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Mathematics for AI I

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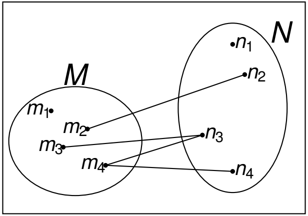
Chapter 1.3: Relations and functions

Relations

Definition

A ***relation*** R between two sets M and N is a subset of the Cartesian product of M and N, i.e.

$$R \subset M \times N.$$



$$\{(m_2, n_2), (m_3, n_3), (m_4, n_3), (m_4, n_4)\}$$

Important types of relations

We will discuss the following important applications of relations:

- We can use relations to indicate that certain elements in M are related/connected to certain elements in N. The prime example for such a relation are ***functions***.
- If R is a relation between M and M, i.e. $R \subset M^2$ we call R a relation ***on*** M. This concept is useful to
  - compare,
  - group,
  - order

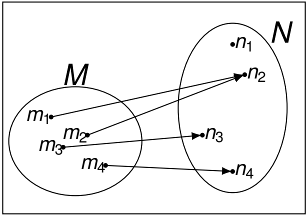
  elements of a given set M.

Functions

Functions

Definition

Let $f \subset M \times N$ be a relation between non-empty sets. We call $f\colon M \to N$ a ***function***, if f each $x \in M$ is assigned exactly one element $f(x) \in N$. M is called the ***domain*** (of definition), N is called the ***codomain*** of f.



$$\{(m_1, n_2), (m_2, n_2), (m_3, n_3), (m_4, n_4)\}$$

Functions

Definition

Let $f\colon M \to N$ be a function and $S \subset M, T \subset N$ be sets. We define

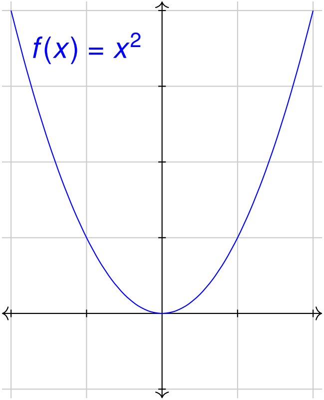
- the ***image*** of S under f as
$$f(S) := \{f(x) : x \in S\} \subset N,$$
- the ***range*** of f as
$$f(M) := \{f(x) : x \in M\} \subset N,$$
- the ***preimage*** of T under f as
$$f^{-1}(T) := \{x : f(x) \in T\} \subset M,$$
- the ***graph*** of f as
$$G_f := \{(x, f(x)) : x \in M\} \subset M \times N.$$

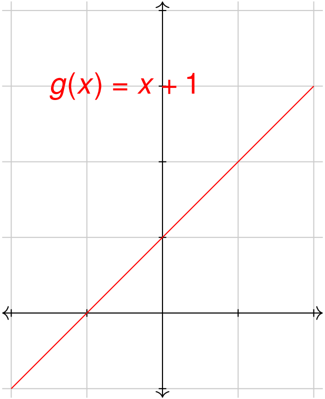
Functions

Example

- Consider $f\colon \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$. This defines a function.
- Consider $g\colon \mathbb{R} \to \mathbb{R}$, defined by $g(x) = x + 1$. This defines a function.
- Consider h, defined for $x \in \mathbb{R}_0^+$ by $h(x) = \pm\sqrt{x}$. This does not define a function.

Functions





Example for a free online tool to plot graphs of functions:
https://www.geogebra.org/

image and preimage

Example

Consider again $f(x) = x^2$ and $g(x) = x + 1$ as introduced before. Let $S = [1, 3]$. Then

$$f(S) = [1, 9] \text{ and } g(S) = [2, 4].$$

The range of f, $f(\mathbb{R}) = R_0^+ = [0, \infty)$, the range of g, $g(\mathbb{R}) = \mathbb{R}$. For $T = [1, 4]$, we have

$$f^{-1}(T) = [-2, -1] \cup [1, 2], \text{ and } g^{-1}(T) = [0, 3].$$

More examples of functions

Example

- $f_1 : \mathbb{R}^2 \to \mathbb{R}$, with $f(\begin{pmatrix} x \\ y \end{pmatrix}) = x + y$,
- $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$, with $f_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \frac{\sqrt{2}}{2}(x - y) \\ \frac{\sqrt{2}}{2}(x + y) \end{pmatrix}$.

https://www.geogebra.org/calculator/ebkwt52u

Note that we often write $f(x, y)$ instead of $f(\begin{pmatrix} x \\ y \end{pmatrix})$.

More examples of functions

Example

Given $A = \{1, 2, 3, 4\}$, we consider the functions

- $f_3 : \mathcal{P}(A) \to \mathbb{N}_0$, with $f_3(X) = |X|$ for $X \subset A$, where we denote by $|X|$ the number of elements contained in X,
- $f_4 : \mathcal{P}(A) \to \mathcal{P}(A)$, with $f_4(X) = A \setminus X = X^C$.

Properties of relations

Definition

Let $R \subset M \times N$ be a relation between non-empty sets. R is called

- ***injective*** iff $\forall (x_1, y_1), (x_2, y_2) \in R,\ x_1 \neq x_2 \Rightarrow y_1 \neq y_2$,
  or equivalently $\forall (x_1, y_1), (x_2, y_2) \in R,\ y_1 = y_2 \Rightarrow x_1 = x_2$,
- ***surjective*** iff $\forall y \in N, \exists x \in M : (x, y) \in R$,
- ***bijective*** iff R is injective and surjective,
- ***functional*** iff $\forall x \in M, \exists y \in N : (x, y) \in R$ and
  $\forall x \in M, y_1, y_2 \in N, ((x, y_1), (x, y_2) \in R) \Rightarrow y_1 = y_2$.

Properties of relations

Definition

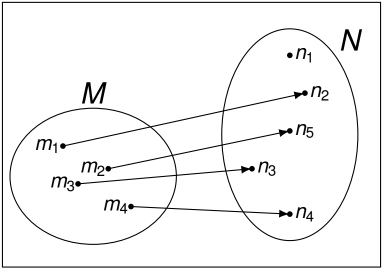
For a function $f\colon M \to N$, the definitions given on the previous slide translate to

- f is ***injective*** iff $\forall x_1, x_2 \in M,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$,

  or equivalently $\forall x_1, x_2 \in M,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$,
- ***surjective*** iff $\forall y \in N,\ \exists x \in M : f(x) = y$,
- ***bijective*** iff $\forall y \in N, \exists! x \in M : f(x) = y$.

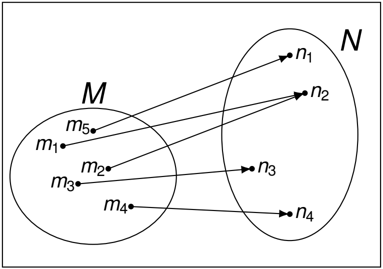
Here $\exists!$ is the unique existence quantifier, which indicates “there exists a unique”.

Injective functions



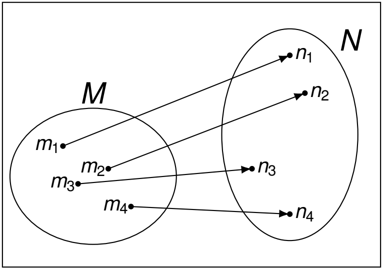
$$\{(m_1, n_2), (m_2, n_5), (m_3, n_3), (m_4, n_4)\}$$

Surjective functions



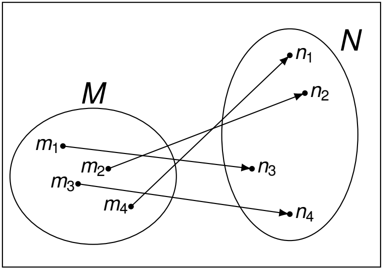
$$\{(m_1, n_2), (m_2, n_2), (m_3, n_3), (m_4, n_4), (m_5, n_1)\}$$

Bijective functions



$$\{(m_1, n_1), (m_2, n_2), (m_3, n_3), (m_4, n_4)\}$$

Bijective functions



$$\{(m_1, n_3), (m_2, n_2), (m_3, n_4), (m_4, n_1)\}$$

Functions

Further important examples for functions:

Example

- Let M, N be non-empty sets and $c \in N$. The function $f: M \to N$, defined by $f(x) = c$ for all $x \in M$ is called *constant* function.
- For a non-empty set M, the function $Id_M: M \to M$, defined by $f(x) = x$ for all $x \in M$ is called *identity* function (on M).

Inverse function

Definition

Let $f\colon M \to N$ and $g\colon N \to M$ be functions with the properties,

$\forall x \in M, g(f(x)) = x$, and

$\forall y \in N, f(g(y)) = y$.

Then f is the ***inverse*** of g and g the ***inverse*** of f.

Theorem

A function $f\colon M \to N$ is invertible, if and only if it is bijective.

Proof.

Later. $\square$

Cardinality of sets

Definition

A finite set M containing n elements, e.g. the set $\{1, 2, \ldots, n\}$, has ***cardinality*** n. We write $|M| = n$.

Exercise

Show that, for two finite sets A and B, the following statements hold.

- $|A| = |B|$ if and only if there is a bijection $f\colon A \to B$.
- $|A| \leq |B|$ if and only if there is an injection $f\colon A \to B$.
- $|A| \geq |B|$ if and only if there is a surjection $f\colon A \to B$.

Cardinality of sets

Bijections allow us (to some extent) to characterize the cardinality of infinite sets.

Definition

Let A be a set.

- If there exists $n \in \mathbb{N}$ such that $|A| = n$, then we call A ***finite***, or a ***finite set***.
- If A is not finite, then we call A ***infinite***, or an ***infinite set***.
- If there exists a bijection $f : \mathbb{N} \to A$, then we call A ***countably infinite***.
- If A is either finite or countably infinite, then we call A ***countable***.
- If A is not countable, then we call A ***uncountable***.

Cardinality of sets

Exercise

- Find a bijective function $f\colon \mathbb{N} \to \mathbb{N}_0$.
- Find a bijective function $g\colon \mathbb{N}_0 \to \mathbb{Z}$.

Function composition

Definition

Let X, Y, Z be non-empty sets, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. We define the ***composition*** of g and f, the function $(g \circ f)\colon X \to Z$, by first applying f and then applying g. That is, we define $(g \circ f)(x) = g(f(x))$.

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x + 1$. Then $(g \circ f)\colon \mathbb{R} \to \mathbb{R}$ with $(g \circ f)(x) = x^2 + 1$.

Function composition

Exercise

Show that function composition is not commutative. I.e. show that not for all functions f, g it holds that $(f \circ g) = (g \circ f)$.

Function composition

Remark

If $f\colon M \to N$ is a function and f^{-1} is its inverse, then, for all $x \in M$ and $y \in N$, we have $f^{-1}(f(x)) = x$, and $f(f^{-1}(y)) = y$, thus $f \circ f^{-1} = Id_N$ and $f^{-1} \circ f = Id_M$.

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions, defined by

$$f(x) = 2x + 1 \text{ and } g(x) = \frac{1}{2}x - \frac{1}{2}.$$

Then $(g \circ f)(x) = g(f(x)) = g(2x + 1) = \frac{1}{2}(2x + 1) - \frac{1}{2} = x$,
and $(f \circ g)(x) = f(g(x)) = f(\frac{1}{2}x - \frac{1}{2}) = 2(\frac{1}{2}x - \frac{1}{2}) + 1 = x$.

Function composition

Exercise

Let $f\colon M \to N$ be a function. We have $\forall x \in M$, $(f \circ Id_M)(x) = f(x) = (Id_N \circ f)(x)$.

Exercise

Function composition is associative, that is, for any functions $f\colon M \to N$, $g\colon N \to O$ and $h\colon O \to P$, it holds that

$$\forall x \in M,\ \Big(h \circ (g \circ f)\Big)(x) = \Big((h \circ g) \circ (f)\Big)(x).$$

Exercise

Let $f\colon M \to N$ be a function. If $g_1, g_2\colon N \to M$ are inverse of f, then $g_1 = g_2$.

Equivalence and order relations

Properties of relations

For a non-empty set M, we call $R \subset M^2$ a relation ***on*** M.

Definition

Let $M \neq \emptyset$ and $R \subset M^2$ be a relation. R is called

- ***reflexive***, iff $\forall\, x \in M$, $(x, x) \in R$,
- ***symmetric***, iff $(x, y) \in R \Rightarrow (y, x) \in R$,
- ***antisymmetric***, iff $(x, y), (y, x) \in R \Rightarrow x = y$,
- ***transitive***, iff $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$,
- ***total***, iff $\forall x, y \in M$, $(x, y) \in R$ or $(y, x) \in R$.

Equivalence and order relations

Definition

- An **equivalence relation** is a relation that is reflexive, symmetric and transitive.
- A **partial order** is a relation that is reflexive, antisymmetric and transitive.
- A **linear order** is a relation that is a partial order and total.

Example

Define $L \subset \mathbb{R}^2$, by $(x, y) \in L \Leftrightarrow x \leq y$.

Example

Define $L \subset \mathbb{R}^2$, as $L = \{(x, x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$.

Partial order

Example

One can define a partial order on a set of sets by using the subset relation $\subset$. For example, for

$$M := \{\emptyset, \{1\}, \{2\}, \{1, 2\}\},$$

we define the relation

$$L = \{(A, B) \in M \times M : A \subset B\}.$$

L is reflexive, antisymmetric and transitive. Hence, it is a partial order on M. However, since $\{1\} \not\subset \{2\}$ and $\{2\} \not\subset \{1\}$, it is not total.

Equivalence classes

Equivalence classes allow us to partition the elements of a set into related chunks.

Definition

Let $R \subset X \times X$ be an equivalence relation. For $x \in X$, we define the **equivalence class of** x by

$$[x]_R := \{y \in X : (x, y) \in R\}.$$

Equivalence classes

Exercise

Prove the following relations are equivalence relations, find the corresponding equivalence classes and if possible, give a geometric interpretation.

- Define $\sim$ on $\mathbb{R}$ by $(x, y) \in \sim$ (or alternatively $x \sim y$), iff $|x| = |y|$.
- Define $\equiv_2$ on $\mathbb{Z}$ by $(x, y) \in \equiv_2$ (or alternatively $x \equiv_2 y$), iff x and y have the same remainder after division by 2.
- More complicated: Let $\mathcal{F}$ be the set of functions from $\mathbb{R}$ to $\mathbb{R}$,
  $$\mathcal{F} := \{f : \mathbb{R} \to \mathbb{R}\}.$$
  Define a relation R on $\mathcal{F}$ by $(f, g) \in R$, iff
  $$\exists c \in \mathbb{R} : \ \forall x \in \mathbb{R}, \ f(x) = g(x) + c.$$

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