Lecture 3
2.2 Separable Equations

Definition 1. A first order differential equation of the form

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

or

$$M(x) \cdot N(y)dx + P(x) \cdot Q(y)dy = 0$$

is said to be **a separable** or to have separable variables.

Method of solution

1. Divide the equation by $N(y) \cdot P(x)$

$$\frac{M(x)}{P(x)}dx + \frac{Q(y)}{N(y)}dy = 0.$$

2. Integrate the last equation directly.

2.3 Linear equations

Definition 2. A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be **a linear differential equation** in the dependent variable y.

Definition 3. When $g(x) \equiv 0$, the linear differential equation is said to be **homogeneous**, otherwise it is **nonhomogeneous**.

Definition 4. A first order linear differential equation of the form

$$\frac{dy}{dx} + p(x)y = f(x)$$

is said to be **a standard form** of differential equation.

$$p(x) = a_0(x)/a_1(x), \qquad f(x) = g(x)/a_1(x).$$

Solution of a first order linear differential equation

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a solution of corresponding homogeneous equation, called **general solution** and $y_p(x)$ is a particular solution of nonhomogeneous equation.

General solution

$$y_c' + p(x)y_c = 0$$

$$\frac{y_c'}{y_c} + p(x) = 0$$

By direct integration, we obtain

$$y_c(x) = ce^{-\int p(x)dx}.$$

Particular solution of nonhomogeneous equation

$$y_p' + p(x)y_p = f(x)$$

By substituting $y_p(x) = C(x) \cdot y_1(x)$, where $C(x)$ is unknown function, into differential equation, we find $C'(x)$ from the equation, that is

$$C'(x) = \frac{f(x)}{y_1(x)}$$

and

$$C(x) = \int \frac{f(x)}{y_1(x)} dx = \int e^{\int p(x)dx} \cdot f(x)dx.$$

Thus

$$y_p(x) = e^{-\int p(x)dx} \cdot \int e^{\int p(x)dx} \cdot f(x)dx.$$

Definition 5. Term $e^{\int p(x)dx}$ is called **integrating factor**.

Method of solution

Way I

1. Put a linear differential equation into a standard form and determine an integrating factor $e^{\int p(x)dx}$.
2. Multiply the differential equation in the standard form by the integrating factor. As a result,

$$\frac{d}{dx}\left[e^{\int p(x)dx} \cdot y(x)\right] = f(x)e^{\int p(x)dx}.$$

3. Integrate both sides of the resulting differential equation.

Way II

1. Put a linear differential equation into a standard form.
2. Find the general solution $y_c(x)$ of the corresponding homogeneous differential equation.
3. Using the method of variation of a parameter, find the particular solution $y_p(x)$ of nonhomogeneous differential equation.

Definition 5. The values of x for which $a_1(x) = 0$ are called **singular points** of the differential equation.