*Winter semester 2021* 03.03.2022, 12:30–15:00 Uhr

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# Second exam (Part 2):

# Mathematics for Artificial Intelligence 1

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Duration: 150 minutes
Maximum of 40 points

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## Rules:

- **Put your name and matriculation nr. on every sheet!**
- Use a new sheet for every exercise!
- Electronic devices are not allowed!
- Do not use pencil or a red pen.
- The whole procedure of solution is required.
- All results must be simplified as far as possible.
- Write your name and matriculation number on each sheet you submit.
- Before you submit the .pdf, make sure everything you want to submit is included.

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Good luck!

EXERCISE 1: **Induction** (4 points)

Prove by induction that

$$\sum_{k=0}^{n-1} 7 \cdot 8^k \;=\; 8^n - 1$$

for all $n \in \mathbb{N}$.

EXERCISE 2: **Inequalities** (4 points)

Determine the set of all real numbers $x \in \mathbb{R}$ with

$$\Big|\,|x+1| - 2\Big| \;\geq\; 1.$$

EXERCISE 3: **Complex numbers** (4 points)

Write $z_1 := \dfrac{6-3\mathrm{i}}{1+2\mathrm{i}}$, $z_2 := (2-2\mathrm{i})^5$ and $z_3 := \mathrm{i}^7 \cdot e^{\mathrm{i}\frac{22\pi}{4}}$ in canonical and polar form.

EXERCISE 4: **Inequalities with complex numbers** (4 points)

Determine all complex numbers $z = x + iy \in \mathbb{C}$ with

$$|\overline{z} + 2 + 4i| \;\leq\; |2 - z|$$

and make a sketch of this set in the complex plane.

EXERCISE 5: **Linear systems** (8 points)

Let

$$A = \begin{pmatrix} -3 & 3 & -3 \\ 2 & t & 0 \\ t & 2 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

For which values of $t \in \mathbb{R}$ does the linear system $A \cdot (x, y, z)^T = b$ have

a) one

b) infinitely many

c) no

solution(s)? Give the set of solutions $L(A, b)$ in the first two cases.

EXERCISE 6: **Sequences** (3+3 points)

Verify if the limits of the following sequences exist, determine the limits, if possible, and otherwise compute the lim inf and lim sup of

a) $a_n := \sqrt[n]{4^{2n} + 2^{5n}}$.

b) $c_n := (2 - (-1)^n)^n$.

EXERCISE 7: **Series I** (3+2 points)

Are the following series convergent or divergent?

$$\text{a)} \sum_{n=1}^{\infty} (-1)^n \left( \left( n + \frac{1}{n} \right)^2 - n^2 \right) \qquad \text{b)} \sum_{n=1}^{\infty} \frac{3^n n}{(\sqrt{n})^n}$$

EXERCISE 8: **Series II** (5 points)

For which $x \in \mathbb{R}$ does the series

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3+1} \left( \frac{4-x^3}{4} \right)^k$$

converge?