LOGIC EXERCISES: SOLUTIONS

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PROBLEM 1. [TIME ALLOWED = 5 MINUTES]

Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

"In the year 2000, Montreal was the capital of Quebec."

SOLUTION. This is a declarative statement, and hence is a proposition. The proposition is false because in the year 2000, Quebec City, not Montreal, was the capital of Quebec.

PROBLEM 2. [TIME ALLOWED = 5 MINUTES]

Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

$$1^{232} \neq 2^{232} \text{ and } \log(1) = 1.$$

SOLUTION. This is a compound proposition in which two atomic propositions are related using the conjunction operator. It is false because the atomic proposition $1^{232} \neq 2^{232}$ is true, but the atomic proposition $\log(1) = 1$ is false.

PROBLEM 3. [TIME ALLOWED = 5 MINUTES]

Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

"Enter your password."

SOLUTION. This is an imperative sentence. It is not a proposition.

PROBLEM 4. [TIME ALLOWED = 5 MINUTES]

Give the negation of the following sentence:

"It is hot today."

SOLUTION. The negation is “It is not that case that it is hot today”, or “It is not hot today.”

Note. A statement and its negation must have opposite truth values. The negation is not “It is cold today,” because the temperature could be neither hot nor cold, making both statements false.

PROBLEM 5. [TIME ALLOWED = 5 MINUTES]

Give the negation of the following sentence:

“2 is negative.”

SOLUTION. The negation states that “It is not the case that 2 is negative”, or “2 is not negative.” This means that 2 is greater than or equal to 0, that is, “2 is nonnegative.”

Note. “2 is positive” is a correct observation, but is not the negation of the given statement. This is because saying that a number is not negative means that the number can be either 0 or positive.

PROBLEM 6. [TIME ALLOWED = 5 MINUTES]

Give the negation of the following sentence:

“The number $\sqrt{2}$ is rational.”

SOLUTION. The negation states that “It is not the case that $\sqrt{2}$ is rational”, or “The number $\sqrt{2}$ is not rational”, or “The number $\sqrt{2}$ is irrational.”

PROBLEM 7. [TIME ALLOWED = 5 MINUTES]

Give the negation of the following sentence:

“$2 + 3 = 6$.”

SOLUTION. The = sign is read “is equal to”. Therefore, “$2 + 3 = 6$.” is read “two plus three is equal to six.” The negation of the sentence is “two plus three is not equal to six.”, or “$2 + 3 \neq 6$.”

PROBLEM 8. [TIME ALLOWED = 5 MINUTES]

Construct a truth table for $p \vee \neg(p \wedge q)$.

SOLUTION.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

PROBLEM 9. [TIME ALLOWED = 5 MINUTES]

Let p and q be propositions. Give the truth value of $(p \vee q) \longrightarrow (p \wedge q)$ when both p and q are false.

SOLUTION. It is given that both p and q are false. Therefore, $(p \vee q)$ is false, and so is $(p \wedge q)$. Hence, $(p \vee q) \longrightarrow (p \wedge q)$ is true.

PROBLEM 10. [TIME ALLOWED = 5 MINUTES]

Express the following propositions using logical connectives:

(a) I will go to the movie if I complete my assignment.
(b) I will go to the movie only if I complete my assignment.
(c) I will not go to the movie if I do not complete my assignment.

SOLUTION. Let

p: I will go to the movie.
q: I complete my assignment.

Then,

(a) p if q. (This could be re-expressed as if q, then p.) Therefore, $q \longrightarrow p$.
(b) p only if q. Therefore, $p \longrightarrow q$.
(c) $\neg p$ if $\neg q$. (This could be re-expressed as if $\neg q$, then $\neg p$.) Therefore, $\neg q \longrightarrow \neg p$.

PROBLEM 11. [TIME ALLOWED = 5 MINUTES]

Express the contrapositive in English of the following sentence:

"I will buy the tickets only if you call."

SOLUTION.

"<u>I will buy the tickets</u> <u>only if</u> <u>you call</u>."

p: I will buy the tickets.
q: You call.

This could be seen as a kind of **natural language text analysis**.

The sentence represents the following implication:

$$p \longrightarrow q.$$

In other words,

"If I buy the tickets, then you call."

The contrapositive of the previous implication is:

$$\neg q \longrightarrow \neg p.$$

In other words,

"If you do not call, then I will not buy the tickets."

PROBLEM 12. [TIME ALLOWED = 5 MINUTES]

Give the contrapositive and converse of the following proposition:

"If it is sunny, then I will go swimming."

SOLUTION.

“If it is sunny, then I will go swimming.”

p: It is sunny.
q: I will go swimming.

Contrapositive:

$\neg q \longrightarrow \neg p$, or “If I do not go swimming, then it is not sunny.”

Converse:

$q \longrightarrow p$, or “If I go swimming, then it is sunny.”

PROBLEM 13. [TIME ALLOWED = 5 MINUTES]

Give the contrapositive, converse, and inverse of the following proposition:

“If the number is positive, then its square is positive.”

SOLUTION.

“If the number is positive, then its square is positive.”

p: The number is positive.
q: The square of a number is positive.

Contrapositive:

$\neg q \longrightarrow \neg p$, or “If the square of a number is not positive, then the number is not positive.”

Converse:

$q \longrightarrow p$, or “If the square of a number is positive, then the number is positive.”

Inverse:

$\neg p \longrightarrow \neg q$, or “If the number is not positive, then the square is not positive.”

PROBLEM 14. [TIME ALLOWED = 5 MINUTES]

Let p and q be the propositions:

> p : I bought a lottery ticket this week.
> q : I won the million dollar jackpot on Friday.

The proposition $\neg(\neg p \wedge \neg q)$ as an English sentence is:

(a) I did not buy a lottery ticket this week, and I did not win the million dollar jackpot on Friday.
(b) Either I bought a lottery ticket this week or I won the million dollar jackpot on Friday.
(c) Not better than good beer.
(d) None of the above.

SOLUTION.

The proposition $\neg(\neg p \wedge \neg q) \equiv p \vee q$. Therefore, the correct choice is (b), where the "or" is understood to be **inclusive-or**.

PROBLEM 15. [TIME ALLOWED = 5 MINUTES]

Let p, q, and r be the propositions:

> p : You have the flu.
> q : You miss the final examination.
> r : You pass the course.

The proposition $\neg q \leftrightarrow r$ as an English sentence is:

(a) You do not miss the final examination if and only if you pass the course.
(b) You do not pass the course if and only if you miss the final examination.
(c) I won the million dollar jackpot on Friday, and so I will not have flu and I want to miss the final examination.
(d) I hope the final examination will not be a lottery.

SOLUTION.

The proposition $\neg q \leftrightarrow r$ is a biconditional in which the first proposition is negated. Therefore, the correct choice is (a).

PROBLEM 16. [TIME ALLOWED = 5 MINUTES]

Show, (1) using a truth table, and (2) using a mathematical proof, that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are logically equivalent.

SOLUTION.

(1)

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$q \wedge \neg p$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

(2)

$\neg(p \vee \neg q)$
$\Leftrightarrow \neg p \wedge \neg\neg q$ (De Morgan's Law)
$\Leftrightarrow \neg p \wedge q$ (Double Negation Law)
$\Leftrightarrow q \wedge \neg p$ (Commutative Law)

PROBLEM 17. [TIME ALLOWED = 5 MINUTES]

State whether "n is divisible by 9" is (a) necessary, (b) sufficient, or (c) neither necessary nor sufficient for "n is divisible by 6", where n is a natural number.

SOLUTION.

In $p \longrightarrow q$, p is sufficient for q, and q is necessary for p. Therefore, a given proposition is **placed on the left for seeking sufficiency**, and **placed on the right for seeking necessity**.

In this problem, the given proposition is "n is divisible by 9".

Test for Sufficiency:

If "n is divisible by 9", then "n is divisible by 6". (If "9 | n", then "6 | n".)

This is false, as, for example, if n = 9, then 9 | 9, but 6 ∤ 9. (It is possible to have other examples, such as n = 27.)

Test for Necessity:

If "n is divisible by 6", then "n is divisible by 9". (If "6 | n", then "9 | n".)

This is false, as, for example, if n = 6, then 6 | 6, but 9 ∤ 6. (It is possible to have other examples, such as n = 12 or n = 24, but it is customary to seek for the smallest, which is also often the simplest, number. 0 would not work, and the next number is 6.)

Therefore, the solution is (c).

PROBLEM 18. [TIME ALLOWED = 5 MINUTES]

Using "laws" of logic, simplify $\neg(p \vee q) \vee (\neg p \wedge q)$.

SOLUTION.

$\neg(p \vee q) \vee (\neg p \wedge q)$
$\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q)$ (De Morgan's Law)
$\Leftrightarrow \neg p \wedge (\neg q \vee q)$ (Distributive Law)
$\Leftrightarrow \neg p \wedge \mathrm{T}$ (Complement Law)
$\Leftrightarrow \neg p$ (Identify Law)

PROBLEM 19. [TIME ALLOWED = 5 MINUTES]

Give a truth table of $\neg[(p \longrightarrow q) \wedge (q \longrightarrow p)]$. Explain.

SOLUTION.

It is preferable to simplify the logical expression, if possible, before proceeding to construct a truth table.

The given expression could be simplified by removing the $\longrightarrow$ operator from the given logical expression.

$\neg[(p \longrightarrow q) \wedge (q \longrightarrow p)]$
$\Leftrightarrow \neg(p \longrightarrow q) \vee \neg(q \longrightarrow p)$
$\Leftrightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$
$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$

Now,

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$q \wedge \neg p$	$(p \wedge \neg q) \vee (q \wedge \neg p)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

This is the same as the truth table of $\oplus$.

PROBLEM 20. [TIME ALLOWED = 5 MINUTES]

This is about the truth value of the statement: "There exist positive integers x, y, and z such that $x^2 + y^2 = z^2$."

Select one of the following:

(a) $x = 0, y = 0, z = 0$.
(b) $x = 1, y = 2, z = 3$.
(c) $x = 2, y = 3, z = 4$.
(d) $x = 3, y = 4, z = 5$.
(e) None of the above.

SOLUTION.

(d).

x, y, and z form the sides of a right-angled triangle.

PROBLEM 21. [TIME ALLOWED = 5 MINUTES]

This is about the truth value of the statement: "There exist positive integers x, y, and z such that $x^3 + y^3 = z^2$."

Select one of the following:

(a) $x = 1, y = 2, z = 3$.
(b) $x = 2, y = 2, z = 4$.
(c) $x = 3, y = 4, z = 5$.
(d) (a), but not (b).
(e) Both (a) and (b), but not (c).

SOLUTION.

(a) or (b) or (e).

PROBLEM 22. [TIME ALLOWED = 5 MINUTES FOR EACH PART]

Let P: $\{1, 2\} \times \{1, 2\} \longrightarrow \{T, F\}$. Express the following using conjunctions and disjunctions only.

(a) $\forall y\ \exists x\ P(x, y)$.
(b) $\exists y\ \forall x\ P(x, y)$.
(c) $\exists x\ \exists y\ P(x, y)$.
(d) $\forall x\ \forall y\ P(x, y)$.
(e) $\forall y\ \forall x\ P(x, y)$.
(f) $\exists x\ \forall y\ \neg P(x, y)$.

SOLUTION.

(a) $\forall y\ \exists x\ P(x, y)$.

$[\exists x\ P(x, 1)] \wedge [\exists x\ P(x, 2)]$
$\Leftrightarrow [P(1, 1) \vee P(2, 1)] \wedge [P(1, 2) \vee P(2, 2)]$.

(b) $\exists y\ \forall x\ P(x, y)$.

$[\forall x\ P(x, 1)] \vee [\forall x\ P(x, 2)]$
$\Leftrightarrow [P(1, 1) \wedge P(2, 1)] \wedge [P(1, 2) \wedge P(2, 2)]$.

(c) $\exists x\ \exists y\ P(x, y)$.

$[\exists y\ P(1, y)] \vee [\exists y\ P(2, y)]$
$\Leftrightarrow [P(1, 1) \vee P(1, 2)] \vee [P(2, 1) \vee P(2, 2)]$
$\Leftrightarrow P(1, 1) \vee P(1, 2) \vee P(2, 1) \vee P(2, 2)$.

(d) $\forall x\ \forall y\ P(x, y)$.

$[\forall y\ P(1, y)] \wedge [\forall y\ P(2, y)]$
$\Leftrightarrow [P(1, 1) \wedge P(1, 2)] \wedge [P(2, 1) \wedge P(2, 2)]$
$\Leftrightarrow P(1, 1) \wedge P(1, 2) \wedge P(2, 1) \wedge P(2, 2)$.

(e) $\forall y\ \forall x\ P(x, y)$.

$[\forall x\ P(x, 1)] \wedge [\forall x\ P(x, 2)]$
$\Leftrightarrow [P(1, 1) \wedge P(2, 1)] \wedge [P(1, 2) \wedge P(2, 2)]$
$\Leftrightarrow P(1, 1) \wedge P(1, 2) \wedge P(2, 1) \wedge P(2, 2)$.

(f) $\exists x\ \forall y\ \neg P(x, y)$.

$[\forall y\ \neg P(1, y)] \vee [\forall y\ \neg P(2, y)]$
$\Leftrightarrow [\neg P(1, 1) \wedge \neg P(1, 2)] \vee [\neg P(2, 1) \wedge \neg P(2, 2)]$.

PROBLEM 23. [TIME ALLOWED = 5 MINUTES FOR EACH PART]

Let P: $\mathbf{Z} \times \mathbf{Z} \longrightarrow \{\mathrm{T}, \mathrm{F}\}$, where $P(x, y)$ denotes "$x + y^2 = 10$".

Give the truth value of the following propositions:

(1) $\forall y\ \exists x\ P(x, y)$.

(a) True.
(b) False.
(c) I do not know yet.
(d) I do not care.

(2) $\exists y \ \forall x \ P(x, y)$.

(a) True.
(b) False.
(c) I do not know.
(d) I do not care yet.

(3) $\exists x \ \exists y \ P(x, y)$.

(a) True.
(b) False.
(c) I know, but I am not going to tell you.
(d) Is this really a question?

SOLUTION.

(1) (a) True.

For any y, it is always possible to find an integer x by setting $x = 10 - y^2$.

(2) (b) False.

For example, $y = \mathrm{n}$ and $x \neq 10 - \mathrm{n}^2$.

(3) (a) True.

For example, $x = 1$ and $y = 3$. (There are other possibilities, such as, $x = 9$ and $y = 1$.)

PROBLEM 24. [TIME ALLOWED = 5 MINUTES]

Simplify $\neg[\exists x \ \exists y \ [P(x, y) \oplus P(y, x)]]$ so that there are no conjunction, disjunction, or negation symbols in the resulting logical expression.

SOLUTION.

$\neg[\exists x\ \exists y\ [P(x, y) \oplus P(y, x)]]$
$\Leftrightarrow \forall x\ \forall y\ \neg[P(x, y) \oplus P(y, x)]$
$\Leftrightarrow \forall x\ \forall y\ \neg[[P(x, y) \wedge \neg P(y, x)] \vee [(P(y, x) \wedge \neg P(x, y)]]$
$\Leftrightarrow \forall x\ \forall y\ [\neg[P(x, y) \wedge \neg P(y, x)] \wedge \neg[(P(y, x) \wedge \neg P(x, y)]]$
$\Leftrightarrow \forall x\ \forall y\ [[\neg P(x, y) \vee P(y, x)] \wedge [\neg P(y, x) \vee P(x, y)]]$
$\Leftrightarrow \forall x\ \forall y\ [[P(x, y) \longrightarrow P(y, x)] \wedge [P(y, x) \longrightarrow P(x, y)]]$
$\Leftrightarrow \forall x\ \forall y\ [P(x, y) \leftrightarrow P(y, x)]$.

PROBLEM 25. [TIME ALLOWED = 5 MINUTES]

Show, (1) using a truth table, and (2) using a mathematical proof, that $[(p \longrightarrow q) \wedge (q \longrightarrow r)] \longrightarrow (p \longrightarrow r)$ is a tautology.

SOLUTION.

(1) In the truth table, all rows in the last column must contain a T. Let $s \equiv (p \longrightarrow q) \wedge (q \longrightarrow r)$.

p	q	r	$p \longrightarrow q$	$q \longrightarrow r$	s	$q \longrightarrow r$	$s \longrightarrow (p \longrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

(2) In a mathematical proof, the goal is to obtain a T at the end of the steps of the proof.

$[(p \longrightarrow q) \wedge (q \longrightarrow r)] \longrightarrow (p \longrightarrow r)$
$\Leftrightarrow \neg[(\neg p \vee q) \wedge (\neg q \vee r)] \vee (\neg p \vee r)$
$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$
$\Leftrightarrow (\neg p \vee r) \vee (p \wedge \neg q) \vee (q \wedge \neg r)$
$\Leftrightarrow [((\neg p \vee r) \vee p) \wedge ((\neg p \vee r) \vee \neg q)] \vee (q \wedge \neg r)$
$\Leftrightarrow [((\neg p \vee p) \vee r) \wedge ((\neg p \vee r) \vee \neg q)] \vee (q \wedge \neg r)$
$\Leftrightarrow [(\mathrm{T} \vee r) \wedge ((\neg p \vee r) \vee \neg q)] \vee (q \wedge \neg r)$
$\Leftrightarrow [\mathrm{T} \wedge ((\neg p \vee r) \vee \neg q)] \vee (q \wedge \neg r)$
$\Leftrightarrow ((\neg p \vee r) \vee \neg q) \vee (q \wedge \neg r)$
$\Leftrightarrow \neg p \vee (\neg q \vee r) \vee (q \wedge \neg r)$
$\Leftrightarrow \neg p \vee [(\neg q \vee r) \vee \neg(\neg q \vee r)]$
$\Leftrightarrow \neg p \vee \mathrm{T}$
$\Leftrightarrow \mathrm{T}.$

PROBLEM 26. [TIME ALLOWED = 5 MINUTES]

Express the negation of the following statements in terms of quantifiers without using the negation operator:

(a) $\forall x\ ((x > -1) \vee (x < 1))$
(b) $\exists x\ (3 < x \leq 7)$

SOLUTION.

(a) $\exists x\ ((x \leq -1) \wedge (x \geq 1))$
(b) $\forall x\ ((x \leq 3) \vee (x > 7))$