Lecture 6 - 7

1.3 Differential Equations as Mathematical Models

Definition 1. The mathematical description of a system or phenomenon is called a **mathematical model**.

Steps in Mathematical Modeling

1. Construction of the model. Translate the physical situation into mathematical terms
2. Analysis of the model
3. Comparison with experiment or observation

2.7 Linear models

Growth and decay models

Example 1. Bacteria growth

A culture initially has P_0 number of bacteria. After 2 hours the number of bacteria has tripled. If the rate of growth of bacteria is proportional to the number of bacteria $P(t)$ present at time t, determine

(a) the time necessary for the number of bacteria to double;

(b) number of bacteria in 10 hours if the number of bacteria is $5 \cdot 10^4$ after 4 hours;

(c) how fast the culture growth at $t = 10$ hours.

Example 2. Half-life of Radioactive Isotope

The radioactive isotope $Pb - 209$ decays at a rate proportional to the amount present at time t. After 5 hours 10% of the initial amount A_0 of the isotope has disintegrated. If the rate of decay is proportional to the amount remaining determine

(a) the half-life of this isotope;

(b) amount of isotope decay after 48 hours if 1.5g of isotope presented initially.

Example 3. Newton law of cooling

When a cake is removed from an oven, its temperature is measured at $300^o F$. Three minutes later its temperature is $200^o F$. How long will it take for the cake to cool off to a room temperature of $70^o F$.

Example 4. Mixture problem.

At time t_0 tank contains Q_0 gal of salt dissolved in 100 gal of water. Assume that water containing 0.25 lb of salt/gal is entering the tank at a rate of r gal/min and that the well-stirred mixture is draining from the tank at the same rate.

(a) Set up the initial value problem that describes this flow process.

(b) Find the amount of salt $Q(t)$ in the tank in any time t.

(c) Find the limiting amount Q_L that is present after long enough time.

(d) If $r = 3$ gal/min and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L.

(e) Find the flow rate that is required if the value of T is not exceed 45 min.

Example 5. Series circuits.

a 30 volt electromotive force is applied to an LR-series circuit in which the inductance is 0.1 henry and resistance is 50 ohms. Find the steady-state current.

Example 6. Logistic Growth

A model for the population $P(t)$ in a suburb of a large city is given by

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \qquad P(0) = 5 \cdot 10^3$$

where t is measured in months.

(a) What is the limiting value of the population?

(b) At what time will the population be equal to a one-half of this limiting value?

Example 7. Second Order Chemical Reaction

Two chemicals **A** and **B** are combined to form a chemical **C**. The rate of the reaction is proportional to the product of instantaneous amount of **A** and **B** not converted to chemical **C**. Initially there are 40 gr of **A** and 50 gr of **B**, and for each gr of **B**, 2 gr of **A** is used. It is observed that 10 gr of **C** is formed in 5 min.

(a) How much is formed in 20 min?

(b) What is the limiting amount of **C** after a long time?

(c) How much chemicals **A** and **B** remains after a long time?

Example 8. Leaking Tank

A tank in the form of a right circular cylinder standing on end is leaking water through a circular hole in its bottom. If the initial height $H = 4$m, diameter of base of the tank is 20cm and diameter of the hole is 1cm, how ling will it take to empty the tank?