Lecture 4
2.4 Exact equations

Definition 1. A differential expression $M(x, y)dx + N(x, y)dy$ is **an exact differential** if it corresponds to the differential of some function $f(x, y)$.

Definition 2. A first order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **an exact differential equation** if the differential expression on the left-hand side is an exact differential.

Criterion for an exact differential

Theorem 1. Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first order derivative in $R : a \leq x \leq b$, $c \leq y \leq d$. Then the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Method of solution

1. Given a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$, determine whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If does then there exist a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y).$
2. From $\frac{\partial f}{\partial x} = M(x,y)$ find $f(x,y) = \int M(x,y)dx + g(y)$.
3. From $\frac{\partial f}{\partial y} = N(x,y)$ determine $g'(y)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y),$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx.$$

4. By integrating with respect to y, find $g(y)$

$$g(y) = \int N(x,y)dy - \int \frac{\partial}{\partial y}\int M(x,y)dxdy.$$