Lecture 10
Ch. 3 Higher-order differential equations
3.1 Theory of linear equations

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\dots a_1(x)y'(x)+a_0(x)y(x)=g(x)$$

There are two kinds of additional conditions:

1. initial conditions;
2. boundary conditions.

Definition 1. Initial value problem for a second order differential equation
Solve

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x), \qquad x \in I$$

subject to

$$y(x_0) = y_0, \qquad y'(x_0) = y_1,$$

where y_0 and y_1 are given.

Definition 2. Boundary value problem for a second order differential equation
Solve

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x), \qquad x \in I$$

subject to

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1,$$
$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

where α_i and β_i γ_i are given.

Homogeneous equations

Definition 3. The linear differential equation

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\dots a_1(x)y'(x)+a_0(x)y(x)=g(x), \;\; x\in$$

is said to be homogeneous if $g(x)\equiv 0$, and nonhomogeneous if $g(x)\neq 0$.

Assumptions

1. coefficients $a_i(x)$ $(i=\overline{0,n})$ are continuous functions on I;
2. $g(x)$ is continuous on I;
3. $a_n(x)\neq 0$ for every $x\in I$.

Differential operators

Definition 4. Operator $D = \frac{d}{dx}$ is called **a differential operator**.

Definition 5. n-th order differential operator

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x)$$

Theorem 1. Superposition principle
Let $y_1(x)$, $y_2(x)$, $\ldots$, $y_n(x)$ be solutions of the homogeneous differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots a_1(x)y'(x) + a_0(x)y(x) = 0.$$

Then the linear combination

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x),$$

where c_i $(i = \overline{1, n})$ are arbitrary constants, is also a solution of the homogeneous differential equation.

Linear dependence/independence

Definition 6. A set of functions $\{f_i(x)\}_{i=1}^n$ is said to be **linearly independent** on interval I if

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

if and only if $c_i = 0$ for $i = \overline{1, n}$.
If there exist constants c_1, c_2, …, c_n not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0,$$

then $\{f_i(x)\}_{i=1}^n$ is said to be **linearly dependent**.

Fundamental set of solutions

Definition 7. Suppose $f_i(x) \in C^{n-1}(I)$ $(i = \overline{1, n})$. The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

is called **the Wronskian** of $\{f_i(x)\}_{i=1}^n$.

Theorem 2. Set of functions $\{f_i(x)\}_{i=1}^n$ is linearly independent if and only if $W(f_1, f_2, \dots, f_n) \neq 0$ for every $x \in I$.

Definition 8. Set $\{y_i(x)\}_{i=1}^n$ of linearly independent solutions of the n-th order differential equation is said to be **a fundamental set of solutions** on the interval I.

Theorem 3. Existence of a set of fundamental solutions
There exists a fundamental set of solutions for the homogeneous linear differential equation on interval I.

Theorem 4. General solution of a homogeneous equation
Let $\{y_i(x)\}_{i=1}^n$ be a fundamental set of of solutions of the homogeneous linear n-th order differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots a_1(x)y'(x) + a_0(x)y(x) = 0$$

on interval I. Then the general solution of the equation on interval I is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x),$$

where c_i $(i = \overline{1, n})$ are arbitrary constants.

Nonhomogeneous equation

$$L(y) = a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\ldots a_1(x)y'(x)+a_0(x)y(x) = g(x$$

Definition 9. Any function $y_p(x)$ free of the arbitrary parameters such that $L(y) = g(x)$ is said to be **a particular solution** of the differential equation $L(y) = g(x)$.

Theorem 5. Solution of nonhomogeneous equation
Let $y_p(x)$ be any particular solution of the nonhomogeneous n-th order differential equation on interval I. And let $\{y_i(x)\}_{i=1}^{n}$ be a fundamental set of solutions of the associated homogeneous differential equation $Ly = 0$. Then the general solution of the nonhomogeneous equation $Ly = g(x)$ on interval I is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x),$$

where c_i $(i = \overline{1, n})$ are arbitrary constants.

Definition 10. The linear combination

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

which is a general solution of homogeneous equation $Ly = 0$ is said to be **a complimentary function** for nonhomogeneous equation $Ly = g(x)$.

Theorem 6. Superposition principle
Let $y_{p1}(x)$, $y_{p2}(x)$, ..., $y_{pk}(x)$ be k particular solutions of the nonhomogeneous linear n-th order differential equation on interval I corresponding k distinct functions $g_1(x)$, $g_2(x)$, ..., $g_k(x)$. Then

$$y_p(x) = y_{p1}(x) + y_{p2}(x) + \ldots + y_{pk}(x)$$

is a particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots a_1(x)y'(x) + a_0(x)y(x) = g_1(x) + g_2(x) + \ldots + g_k(x).$$

Dynamical systems

Definition 11. A dynamical system whose mathematical model is

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

is said to be **a linear dynamical system**, $y(t)$, $y'(t)$, ..., $y^{(n-1)}(t)$ are **the state variables** of the system, $g(t)$ is **the input function**, $y(t)$ is **a response** of the system.