Confidence Intervals

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Outlines

- What I mean by confidence intervals
- Estimating the population mean μ given s.t.d σ is known
- Estimating the population mean μ given s.t.d σ is unknown
- Estimating the std σ is known

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Procedure A: Estimating population mean μ given s.t.d σ is known

- Step 1: Find the sample mean $\overline{X}$
- Step 2: Find Z_α, and $Z_{\alpha/2}$
  - $Z_\alpha = \Phi^{-1}(1-\alpha)$
  - $Z_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$
- Step 3: The two-sided confidence interval is given as follows:

$$\mu_x \in [\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}] \tag{1}$$

- Step 4: The lower confidence interval is given as follows:

$$\mu_x \in (-\infty, \overline{X} + Z_\alpha \frac{\sigma}{\sqrt{n}}] \tag{2}$$

- Step 5: The upper confidence interval is given as follows:

$$\mu_x \in [\overline{X} - Z_\alpha \frac{\sigma}{\sqrt{n}}, \infty) \tag{3}$$

Question 1

The PCB concentration of a fish caught in Lake Michigan was measured by a technique that is known to result in an error of measurement that is normally distributed with a standard deviation of .08 ppm (parts per million). Suppose the results of 10 independent measurements of this fish are

11.2, 12.4, 10.8, 11.6, 12.5, 10.1, 11.0, 12.2, 12.4, 10.6

(a) Give a 95 percent confidence interval for the PCB level of this fish.

(b) Give a 95 percent lower confidence interval.

(c) Give a 95 percent upper confidence interval.

Question 1

- Summary: $\sigma = 0.08$, $n = 10$, and $\alpha = 0.05$
- S.t.d is given ($\sigma = 0.08$)
- Estimate the population mean μ_x using confidence intervals

Question 1

- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{11.2+\ldots+10.6}{10} = 11.48$

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  - $\overline{X} = \frac{11.2+...+10.6}{10} = 11.48$
- Step2: Find Z_α, and $Z_{\alpha/2}$
  - $Z_{0.05} = \Phi^{-1}(1-0.05) = 1.65$
  - $Z_{0.025} = \Phi^{-1}(1-0.025) = 1.96$

Question 1

- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{11.2 + \ldots + 10.6}{10} = 11.48$
- Step2: Find Z_{α}, and $Z_{\alpha/2}$
  - $Z_{0.05} = \Phi^{-1}(1 - 0.05) = 1.65$
  - $Z_{0.025} = \Phi^{-1}(1 - 0.025) = 1.96$
- Step3: Use the two-sided confidence interval:
  - $\mu_x \in [\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$
    $\Rightarrow \mu_x \in [11.48 - 1.96 * \frac{0.08}{\sqrt{10}}, 11.48 + 1.96 * \frac{0.08}{\sqrt{10}}]$
    $\Rightarrow \mu_x \in [11.43, 11.53]$

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- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{11.2+...+10.6}{10} = 11.48$
- Step2: Find Z_α, and $Z_{\alpha/2}$
  - $Z_{0.05} = \Phi^{-1}(1-0.05) = 1.65$
  - $Z_{0.025} = \Phi^{-1}(1-0.025) = 1.96$
- Step3: Use the two-sided confidence interval:
  - $\mu_x \in [\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$
    $\Rightarrow \mu_x \in [11.48 - 1.96 * \frac{0.08}{\sqrt{10}}, 11.48 + 1.96 * \frac{0.08}{\sqrt{10}}]$
    $\Rightarrow \mu_x \in [11.43, 11.53]$
    $\Rightarrow$ we are sure 95% that the population mean of PCB level will be between 11.43 and 11.53
    $\Rightarrow P(11.43 \leq \mu_x \leq 11.53) = 95\%$

Question 1

- Step4: Use the lower confidence interval
  - $\mu_x \in (-\infty, \overline{X} + Z_\alpha \frac{\sigma}{\sqrt{n}}]$
    $\Rightarrow \mu_x \in (-\infty, 11.48 + 1.65 * \frac{0.08}{\sqrt{10}}]$
    $\Rightarrow \mu_x \in (-\infty, 11.52]$
    $\Rightarrow$ we are sure 95% that the population mean of PCB level will not be greater than 11.52
    $\Rightarrow P(\mu_x \leq 11.52) = 95\%$

Question 1

- Step5: Use the upper confidence interval
  - $\mu_x \in [\overline{X} - Z_\alpha \frac{\sigma}{\sqrt{n}}, \infty)$
    $\Rightarrow \mu_x \in [11.48 - 1.65 * \frac{0.08}{\sqrt{10}}, \infty)$
    $\Rightarrow \mu_x \in [11.44, \infty)$
    $\Rightarrow$ we are sure 95% that the population mean of PCB level will not be less than 11.44
    $\Rightarrow P(11.44 \leq \mu_x) = 95\%$

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- Estimating the s.t.d σ

Procedure A: Estimating population mean μ given s.t.d σ is unknown

- Step 1: Find the sample mean $\overline{X}$
- Step 2: Find $t_{\alpha}, n-1$, and $t_{\alpha/2}, n-1$
- Step 3: Find the sample variance using the following equation:

$$S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1} \tag{4}$$

- Step 4: The two-sided confidence interval is given as follows:

$$\mu_x \in [\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}] \tag{5}$$

- Step 5: The lower confidence interval is given as follows:

$$\mu_x \in (-\infty, \overline{X} + t_{\alpha}\frac{S}{\sqrt{n}}] \tag{6}$$

- Step 6: The upper confidence interval is given as follows:

$$\mu_x \in [\overline{X} - t_{\alpha}\frac{S}{\sqrt{n}}, \infty) \tag{7}$$

Question 2

The following data resulted from 24 independent measurements of the melting point of lead.

330	322	345
328.6	331	342
342.4	340.4	329.7
334	326.5	325.8
337.5	327.3	322.6
341	340	333
343.3	331	341
329.5	332.3	340

Assuming that the measurements can be regarded as constituting a normal sample whose mean is the true melting point of lead,
(a) determine a 95 percent two-sided confidence interval for this value.
(b) Also determine a 99 percent two-sided confidence interval.

Question 2

- Summary: $n = 24$, and $\alpha = 0.05$ for part a, and $\alpha = 0.01$ for part b
- S.t.d is not given $(\sigma) \Rightarrow$ use sample s.t.d (S)
- Estimate the population mean μ_x using two-sided confidence intervals

Question 2

- Step1: Find the sample mean $\overline{X}$
    - $\overline{X} = \frac{330+\ldots+340}{24} = 334$

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- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{330+...+340}{24} = 334$
- Step2: Find the sample variance $\overline{X}$
  - $S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$
    $\Rightarrow S^2 = \frac{(330-334)^2}{23} + ... + \frac{(340-334)^2}{23}$
    $\Rightarrow S = 6.96$

Question 2

- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{330+...+340}{24} = 334$
- Step2: Find the sample variance $\overline{X}$
  - $S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$
    $\Rightarrow S^2 = \frac{(330-334)^2}{23} + ... + \frac{(340-334)^2}{23}$
    $\Rightarrow S = 6.96$
- Step3: Find $t_{\alpha/2,n-1}$
  - $\Rightarrow t_{0.025,23} = 2.07$

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  - $\overline{X} = \frac{330+...+340}{24} = 334$
- Step2: Find the sample variance $\overline{X}$
  - $S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$
    $\Rightarrow S^2 = \frac{(330-334)^2}{23} + ... + \frac{(340-334)^2}{23}$
    $\Rightarrow S = 6.96$
- Step3: Find $t_{\alpha/2,n-1}$
  - $\Rightarrow t_{0.025,23} = 2.07$
- Step4: Use the two-sided confidence interval using t-distribution
  - $\mu_x \in [\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \overline{X} + Z_{\alpha/2,n-1}\frac{S}{\sqrt{n}}]$
    $\Rightarrow \mu_x \in [334 - 2.07 * \frac{6.96}{\sqrt{24}}, 334 + 2.07 * \frac{2.07}{\sqrt{24}}]$

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Procedure A: Estimating population std σ

- Step 1: Find the sample mean $\overline{X}$
- Step 2: Find $(\chi_{\alpha/2,n-1})$, $(\chi_{1-\alpha/2,n-1})$, $(\chi_{\alpha,n-1})$, and $(\chi_{1-\alpha,n-1})$
- Step 3: Find the sample variance using the following equation:

$$S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1} \tag{8}$$

- Step 4: The two-sided confidence interval is given as follows:

$$\sigma^2 \in [\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}] \tag{9}$$

- Step 5: The lower confidence interval is given as follows:

$$\sigma^2 \in (-\infty, \frac{(n-1)S^2}{\chi^2_{1-\alpha,n-1}}] \tag{10}$$

Procedure A: Estimating population std σ

- Step 6: The upper confidence interval is given as follows:

$$\sigma^2 \in [\frac{(n-1)S^2}{\chi^2_{\alpha,n-1}}, \infty) \tag{11}$$

Question 3

The amount of beryllium in a substance is often determined by the use of a photometric filtration method. If the weight of the beryllium is μ, then the value given by the photometric filtration method is normally distributed with mean μ and standard deviation σ. A total of eight independent measurements of 3.180 mg of beryllium gave the following results.

3.166, 3.192, 3.175, 3.180, 3.182, 3.171, 3.184, 3.177

Use the preceding data to

(a) estimate σ (i.e., give a point estimate).

(b) Find a 90 percent confidence interval estimate of σ.

Question 3

- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{3.166+...+3.184}{8} = 3.180$
- Step2: Find the sample variance $\overline{X}$
  - $S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$
    $\Rightarrow S^2 = \frac{(3.166-3.180)^2}{7} + ... + \frac{(3.177-3.180)^2}{7}$
    $\Rightarrow S^2 = \frac{0.000475}{7}$

Question 3

- Step1: Find the sample mean $\overline{X}$
  - $\overline{X} = \frac{3.166 + ... + 3.184}{8} = 3.180$
- Step2: Find the sample variance $\overline{X}$
  - $S^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$

    $\Rightarrow S^2 = \frac{(3.166 - 3.180)^2}{7} + ... + \frac{(3.177 - 3.180)^2}{7}$

    $\Rightarrow S^2 = \frac{0.000475}{7}$
- Step3: Find $\chi^2_{\alpha/2, n-1}$ and $\chi^2_{1-\alpha/2, n-1}$

  $\Rightarrow \chi^2_{0.05,7} = 14.067$

  $\Rightarrow \chi^2_{0.95,7} = 2.167$

Question 3

- Use the two-sided confidence interval
  - $\sigma^2 \in [\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}]$

    $\Rightarrow \sigma^2 \in [\frac{0.000475}{14.067}, \frac{0.000475}{2.167}]$