Lectures 8 - 9
Ch. 17 Functions of a complex variable
17.1 Complex numbers

Definition 1. A complex number is any number of the form

$$z = a + ib$$

where a and b are real numbers, and $i = \sqrt{-1}$ is **the imaginary unit**.

$$Re(z) = a, \quad Im(z) = b.$$

Definition 2. A real constant multiply by i is called **pure imaginary number**.

Definition 3. Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal $z_1 = z_2$ if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$

Arithmetic operations

$$z_1 = a_1 + ib_1, \qquad z_2 = a_2 + b_2$$

1. Addition $\quad z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
2. Subtraction $\quad z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
3. Multiplication $\quad z_1 \cdot z_2 = a_1 a_2 - b_1 b_2 + (b_1 a_2 + a_1 b_2)i$
4. Division $\quad \frac{z_1}{z_2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$

Algebraic laws

1. Commutative laws $$z_1 + z_2 = z_2 + z_1$$ $$z_1 \cdot z_2 = z_2 \cdot z_1$$
2. Associative laws $$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$ $$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$
3. Distributive laws $$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$ $$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$

Definition 4. If $z = a + ib$ is a complex number, then $\overline{z} = a - ib$ is **the complex conjugated** to z.

Definition 5. The absolute value of a complex number $z = a + ib$ is the real number number $|z| = \sqrt{a^2 + b^2}$.

17.2 Powers and roots

$$z = a + ib$$

Definition 6. $z = r(\cos\varphi + i\sin\varphi)$ is said to be **a polar form** of complex number z.

$$a = r\cos\varphi, \qquad b = r\sin\varphi$$

$$r = \sqrt{a^2 + b^2}, \qquad \varphi = \arctan\frac{b}{a}$$

Multiplication and division

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1), \qquad z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

$$z_1 \cdot z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$$

Powers of z

$$z = r(\cos\varphi + i\sin\varphi)$$

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

De Moivre's formula

$$z = \cos\varphi + i\sin\varphi$$

$$z^n = \cos(n\varphi) + i\sin(n\varphi)$$

Roots

$$z = r(\cos\varphi + i\sin\varphi)$$

Definition 7. A number ω is said to be **an n-th root** of nonzero complex number z if $\omega^n = z$.

$$\omega_k = r^{1/n}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right), \quad k = \overline{0, n-1}.$$

Euler's formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi, \qquad e^{-i\varphi} = \cos\varphi - i\sin\varphi$$