RELATIONS

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1. INTRODUCTION

The study of relations is based on the idea that **entities in the universe are not isolated** and, indeed, can be related to each other in intricate ways.

This document provides an introduction to (mostly **binary**) relations, including properties of those relations.

2. THE SIGNIFICANCE OF RELATIONS TO COMPUTER SCIENCE AND SOFTWARE ENGINEERING

The notion of a relation has a number of applications in computer science and software engineering [Devlin, 2004, Chapter 5; O'Donnell, Hall, Page, 2006, Chapter 10; Vatsa, Vatsa, 2009, Chapter 3; Roberts, 2010, Chapter 4; Gallier, 2011, Chapter 2; Cunningham, 2012, Chapter 7; Lehman, Leighton, Meyer, 2012, Section 4.4; Makinson, 2012, Chapter 2; Pace, 2012, Chapter 5, Chapter 6; Rosen, 2012, Chapter 9; Wallis, 2012, Chapter 4; Jenkyns, Stephenson, 2013, Chapter 6; Moller, Struth, 2013, Chapter 7; O'Regan, 2013, Chapter 2; O'Regan, 2016, Chapter 2; Kurgalin, Borzunov, 2018, Chapter 3; Sundstrom, 2018, Chapter 7].

2.1. USES OF RELATIONS

The significance of relations to computer science and software engineering can be seen from the following examples.

PROJECTS

The different activities in a software project need not be isolated, and may indeed depend on each other [Strode, Huff, 2012; Strode, 2015]. Figure 1 shows a **taxonomy of dependencies that can arise in software projects based on agile methodologies**. For **effective coordination**, it is important to understand and manage dependencies in software projects.

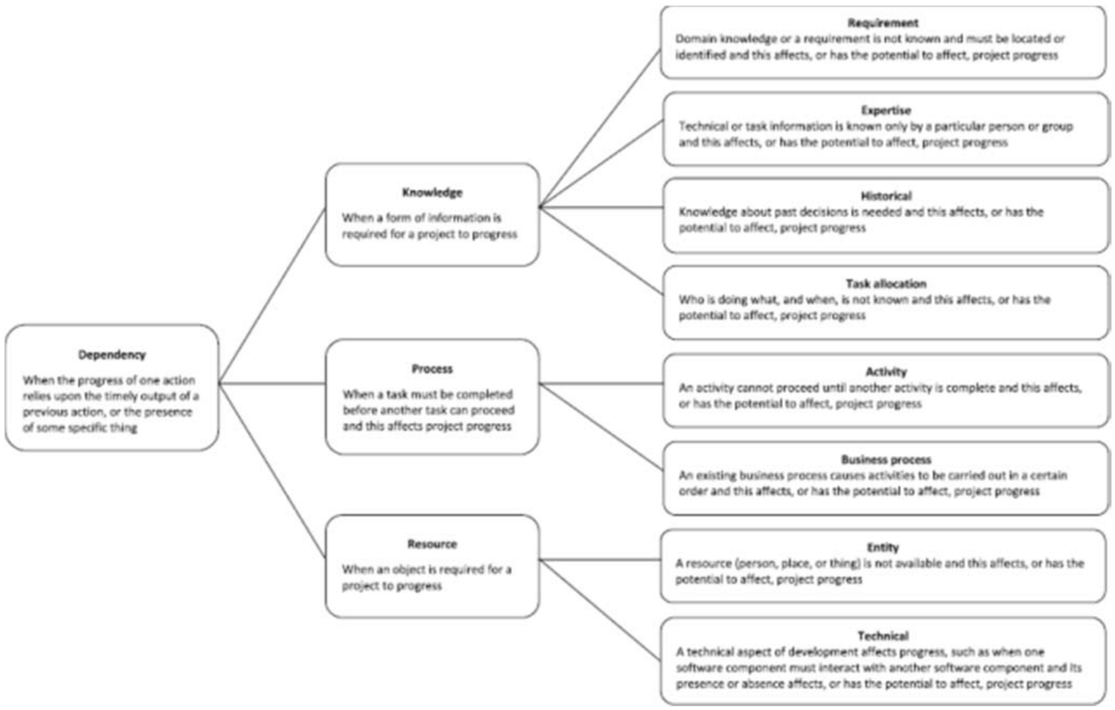


Figure 1. A taxonomy of dependencies in agile software projects.

REQUIREMENTS

The (software) requirements in (software) requirements specification need not be isolated, and can indeed be mutually related in some manner [Robinson, Pawlowski, Volkov, 2003; Trkman, Mendling, Krisper, 2016].

For example, one requirement, R_i may **depend** on another requirement R_j in different ways: R_i may need to be **developed** before R_j (**development order dependency**), or R_i need to be **executed** before R_j (**execution order dependency**).

HIGH-LEVEL DESIGN

The **components** in a software architecture are interrelated by **connectors**.

LOW-LEVEL DESIGN

The classes in an **object-oriented design (OOD)** are related to each other. The OOD could be expressed in some modeling language, such as the **Unified Modeling Language (UML)**. Then, the **relationships between classes in the OOD based on UML need to be articulated precisely**. For the sake of **syntactic and semantic quality** of the model, **salient properties of relationship types** possible in UML need to be understood.

INFORMATION

The components of information are often **related** to each other. In some cases, these components need to be **compared**. These components may also need to be **organized**. The **relational databases** standardize organization of structure for large databases [Bagdasar, 2013, Chapter 2].

In a **relational database model**, a database consists of **records**. A record is an n-tuple, made up of fields. The fields are the entries of an n-tuple. **The relational data model represents a database of records as an n-ary relation.** For example, a database of student records may be made up of fields containing the name, number, major, and grade point average of the student. Then, the student records are represented as 4-tuples of the form (`Name`, `IdentificationNumber`, `Major`, `GPA`).

However, to be clear, **not** all databases are relational. For example, certain legacy database systems and Extensible Markup Language (XML) databases are not relational. In such cases, **transformation across databases** is desirable. The formulation of a suitable transformation is based on the **understanding of relations**.

SENTENCES

A sentence is a textual unit consisting of one or more words in a natural language. In natural languages which impose a left-to-right reading order, a word on the left is linguistically-related to a word on the right, as shown in Figure 2.

Figure 2. A pair of sentences which should be read left-to-right, at least by pessimists.

NETWORKS

There are at least three important types of networks: (1) **social networks**, which show relationships between people (specifically, between individuals or between groups of people), (2) **computer networks**, which show relationships between computers, and (3) **information networks**, which show relationships between resources (documents, images, videos, and so on) [Easley, Kleinberg, 2010].

ORTHOGONALITY

$\perp$

In some cases, it is desirable that **two things, A and B, be not related in some way**. For example, a use of A does not require the use of B, and conversely. For another example, a change in A does not affect B, and conversely.

The notion of **orthogonality** is a generalization of **perpendicularity**, and is important in many areas including classification, comparison, linear algebra, programming language design, software process design, specification of software requirements, software design, and software testing.

There are several advantages of orthogonality, including **avoiding redundancy and avoiding the creation or propagation of unintended side-effects**.

It could be said that **separation of concerns, mutual exclusion, and encapsulation** are practical realizations of orthogonality.

2.2. THE RELATIONSHIP OF RELATIONS TO SOFTWARE ENGINEERING

SWEBOK®

The **Guide to the Software Engineering Body of Knowledge (SWEBOK)** “describes the sum of knowledge within the profession of software engineering” [IEEE, 2014]. In SWEBOK, there are a number of Knowledge Areas (KAs). The study of relations is part of **Mathematical Foundations KA** of the SWEBOK3, as shown in Figure 3.

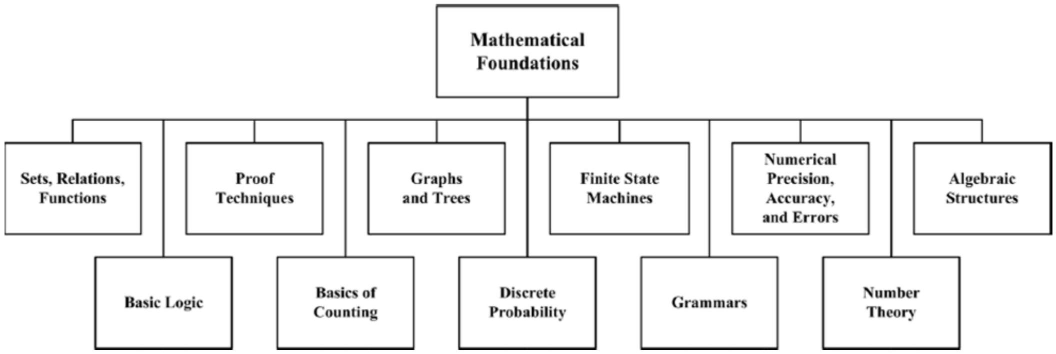


Figure 3. Mathematical Foundations is a Knowledge Area in the Guide to the Software Engineering Body of Knowledge (SWEBOK). (Source: SWEBOK [IEEE, 2014].)

3. RELATIONS AS GENERALIZED FUNCTIONS

The **notion of relation** can be viewed as a **generalization of the notion of function**. In other words, every function is a relation. However, the converse is **not** necessarily true. In particular, (1) a relation does not have to be defined for every element in the domain, and (2) can have multiple images.

The additional property that ensures that a relation is a function is the **vertical line test**. For **every** a in A_1, there is a **unique** b in A_2 for which (a, b) is in the relation. Let A_1 be on the x-axis, A_2 on the y-axis, and the relation be represented by a graph.

4. RELATIONS AS CARTESIAN PRODUCTS

A relation can be represented as a Cartesian product of sets.

EXAMPLE

Database $\subseteq$ Set of Names × Set of Colors × Set of Occupations.

EXAMPLE

Siblinghood $\subseteq$ Set of People × Set of People.

5. BINARY RELATION

Definition [Binary Relation]. Let A and B be sets. A **binary relation** from A to B is a subset of $A \times B$.

The above definition can be generalized to **n-ary relations**.

EXAMPLE

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then, $\{(0, a), (0, b), (1, a), (2, b)\}$ is a binary relation from A to B, and can be shown pictorially, as in Figure 4.

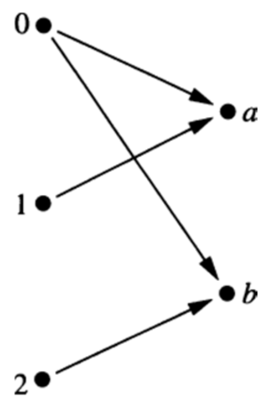


Figure 4. An example of a relation.

Board Time!

Give an example of something that is not a binary relation from A to B, where A and B are given in the previous example.

EXAMPLE

Find the number of binary relations on sets A_1 and A_2.

Solution.

The relations on A_1 and A_2 are subsets of $A_1 \times A_2$. Therefore, the number of binary relations on sets A_1 and A_2 is the **number of subsets** of $A_1 \times A_2$ or $2^{|A_1|\cdot|A_2|}$.

6. INVERSE RELATION

Definition [Inverse Relation]. Let R be a relation from set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs (b, a), where $(a, b) \in R$.

REMARKS

A relation, **unlike a function**, is **always** invertible.

EXAMPLE

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then, if R be a relation from A to B given by

$$R = \{(1, a), (1, b), (2, a), (3, b)\},$$

then R^{-1} is a relation from B to A given by

$$R^{-1} = \{(a, 1), (a, 2), (b, 1), (b, 3)\}.$$

EXAMPLE

Let R be defined on **R** by xRy if and only if $y = x^2$. Find the inverse R^{-1}.

Solution.

R is the **square relation**, xRy if and only if $y = x^2$, so R^{-1} is **square root relation**, $xR^{-1}y$ if and only if $y = \pm\sqrt{x}$, where x is non-negative. In other words, R^{-1} is the **union** of the two **square root branches**. Figure 5 shows a visualization of $x = y^2$ as the graph showing (both) branches of the solution.

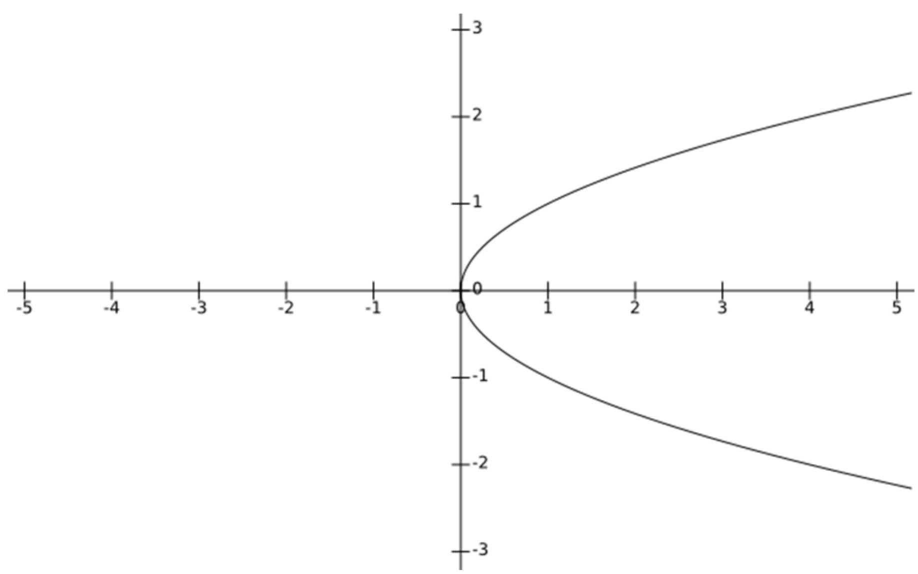


Figure 5. The graph of $x = y^2$.

7. RELATION ON A SET

Definition [Relation on a Set]. A relation on the set A is a subset of $A \times A$.

EXAMPLE

The **"is sibling of"** is a relation on set of people.

EXAMPLE

Let $A = \{-1, 2, 3, 4\}$. List the ordered pairs that are in the relation $R = \{(a, b) \mid a^2 < b\}$.

Solution. Improvise.

EXAMPLE

Find the number of relations on a set A.

Solution.

The number of relations on a set A is the **number of subsets** of $A \times A$, that is, $2^{|A| \cdot |A|}$.

8. SET OPERATIONS ON RELATIONS

A relation is a **subset**. Therefore, all the **usual set-theoretic operations** can be defined between relations that belong to the **same** Cartesian product.

EXAMPLE

Let R and S be relations on the set $\{1, 2\}$ given by

$$R = \{(1, 1), (2, 2)\} \text{ and } S = \{(1, 1), (1, 2)\}.$$

Find:

(a) The union $R \cup S$.
(b) The intersection $R \cap S$.
(c) The symmetric difference $R \oplus S$.
(d) The difference $R - S$.
(e) The complement R^c.

Solution.

(a) $R \cup S = \{(1, 1), (1, 2), (2, 2)\}$.
(b) $R \cap S = \{(1, 1)\}$.
(c) $R \oplus S = \{(1, 2), (2, 2)\}$.
(d) $R - S = \{(2, 2)\}$.
(e) $R^c = \{(1, 2), (2, 1)\}$.

EXAMPLE

The following relations on the set of real numbers are given:

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the "greater than" relation,
$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the "greater than or equal to" relation,
$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the "less than" relation,
$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the "less than or equal to" relation,
$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the "equal to" relation,
$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the "unequal to" relation.

Find:

(a) $R_1 \cup R_3$.
(b) $R_1 \cup R_5$.
(c) $R_3 \cap R_5$.

Solution.

(a) The union of two relations is the union of these sets. Thus, $R_1 \cup R_3$ holds between two real numbers if R_1 holds or R_3 holds (or both hold). In this case, it means that the first number is greater than the second, or vice versa. In other words, that the two numbers are not equal. (This, incidentally, is just the relation R_6.)

(b) For (a, b) to be in $R_1 \cup R_5$, it should be the case that $a > b$ or $a = b$. This is the case if $a \geq b$. (This, incidentally, is just the relation R_2.)

(c) For (a, b) to be in $R_3 \cap R_5$, it should be the case that $a < b$ and $a = b$. It is impossible for $a < b$ and $a = b$ to hold simultaneously. Therefore, the answer is $\varnothing$, that is, **the relation that never holds**.

EXAMPLE

Let R_1 and R_2 be relations on the set of all positive integers given by:

$R_1 = \{(a, b) \mid a \text{ divides } b\}$.
$R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$.

Find:

(a) $R_1 \cup R_2$.
(b) $R_1 \cap R_2$.

Solution.

It could be noted that R_1 and R_2 be are **inverses of each other**, as a is a multiple of b if and only if b divides a.

(a) The union of two relations is the union of these sets. Thus, $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both hold). Therefore, $(a, b) \in R_1 \cup R_2$ if and only if $a \mid b$ or $b \mid a$.

(b) The intersection of two relations is the intersection of these sets. Thus, $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus, $(a, b) \in R_1 \cap R_2$ if and only if $a \mid b$ and $b \mid a$. This happens if and only if $a = b$ and $a \neq 0$.

Board Time!
Explain whether the solution of the previous example will change if the relations are on the set of all integers.

9. PROPERTIES OF BINARY RELATIONS

There are special properties for a relation R on a set A. The most common properties are:

- Reflexive
- Irreflexive
- Symmetric
- Asymmetric
- Antisymmetric
- Transitive

10. REFLEXIVE RELATION

Definition [Reflexive]. A relation R on a set A is reflexive if each element is related to itself, that is, aRa for all $a \in A$.

EXAMPLE

The **“equal to”** (=) relation on any set of numbers is reflexive.

EXAMPLE

The **“less than or equal to”** (≤) relation on **Z** is reflexive.

EXAMPLE

The **“divides”** (|) relation on $\mathbf{Z}^+$ is reflexive.

EXAMPLE

The **“is sibling of”** relation on the set of people is **not** reflexive. (This also applies to the **generalization/specialization relationship between classes in object-oriented design**.)

EXAMPLE

The **“can overtake”** relation on the set of cars is **not** reflexive.

EXAMPLE

Find the number of different reflexive relations that can be defined on a set A containing n elements.

Solution.

The set $A \times A$ contains n^2 elements. The relations on R are subsets of $A \times A$. Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements. Thus, there are $2^{n \cdot n}$ relations.

However, a reflexive relation must contain the n elements of the type (a, a), for every $a \in A$. Therefore, reflexive relations can be generated by choosing subsets of $n^2 - n = n(n-1)$ elements. Thus, there are $2^{n(n-1)}$ different reflexive relations on a set A containing n elements.

11. IRREFLEXIVE RELATION

Definition [Irreflexive]. A relation R on a set A is irreflexive if it is **never** the case that aRa holds.

REMARKS

It could be noted that irreflexive is **not** equivalent to “not reflexive”.

EXAMPLE

A **“strict inequality”** (< or >) relation over any set of numbers is irreflexive.

It is possible for a relation on a set be **neither reflexive, nor irreflexive**. This occurs when aRa for some, but **not** all, $a \in A$.

EXAMPLE

Give an example of a relation on a set that is neither reflexive, nor irreflexive.

Solution.

For instance, the **“is proud of”** relation on the set of people is neither reflexive, nor irreflexive.

For instance, the **“product of *x* and *y* is even”** relation on the **set of even numbers** is reflexive, on the **set of odd numbers** is irreflexive, and on the **set of natural numbers** is neither reflexive, nor irreflexive.

EXAMPLE

Use quantifiers to express the meaning in logic of a relation R to be irreflexive.

Solution.

$\forall x\ ((x, x) \notin R)$.

12. SYMMETRIC RELATION

Definition [Symmetric]. A relation R on a set A is symmetric if for all $a, b \in A$, aRb if and only if bRa.

In a symmetric relation, **order** is **irrelevant**.

EXAMPLE

(In number theory, a pair of integers that do not have any common factor other than 1 are called **co-prime**. For example, 2 and 3 are co-prime.)

The property of being co-prime in the set of natural numbers is **symmetric**. If mRn means that m and n are co-prime, then nRm, that is, are n and m are co-prime.

EXAMPLE

The relation R on the real numbers defined by xRy if and only if $x^2 + y^2 \leq 1$ is symmetric.

Solution.

Improvise. (It can be useful to understand the nature of the relation, which is shown in Figure 6.)

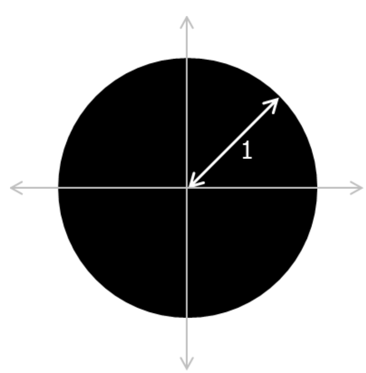


Figure 6. $x^2 + y^2 \leq 1$.

EXAMPLE

The **“divides”** (|) relation on $\mathbf{Z}^+$ is **not** symmetric.

EXAMPLE

The **“proper subset”** relation over any set is **not** symmetric. The “strict inequality” (< or >) relation over any set of numbers is also **not** symmetric. In general, any **strict ordering** relation over any set is **not** symmetric.

13. ASYMMETRIC RELATION

Definition [Asymmetric]. A relation R on a set A is asymmetric if it is **never** the case that both aRb and bRa hold.

REMARKS

It could be noted that asymmetric is **not** equivalent to “not symmetric”. (This is in contrast to common English usage.)

EXAMPLE

The **“is a father of”** relation on the set of people is asymmetric.

EXAMPLE

The **“divides”** (|) relation on $\mathbf{Z}^+$ is **not** asymmetric.

EXAMPLE

Explain how the matrix representing a relation R on a set A can be used to determine whether the relation is asymmetric.

Solution.

In a matrix for an asymmetric relation, there are no 1’s on the main diagonal (that is, $m_{ii} \neq 1$, for every i), and there is no pair of 1’s symmetrically placed around the main diagonal (that is, $m_{ij} \neq m_{ji} = 1$, for every i and j).

14. ANTISYMMETRIC RELATION

Definition [Antisymmetric]. A relation R on a set A is antisymmetric if, for $a \neq b$, it is **never** the case that both aRb and bRa hold.

REMARKS

It could be noted that antisymmetric is **not** equivalent to "not symmetric". (This is in contrast to common English usage.)

EXAMPLE

The **"divides"** (|) relation on $\mathbf{Z}^+$ is antisymmetric.

EXAMPLE

Determine whether the **"divides"** (|) relation is antisymmetric on **Z**.

Solution.

Improvise. (The solution illustrates that the **set** underlying the relation **is** relevant.)

EXAMPLE

(a) Give an example of a relation on a set that is both symmetric and antisymmetric.
(b) Give an example of a relation on a set that is neither symmetric, nor antisymmetric.

Solution.

(a) $A = \{a\}$ and $R = \varnothing$. (R is called an **empty relation**. R is **vacuously** symmetric as well as antisymmetric. In fact, R is also **vacuously** asymmetric.)

(b) $A = \{a, b, c\}$ and $R = \{(a, b), (b, a), (a, c)\}$.

15. ASYMMETRIC VERSUS ANTISYMMETRIC RELATIONS

The definition of asymmetry is **identical, but stricter** than that of antisymmetry. Therefore, **if a relation on a set is antisymmetric, then it does not have to be asymmetric**.

However, **if a relation on a set is asymmetric, then it has to be antisymmetric**. In fact, a relation is asymmetric if and only if it is **antisymmetric and irreflexive**.

EXAMPLE

Give an example of a relation that is antisymmetric but not asymmetric.

Solution.

Let R be a relation on $A = \{a, b, c\}$. If aRa but both aRb and bRa do not hold for $a \neq b$, then there is antisymmetry but not asymmetry. For instance, $R = \{(a, a), (a, b), (b, c)\}$ is antisymmetric but not asymmetric.

EXAMPLE

Give an example of a relation that is both asymmetric and antisymmetric.

Solution.

The **"is a father of"** relation on the set of people.

16. TRANSITIVE RELATION

Definition [Transitive]. A relation R on a set A is transitive if whenever a is related to b and b is related to c, then a is related to c. In other words, for all $a, b, c \in A$,

$$aRb \text{ and } bRc \text{ implies } aRc.$$

EXAMPLE

The **"divides"** (|) relation on $\mathbf{Z}^+$ is transitive.

EXAMPLE

An inequality ($<, >, \leq$, or $\geq$) relation on numbers is transitive.

EXAMPLE

The **"*x* + *y* is a prime number"** relation on $\mathbf{Z}^+$ is **not** transitive.

For example, let $a = 2$, $b = 3$, and $c = 4$, then $a + b = 5$ is a prime number, and $b + c = 7$ is a prime number. However, $a + c = 6$ is **not** a prime number, and therefore R is **not** transitive.

EXAMPLE

The **"depends on"** relation is transitive.

For instance, let A, B, and C be three modules of a software system. If (1) A depends on B, and (2) B depends on C, then, by transitivity, (3) A depends on C.

(1) (and (2)) is called **direct dependency**; (3) is called **indirect dependency**.

The **indirect dependencies can pose a challenge** to a proper design of a software system [MacCormack, Sturtevant, 2016]. This generalizes to indirect dependencies between other types of software project artifacts [Ghazi, Glinz, 2017].

EXAMPLE

The **"is a friend of"** relation is **not** transitive.

EXAMPLE

In the following list of relations on $\{1, 2, 3, 4\}$, state the ones that are transitive:

(a) $R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$.
(b) $S = \{(1, 3), (3, 2), (2, 1)\}$.
(c) $T = \{(2, 4), (4, 3), (2, 3), (1, 4)\}$.

Solution. Improvise.

EXAMPLE

Give, if possible, a situation where the **"is sibling of"** relation fails to be transitive.

Solution.

This is that case when the **"is sibling of"** relation is that of half-brothers (or half-sisters).

EXAMPLE

Give the different relations on the set $\{0, 1\}$. List those relations that are (a) reflexive, (b) irreflexive, (c) symmetric, and (d) transitive.

Solution.

(The relations are the $2^4 = 16$ different subsets of $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.)

(a) Reflexive: $\{(0, 0), (1, 1)\}$, $\{(0, 0), (0, 1), (1, 1)\}$, $\{(0, 0), (1, 0), (1, 1)\}$, $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

(b) Irreflexive: $\varnothing$, $\{(0, 1)\}$, $\{(1, 0)\}$, $\{(0, 1), (1, 0)\}$.

(c) Symmetric: Improvise.

(d) Transitive: Improvise.

17. MATRICES

Definition [Matrix]. A **two-dimensional structure**, similar to a table, with rows and columns that contain entries (usually numbers) that can be manipulated.

17.1. BINARY RELATIONS AND BOOLEAN MATRICES

A binary relation can be represented using **Boolean matrices**.

A Boolean matrix is a **zero-one (or, equivalently, 0-1) matrix**, that is, a matrix consisting of **0's and 1's**.

For a relation R from A to B, the matrix $\mathbf{M}_R$ is defined by the following:

- There is one row for each element of A.
- There is one column for each element of B.
- The value at i^{th}-row and j^{th}-column is:
  - 1, if i^{th}-element of A is related to j^{th}-element of B, and
  - 0, otherwise.

REMARKS

A matrix used to represent a binary relation is also called as a **relation matrix** (or, equivalently, a **transition matrix**).

EXAMPLE

The following are relations on $\{1, 2, 3\}$:

(a) $\{(1, 1), (1, 2), (1, 3)\}$.
(b) $\{(1, 3), (3, 1)\}$.

Represent each of these relations using a matrix.

Solution.

In this case, $A = B = \{1, 2, 3\}$.

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

EXAMPLE

For each of the given matrices, where the rows and columns correspond to the **integers listed in increasing order**, list the ordered pairs in the relations on $\{1, 2, 3, 4\}$.

(a)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Solution.

In this case, $A = B = \{1, 2, 3, 4\}$.

(a) It can be seen that (1, 1)$^{\text{th}}$ entry is a 1, and so (1, 1) is in the relation; (1, 3)$^{\text{th}}$ entry is a 0, and so (1, 3) is not in the relation; and so on. Therefore, the relation contains (1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), and (4, 4).

(b) The relation contains (1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), and (4, 4).

(c) The relation contains (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1), and (4, 3).

CONVENTION

Even if some entries of the matrix are 0s, they are included, and, usually, the entire block of elements is parenthesized.

If R is a relation on A, then it is the **same set** in the Cartesian product. In this case, shape of M_R is a **square**, that is, the number of rows is **equal** to the number of columns.

BOOLEAN OPERATIONS

The Boolean operations **disjunction and conjunction** can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

- **Disjunction.** To obtain the disjunction of two 0-1 matrices, apply the Boolean **“or”** function to all corresponding elements in the matrices.

- **Conjunction.** To obtain the conjunction of two 0-1 matrices, apply the Boolean **“and”** function to all corresponding elements in the matrices.

EXAMPLE

Let the relations R_1 and R_2 be represented by the following matrices:

$$\mathrm{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$\mathrm{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Give the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$.

Solution. The desired matrices are given by

$$\mathrm{M}_{R_1 \cup R_2} = \mathrm{M}_{R_1} \vee \mathrm{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and

$$\mathrm{M}_{R_1 \cap R_2} = \mathrm{M}_{R_1} \wedge \mathrm{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

17.2. PROPERTIES OF BINARY RELATIONS AND MATRICES

The different properties of binary relations lend **different structures** to the Boolean matrices. In fact, these special structures can help identifying whether the corresponding binary relations have those properties.

Reflexivity: There should be all 1's on the main diagonal.

Irreflexivity: There should be all 0's on the main diagonal.

Symmetry: The matrix should be symmetric about the main diagonal (equivalently, the matrix equals its **transpose**).

Antisymmetry: It should **never** be the case that two 1's are symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the main diagonal).

Transitivity: The Boolean square of the matrix (that is, the Boolean product of the matrix with itself) should be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.

EXAMPLE

For the relation represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

determine whether the relation is reflexive, irreflexive, symmetric, antisymmetric, and/or transitive, where the rows and columns correspond to the integers listed in increasing order and the relations are on $\{1, 2, 3, 4\}$.

Solution.

- Reflexive: No. There are some 1's and some 0's on the main diagonal.

- Irreflexive: No. There are some 1's and some 0's on the main diagonal.

- Symmetric: Yes. The matrix is symmetric about the main diagonal.

- Antisymmetric: No. For example, see positions (1, 2) and (2, 1).

- Transitive: No. For example, the 1's in positions (1, 2) and (2, 3) would require a 1 in position (1, 3) if the relation were to be transitive.

18. DIRECTED GRAPHS

Definition [Directed Graph]. A set of **vertices** V together with a set E of **edges** (ordered pairs of elements) of V.

For an ordered pair (a, b): (1) a is the **initial vertex** of (a, b), and (2) b is the **terminal vertex** of (a, b). The edge (a, b) is also denoted by $a \longrightarrow b$.

REMARKS

In literature on graph theory, a directed graph is abbreviated as **digraph,** vertices are also known as **nodes**, and, in some cases, 'a' is called the **source** of the edge, while 'b' is called the **target** of the edge.

18.1. RELATIONS AND DIRECTED GRAPHS

The set A is represented by **vertices**.

If a is related to b (that is, aRb), a **directed edge** (or arrow) $a \longrightarrow b$ is created.

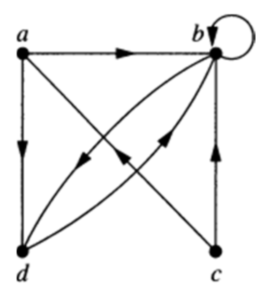
A self-directing edge (or **loop**) is used to represent aRa. The direction of the arrow is, in general, **counter-clockwise**.

The relation on a set A can be represented by a directed graph that has elements of A as vertices and the ordered pairs (a, b) as edges, where (a, b) are in R.

EXAMPLE

Draw the directed graph that represents the relation {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)} on the set {a, b, c, d}.

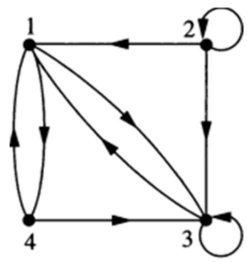
Solution.



EXAMPLE

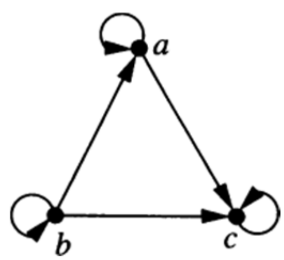
Draw the directed graph that represents the relation {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)} on the set {1, 2, 3, 4}.

Solution.



EXAMPLE

List the elements that represent the relation shown by the following directed graph:



Solution.

{(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)} on the set {a, b, c}.

18.2. PROPERTIES OF RELATIONS AND DIRECTED GRAPHS

The different properties of binary relations lend different graph structures. In fact, these special structures can help identifying whether the corresponding binary relations have those properties.

- R is **reflexive** if and only if there is a loop at every vertex of the directed graph of R.

- R is **irreflexive** if and only if there is no loop at every vertex of the directed graph of R.

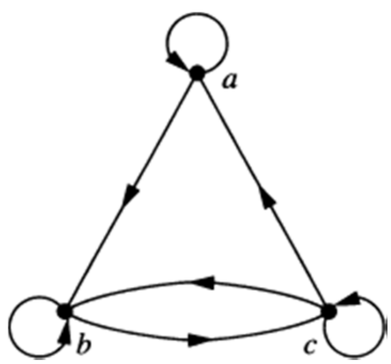
- R is **symmetric** if and only if for every edge between distinct vertices in its directed graph, there is an edge in the reverse direction.

- R is **antisymmetric** if and only if there are **never** two edges in reverse directions between two **distinct** vertices in its directed graph.

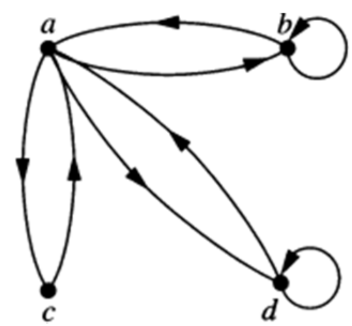
- R is **transitive** if and only if whenever there is an edge from a vertex a to a vertex b, and an edge from the vertex b to a vertex c in its directed graph, there is an edge from a to c.

EXAMPLE

Determine whether the relations for the directed graphs shown in the following figure are reflexive, symmetric, antisymmetric, and/or transitive.



Directed Graph of R



Directed Graph of S

Solution.

R:

Reflexive: Yes.
Symmetric: No.
Antisymmetric: No.
Transitive: No.

S:

Reflexive: No.
Symmetric: Yes.
Antisymmetric: No.
Transitive: No.

EXAMPLE

Suggest a scheme to represent the following on a directed graph:

- Intersection
- Symmetric Difference

Solution.

- Take the directed graph on the same vertices. Then, place an edge from *a* to *b* whenever there is an edge from *a* to *b* in **both** the directed graphs.

- Take the directed graph on the same vertices. Then, place an edge from *a* to *b* whenever there is an edge from *a* to *b* in one, but **not** both, the directed graphs.

18.3. PATHS IN DIRECTED GRAPHS

Let *G* be a directed graph. A **path** in *G* is a **sequence** of one or more edges in *G* such that the terminal vertex of an edge is the **same** as the initial vertex in the next edge.

A path $x_0, x_1, \ldots, x_n$ is said to be of **length** n.

A **circuit** is a path which begins and ends at the same vertex.

19. EQUIVALENCE RELATION

Definition [Equivalence Relation]. A relation on *A* that is reflexive, symmetric, and transitive.

REMARKS

- An equivalence relation **generalizes** the conventional notion of "equals".

- For a relation to be **not** an equivalence relation, it must **fail** to be (**at least**) one of **reflexive, symmetric, or transitive**.

EXAMPLE

The $R = \varnothing$ relation on a set $A = \varnothing$ is an equivalence relation. This is because R is reflexive, symmetric, and transitive.

EXAMPLE

The **"equal to"** (=) relation on a set of numbers is an equivalence relation.

EXAMPLE

The **"similar to"** relation on a set of geometrical figures (such as, triangles and rectangles) is an equivalence relation. Figure 7 shows similar rectangles.

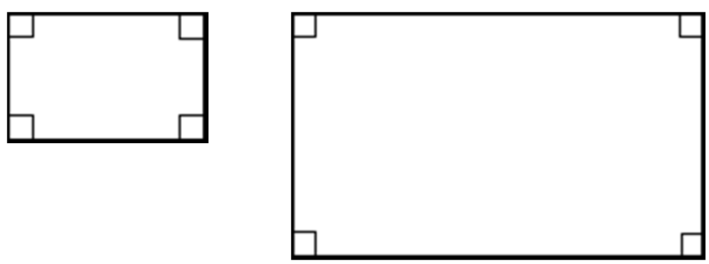


Figure 7. Two similar rectangles.

EXAMPLE

Let R be a relation on $\mathbf{Z} \times \mathbf{Z}$ defined by

$$(a_1, b_1)R(a_2, b_2) \text{ if and only if } a_1 - b_1 = a_2 - b_2.$$

Show that R an equivalence relation.

Solution.

Reflexivity: $(a, b)R(a, b)$ as $a - b = a - b$.

Symmetry: If $(a_1, b_1)R(a_2, b_2)$, then $a_1 - b_1 = a_2 - b_2$. Therefore, $a_2 - b_2 = a_1 - b_1$, and so $(a_2, b_2)R(a_1, b_1)$.

Transitivity: If $(a_1, b_1)R(a_2, b_2)$ and $(a_2, b_2)R(a_3, b_3)$, then $a_1 - b_1 = a_2 - b_2$ and $a_2 - b_2 = a_3 - b_3$. Therefore, $a_1 - b_1 = a_3 - b_3$, and so $(a_1, b_1)R(a_3, b_3)$.

The **equivalence classes** of R:

$$\{(a, b) \mid a - b = k, \text{ where } k \text{ is some constant integer}\}.$$

For example,

$$[(2, 1)]_R = \{\ldots, (-1, -2), (0, -1), (1, 0), (2, 1), \ldots\}.$$

Board Time!
Give $[(3, 7)]_R$.

EXAMPLE

Show that the relation $R = \varnothing$ on a nonempty set A is symmetric and transitive, but not reflexive.

Solution.

Approach 1: If $R = \varnothing$, then the **hypotheses** of the conditional statements in the definitions of symmetric and transitive are **never true**. Thus, the conclusion is always true, and R is always symmetric as well as transitive. (This follows from the truth table of implication.)

Approach 2: If $R = \varnothing$, then there is no reason for the definitions of symmetry or transitivity to be false. Thus, R is always symmetric as well as transitive. (This is similar to a **vacuous proof**.)

As $A \neq \varnothing$, the statement $(a, a) \in R$ is false for an element a of A. Thus, R is not reflexive.

EXAMPLE

For the following relations on the set of all people, determine the ones that are equivalence relations:

(a) $\{(a, b) \mid a$ and b are the same age$\}$.
(b) $\{(a, b) \mid a$ and b share a common parent$\}$.
(c) $\{(a, b) \mid a$ and b have met$\}$.

Solution.

(a) This is an equivalence relation.

(b) This is not an equivalence relation, as it need not be transitive, assuming that biological parentage is at issue.

(c) This is not an equivalence relation, as it need not be transitive.

EXAMPLE

Let R be the relation on the set of ordered pairs of positive integers such that

$$((a, b), (c, d)) \in R \text{ if and only if } ad = bc.$$

Show that R is an equivalence relation.

Solution.

- R is Reflexive: $((a, b), (a, b)) \in R$ because $ab = ba$.

- R is Symmetric: If $((a, b), (c, d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c, d), (a, b)) \in R$.

- R is Transitive: If $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, then $ad = bc$ and $cf = de$. Then, multiplying these equations yields $acdf = bcde$. Furthermore, because all these numbers are nonzero, $af = be$. Therefore, $((a, b), (e, f)) \in R$.

20. CONGRUENCE RELATION

Definition [Congruence Relation]. Let d be a positive integer with $d > 1$. Let $a \equiv b \pmod{d}$ if and only if d divides $a - b$.

The relation

$$R = \{(a, b) \mid a \equiv b \pmod{d}\}$$

is called as **congruence modulo *d***. The number d is called the **modulus** of the congruence.

REMARKS

The congruence relation is used for keeping the 12-hour clock time convention.

EXAMPLE

$24 \equiv 10 \pmod{7}$ because 7 divides $24 - 10 = 14$.

Board Time!
Show that the same argument can be applied to conclude that (a) $-3 \equiv 7 \pmod{5}$, (b) $2 \equiv -3 \pmod{5}$, and (c) $-3 \equiv -8 \pmod{5}$.

EXAMPLE

Prove that congruence modulo d is an equivalence relation on the set of integers.

Solution.

Reflexivity: $a - a = 0$ is divisible by d. It follows that, $a \equiv a \pmod{d}$. Thus, congruence modulo d is reflexive.

Symmetry: If $a \equiv b$ (mod d), then d divides $a - b$, that is, $a - b = \text{k}d$, for some integer k. It follows that $b - a = (-\text{k})d$, and so $b \equiv a$ (mod d). Thus, congruence modulo d is symmetric.

Transitivity: If $a \equiv b$ (mod d), then d divides $a - b$, that is, $a - b = \text{k}d$, for some integer k. If $b \equiv c$ (mod d), then d divides $b - c$, that is, $b - c = \text{p}d$, for some integer p. It follows that $a - c = (b + \text{k}d) - (b - \text{p}d) = (\text{k} + \text{p})d$, and so $a \equiv c$ (mod d). Thus, congruence modulo d is transitive.

20.1. PROPERTIES OF THE CONGRUENCE RELATION

Let $a \equiv b$ (mod d), $u \equiv v$ (mod d), and c be an integer. Then, the following properties hold.

PROPERTY 1

There is **compatibility with translation**, that is,

$$a + c \equiv (b + c) \pmod{d}.$$

EXAMPLE

It is given that $24 \equiv 10$ (mod 7). Then,

$$24 + 3 = 27 \equiv (10 + 3) \pmod{7} = 13 \pmod{7}.$$

PROPERTY 2

There is **compatibility with scaling**, that is,

$$ac \equiv bc \pmod{d}.$$

EXAMPLE

It is given that $24 \equiv 10$ (mod 7). Then,

$$24 \cdot 2 = 48 \equiv (10 \cdot 2) \pmod{7} = 20 \pmod{7}.$$

PROPERTY 3

There is **compatibility with addition**, that is,

$$a + u \equiv (b + v) \pmod{d}.$$

EXAMPLE

It is given that $24 \equiv 10 \pmod{7}$ and $-3 \equiv 4 \pmod{7}$. Then,

$$24 + (-3) = 21 \equiv (10 + 4) \pmod{7} = 14 \pmod{7}.$$

PROPERTY 4

There is **compatibility with subtraction**, that is,

$$a - u \equiv (b - v) \pmod{d}.$$

EXAMPLE

It is given that $24 \equiv 10 \pmod{7}$ and $-3 \equiv 4 \pmod{7}$. Then,

$$24 - (-3) = 27 \equiv (10 - 4) \pmod{7} = 6 \pmod{7}.$$

PROPERTY 5

There is **compatibility with multiplication**, that is,

$$au \equiv bv \pmod{d}.$$

EXAMPLE

It is given that $24 \equiv 10 \pmod{7}$ and $-3 \equiv 4 \pmod{7}$. Then,

$$24 \cdot (-3) = -72 \equiv (10 \cdot 4) \pmod{7} = 40 \pmod{7}.$$

PROPERTY 6

There is **compatibility with exponentiation**, that is, for any non-negative integer n,

$$a^n \equiv b^n \pmod{d}.$$

EXAMPLE

It is given that $7 \equiv 3 \pmod{4}$. Then,

$$7^3 = 343 \equiv (3^3) \pmod{4} = 27 \pmod{4}.$$

21. COMPOSING RELATIONS

A binary relation, like a function, can also be composed. A function is a **special case** of a **relation** (namely, it maps each element in the domain to **exactly one** element in the co-domain).

If two functions are formally converted into relations, that is, expressed as **sets of ordered pairs**, the composite of these relations will be exactly the same as the composite of the functions.

Definition [Composition of Relations]. Let A, B, and C be sets. Let R be a relation from A to B, and S be a relation from B to C. The **composite** of R and S is the relation $S \circ R$ (or SR) consisting of ordered pairs (a, c), where $a \in A$ and $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

EXAMPLE

Let R be a relation defined on **N** by

$$xRy \text{ if and only if } y = x^2,$$

and S be a relation defined on **N** by

$$xSy \text{ if and only if } y = x^3.$$

Find the composite $S \circ R$.

Solution.

R and S are functions (**squaring and cubing**, respectively), so the composite $S \circ R$ is the function composition (raised to the **sixth power**):

$$xSRy \text{ if and only if } y = x^6.$$

REMARKS

Incidentally, in this **exceptional** case, $R \circ S = S \circ R$. However, composition of relations is, in general, **not commutative**.

EXAMPLE

Let $A = \{1, 2, 3\}$, $B = \{2, 6\}$, and $C = \{1, 9, 15\}$, and let the relations R from A to B and S from B to C be defined, respectively, by

$$aRb \text{ if and only if } a \mid b,$$

and

$$bSc \text{ if and only if } b + c \text{ is a prime number.}$$

Define

(a) $S \circ R$.
(b) $R \circ S$.

Solution.

(a)

$R = \{(1, 2), (1, 6), (2, 2), (2, 6), (3, 6)\}$, and
$S = \{(2, 1), (2, 9), (2, 15), (6, 1)\}$.

Therefore,

$$S \circ R = \{(1, 1), (1, 9), (1, 15), (2, 1), (2, 9), (2, 15), (3, 1)\}.$$

(b) Improvise.

Definition [General Composition of a Relation]. Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by:

$$R^1 = R$$
$$R^{n+1} = R^n \circ R, \text{ where } R^n = R \circ R \circ \ldots \circ R \text{ (n times).}$$

R^n, for n > 1, are called as **composite relations**.

EXAMPLE

Let $A = \{2, 3, 4, 8, 9, 12\}$, and let the relation R be defined by

$$aRb \text{ if and only if } (a \mid b \wedge a \neq b).$$

Define R, R^2, and R^3.

Solution.

$R = \{(2, 4), (2, 8), (2, 12), (3, 9), (3, 12), (4, 8), (4, 12)\}$.

$R^2 = R \circ R = \{(2, 8), (2, 12)\}$.

$R^3 = R^2 \circ R = R \circ R \circ R = \{ \}$.

EXAMPLE

Let R be the relation on the set of real numbers defined by

$$xRy \text{ if and only if } xy = 1.$$

(x and y are **reciprocals**.) Using definitions of R^2 and R^3, predict a definition of R^n.

Solution.

(a) R^2.

xR^2z
$\Leftrightarrow \exists y : xRy$ and yRz
$\Leftrightarrow \exists y : xy = 1$ and $yz = 1$
$\Leftrightarrow x = z$ and $x \neq 0$.

If $xy = 1$ and $yz = 1$, then x and z cannot be 0. Therefore, $y = 1/x$ and $y = 1/z$. (The purpose is to obtain an expression involving **only** x and z, and thus the need to **eliminate** y.) This yields $x = z$ and $x \neq 0$.

If x and z are real numbers such that $x = z$ and $x \neq 0$, then set $y = 1/x$. This yields $xy = 1$ and $yz = 1$.

(b) R^3.

xR^3z
$\Leftrightarrow \exists y : xR^2y$ and yRz
$\Leftrightarrow \exists y : x = y$ and $x \neq 0$ and $yz = 1$
$\Leftrightarrow xz = 1$.

Therefore, $R^3 = R$.

Furthermore,

$$R^4 = R^3 \circ R = R \circ R = R^2,$$

and

$$R^5 = R^4 \circ R = R^2 \circ R = R^3 = R.$$

In other words, there is a **pattern** for **even and odd powers** of R. This can be used to give a definition of R^n.

The **transitive closure** of R is

$$xR^*y \text{ if and only if } (xy = 1) \vee (x = y \wedge x \neq 0).$$

EXAMPLE

Let R be the relation on the set of real numbers defined by

$$xRy \text{ if and only if } x^2 + y^2 \leq 1.$$

Show that $R^2 = R^3$.

Solution.

(a) R^2.

xR^2z
$\Leftrightarrow \exists y : xRy$ and yRz
$\Leftrightarrow \exists y : x^2 + y^2 \leq 1$ and $y^2 + z^2 \leq 1$
$\Leftrightarrow x^2 \leq 1$ and $z^2 \leq 1$
$\Leftrightarrow | x | \leq 1$ and $| z | \leq 1$.

(b) R^3.

xR^3z
$\Leftrightarrow \exists y : xR^2y$ and yRz
$\Leftrightarrow \exists y : | x | \leq 1$ and $| y | \leq 1$ and $y^2 + z^2 \leq 1$
$\Leftrightarrow \exists y : | x | \leq 1$ and $y^2 + z^2 \leq 1$
$\Leftrightarrow | x | \leq 1$ and $| z | \leq 1$.

Therefore, $R^3 = R^2$.

The **transitive closure** of R is

$$xR^*y \text{ if and only if } (| x | \leq 1 \text{ and } | y | \leq 1) \vee (x^2 + y^2 \leq 1),$$

or, to simplify,

$$xR^*y \text{ if and only if } (| x | \leq 1 \text{ and } | y | \leq 1).$$

EXAMPLE

Let a relation R on a set A is irreflexive. Determine whether it is necessarily the case that R^2 is irreflexive.

Solution.

No; R^2 need not be irreflexive.

(If there exists an $(a, a) \in R^2$, for some $a \in A$, then R^2 cannot be irreflexive.)

Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$. Then, R is irreflexive, but $(1, 1) \in R^2$.

Proposition. Let A, B, C, and D be sets, and let R, S, and T be relations defined by $R : A \longrightarrow B$, $S : B \longrightarrow C$, and $T : C \longrightarrow D$. The composition of relations is **associative**, that is,

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

22. CLOSURES OF RELATIONS

Let R be a relation on a set A. R may **not** satisfy a certain property P such as reflexivity, symmetry, transitivity, and so on.

The idea behind a closure is to seek: (1) **smallest** (minimal) relation S (2) **containing** R that has the property P.

If such an S exists, then S is called a **closure** of R with respect to P.

In other words, the closure of a binary relation R with respect to property P is the **relation obtained by adding the minimum number of ordered pairs** to R to obtain property P.

22.1. PROPERTIES OF A CLOSURE

R is a subset of its closure S.

Let Q be a relation on a set A. If Q has property P, and R is a subset of Q, then S is a subset of Q. This is what it means by S being "minimal".

22.2. REFLEXIVE CLOSURE OF RELATIONS

Let R be a relation on a set A. The **reflexive closure of R** is the **smallest relation** containing R that is reflexive.

- **Set:** In terms of the **set** representation of R, the reflexive closure of R can be formed by **adding** to R all pairs of the form (a, a), where a is in A, that are **not** in R.

- **Matrix:** In terms of the **zero-one matrix** representation, the reflexive closure can be formed by putting 1's on the diagonal.

- **Graph:** In terms of the **directed graph** representation of R, the reflexive closure can be formed by adding loops to all vertices.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$, and let R be the relation on A defined by

$$R = \{(1, 4), (2, 1), (2, 2), (3, 2), (4, 1)\}.$$

Then, R is not reflexive.

However,

$$S = \{\mathbf{(1, 1)}, (1, 4), (2, 1), (2, 2), (3, 2), \mathbf{(3, 3)}, (4, 1), \mathbf{(4, 4)}\},$$

is reflexive.

EXAMPLE

Let R be a relation of the set of integers given by $\{(a, b) \mid a \neq b\}$. Find the reflexive closure of R.

Solution. Upon adding all pairs of the form (a, a), the result is $\mathbf{Z} \times \mathbf{Z}$.

EXAMPLE

Let R be the relation $\{(a, b) \mid a \neq b\}$ on $\mathbf{Z}$. Give the reflexive closure of R.

Solution.

Upon adding all the pairs of the form (a, a) to the given relation yields $\mathbf{Z} \times \mathbf{Z}$, which is always reflexive.

22.3. SYMMETRIC CLOSURE OF RELATIONS

Let R be a relation on a set A. The **symmetric closure of R** is the smallest relation containing R that is symmetric.

- **Set:** In terms of the **set** representation of R, the symmetric closure of R can be formed by **adding** to R all ordered pairs of the form (b, a), where (a, b) is in R, that are not already in R. (The **inverse relation** R^{-1} is added to R.)

- **Matrix:** In terms of the **zero-one matrix** representation, the symmetric closure of R can be formed by adding 1's to the pairs across the diagonals that differ in value.

- **Graph:** In terms of the **directed graph** representation of R, the symmetric closure of R can be formed by adding arcs in the opposite direction to find the symmetric closure.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$, and let R be the relation on A defined by

$$R = \{(1, 4), (2, 1), (2, 2), (3, 2), (4, 1)\}.$$

Then, R is not symmetric.

However,

$$S = \{\mathbf{(1, 2)}, (1, 4), (2, 1), (2, 2), \mathbf{(2, 3)}, (3, 2), (4, 1)\},$$

is symmetric.

EXAMPLE

Let R be a relation of the set of integers given by $\{(a, b) \mid a \text{ divides } b\}$. Find the symmetric closure of R.

Solution. $\{(a, b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\}$.

EXAMPLE

Find the smallest relation containing the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers that is both reflexive and symmetric.

Solution.

The symmetric closure of R is $\{(a, b) \mid a \neq b\}$. If this relation is made reflexive as well, then the result is a relation that both reflexive and symmetric.

22.4. TRANSITIVE CLOSURE OF RELATIONS

Let R be a relation on a set A. The transitive closure of R the smallest relation containing R that is transitive.

[Void] The transitive closure of R can be formed by **adding** to R all ordered pairs of the form (a, c), where (a, b) and (b, c) are in r, that are not already in R.

This **one step process** of obtaining transitivity **may not work**. (Thus, the above process is void.) This is because **transitivity, unlike reflexivity and symmetry**, is **not a 'local' property**.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$, and let R be the relation on A defined by

$$R = \{(1, 2), (2, 3), (3, 2), (3, 4)\}.$$

Then, R is **not** transitive.

The addition of (1, 3), (2, 2), (2, 4), and (3, 3) should make R transitive, but it **does not**.

Let

$$R^1 = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 2), (3, 4), (3, 3)\}.$$

Then, R^1 is **not** transitive as (1, 4) is missing.

If R^1 is not transitive, **repeat the process** on R^1 and obtain R^2; if R^2 is not transitive, repeat the process on R^2 and obtain R^3; and so on.

Transitive Closure of *R*

{
(a, b) |

(a, b) is in R

or

(a, b) is in R^1

or

(a, b) is in R^2

or

…

}

The process **will stop** when it reaches $A \times A$, which is transitive. The result will be transitive as a transitive closure is **always** a transitive relation.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$, and let R be the relation on A defined by

$$R = \{(1, 4), (2, 1), (2, 2), (3, 2), (4, 1)\}.$$

Then, R is **not** transitive.

However,

$$R^1 = \{\mathbf{(1, 1)}, (1, 4), (2, 1), (2, 2), \mathbf{(2, 4)}, \mathbf{(3, 1)}, (3, 2), \mathbf{(3, 4)}, (4, 1), \mathbf{(4, 4)}\},$$

is transitive.

22.5. RELATIONSHIP BETWEEN DIFFERENT CLOSURES OF RELATIONS

It should be evident from foregoing examples that the closure of one type (say, reflexive) does **not** necessarily lead the closure of another type (say, symmetric).

Furthermore, the **order in which closures are created** can matter.

A transitive closure of a symmetric closure of a reflexive closure of a relation will be reflexive, symmetric, and, obviously, transitive.

However, it is **not** automatic that the symmetric closure of the reflexive closure of the transitive closure of a relation is transitive. (This, again, shows the uniqueness of transitive closures.)

EXAMPLE

Let $A = \{a, b, c\}$ and let $R = \{(a, b), (c, b)\}$.

Then, R is transitive. The transitive closure of R is R.

Now,

$$S = \{\boldsymbol{(a, a)}, (a, b), \boldsymbol{(b, b)}, (c, b), \boldsymbol{(c, c)}\},$$

is reflexive, and

$$T = \{(a, a), (a, b), \boldsymbol{(b, a)}, (b, b), \boldsymbol{(b, c)}, (c, b), (c, c)\},$$

is symmetric.

However, T is **not** transitive because it has (a, b) and (b, c), but not (a, c).

In other words, R has **'lost'** the property of transitivity due to the addition of new elements.

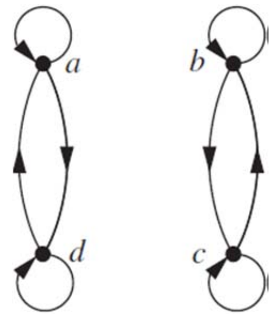
23. EQUIVALENCE CLASSES

Definition [Equivalence Class]. Let A be a set and let R be an equivalence relation on A. Let $a \in A$. Then, **equivalence class of *a***, denoted by $[a]_R$, is the set of all elements in A that are **related** to a.

In the case the underlying relation is obvious, the equivalence class of a is denoted by $[a]$.

EXAMPLE

For the relation with the directed graph shown below, determine whether it is an equivalence relation. If it is, give its equivalence classes.



Solution.

$A = \{a, b, c, d\}$. The relation is an equivalence relation, satisfying all three desired properties. The equivalence classes are $[a] = [d] = \{a, d\}$ and $[b] = [c] = \{b, c\}$.

EXAMPLE

Give the equivalence classes of the following equivalence relations on $\{0, 1, 2, 3\}$:

(a) $R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$.
(b) $S = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$.

Solution.

(a) In this case, each element is in an equivalence class by itself, that is, $[0]_R = \{0\}$, $[1]_R = \{1\}$, $[2]_R = \{2\}$, and $[3]_R = \{3\}$.

(b) In this case, the elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class, that is, $[0]_S = \{0\}$, $[1]_S = \{1, 2\}$, $[2]_S = \{1, 2\}$, $[3]_S = \{3\}$.

EXAMPLE

Give the equivalence classes of the following equivalence relations on the set of all functions from **Z** to **Z**:

(a) $\{(f, g) \mid f(1) = g(1)\}$.
(b) $\{(f, g) \mid \text{for some } c \in \mathbf{Z}, \text{for all } x \in \mathbf{Z}, f(x) - g(x) = c\}$.

Solution.

(a) In this case, there is one equivalence class for each n ∈ **Z**, and it contains all those functions whose value at 1 is n.

(b) In this case, there is no single way to describe the equivalence classes, as the set of equivalence classes is **uncountable**. For each function f from **Z** to **Z**, there is the equivalence class consisting of all those functions g for which there is a constant c such that

$$g(\mathrm{n}) = f(\mathrm{n}) + c, \text{ for all } \mathrm{n} \in \mathbf{Z}.$$

EXAMPLE

Let set $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\}$ be an equivalence relation on A. Find the equivalence classes of R.

Solution.

The equivalence classes of R:

$[1]_R = \{1, 3\}$.
$[2]_R = \{2\}$.
$[3]_R = \{1, 3\}$.
$[4]_R = \{4\}$.
$[5]_R = \{5\}$.

EXAMPLE

Let $A = \{1, 2, \ldots, 10\}$. Let aRb mean $3 \mid a - b$. (This is the same as saying $a \equiv b \pmod 3$.)

(a) Show that R is an equivalence relation.
(b) Find the equivalence classes of R.
(c) Give the meaning of ‘equivalence’.

Solution.

(a) R is an equivalence relation.

Reflexivity:

$3 \mid a - a = 0$.

Symmetry:

If 3 | $a - b$, then 3 | $b - a$.

Transitivity:

If 3 | $a - b$ and 3 | $b - c$, then 3 | $a - b + b - c = a - c$.

(b) The equivalence classes of R:

$[1]_R = \{1, 4, 7, 10\}$,
$[2]_R = \{2, 5, 8\}$,
$[3]_R = \{3, 6, 9\}$,

and so on.

Board Time!
Give $[7]_R$.

(c) The equivalence is "has the **same remainder** when divided by 3."

The **union** of sets $[1]_R$, $[2]_R$, and $[3]_R$ yields A. The **pairwise intersections** of $[1]_R$, $[2]_R$, and $[3]_R$ yield $\varnothing$. Therefore, the sets $[1]_R$, $[2]_R$, and $[3]_R$ form a **partition** of A.

EXAMPLE

The **congruence modulo *d*** is an equivalence relation on the set of integers. The equivalence classes of the congruence modulo d relation are called the **congruence classes modulo *d***.

Give the equivalence classes of congruence modulo $[4]_d$ for (a) $d = 2$, and (b) $d = 3$.

Solution.

(a) This set is those integers b such that $4 \equiv b \pmod 2$. Therefore:

$$\{\ldots, -2, 0, 2, \ldots\} = \{4 + 2n \mid n \in \mathbf{Z}\}.$$

(b) This set is those integers b such that $4 \equiv b \pmod 3$. Therefore:

$$\{\ldots, -2, 1, 4, \ldots\} = \{4 + 3n \mid n \in \mathbf{Z}\}.$$

REMARKS

The concept of an equivalence class is useful in several places in **software engineering**. For example, it is useful for **modeling users** (by reducing the number of users to model) or for **devising test cases** (by reducing the number of test cases to consider) [Burnstein, 2003, Section 4.5].

The previous example motivates the following:

Fundamental Theorem of Equivalence Relations. An equivalence relation R on a set A partitions A, and, conversely, corresponding to any partition of A, there exists an equivalence relation R on A.

24. ORDERING

There are a number of purposes for ordering:

- An **ordering of things** is about **comparison** based on an **implicit assumption** that the things are "not all the same".
- An ordering is about **organization** of things, based on **comparison** in some sense.
- An ordering of things is about supporting **understanding and finding** of things. (If a large number of things are presented simultaneously, then it can become difficult to understand or find them.)

24.1. ORDERING BASED ON SPACE AND TIME

There can be different kinds of ordering, including ordering based on **space** (**spatial ordering**) and ordering based on **time** (**temporal ordering**).

SPATIAL ORDERING

For example, consider the case of **spatial ordering**. There are cases where certain elements need to be placed before others:

- In user interfaces of certain software systems, the **`File`** Menu is placed before the **`Help`** Menu.

- In books, the **Preface** is placed before any **Chapter**, and the **Index** is placed after any **Chapter**.

- In a theater, the seats in a private box are placed in a manner that provides a better view of the event than other seats.

Board Time!
Give yet another example of spatial ordering.

TEMPORAL ORDERING

For example, consider the case of **temporal ordering**. There are cases where certain activities that cannot start until others are completed:

- In a typical software process, implementation follows design, and design follows requirements.

- In a business-to-business supply-chain integration, the activities are ordered sequentially.

- In business-to-consumer electronic commerce, payment transaction follows shopping.

(It is assumed that concurrency is not always an option.)

Board Time!
Give yet another example of temporal ordering.

24.2. PARTIAL ORDERING

Definition [Partial Order]. A relation R on set A is called a **partial order** if it has the following properties:

- R is reflexive.
- R is antisymmetric.
- R is transitive.

REMARKS

- A partial ordering means that, although **it is possible to compare some elements** against others, **there are other elements that cannot be compared**. Thus, the property of **symmetry** is removed from partial ordering and replaced by **antisymmetry**.

- A set A together with a partial order R is called a **partially ordered set**, or, equivalently, a **poset**, and is denoted by (A, R).

- A relation R that is a partial order is usually denoted by $\preccurlyeq$.

EXAMPLE

The $R = \varnothing$ relation on a set $A = \varnothing$ is a partial order. This is because R is reflexive, antisymmetric, and transitive.

EXAMPLE

The **"divides"** (|) relation on $\mathbf{Z}^+$ is a partial order.

EXAMPLE

Determine whether the "≥" relation on the set of integers is a partial order.

Solution.

Reflexivity: $\geq$ is reflexive, as $a \geq a$ for every integer a.

Antisymmetry: $\geq$ is antisymmetric, as if $a \geq b$ and $b \geq a$, then $a = b$.

Transitivity: $\geq$ is transitive, as $a \geq b$ and $b \geq c$ implies $a \geq c$.

Therefore, "$\geq$" is a partial order on **Z**, and (**Z**, $\geq$) is a partially ordered set.

EXAMPLE

In each of the following cases, determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive:

(a) $(x, y) \in R$ if and only if $x = \pm y$.
(b) $(x, y) \in R$ if and only if $x - y$ is a rational number.
(c) $(x, y) \in R$ if and only if $x = 2y$.
(d) $(x, y) \in R$ if and only if $xy \geq 0$.
(e) $(x, y) \in R$ if and only if $xy = 0$.
(f) $(x, y) \in R$ if and only if $x = 1$.
(g) $(x, y) \in R$ if and only if $x = 1$ or $y = 1$.

Solution.

(a)

- Reflexive: Yes. $x = \pm x$ (choosing the plus sign).
- Symmetric: Yes. $x = \pm y$ if and only if $y = \pm x$.
- Antisymmetric: No. $(1, -1)$ and $(-1, 1)$ are both in R.
- Transitive: Yes. The product of 1's and −1's is ±1.

(b)

- Reflexive: Yes. $x - x = 0$ is a rational number.
- Symmetric: Yes. If $x - y$ is a rational number, then so is $-(x - y) = y - x$.
- Antisymmetric: No. $(1, -1)$ and $(-1, 1)$ are both in R.
- Transitive: Yes. If $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore, their sum $x - z$ is a rational number, and that means that $(x, z) \in R$.

(c)

- Reflexive: No. $1 \neq 2 \cdot 1$.
- Symmetric: No. $(2, 1) \in R$, but $(1, 2) \notin R$.

- Antisymmetric: Yes. If $x = 2y$ and $y = 2x$, then $y = 4y$, from which it follows that $y = 0$ and so $x = 0$. Thus, the only time that (x, y) and (y, x) are both is R is when $x = y$ (and both are 0).
- Transitive: No. $(4, 2) \in R$ and $(2, 1) \in R$, but $(4, 1) \notin R$.

(d)

- Reflexive: Yes. The squares are always nonnegative.
- Symmetric: Yes. The roles of x and y in the statement are interchangeable.
- Antisymmetric: No. $(2, 3)$ and $(3, 2)$ are both in R.
- Transitive: No. $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

(e)

- Reflexive: No. $(1, 1) \notin R$.
- Symmetric: Yes. The roles of x and y in the statement are interchangeable.
- Antisymmetric: No. $(2, 0)$ and $(0, 2)$ are both in R.
- Transitive: No. $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

(f)

- Reflexive: No. $(2, 2) \notin R$.
- Symmetric: No. $(1, 2) \in R$ but $(2, 1) \notin R$.
- Antisymmetric: Yes. If $(x, y) \in R$ and $(y, x) \in R$, then $x = 1$ and $y = 1$, so $x = y$.
- Transitive: Yes. If $(x, y) \in R$ and $(y, z) \in R$, then $x = 1$ (and $y = 1$, although that does not matter), so $(x, z) \in R$.

(g)

- Reflexive: No. $(2, 2) \notin R$.
- Symmetric: Yes. The roles of x and y in the statement are interchangeable.
- Antisymmetric: No. $(2, 1)$ and $(1, 2)$ are both in R.
- Transitive: No. $(3, 1) \in R$ and $(1, 7) \in R$, but $(3, 7) \notin R$.

REMARKS

This example shows that, for a given relation, the properties of equivalence or partial ordering are neither automatic, nor easy to come by.

EXAMPLE

Let R be the following relation on the set of real numbers: aRb if and only if $\lfloor a \rfloor = \lfloor b \rfloor$. Determine whether R is a partial order.

Solution.

R is not antisymmetric, as one can have aRb and bRa for distinct $a \neq b$. For example, $\lfloor 1.1 \rfloor = \lfloor 1.2 \rfloor$. Therefore, R is not a partial order.

REMARKS

It could be checked that R given previously is, however, reflexive, symmetric, and transitive. Therefore, R is an **equivalence relation**.

EXAMPLE

Determine whether the relation with the directed graph shown below is a partial order.

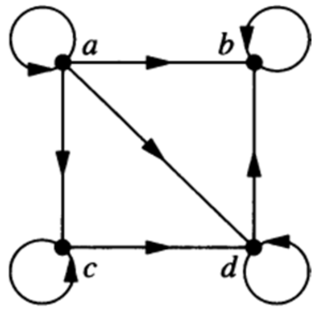
Solution.

$A = \{a, b, c, d\}$.

The relation is a partial order. It is reflexive (there is a loop at each vertex), it is antisymmetric (there are no multiple edges in opposite directions), and transitive (there is **no reason for it to be not transitive**).

EXAMPLE

Determine whether the relation with the directed graph shown below is a partial order.



Solution.

$A = \{a, b, c, d\}$.

This relation is **not** transitive. (There are directed edges from c to d and from d to b, but there is no directed edge from c to b. Therefore, it is not a partial order.)

EXAMPLE

For each of the following relations on $\{0, 1, 2, 3\}$, determine whether any of them is a partial order:

(a) $R = \{(0, 0), (2, 2), (3, 3)\}$.
(b) $R = \{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$.

Solution.

(a) R is not a partial order. R is not reflexive because 1 is not related to itself. (R is antisymmetric and R is transitive.)

(b) R is not a partial order. R is not transitive because $3R1$ and $1R2$, but 3 is not related to 2. (R is reflexive and R is antisymmetric.)

EXAMPLE

For each of the following relations represented by 0-1 matrices, determine whether the relations are partial orders:

(a)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Solution.

(a) This relation is $\{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\}$. It is clearly reflexive and antisymmetric. However, it is not transitive, because (2, 1) and (1, 3) are in the relation, but (2, 3) is not. Therefore, the relation is not a partial order.

(b) Improvise.

(c) This relation is not transitive, because (1, 3) and (3, 4) are present, but (1, 4) is not. Therefore, the relation is not a partial order.

EXAMPLE

Give an example of a relation on a set that is both an equivalence relation and a partial order.

Solution.

Let $A = \{a\}$ and $R = \{(a, a)\}$. Then, R is reflexive, symmetric, antisymmetric, and transitive.

24.3. LEXICOGRAPHIC ORDERING

The term 'lexicographic order' originally comes from **natural language alphabetic order of words** in a **dictionary**. It has since been applied to related contexts, including **glossary, index, and table of contents**.

Let $(S, \preccurlyeq_1)$ and $(T, \preccurlyeq_2)$ be two partially ordered sets.

The lexicographic ordering $\preccurlyeq$ on the Cartesian product $S \times T$ is, for any (a, b) and $(c, d) \in S \times T$, defined by:

$$(a, b) \preccurlyeq (c, d),$$

when either

(1) $a \preccurlyeq_1 c$, or
(2) both $a = c$ and $b \preccurlyeq_2 d$.

Theorem. The lexicographic order is a partial ordering on the Cartesian products of two partially ordered sets.

24.4. ORDERING IN HIGHER DIMENSIONS

The lexicographic ordering allows a **generalization** of the notion of $\leq$ in **higher dimensions**.

In particular, it is possible to compare points in the Cartesian product $\mathbf{Z}^+ \times \mathbf{Z}^+$ using the lexicographic ordering $\preccurlyeq$ constructed from the usual (less than or equal to) relation $\leq$.

EXAMPLE

Consider $\mathbf{Z}^+ \times \mathbf{Z}^+$ using the lexicographic ordering $\preccurlyeq$ constructed from the usual relation $\leq$. Determine whether the following are true:

- $(2, 7) \preccurlyeq (1, 8)$
- $(3, 6) \preccurlyeq (5, 3)$
- $(1, 2) \preccurlyeq (1, 4)$

Solution. Improvise.

24.5. TOTAL ORDERING

Definition [Comparable Elements]. The elements a and b of a poset $(A, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. If a and b are elements of A such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called **incomparable**.

EXAMPLE

The $(\mathbf{Z}^+, |)$ is a partially ordered set in which 1 and any other number are comparable, 2 and any other even number are comparable, but two numbers that are co-prime are not comparable.

The term **"partial"** is used to describe partial orderings because there may be pairs of elements that are incomparable [Rosen, 2012]. The possibility that every two elements in the set are comparable motivates the following.

Definition [Total Order]. A partially ordered set, $(A, \preccurlyeq)$, such that every two elements in A are comparable is called as a totally ordered set, and $\preccurlyeq$ is called a total order.

EXAMPLE

$(\mathbf{Z}, \leq)$ is a totally ordered set. This is because $a \leq b$ or $b \leq a$ whenever a and b are integers.

EXAMPLE

$(\mathbf{Z}^+, |)$ is not a totally ordered set. This is because not every two elements in $\mathbf{Z}^+$ are comparable. For instance, two prime numbers are not comparable. For another instance, two consecutive numbers, both > 1, are not comparable.

25. HASSE DIAGRAMS

It is possible to represent relations using directed graphs. This motivates the use of graphs to represent partially ordered sets.

ISSUE

The issue is about **scale**. The graphs of partially ordered sets become **cluttered quickly** as the **size of the underlying set increases**. (The large number of edges and the symbols for direction **obscure the basic structure of the graph**.)

RESOLUTION

It can be observed that a partially ordered set is, **by definition**, reflexive, antisymmetric and transitive. Thus, the **graphical representation** for a partially ordered set can be **minimized** in a compact manner:

1. All self-loops are omitted.

2. The vertices are arranged in the vertical dimension so the directional arrows are implied (always upwards) and are hence omitted.

3. All edges implied by transitivity are omitted.

25.1. CHARACTERISTICS OF HASSE DIAGRAMS

The Hasse diagrams have a number of defining characteristics, including the following:

- They are an effort to **simplify a graph** by (1) eliminating parts of it that are **obvious**, and (2) those that are easily reconstructed by "visual examination".

- They are a graphical representation of a partially ordered set. Every **finite** partially ordered set can be represented by a Hasse diagram.

- They are also called **upward drawings** (because of their **implied** upward orientation).

25.2. HASSE DIAGRAM CONSTRUCTION

The process for constructing a Hasse diagram is based on **increasing the level of abstraction**, as seen in the following:

1. Start with the directed graph of the partial order.

2. Remove the loops at each vertex because of the reflexivity.

3. Remove all edges that must be present because of the transitivity.

4. Arrange each edge so that all arrows point upwards.

5. Remove all arrowheads.

EXAMPLE

Construct the Hasse diagram for ({1, 2, 3, 4}, ≤).

Solution.

Figure 8 shows the directed graph of ({1, 2, 3, 4}, ≤) and its **transformation** to the Hasse diagram.

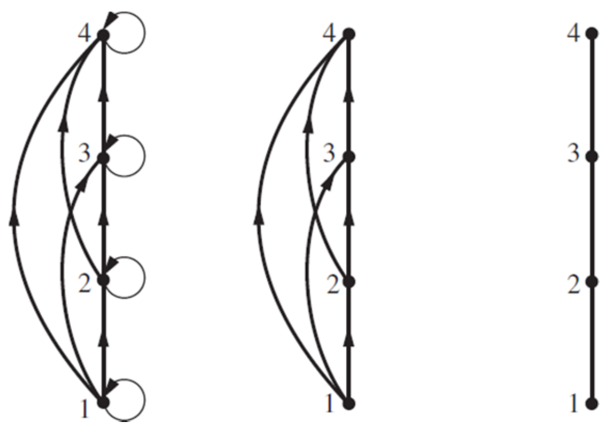


Figure 8. The stages in the construction of the Hasse diagram for ({1, 2, 3, 4}, ≤). (Source: [Rosen, 2012].)

({1, 2, 3, 4}, ≤) is reflexive, and therefore each vertex of its directed graph has loops, which can be removed.

({1, 2, 3, 4}, ≤) is transitive, and therefore its directed graph has edges (1, 3), (1, 4), and (2, 4), which can be removed.

The directed graph of ({1, 2, 3, 4}, ≤) shown in Figure 8 is such that all of its arrows point upwards. Therefore, all arrowheads can be removed.

EXAMPLE

(a) Construct the Hasse diagram for ({1, 2, 3}, ≤).
(b) Construct the Hasse diagram for ({1, 2, 3}, ≥).

Solution. Improvise. (They are **not identical**!)

EXAMPLE

Construct the Hasse diagram for (*P*({0, 1}, ⊆). (Let *S* = {0, 1}. Then, *P*(*S*) with ⊆-relation is indeed a partially ordered set.)

Solution.

Let e denote the empty set.

The following 9 edges are present in the relation:

> (e, e), (e, {0}), (e, {1}), (e, {0, 1}), ({0}, {0}), ({0}, {0, 1}), ({1}, {1}), ({1}, {0, 1}), ({0, 1}, {0, 1}).

The following 4 edges are eliminated because they are implied by the reflexive property:

> (e, e), ({0}, {0}), ({1}, {1}), ({0, 1}, {0, 1}).

The following edge is eliminated because it is implied by the transitive property:

> (e, {0, 1}).

25.3. PROPERTIES OF ELEMENTS OF HASSE DIAGRAMS

The elements of a Hasse diagram can have a number properties terminology:

- Maximal Element
- Minimal Element
- Greatest Element
- Least Element
- Upper Bound
- Lower Bound
- Least Upper Bound
- Greatest Lower Bound

It can be useful to consider the connection between **maximal element, greatest element, upper bound, and least upper bound**.

It can be useful to consider the **connection** between minimal element, least element, lower bound, and greatest lower bound.

In particular, in each case, it can be useful to ask whether they are the same, whether they are different, and if so, is that always, sometimes, or never.

MAXIMAL ELEMENTS

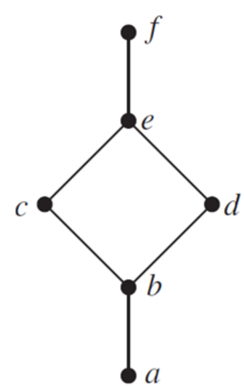
Let $(S, \preccurlyeq)$ be a partially ordered set. An element $a \in S$ is **maximal** in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.

In other words, there is nothing above a. This constitutes the **top** of the Hasse diagram.

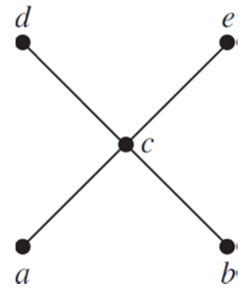
It could be noted that a maximal element **need not be unique**.

EXAMPLE

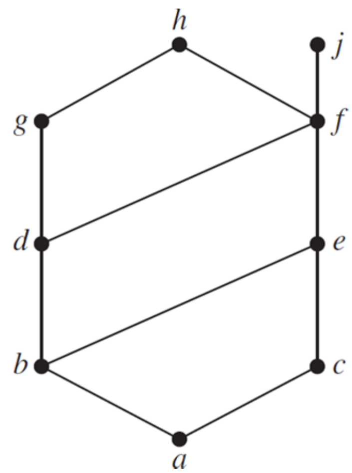
The Hasse diagram shown in Figure 9(a) has one maximal element, and the Hasse diagram shown in Figure 9(b) **two** maximal elements.



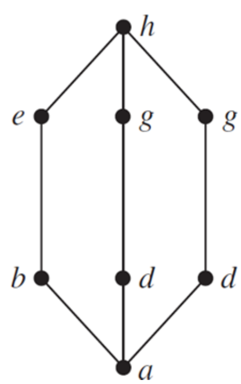
(a)



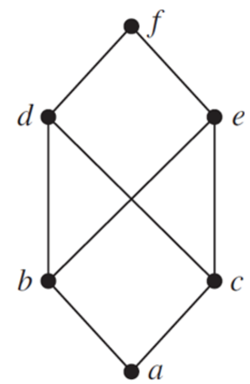
(b)



(c)



(d)



(e)

Figure 9. A collection of Hasse diagrams. (Source: [Rosen, 2012].)

MINIMAL ELEMENTS

Let $(S, \preccurlyeq)$ be a partially ordered set. An element $a \in S$ is **minimal** in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \prec a$.

In other words, there is nothing above a. This constitutes the **bottom** of the Hasse diagram.

It could be noted that a minimal element **need not be unique**.

EXAMPLE

The Hasse diagram shown in Figure 9(c) has one minimal element, and the Hasse diagram shown in Figure 9(b) has **two** minimal elements.

MAXIMAL AND MINIMAL ELEMENTS

For a partially ordered set, **each of the following is possible**:

- No minimal element.
- No maximal element.
- No minimal element and no maximal element.
- Multiple minimal elements.
- Multiple maximal elements.
- Multiple minimal elements and multiple maximal elements.

EXAMPLE

(a) Give an example of a partially ordered set that has a minimal element but no maximal element
(b) Give an example of a partially ordered set that has a maximal element but no minimal element.
(c) Give an example of a partially ordered set that has no minimal element and no maximal element.

Solution.

The following is one possible **strategy**. For (a) and (b), look for **infinite sets that are bound at one end**. For (c), look for **infinite sets that are unbounded** in both directions.

Then, one possible solution is:

For (a), an example of a partially ordered set that has a minimal element but no maximal element is (Natural Numbers, ≤).

For (b), an example of a partially ordered set that has a maximal element but no minimal element is (Negative Integers, ≤).

For (c), an example of a partially ordered set that has no minimal element and no maximal element is (Integers, ≤).

GREATEST ELEMENTS

Let $(S, \preccurlyeq)$ be a partially ordered set. An element $a \in S$ is the **greatest element** in $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$.

It could be noted that the greatest element, **if it exists**, must be **unique**.

LEAST ELEMENTS

Let $(S, \preccurlyeq)$ be a partially ordered set. An element $a \in S$ is the **least element** in $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.

It could be noted that the least element, **if it exists**, must be **unique**.

UPPER BOUNDS

Let A be a subset of a partially ordered set $(S, \preccurlyeq)$.

If $u \in S$ such that $a \preccurlyeq u$ for all $a \in A$, then u is called an **upper bound** of A.

It could be noted that there must be a **clear path** from u to all $a \in A$.

LOWER BOUNDS

Let A be a subset of a partially ordered set $(S, \preccurlyeq)$.

If $l \in S$ such that $l \preccurlyeq a$ for all $a \in A$, then u is called an **upper bound** of A.

It could be noted that there must be a **clear path** from l to all $a \in A$.

LEAST UPPER BOUNDS

Let A be a subset of a partially ordered set $(S, \preccurlyeq)$.

If x is an **upper bound** of A and $x \preccurlyeq z$ whenever z is an **upper bound** of A, Then x is called the **least upper bound (LUB)** of A.

This is simply saying that one looks for the smallest of the upper bounds.

It could be noted that the least upper bound, **if it exists**, must be **unique**.

GREATEST LOWER BOUNDS

Let A be a subset of a partially ordered set $(S, \preccurlyeq)$.

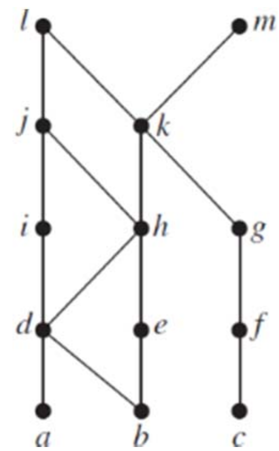
If y is a **lower bound** of A and $z \preccurlyeq y$ whenever z is a **lower bound** of A, Then y is called the **greatest lower bound (GLB)** of A.

This is simply saying that one looks for the largest of the lower bounds.

It could be noted that the greatest lower bound, **if it exists**, must be **unique**.

EXAMPLE

The following Hasse diagram is given:



(a) Find the maximal elements.
(b) Find the minimal elements.
(c) Find the greatest element, if any.
(d) Find the least element, if any.
(e) Find all upper bounds of $\{a, b, c\}$.
(f) Find the LUB of $\{a, b, c\}$, if it exists.
(g) Find all lower bounds of $\{f, g, h\}$.
(h) Find the GLB of $\{f, g, h\}$, if it exists.

Solution.

(a) The maximal elements are the ones with no other elements above them, namely l and m.

(b) The minimal elements are the ones with no other elements below them, namely a, b, and c.

(c) There is no greatest element, as neither l nor m is greater than the other.

(d) There is no least element, as neither a nor b is less than the other.

(e) There is a need to find elements from which downward paths to each of a, b, and c can be found. k, l, and m are the elements satisfy this criterion.

(f) As k is less than both l and m, it is the LUB of a, b, and c.

(g) No element is less than both f and h, and so there are no lower bounds.

(h) As there are no lower bounds, there can be no GLB.

26. LATTICES

A partially ordered set in which **every pair** of elements has both a **least upper bound** and **greatest lower bound** is called a **lattice**.

EXAMPLE

The partially ordered set represented by the Hasse diagram shown in Figure 9(a) is a lattice. This is because every pair of elements in it has both the LUB and the GLB.

EXAMPLE

The partially ordered set represented by the Hasse diagram shown in Figure 9(c) is not a lattice. This is because $\{h, j\}$ does not have an upper bound.

The partially ordered set represented by the Hasse diagram shown in Figure 9(e) is not a lattice. This is because $\{b, c\}$ does not have the LUB.

EXAMPLE

(a) Show that if (S, R) is a partially ordered set, then (S, R^{-1}), where R^{-1} is the inverse of R, is also a partially ordered set.

(b) Show that if (S, R) is a lattice, then (S, R^{-1}), is also a lattice.

Solution.

(a) Use reflexivity, antisymmetry, and transitivity of (S, R).

(b) Due to the definition of inverse, the GLB and the LUB under (S, R) become the LUB and the GLB under (S, R^{-1}), respectively.

27. TOPOLOGICAL SORTING

A **total ordering** $\preccurlyeq$ is said to be **compatible** with the **partial ordering** R if $a \preccurlyeq b$ whenever aRb.

The process of **constructing a compatible total ordering from a partial ordering** is called **topological sorting**. (In computer science, a **topology** is any arrangement of objects that can be connected with edges.)

The following result is needed to construct a topological sort:

Lemma. Every **finite nonempty partially ordered** set $(S, \preccurlyeq)$ has **at least one** minimal element.

EXAMPLE

$(\{1, 2, 4, 5, 12, 20\}, |)$ is a finite partially ordered set.

Figure 10 demonstrates the process of constructing a compatible total ordering for $(\{1, 2, 4, 5, 12, 20\}, |)$.

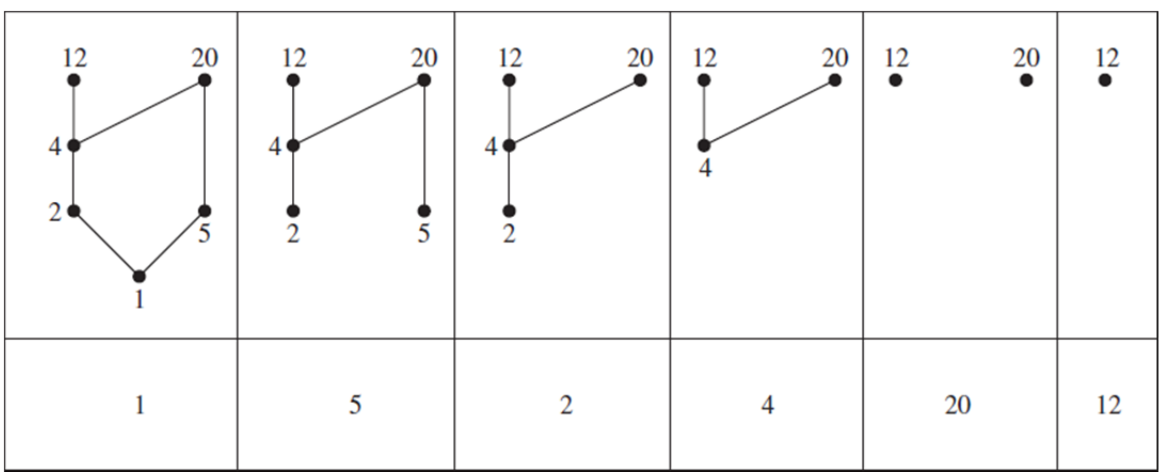


Figure 10. A topological sort for $(\{1, 2, 4, 5, 12, 20\}, |)$. (Source: [Rosen, 2012].)

Step 1: A minimal element is selected from $(\{1, 2, 4, 5, 12, 20\}, |)$. There is only one minimal element in this partially ordered set, namely 1. 1 is 'selected'.

Step 2: A minimal element is selected from

$$(\{1, 2, 4, 5, 12, 20\} - \{1\}), |) := (\{2, 4, 5, 12, 20\}, |).$$

There are two minimal elements in this partially ordered set, namely, 2 and 5. 5 is selected. (A partially ordered set **does not associate any significance indicators** with its minimal elements, and so in case of ties a selection can appear random.)

Step 3: A minimal element is selected from

$$(\{2, 4, 5, 12, 20\} - \{5\}), |) := (\{2, 4, 12, 20\}, |).$$

. . .

The process continues in a similar manner until all elements of {1, 2, 4, 5, 12, 20} are selected.

The resulting topological sort, shown in Figure 10, is:

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

Yet another topological sort can be:

$$1 \prec 2 \prec 5 \prec 4 \prec 12 \prec 20.$$

27.1. THE KAHN ALGORITHM

The following is a compact and simplified incarnation of the Kahn Algorithm:

Input: ($R = \{a_1, a_2, \ldots, a_n\}$, $\preccurlyeq$, a partial ordering
Output: ($S = \{b_1, b_2, \ldots, b_n\}$, $\preccurlyeq$), a compatible total ordering of R
Process:

1. k ⟵ 1, where ⟵ denotes assignment (or substitution)
2. Until $S \neq \varnothing$, repeat the following steps:
   (a) $a_k \longleftarrow$ a minimal element of R
   (b) $R \longleftarrow R - \{a_k\}$
   (c) k ⟵ k + 1

EXAMPLE

A project at a company requires the completion of seven tasks. These tasks are not independent. From these tasks, some tasks can be started only after other tasks are finished. The tasks are equipped with the following a partial ordering:

$$\text{task } X \prec \text{task } Y$$

if

$$\text{task } Y \text{ cannot be started until task } X \text{ has been completed.}$$

The Hasse diagram for the seven tasks, with respect to the above partial ordering, is shown in Figure 11. Find an order in which these tasks can be carried out to complete the project.

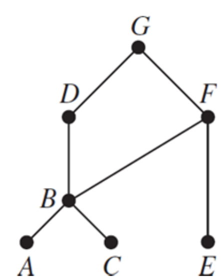


Figure 11. The Hasse diagram for a project consisting of seven tasks. (Source: [Rosen, 2012].)

Solution.

An ordering of the seven tasks can be obtained by performing a topological sort.

Let A be chosen as the minimal element for a topological sort. Then, the steps for one possible topological sort are illustrated in Figure 12.

The result of this sort,

$$A \prec C \prec B \prec E \prec F \prec D \prec G,$$

gives one possible order for the tasks.

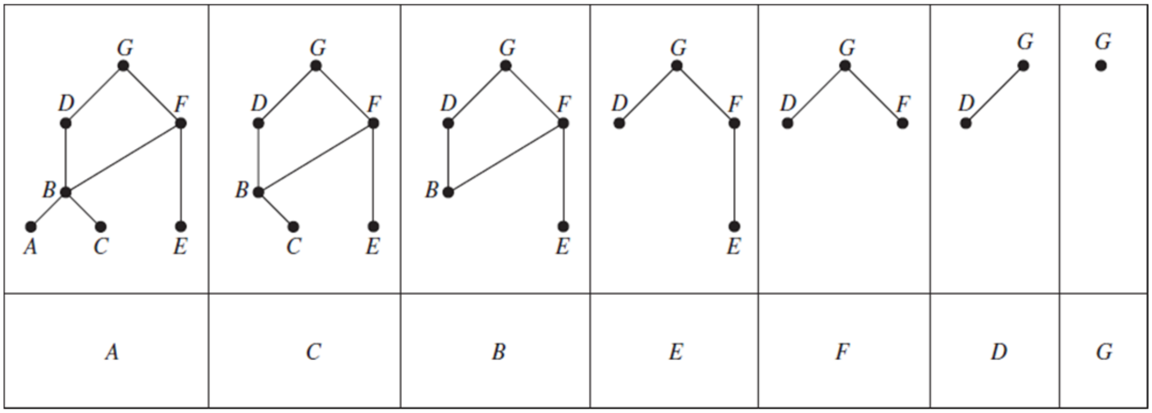


Figure 12. A topological sort for a project consisting of seven tasks. (Source: [Rosen, 2012].)

Let E be chosen as the minimal element for a topological sort. Then, it is possible to have another topological sort, the result of which can be,

$$E \prec C \prec A \prec B \prec D \prec F \prec G,$$

and gives another possible order for the tasks.

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The inclusion of images from external sources is only for non-commercial educational purposes, and their use is hereby acknowledged.

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