Concordia University
Faculty of Engineering and Computer Science
Applied Advanced Calculus - ENGR233/4-X
2020 Final Examination
April 17, 2020

Important points

- Professor: Alexey Kokotov
- Start: **14-00**. Finish: **17-00**
- Total marks: 118. Evaluation out of 100. The exam implies 18% bonus. All questions should be answered.
- There are 14 problems. Some problems consist of several parts; mark related to each part is given in brackets.
- There is an answer sheet attached to this exam. **Please write your answer to each question in the answer sheet.**
- You can also provide your detailed solutions of **up to three questions as optional supplementary attachments**.
- If you choose to submit optional solutions to three questions, solution to each question should be submitted as one separate file
- Your **mandatory answer sheet** and your **optional supplementary attachments** should be sent as **one** email with **up to four** attachments to

  engr233.x@gmail.com

  and copy to

  alexey.kokotov@concordia.ca

  The answer sheet can be filled and sent in any electronic format. Alternatively, if can be printed, filled and photographed/scanned.
- Students are given an **additional 20 minutes administration time from 17:00 to 17:20** to prepare their email attachments and submit their exam.
- Exams that are **received after 17:21 PM will not be marked**.

Problems

1. Consider the following line integral of the conservative vector field:

$$\int_C (y^2 \sin z - z)dx + 2xy \sin z \, dy + (xy^2 \cos z - x)dz$$

where C is the contour given by $\mathbf{r}(t) = \langle t^3, 2t^2 - 1, \pi t \rangle$, $0 \le t \le 1/2$.

a. [4] Find the potential f of the vector field satisfying the condition $f(1,1,0) = 0$.

b. [5] Compute the line integral.

2. ([8]) Find the point on the surface $z = x^2 + 2y^2$ where the tangent plane is orthogonal to the line connecting the points $(3,0,1)$ and $(1,4,0)$.

3. Consider the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^3\mathbf{k}$ where C is the triangle with vertices $(0,0,3)$, $(1,1,4)$, $(2,0,0)$, oriented counterclockwise if viewed from above.

a. [2] Represent the equation of the plane containing the triangle in the form $z = ax + by + c$ and find a, b and c.

b. [2] Compute curl $\mathbf{F}$.

c. [4] Compute the original line integral using Stokes' theorem.

4. The line l_1 has the direction vector $\langle 1, 0, -1 \rangle$ and passes through the point $(0,-1,-1)$. The line l_2 passes through the points $(1,2,3)$ and $(1,3,2)$.

a. [2] What is the angle between l_1 and l_2 in radians? The answer should lie between 0 and $\pi/2$.

b. [6] What is the distance between l_1 and l_2?

5. Consider the integral $\oint_C xy \, dx + x^2 \, dy$ where C is the boundary of the region bounded by $x = 0$, $x = y$, $x^2 + y^2 = 64$, $x, y \ge 0$, taken in clockwise direction.

a. [3] Apply the Green's theorem and rewrite the obtained double integral in polar coordinates. Give the range for the polar angle θ.

b. [6] Compute the original line integral using Green's theorem.

6. The temperature T at a point (x,y,z) in space is inversely proportional to the square of the distance from (x,y,z) to the origin. It is known that $T(0,0,1) = 500$.

a. [2] Compute $T(2,0,0)$.

b. [3] Find the rate of change of T at the point $(2,3,3)$ in the direction of the point $(3,1,1)$.

c. [3] What is the maximal rate of change of T at the point $(2,3,3)$?

7. Suppose $\mathbf{r}(t) = t^2\mathbf{i} + (t^3 - t)\mathbf{j} + (t-1)^2\mathbf{k}$ is the position vector of a moving particle.

a. [2] At what point does the particle pass through the xy-plane?

b. [2] What is its velocity vector at this point?

c. [4] What is the radius of curvature of the trajectory at this point?

8. Consider the surface which is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the parabolic cylinder $y = x^2$.

   a. [2] Represent the scalar surface element dS in the form $a\,dx\,dy$ and find a.

   b. [6] Compute the area of the surface.

9. Consider the sum of double integrals

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx\ .$$

   a. [4] Combine into one integral and describe the domain of integration in terms of polar coordinates. Give the range for the radius r.

   b. [4] Compute the integral.

10. Consider the *inward* flux $\int\int_S (\mathbf{F}\cdot\mathbf{n})dS$ of the vector field $\mathbf{F} = y^2\mathbf{i} + xz^3\mathbf{j} + z^2\mathbf{k}$ where S is the surface of the region D bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 1$, $z = 5$, $x = \sqrt{3}y$, $y = 0$, $x, y \geq 0$.

    a. [2] Compute the divergence of the vector field $\mathbf{F}$ at the point $(1, 1, -1)$.

    b. [7] Transform the surface integral into the triple integral using the divergence theorem and evaluate.

11. ([9]) Use spherical coordinates to find the volume of the solid situated below $x^2 + y^2 + z^2 = 1$ and above $z = \sqrt{x^2 + y^2}$ and lying in the first octant.

12. Consider the plane curve $y = \frac{x^3}{\sqrt{45}}$ for $0 \leq x < \infty$.

    a. [4] Find the x-coordinate of the point where the curvature of the curve is minimal.

    b. [4] Find the x-coordinate of the point where the curvature of the curve is maximal.

    Useful formula: The curvature of the plane curve $y = f(x)$ is given by $\kappa(x) = |f''|(1 + f'^2)^{-3/2}$.

13. a. [2] Compute the divergence of vector field $\mathbf{F} = x^3y^2\mathbf{i} + y\mathbf{j} - 3zx^2y^2\mathbf{k}$

    b. [7] Use divergence theorem to compute the outward flux of the vector field $\mathbf{F}$ through the surface of the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 2y$.

14. ([8]) Find the work done by the force field $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ acting along the curve given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(8, 4, 2)$.