outcome. How many different ways can the coach select five (5) players to take the penalty kicks?

(b) A couple of weeks later, the coach wants to form two (2) teams of five (5) from the eleven players on the team for a scrimmage game (i.e., just a small practice game where player positions are not important). The eleventh player will act as the referee. How many ways can the coach divide the team into two teams of five players?

(c) Another week later, the coach wants to test the players to be able to select a captain for the team. Therefore, again the coach wants to form two (2) teams of five (5) from the eleven players on the team for a scrimmage game, with the eleventh player again acting as the referee, but with a small change. The first person chosen for a team of five will be the captain of the team and will have extra responsibilities. For the rest of the players, their roles and positions are not important. How many ways can the coach divide the team into two teams of five players with one captain for each team?

Solution:

a) $P_5^{11} = \frac{11!}{(11-5)!} = 55440$ ways

b) $C_{10}^{11} = \frac{11!}{(11-10)!10!} = 11$

c) $\binom{11}{1}\binom{10}{1}\binom{9}{1}\binom{8}{4}\binom{4}{4} = 69300$ ways

Question 4 A red die, a blue die, and a yellow die (all six-sided) are rolled. We are interested in the probability that the number appearing on the blue die is less than that appearing on the yellow die which is less than that appearing on the red die. (That is, if B (R) [Y] is the number appearing on the blue (red) [yellow] die, then we are interested in P(B < Y < R).)

(a) What is the probability that no two of the dice land on the same number?
(b) Given that no two of the dice land on the same number, what is the conditional probability that B < Y < R?
(c) What is P(B < Y < R)?
(d) If we regard the outcome of the experiment as the vector B, R, Y, how many outcomes are there in the sample space?
(e) Without using the answer to (c), determine the number of outcomes that result in B < Y < R.
(f) Use the results of parts (d) and (e) to verify your answer to part (c).

Solution:

a) $1 \cdot \frac{5}{6} \cdot \frac{4}{6} = \frac{20}{36}$

b) $P(B|Y|R) = \frac{P\left(B \cdot P(Y|R)\right)}{P(Y|R)} = \frac{P\left(B \cdot \frac{P(YR)}{P(R)}\right)}{\frac{P(YR)}{P(R)}} = \frac{P\left(B \cdot \frac{\frac{4}{6} \cdot \frac{5}{6}}{\frac{5}{6}}\right)}{\frac{\frac{4}{6} \cdot \frac{5}{6}}{\frac{5}{6}}} = \frac{P\left(\frac{\frac{4}{6} \cdot \frac{5}{6}}{\frac{5}{6}}\right)}{\frac{\frac{4}{6} \cdot \frac{5}{6}}{\frac{5}{6}}}$

c) Type equation here.

d)

e) $\binom{6}{6}\binom{6}{5}\binom{6}{4}$

f)

Question 5 Suppose that distinct integer values are written on each of three (3) cards. Suppose that you are to be offered these cards in a random order. When you are offered a card you must immediately either accept it or reject it. If you accept a card, the process ends. If you reject a card, then the next card (if a card remains) is offered. If you reject the first two cards offered, then you must accept the final card.

(a) If you plan to accept the first card offered, what is the probability that you will accept the highest valued card?
(b) If you plan to reject the first card offered, and to then accept the second card if and only if its value is greater than the value of the first card, what is the probability that you will accept the highest valued card?

Solution:

a) $\frac{1}{3}$

b) Let A: Probability to Accept second card
Let H: Probability that second Card is higher than the first one

$$P(A|H) = \frac{P(H|A)P(A)}{P(H)} = \frac{\frac{P(HA)}{P(A)}P(A)}{P(H)}$$

$$P(A|H) = \frac{P(H|A)P(A)}{P(H)} = \frac{\frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{2}}\cdot\frac{1}{2}}{\frac{1}{3}} = \frac{1}{2}$$

Question 6 The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) For what value of λ is *f(x)* a probability density function?
(b) What is the probability that a computer will function between 50 and 150 hours before breaking down?
(c) What is the probability that it will function less than 100 hours?

Solution:

a) We observe that f is non-negative if λ is so:

$$\int_{-\infty}^{\infty} f(x)dx = 1 \rightarrow \int_{0}^{\infty} \lambda e^{-\frac{x}{100}}dx = 1$$

$$\rightarrow -100\lambda e^{\frac{-x}{100}} + C = 1$$

$$\rightarrow -100\lambda e^{\frac{-x}{100}} = 1 \rightarrow 100\lambda = 1 \rightarrow \lambda = \frac{1}{100}$$

For $\lambda = \frac{1}{100}$ this function will be a PDF

b) $\int_{50}^{150} \frac{1}{100} e^{-\frac{x}{100}}dx = 0.38340$

c) $\int_{0}^{100} \frac{1}{100} e^{-\frac{x}{100}}dx = 0.63212$

Question 7 The joint probability density function of random variables X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \qquad 0 < x < 1, \qquad 0 < y < 2$$

(a) Verify that this is indeed a joint density function.
(b) Compute the density function of X.
(c) Find P{X > Y}.

a) $f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$ is a joint density function if the area under the curve is = to 1. Therefor:

$$\int_0^2 \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dxdy$$

$$\int_0^2 \frac{2x^3}{7} + \frac{3yx^2}{14}\bigg|_0^1 dy \ = 1$$

$$\int_0^2 \frac{2}{7} + \frac{3y}{14} dy = 1 \rightarrow \left. \frac{2y}{7} + \frac{3y^2}{28} \right|_0^2 = 1$$

$$\frac{4}{7} + \frac{12}{28} = 1$$

Hence this function is a joint density function

b) Integrate $f(x, y)$ over y:

$$\int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy = \left. \frac{6x^2y}{7} + \frac{6xy^2}{28} \right|_0^2 = \frac{12x^2}{7} + \frac{6x}{7}$$

c) $P\{X > Y\} = 1 - P\{X < Y\}$. And thus, we have:

$$1 - \int_0^y \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx$$

$$= 1 - \int_0^y \frac{12x^2}{7} + 144x\, dx$$

$$1 - \left. \frac{12x^3}{21} + 72x^2 \right|_0^y = 1 - \frac{12y^3}{21} + 72y^2$$