Mathematics for AI 1

3. Sequences and Series

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Convergence tests

Comparison Test

Theorem

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series.

1. *If $\sum_{k=1}^{\infty} b_k$ is absolutely convergent and $|a_k| \leq |b_k|$ holds for all but finitely many $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} a_k$ is also absolutely convergent.*
2. *If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are real-valued sequences such that $0 \leq b_k \leq a_k$ holds for all $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} b_k = \infty$, then $\sum_{k=1}^{\infty} a_k = \infty$.*

Proof

1. It follows from the hypothesis that there exists $k_0 \in \mathbb{N}$ such that,

$$|a_k| \le |b_k|, \quad \forall\, k \ge k_0.$$

Since $\sum_{k=1}^{\infty} b_k$ converges absolutely, it follows that the sequence of partial sums $s_n = \sum_{k=1}^{n} |a_k|$ is bounded. Indeed, for any $n \ge k_0$, we have

$$\begin{aligned}\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{k_0-1} |a_k| + \sum_{k=k_0}^{n} |a_k| &\le \sum_{k=1}^{k_0-1} |a_k| + \sum_{k=k_0}^{n} |b_k| \\ &\le \sum_{k=1}^{k_0-1} |a_k| + \sum_{k=k_0}^{\infty} |b_k|.\end{aligned}$$

Since s_n is bounded and monotone it follows that $\sum_{k=1}^{\infty} |a_k|$ converges.

Proof

2. Let

$$s_n = \sum_{k=1}^{n} a_n, \quad t_n = \sum_{k=1}^{n} b_n$$

denote the sequences of partial sums of the two series. Since $b_n \geq 0$, we know (by monotonicity of (t_n)) that the sequence (t_n) is not bounded. But also, $s_n \geq t_n$ for all $n \in \mathbb{N}$, and so it must also be the case that the sequence (s_n) is not bounded. Since $s_n > 0$ for $n \in \mathbb{N}$ and $(s_n)_{n\in\mathbb{N}}$ is an unbounded monotone sequence we know that $lim_{n\to\infty} s_n = \infty$.

Example

We will now use Theorem from the last slide to prove that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is (absolutely) convergent. For any $k \in \mathbb{N}$, we have $k + 1 \leq 2k$ and thus

$$\frac{1}{k^2} = \frac{k+1}{k} \cdot \frac{1}{k(k+1)} \leq 2 \cdot \frac{1}{k(k+1)}.$$

Since both sides of this inequality are non-negative, it follows that

$$\left|\frac{1}{k^2}\right| \leq \left|\frac{2}{k(k+1)}\right|.$$

Example

Note that the series $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ is absolutely convergent. Indeed, we showed earlier that the sequence $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges. Therefore the series $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ is also convergent, and moreover

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)} = 2 \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 2.$$

Since the corresponding sequence consists of non-negative real numbers, it follows that $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ is also absolutely convergent. Apply the first part of comparison test with $a_k = \frac{1}{k^2}$ and $b_k = \frac{2}{k(k+1)}$. It follows that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is absolutely convergent.

Example

For $c > 2$ and $k \in \mathbb{N}$ we have $k^c \geq k^2$. Therefore, for $k \in \mathbb{N}$

$$\frac{1}{k^c} \leq \frac{1}{k^2}.$$

Since series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent by comparison test we have

$$\sum_{k=1}^{\infty} \frac{1}{k^c} < \infty.$$

Example

Similarly for $c < 1$ we have

$$\frac{1}{n} \leq \frac{1}{n^c}.$$

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. By comparison test the series $\sum_{k=1}^{\infty} \frac{1}{k^c}$ is also divergent for $c \leq 1$.

Soon we will show that series $\sum_{k=1}^{\infty} \frac{1}{k^c}$ is convergent if and only if $c > 1$.

Example

A potentially more complicated example $\sum_{k=1}^{\infty} \frac{3k^3+2k^2-k-1}{k^6+k^4-k}$. First we estimate terms of a sequence corresponding to the series. Clearly

$$3k^3 + 2k^2 - k - 1 \geq 3k^3 > 0 \qquad \text{and} \qquad k^6 + k^4 - k \geq k^6 > 0$$

for $k \in \mathbb{N}$. Therefore

$$\frac{3k^3 + 2k^2 - k - 1}{k^6 + k^4 - k} > 0$$

for $k \in \mathbb{N}$. Now we observe that

$$3k^3 + 2k^2 - k - 1 \leq 3k^3 + 2k^2 \leq 3k^3 + 2k^3.$$

Example

Therefore

$$0 < \frac{3k^3 + 2k^2 - k - 1}{k^6 + k^4 - k} \leq \frac{5k^3}{k^6} = \frac{5}{k^3}.$$

A moment ago we have proven that the series $\sum_{k=1}^{\infty} \frac{5}{k^3}$ is absolutely convergent. By comparison test $\sum_{k=1}^{\infty} \frac{3k^3+2k^2-k-1}{k^6+k^4-k}$ is convergent.

Example

We study similar example: $\sum_{k=1}^{\infty} \frac{k^5-k^4-k-3}{k^7-k^4-k}$. Let's do this using limits. Observe that

$$\frac{k^5-k^4-k-3}{k^7-k^4-k} = \frac{k^5}{k^7}\frac{1-\frac{1}{k}-\frac{1}{k^4}-\frac{3}{k^5}}{1-\frac{1}{k^3}-\frac{1}{k^6}} = \frac{1}{k^2}\frac{1-\frac{1}{k}-\frac{1}{k^4}-\frac{3}{k^5}}{1-\frac{1}{k^3}-\frac{1}{k^6}}.$$

Note the that

$$\lim_{k\to\infty}\frac{1-\frac{1}{k}-\frac{1}{k^4}-\frac{3}{k^5}}{1-\frac{1}{k^3}-\frac{1}{k^6}} = 1.$$

Example

By the definition of a limit ($\varepsilon = \frac{1}{2}$) there exists $N \in \mathbb{N}$ such that for $k > N$ we have

$$\frac{1}{2} \leq \frac{1 - \frac{1}{k} - \frac{1}{k^4} - \frac{3}{k^5}}{1 - \frac{1}{k^3} - \frac{1}{k^6}} \leq \frac{3}{2}.$$

Thus,

$$0 \leq \frac{1}{2}\frac{1}{k^2} \leq \frac{k^5 - k^4 - k - 3}{k^7 - k^4 - k} \leq \frac{3}{2}\frac{1}{k^2}.$$

Since the series $\sum_{k=1}^{\infty} k^{-2}$ is convergent then by comparison test

$$\sum_{k=1}^{\infty} \frac{k^5 - k^4 - k - 3}{k^7 - k^4 - k}$$

is convergent.

Exercise

Investigate for absolute convergences of series using the comparison test:

a) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)(n+3)(n+4)(n+5)}}$,

b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[6]{(n+1)(n+2)(n+3)(n+4)(n+5)}}$,

c) $\sum_{k=1}^{\infty} \frac{k^3+k}{k^5+1}$.

Theorem (Root test)

Let $\sum_{k=1}^{\infty} a_k$ be a series.

1. *If there exists a real number $c < 1$ such that*

$$\sqrt[k]{|a_k|} \leq c$$

   holds for all but finitely many $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

2. *Conversely, if*

$$\sqrt[k]{|a_k|} \geq 1$$

   holds for infinitely many $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} a_k$ is divergent.

Proof 1.

The series $\sum_{k=1}^{\infty} c^k$ is absolutely convergent for $|c| < 1$. By the hypothesis of the theorem $|a_k| \leq c^k = |c^k|$ holds for all but finitely many $k \in \mathbb{N}$. Therefore, by comparison test, $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof 2.

It follows from the condition that $|a_k| \geq 1$ for infinitely many $k \in \mathbb{N}$. In particular, the sequence $(a_k)_{k\in\mathbb{N}}$ is not a null sequence. By Cauchy's criterion the series $\sum_{k=1}^{\infty} a_k$ does not converge.

Remark

The root test is usually most helpful when a the kth term of the corresponding sequence involves a kth power.

Root test - limits version

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence.

1. If
$$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$$
then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.
2. If
$$\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$$
then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Proof of 1.

Since

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$$

it follows that there is some $c < 1$ such that $\sqrt[k]{|a_k|} < c$ holds for all k sufficiently large. It then follows from part 1 of root test that the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof of 2.

If

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$$

then it follows that $\sqrt[k]{|a_k|} > 1$ holds for all k sufficiently large. Part 2 of root test then implies that the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Example

Consider this scary looking series

$$\sum_{k=1}^{\infty} \sin(k) \frac{k^{100}}{2^{k/2}}.$$

Note that, for all $k \in \mathbb{N}$,

$$\left| \sin(k) \frac{k^{100}}{2^{k/2}} \right| \leq \left| \frac{k^{100}}{2^{k/2}} \right|$$

Therefore, by comparison test it will be sufficient to prove that the series

$$\sum_{k=1}^{\infty} \frac{k^{100}}{2^{k/2}}$$

is absolutely convergent.

Example

We use the root test. Observe that

$$\sqrt[k]{\left|\frac{k^{100}}{2^{k/2}}\right|} = \frac{1}{\sqrt{2}}\sqrt[k]{k^{100}}.$$

Recall that $\lim_{n\to\infty} \sqrt[k]{k^{100}} = 1$. Therefore, we get

$$\lim_{n\to\infty} \sqrt[k]{\left|\frac{k^{100}}{2^{k/2}}\right|} = \frac{1}{\sqrt{2}} < 1.$$

Example

Consider the series

$$\sum_{k=1}^{\infty} \frac{k^{k/4}}{3^{2+3k}}.$$

For the root test, we study the terms

$$\sqrt[k]{\left|\frac{k^{k/4}}{3^{2+3k}}\right|} = \frac{k^{1/4}}{27 \cdot \sqrt[k]{9}}.$$

Recall that $\lim_{k\to\infty} \sqrt[k]{9} = 1$. Clearly

$$\lim_{k\to\infty} \frac{k^{1/4}}{27 \cdot \sqrt[k]{9}} = +\infty$$

Example

Therefore for infinitely many $k \in \mathbb{N}$

$$\frac{k^{k/4}}{3^{2+3k}} > 1$$

It follows from root test that the series

$$\sum_{k=1}^{\infty} \frac{k^{k/4}}{3^{2+3k}}.$$

is divergent.

Example

The root test does not always provide a definite answer regarding the convergence or divergence of a series. To see this, we fix $m \in \mathbb{N}$ and try and to apply the ratio test for the series $\sum_{n=1}^{\infty} \frac{1}{n^m}$.

Observe that

$$\sqrt[k]{|a_k|} = \sqrt[k]{k^{-m}} \overset{k\to\infty}{\longrightarrow} 1.$$

It follows that neither of the two conditions in root test (slide 15 and slide 18) hold, and we do not get any information about the series $\sum_{n=1}^{\infty} \frac{1}{n^m}$ by this method.

Example

Moreover we know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{is divergent}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent.}$$

Exercise

Find all parameters $b \in \mathbb{R}$ such that the series

$$\sum_{k=1}^{\infty} \frac{(b^2 + 2b)^k}{k^2}$$

is convergent. **Hint.** Check the range of the function $f(x) = x^2 + 2x$ and use the root test.

Exercise

Let $(F_n)_{n\in\mathbb{N}}$ be a Fibonacci sequence. Show that the series $\sum_{k=1}^{\infty} \frac{1}{F_k}$ is convergent. **Hint.** Use the definition of the Fibonacci sequence.

Answer to the question from the audience

If we have $b_k \leq a_k$ for $k > k_0$ and $\sum_{k=1}^{\infty} b_k = \infty$ then for $k > k_0$:

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \sum_{k=k_0+1}^{n} a_k + \sum_{k=1}^{k_0} a_k \\ &\geq \sum_{k=k_0+1}^{n} b_k + \sum_{k=1}^{k_0} a_k \\ &= \sum_{k=1}^{n} b_k - \sum_{k=1}^{k_0} b_k + \sum_{k=1}^{k_0} a_k.\end{aligned}$$

Answer to the question from the audience

Since $-\sum_{k=1}^{k_0} b_k + \sum_{k=1}^{k_0} a_k$ is a fixed number we get

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} b_k - \sum_{k=1}^{k_0} b_k + \sum_{k=1}^{k_0} a_k\right) = +\infty.$$

By sandwich rule for definitely divergent (to $+\infty$) sequences we have:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty}\sum_{k=1}^{n} a_k = +\infty.$$

Answer to the question from the audience

Remember that just having $\sum_{k=1}^{\infty} b_k$ diverges and $b_k \leq a_k$ for $k > k_0$ is not enough.

Example take $b_k = (-1)$ and $a_k = \frac{-1}{k^2}$. Clearly $b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges. However $\sum_{k=1}^{\infty} \frac{-1}{k^2}$ is absolutely convergent.

Another example: We take $a_k = \frac{1}{k^3}$ and $b_k = \frac{-1}{k}$ we have $b_k \leq a_k$ but $\sum_{k=1}^{\infty} b_k$ diverges and $\sum_{k=1}^{\infty} a_k$ converges.

Theorem (Ratio test)

Let $\sum_{k=1}^{\infty} a_k$ be a series.

1. *If there exists some real number $c < 1$ such that, for all but finitely many $k \in \mathbb{N}$,*

$$a_k \neq 0, \quad \text{and} \quad \left|\frac{a_{k+1}}{a_k}\right| \leq c,$$

   then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

2. *Conversely, if for all but finitely many $k \in \mathbb{N}$,*

$$a_k \neq 0, \quad \text{and} \quad \left|\frac{a_{k+1}}{a_k}\right| \geq 1,$$

   then $\sum_{k=1}^{\infty} a_k$ is divergent.

Proof of 1.

There exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\left|\frac{a_{k+1}}{a_k}\right| \leq c.$$

It follows by induction that, for all $m \in \mathbb{N}$,

$$|a_{k_0+m}| \leq c^m |a_{k_0}|.$$

Let

$$b_n := c^n \cdot \frac{|a_{k_0}|}{c^{k_0}}.$$

Proof of 1.

The series $\sum_{k=1}^{\infty} b_k$ is absolutely convergent. This follows from rules of calculation for series and absolute convergence of geometric series for $|q| < 1$. For all $k \geq k_0$,

$$|a_k| = |a_{m+k_0}| \leq c^m |a_{k_0}| = c^{k-k_0} |a_{k_0}| = |b_k|.$$

It follows from part 1 of comparison test that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof of 2.

There exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\left| \frac{a_{k+1}}{a_k} \right| \geq 1.$$

It follows by induction that, for all $k \geq k_0$,

$$|a_k| \geq |a_{k_0}|.$$

In particular, the sequence $(a_k)_{k \in \mathbb{N}}$ is not a null sequence. It therefore follows from Cauchy's criterion that the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Ratio test - limits version

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence.

1. If
$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$$
then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.
2. If
$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$$
then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Proof.

This is left as an exercise. □

Example

We can use the ratio test to prove that the series $\sum_{k=1}^{\infty} a_k$ given by

$$a_k = \frac{k^3}{k!}$$

is convergent. Indeed, for all $k \in \mathbb{N}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^3}{(k+1)!} \cdot \frac{k!}{k^3}\right| = \frac{1}{k+1} \cdot \left(\frac{k+1}{k}\right)^3 \leq \frac{8}{k+1}.$$

The last inequality is just an application of the fact that $\frac{k+1}{k} \leq 2$ holds for all $k \in \mathbb{N}$.

Example

It therefore follows that, for all $k \geq 15$ we have

$$\left|\frac{a_{k+1}}{a_k}\right| \leq \frac{1}{2}.$$

The ratio test implies that the series $\sum_{k=1}^{\infty} \frac{k^3}{k!}$ is convergent.

Example

We study convergence of another series $\sum_{k=1}^{\infty} a_k$ given by

$$a_k = \frac{1}{k^k}.$$

By now you know several ways to prove that the series above is convergent. We will use the ratio test. Indeed, for all $k \in \mathbb{N}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{k^k}{(k+1)^{k+1}}\right| = \frac{1}{k} \cdot \left(\frac{k}{k+1}\right)^{k+1} = \left(1 - \frac{1}{k+1}\right)^{k+1} \frac{1}{k}.$$

Example

Recall that

$$\lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)^{k+1}=e^{-1}\qquad\text{and}\qquad\lim_{k\to\infty}\frac{1}{k}=0$$

It therefore follows that

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=0.$$

By the ratio test the series $\sum_{k=1}^{\infty}\frac{1}{k^k}$ is convergent.

Example

The ratio test does not always provide a definite answer regarding the convergence or divergence of a series. To see this, we fix $m \in \mathbb{N}$ and try and to apply the ratio test for the series $\sum_{n=1}^{\infty} \frac{1}{n^m}$.

Observe that
$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{k^m}{(k+1)^m} \overset{k\to\infty}{\longrightarrow} 1.$$

It follows from the above that neither of the two conditions in ratio test (slide 30 and slide 34) hold, and we do not get any information about the series $\sum_{n=1}^{\infty} \frac{1}{n^m}$ by this method.

Example

Recall we know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{is divergent}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent.}$$

Exercise

Determine the convergence of the following series

a) $\sum_{k=1}^{\infty}(1-\frac{1}{k^2})^{k^3}$,

b) $\sum_{k=1}^{\infty} \frac{3^k}{k!}$,

c) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$,

d) $\sum_{k=1}^{\infty} \frac{k+k^k}{k^{2k}}$.

Remark

Last two convergence tests we will discuss you will find in Mario Ulrich notes p.126-127.

Theorem (Cauchy's condensation lemma)

Let $\sum_{k=1}^{\infty} a_k$ *be a series with* $0 \leq a_{k+1} \leq a_k$ *for all* $k \in \mathbb{N}$. *Then,*

$$\sum_{k=1}^{\infty} a_k \text{ is convergent } \Leftrightarrow \quad \sum_{k=1}^{\infty} 2^k a_{2^k} \text{ is convergent.}$$

Proof

Let $s_n = \sum_{k=1}^{n} a_k$. Since $a_k \geq 0$ then (s_n) is a monotone sequence.

Thus, (s_n) is convergent if and only if (s_n) is bounded.

Similarly for $t_n = \sum_{k=1}^{n} 2^k a_{2^k}$. Sequence (t_n) monotone.

Therefore the sequence (t_n) is convergent if and only if the sequence (t_n) is bounded.

So it is enough that we show that the sequence (s_n) is bounded if and only if the sequence (t_n) is bounded.

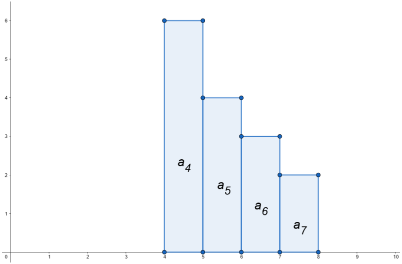
Proof

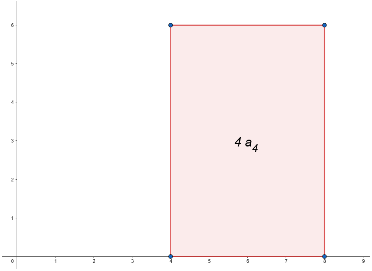
First we observe that for

$$2^k a_{2^{k+1}} \leq \sum_{j=2^k}^{2^{k+1}-1} a_j \leq 2^k a_{2^k}$$

Indeed, if we put in the sum a_{2^k} instead of a_j for every j in range we increase the value of the sum. If we put in the sum $a_{2^{k+1}}$ instead of a_j for every j in range we decrease the value of the sum .

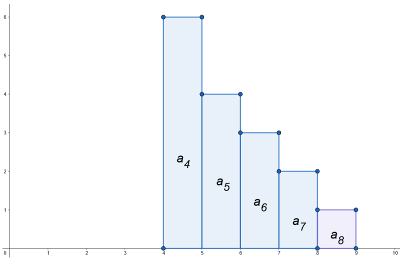
Visualization for $k = 2$

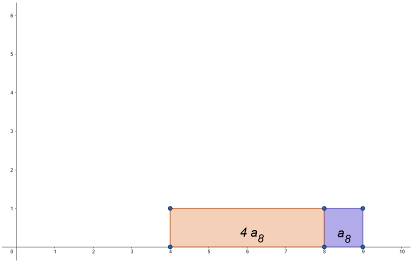




$$\sum_{j=2^k}^{2^{k+1}-1} a_k = \color{blue}{\text{Area}} \leq \color{red}{\text{Area}} = 2^k a_{2^k}$$

Visualization for $k = 2$





$$\sum_{j=2^k}^{2^{k+1}-1} a_k = {\color{blue}\text{Area}} \geq {\color{orange}\text{Area}} = 2^k a_{2^{k+1}}$$

Proof

Therefore, we get

$$\frac{1}{2}\sum_{k=0}^{n} 2^{k+1} a_{2^{k+1}} \leq \sum_{k=1}^{2^{n+1}-1} a_k = \sum_{k=0}^{n} \sum_{j=2^k}^{2^{k+1}-1} a_j \leq \sum_{k=0}^{n} 2^k a_{2^k}$$

which finishes the proof.

Example

We will prove that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$$

is convergent if and only if $\alpha > 1$. We have discuss that for $\alpha < 0$ the series is divergent. For $\alpha > 0$ sequence $(\frac{1}{k^{\alpha}})_{k \in \mathbb{N}}$ is decreasing and has non-negative values. We can use the Cauchy's condensation lemma.

$$\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} \quad \text{is convergent} \Leftrightarrow \sum_{k=1}^{\infty} 2^{k} \frac{1}{2^{\alpha k}} \quad \text{is convergent}$$

Example

The series on the right hand side is a geometric series. Indeed, we have

$$2^k \frac{1}{2^{\alpha k}} = (2^{1-\alpha})^k$$

The geometric series is convergent if and only if $|q|<1$. However,

$$2^{1-\alpha} < 1 \Leftrightarrow 1-\alpha < 0 \Leftrightarrow 1 < \alpha.$$

By the Cauchy's condensation lemma we get that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$$

is convergent if and only if $\alpha > 1$.

Exercise

Proof that the series

$$\sum_{k=1}^{\infty} \frac{1}{k \ln^{\alpha} k}$$

is convergent if and only if $\alpha > 1$.

Theorem (Leibniz criterion)

Let $(a_k)_{k\in\mathbb{N}}$ be a monotone null sequence. Then, the series

$$\sum_{k=1}^{\infty}(-1)^k a_k$$

is convergent.

Proof

Without loss of generality we assume that the sequence (a_k) is non-increasing (otherwise we work with $(-a_k)$). Observe that $a_k \geq 0$ for every $k \in \mathbb{N}$. Since (a_k) is decreasing then

$$s_{2n+2} = s_{2n} - a_{2k+1} + a_{2k+2} \leq s_{2n}$$

and

$$s_{2n+3} = s_{2n+1} + a_{2k+2} - a_{2k+3} \geq s_{2n+1}$$

Moreover

$$s_{2n} = s_{2n+1} + a_{2k+1} \geq s_{2n+1} \geq s_{2n-1} \geq \cdots \geq s_1$$

Proof

The sequence $(s_{2n})_{n\in\mathbb{N}}$ is bounded and monotone. Thus, it is convergent. Moreover from $\lim_{n\to\infty} a_n = 0$ we get

$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n} - a_{2n+1} = \lim_{n\to\infty} s_{2n+1}.$$

The sequence s_{2n+1} is convergent. By the definition the series

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

is convergent.

Example

The sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ is a monotone null sequence. Thus, by Leibniz criterion the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

is convergent.

Example

It is crucial that the sequence a_k is monotone. We define $a_{2k} = \frac{1}{k}$ and $a_{2k+1} = -2^{-k}$. The sequence (a_k) is a null sequence. However

$$\sum_{k=1}^{2n} (-1)^k a_k = \sum_{k=1}^{n} a_{2k} - a_{2k+1} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

Thus, the series $\sum_{k=1}^{\infty} (-1)^k a_k$ is divergent.

Fact

Sum of the series which is convergent but not absolutely convergent depends on the order of terms!!!

Example:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \cdots = \ln 2$$

but

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} = (1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + \cdots = \frac{1}{2} \ln 2$$

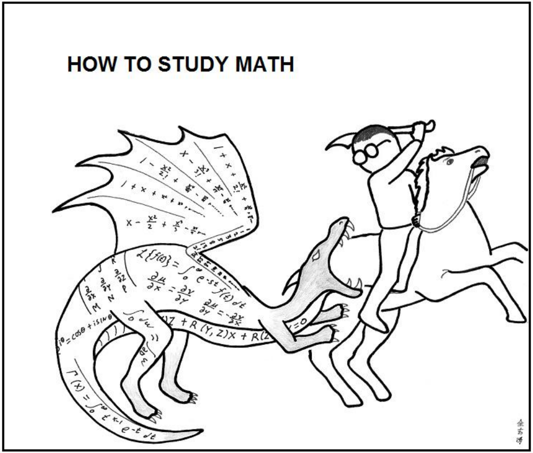
Exercise

Determine the convergence of the following alternating series

a) $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k^5}$,

b) $\sum_{k=1}^{\infty}(-1)^k \frac{k}{k+1}$,

c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$.



Don't just read it; fight it!

--- Paul R. Halmos

source: Abstruse Goose