Sheet 5

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25. a) $\{x \in \mathbb{R} : 3|x+2| - 4x + 3 \leq 5|x-1|\}$

case distinction.

case 1. $x+2 \geq 0$ and $x-1 \geq 0$

$L_1 = \{x \in \mathbb{R} : 3(x+2) - 4x + 3 \leq 5(x-1), x+2 \geq 0, x-1 \geq 0\}$

$= \{x \in \mathbb{R} : 9 - x \leq 5x - 5, x \geq -2, x \geq 1\}$

$= \{x \in \mathbb{R} : 6x \geq 14\} = \cancel{[\frac{7}{3}, \infty)} = [\frac{7}{3}, \infty)$

case 2. $x+2 \geq$ and $x-1 < 0$

$L_2 = \{x \in \mathbb{R} : 3(x+2) - 4x + 3 \leq 5(-(x-1)), x+2 \geq 0, x-1 < 0\}$

$= \{x \in \mathbb{R} : -x + 9 \leq -5x + 5, x \geq -2, x < 1\}$

$= \{x \in \mathbb{R} : 4x \leq -4, x \geq -2, x < 1\} = [-2, -1]$

case 3. $x+2 < 0$ and $x-1 \geq 0$

$L_3 = \{x \in \mathbb{R} : 3(-x-2) - 4x + 3 \leq 5(x-1), x+2 < 0, x-1 \geq 0\}$

$= \{x \in \mathbb{R} : -7x - 3 \leq 5x - 5, x < -2, x \geq 1\}$

$= \{x \in \mathbb{R} : x \geq \frac{1}{6}, x < -2, x \geq 1\}$

$= \emptyset$

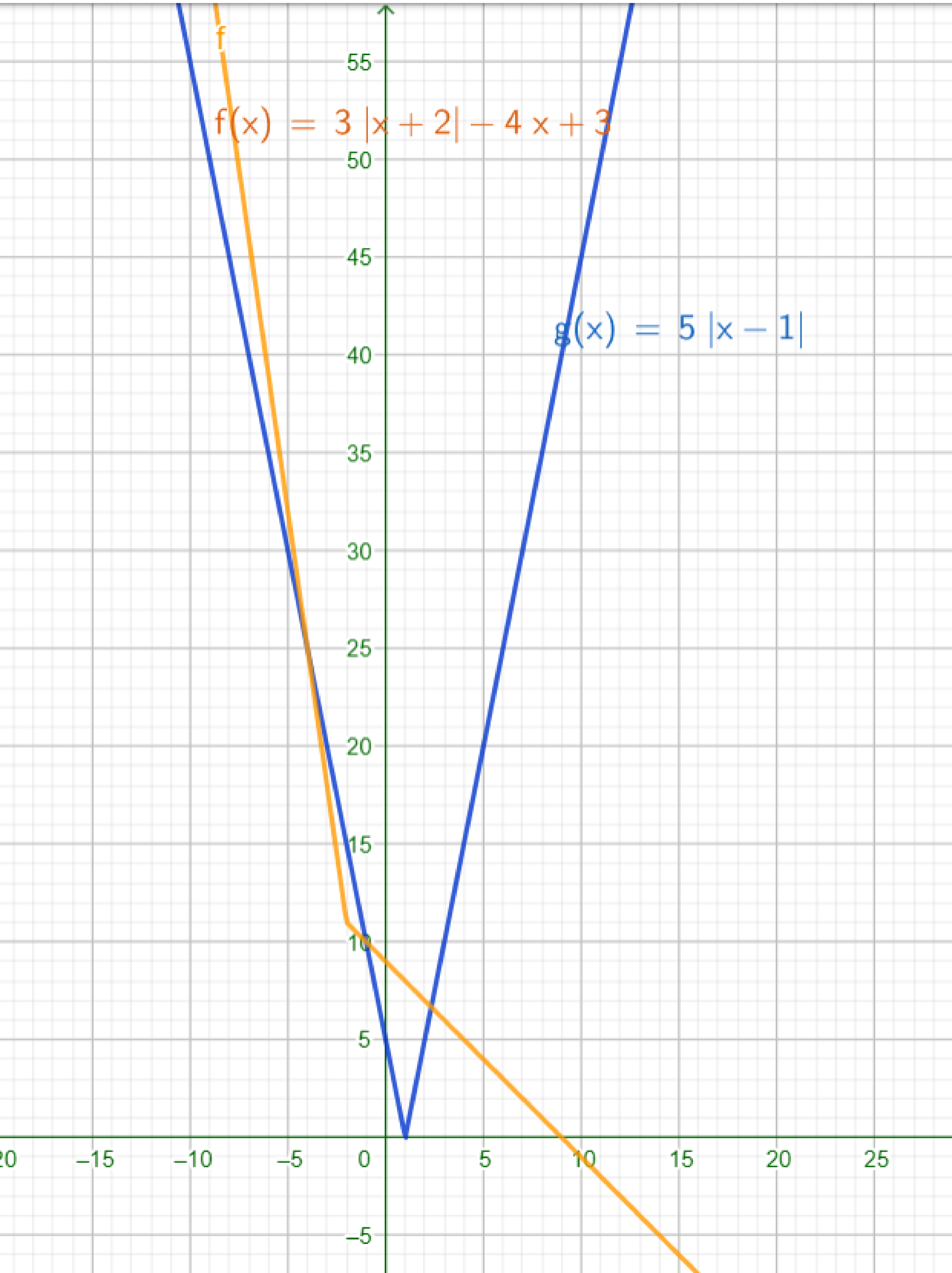
case 4. $x+2 < 0$ and $x-1 < 0$

$L_4 = \{x \in \mathbb{R} : 3(-x-2) - 4x + 3 \leq 5(-x+1), x+2 < 0, x-1 < 0\}$

$= \{x \in \mathbb{R} : -3x - 6 - 4x + 3 \leq -5x + 5, x < -2, x < 1\}$

$= \{x \in \mathbb{R} : x \geq -4, x < -2, x < 1\} = [-4, -2)$

$\# L = L_1 \cup L_2 \cup L_3 \cup L_4 = [\frac{7}{3}, \infty) \cup [-2, -1] \cup \emptyset \cup [-4, -2) = [-4, \infty)$


f
$f(x) = 3|x+2| - 4x + 3$
$g(x) = 5|x-1|$
55
50
45
40
35
30
25
20
15
10
5
−5
−20
−15
−10
−5
0
5
10
15
20
25

25. b) $\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i| \quad x_i \in \mathbb{R},\ i \in \{1, \dots, n\}$

Base case. $n=1$

$$\left|\sum_{i=1}^{1} x_i\right| \leq \sum_{i=1}^{n} |x_i|$$

$|x_1| \leq |x_1|$ True (Let this be true)

Induction step, prove for $n+1$, it's true.

$$\left|\sum_{i=1}^{n+1} x_i\right| \leq \sum_{i=1}^{n+1} |x_i|$$

$$\left|\sum_{i=1}^{n} x_i + x_{n+1}\right| \leq \sum_{i=1}^{n} |x_i| + |x_{n+1}| \quad - (i)$$

$$x_{n+1} \leq |x_{n+1}|$$

Hence, (i) is true.

So, $\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i| \quad \square$

c) $z_1 \cdot z_2$

$= 2\left(\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}\right) \cdot 2\left(\cos \frac{\pi}{3} + i\sin \frac{\pi}{3}\right)$

Using De-Moivre's formula,

$= 4\left(\cos \pi + i\sin \pi\right) =$

$= 4\left(-1 + i \cdot 0\right) = -4 \; -1$

comparing with $z_1 \cdot z_2 = r_1 \cdot r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$

$$\boxed{\begin{array}{ll} \text{Absolute value} & = r_1 \cdot r_2 = 4 \\ \text{Argument} & = \theta_1 + \theta_2 = \pi \end{array}}$$

29. a) $\left(\frac{1}{2}\sqrt{3} + \frac{1}{2} i\right)^{12}$

$= \left(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6}\right)^{12}$

Using De Moivre's formula,

$= \left(\cos 12 \times \frac{\pi}{6} + i\sin 12 \times \frac{\pi}{6}\right)$

$= \left(\cos 2\pi + i\sin 2\pi\right)$

$= \underline{1}$

26. a) • $\{x \in (4,\infty) : 4(\sqrt{3})^{2x-1} \cdot 5^{4x+3} = 3\pi^{x-4}\}$

or, $4 \cdot \frac{3^x}{\sqrt{3}} \cdot 5^{4x} \cdot 125 = \frac{3\pi^x}{\pi^4}$

or, $\frac{500\pi^4}{\sqrt{3}}(3^x \cdot 5^{4x}) = 3\pi^x$

or, $500\pi^4(1875^x) = 3\sqrt{3}\pi^x$

or, $\left(\frac{1875}{\pi}\right)^x = \left(\frac{3\sqrt{3}}{500\pi^4}\right)$

Taking $\log_{\frac{1875}{\pi}}$,

$\log_{\frac{1875}{\pi}}\left(\frac{1875}{\pi}\right)^x = \log_{\frac{1875}{\pi}}\left(\frac{3\sqrt{3}}{500\pi^4}\right) \approx -1.4308 \notin (4,\infty)$

Hence $\{\emptyset\}$.

• $\{x \in (0,\infty) : \log_3 x = 5\}$

$= \{x \in (0,\infty) : x = 3^5\}$

$= \{3^5\}$

• $\{x \in (-1,\infty) : 2\ln(x+3) - 3\ln(x+2) + \ln(x+1) = 0\}$

$= \{x \in (-1,\infty) : \ln\left(\frac{(x+3)^2(x+1)}{\cancel{x+1}}\right) = \ln(x+2)^3\}$

~~$= \{x \in (-1,\infty) : \frac{(x+3)^2(x+1)}{x+1} = (x+2)^3\}$~~

$= \{x \in (-1,\infty) : (x+3)^2(x+1) = (x+2)^3\}$

$= \{x \in (-1,\infty) : x^3 + 7x^2 + 15x + 9 = x^3 + 6x^2 + 12x + 8\}$

$= \{x \in (-1,\infty) : x^2 + 3x + 1 = 0\}$

$= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 1}}{2}$

$\frac{-3+\sqrt{5}}{2} \in (-1,\infty)$ $\qquad$ $\frac{-3-\sqrt{5}}{2} \notin (-1,\infty)$

$\therefore \left\{\frac{-3+\sqrt{5}}{2}\right\}$

$f(x) = x \, (x \geq 0)$

$F(x) = f(x) - 1$

$= x \, (x \geq 0) - 1$

$h(x) = f(x) + 2$

$= x \, (x \geq 0) + 2$

$g(x) = f(x) + 0.5$

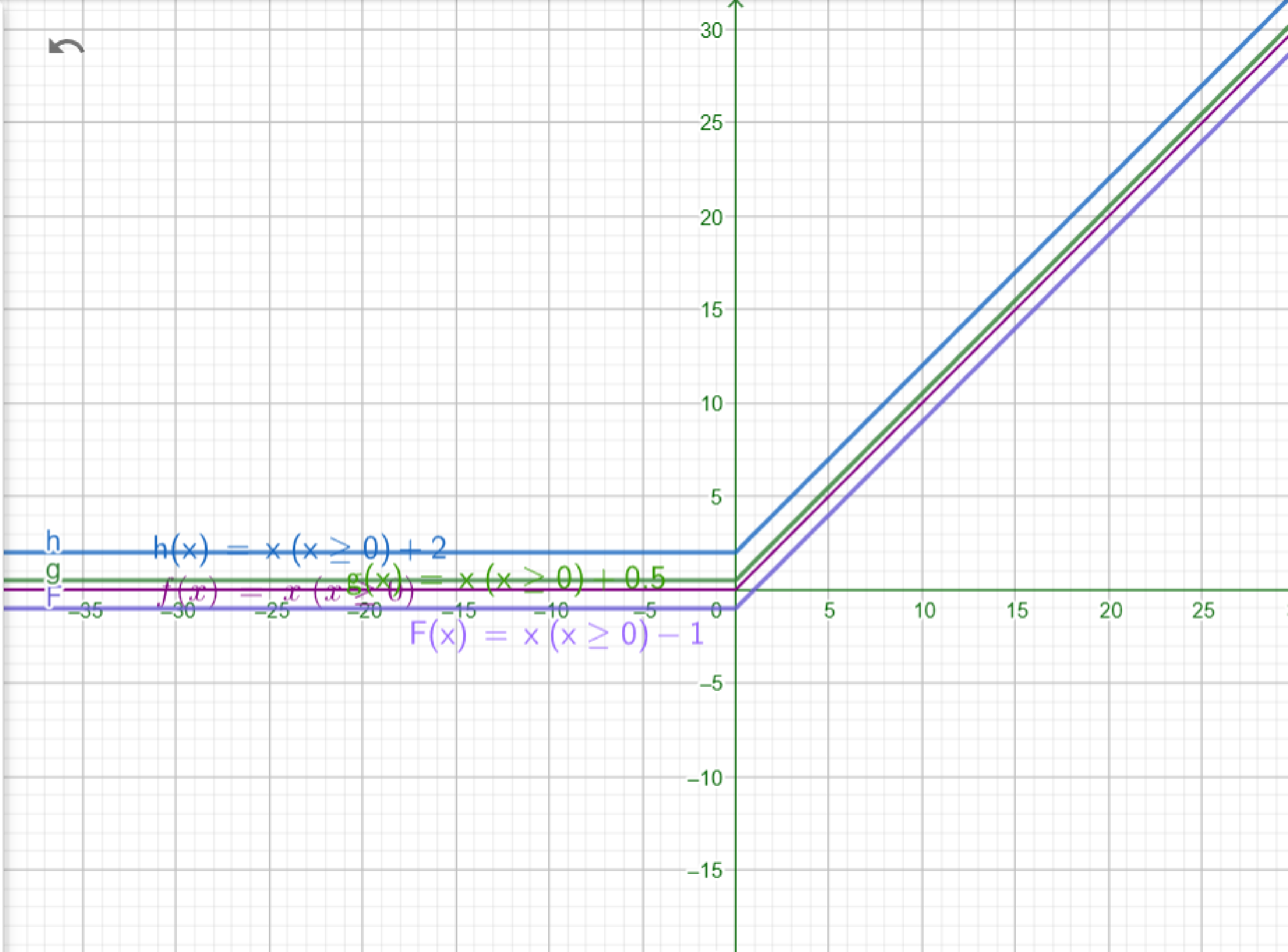
$= x \, (x \geq 0) + 0.5$

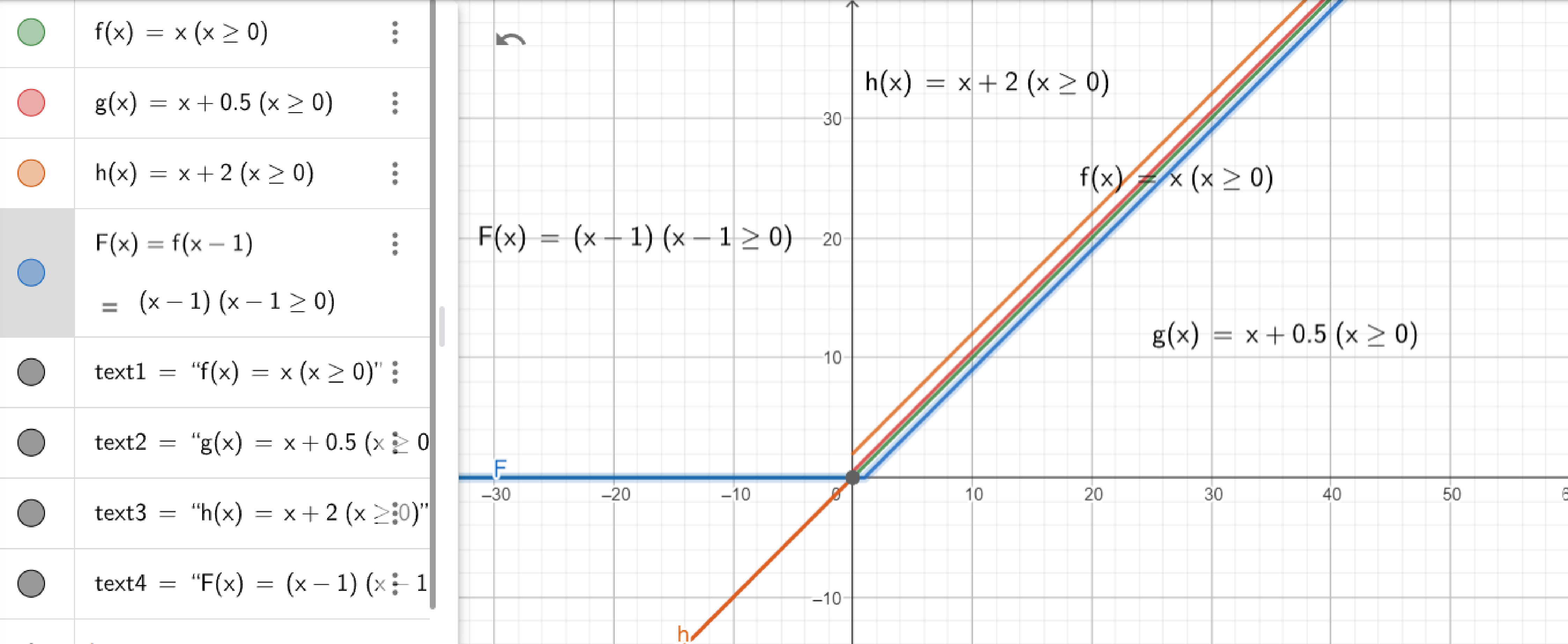
text1 = "h(x) = x (x ≥ 0) + 2"

text2 = "g(x) = x (x ≥ 0) + 0.5

text3 = "F(x) = x (x ≥ 0) − 1"

Input...





f(x) = x (x ≥ 0)
g(x) = x + 0.5 (x ≥ 0)
h(x) = x + 2 (x ≥ 0)
F(x) = f(x − 1)
= (x − 1) (x − 1 ≥ 0)
text1 = "f(x) = x (x ≥ 0)"
text2 = "g(x) = x + 0.5 (x ≥ 0
text3 = "h(x) = x + 2 (x ≥ 0)"
text4 = "F(x) = (x − 1) (x − 1
h(x) = x + 2 (x ≥ 0)
f(x) = x (x ≥ 0)
F(x) = (x − 1) (x − 1 ≥ 0)
g(x) = x + 0.5 (x ≥ 0)
30
20
10
−10
F
h
−30
−20
−10
0
10
20
30
40
50

$f(x) = x\ (x \geq 0)$

$g(x) = 0.5\ f(x)\ (x \geq 0)$

$= 0.5\ x\ (x \geq 0)\ (x \geq 0)$

$h(x) = 2\ f(x)\ (x \geq 0)$

$= 2\ x\ (x \geq 0)\ (x \geq 0)$

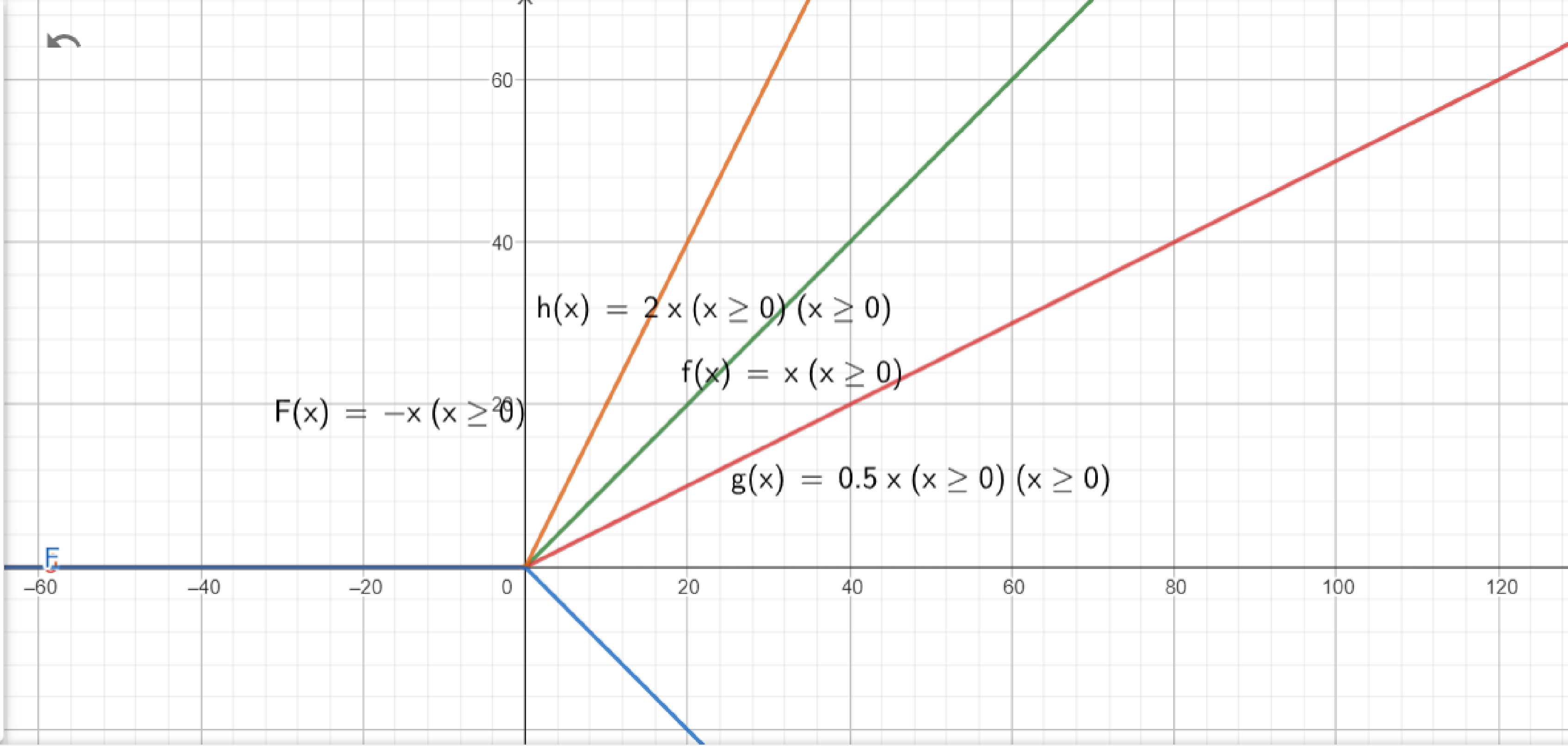
$F(x) = -1\ f(x)$

$= -1\ x\ (x \geq 0)$

text1 = "f(x) = x (x ≥ 0)"

text2 = "g(x) = 0.5 x (x ≥ 0) (

text3 = "h(x) = 2 x (x ≥ 0) (x



f(x) = x (x ≥ 0)

g(x) = x · 0.5 (x · 1 ≥ 0)

h(x) = x · 2 (x · 1 ≥ 0)

F(x) = f(x (−1))

= x (−1) (x (−1) ≥ 0)

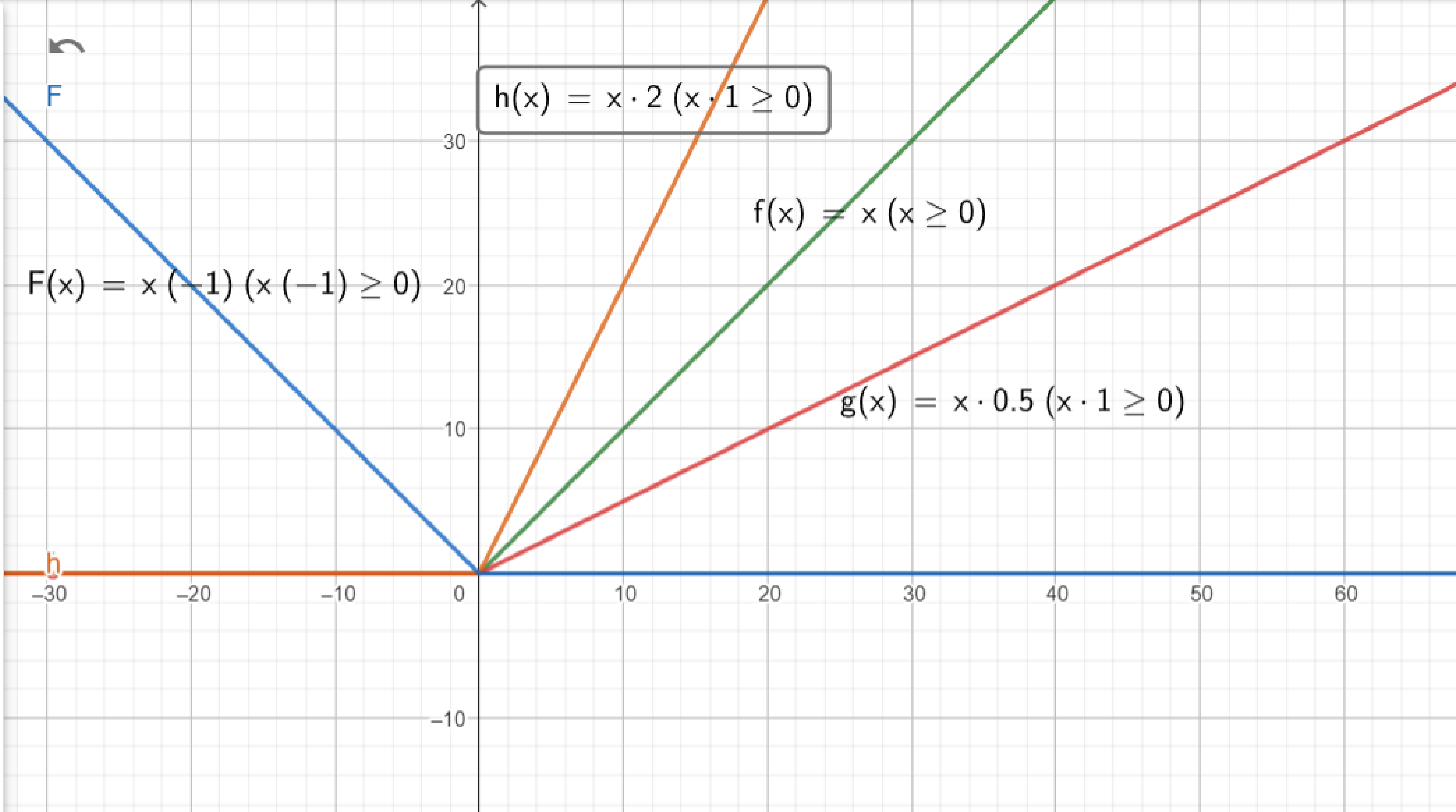
text1 = "f(x) = x (x ≥ 0)"

text2 = "g(x) = x · 0.5 (x · 1 ≥

text3 = "h(x) = x · 2 (x · 1 ≥ 0)

text4 = "F(x) = x (−1) (x (−1)

Input...



$f(x) = x \, (x \geq 0)$

$g(x) = 0.5 \, f(x) \, (x \geq 0)$

$= 0.5 \, x \, (x \geq 0) \, (x \geq 0)$

$h(x) = 2 \, f(x) \, (x \geq 0)$

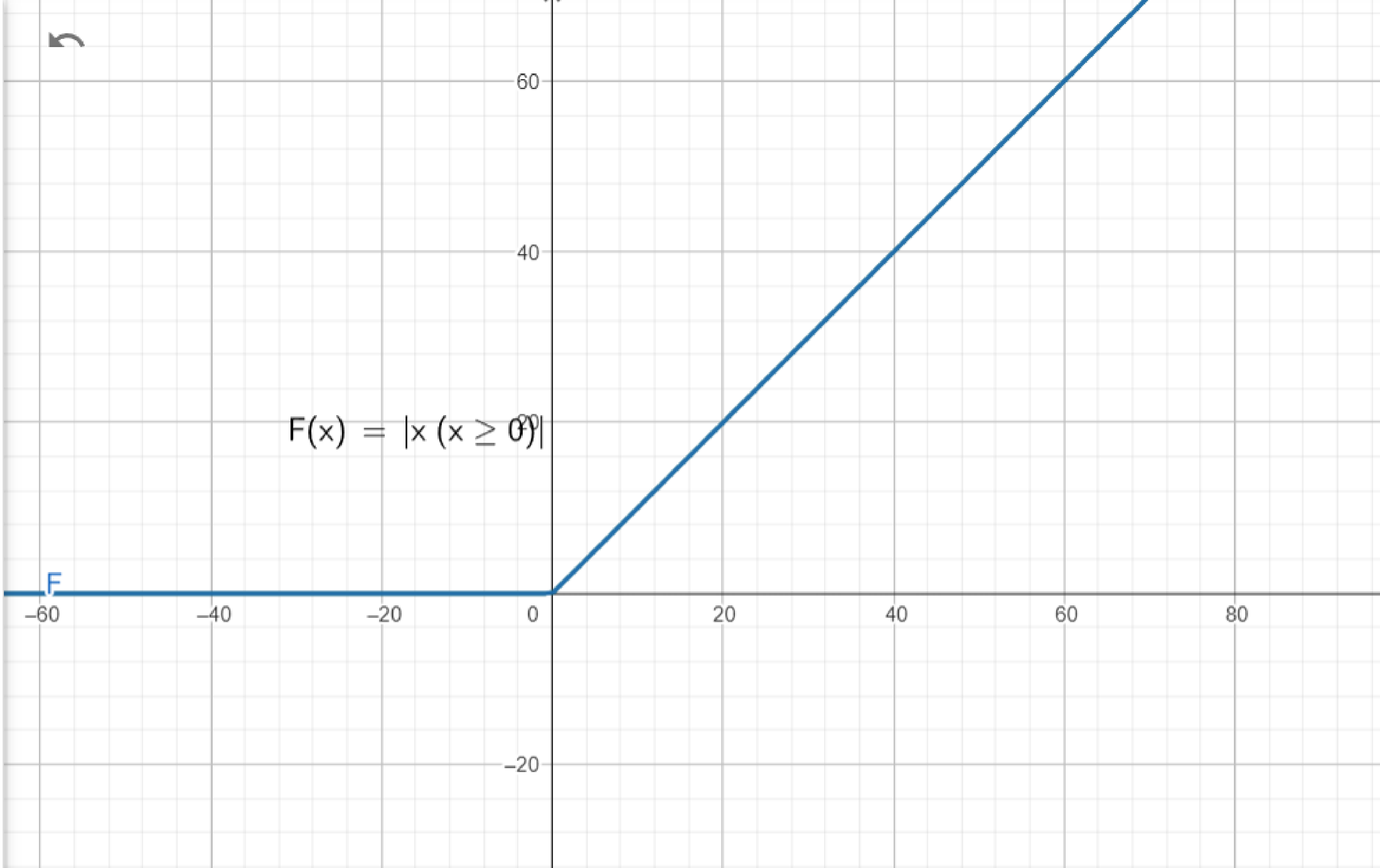
$= 2 \, x \, (x \geq 0) \, (x \geq 0)$

$F(x) = |f(x)|$

$= |x \, (x \geq 0)|$

text4 = "F(x) = |x (x ≥ 0)|"

Input...



$f(x) = x \, (x \geq 0)$

$g(x) = 0.5 \, f(x) \, (x \geq 0)$

$= 0.5 \, x \, (x \geq 0) \, (x \geq 0)$

$h(x) = 2 \, f(x) \, (x \geq 0)$

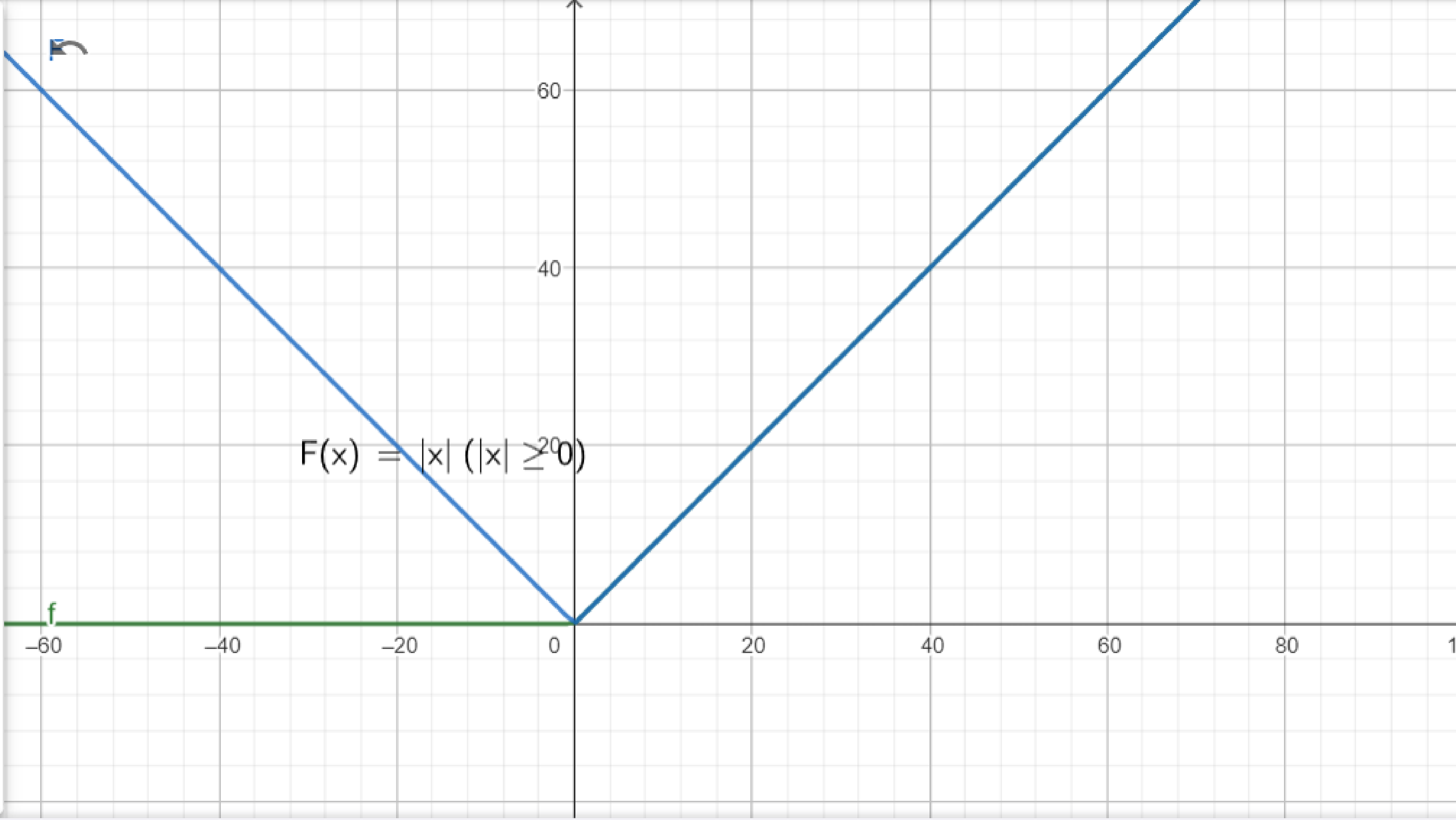
$= 2 \, x \, (x \geq 0) \, (x \geq 0)$

$F(x) = f(|x|)$

$= |x| \, (|x| \geq 0)$

text4 = "F(x) = |x| (|x| ≥ 0)"

+ Input...



123 f(x) ABC #&¬ •••

sin cos tan % ! $ °

26 b) $\left\{ x \in [-\pi, \pi] : \cos x = \frac{\sqrt{3}}{2} \right\}$

$= \left\{ x \in [-\pi, \pi] : \cos x = \cos \frac{\pi}{6} \right\}$

$= \left\{ x \in [-\pi, \pi] : x = \frac{\pi}{6}, -\frac{\pi}{6} \right\}$

$= \left[\frac{\pi}{6}, -\frac{\pi}{6} \right]$

· $\left\{ x \in [\pi, \pi] : \sin^2 x = \frac{1}{2} \right\}$

$= \left\{ x \in [-\pi, \pi] : \sin^2 x = \sin^2 \frac{\pi}{4} \right\}$

$= \left\{ x \in [-\pi, \pi] : x = n\pi \pm \frac{\pi}{4} \right\}$

For $n = 0$ and 1

$= \left\{ \frac{\pi}{4}, -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4} \right\}$

28. $z_1 = \frac{\sqrt{3}}{2} i + \frac{\sqrt{3}i - i}{(\sqrt{2}+\sqrt{2}i)^2}$

$z_2 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i$

Now,

$z_1 = \frac{\sqrt{3}}{2} i + \frac{\sqrt{3} - i}{(\sqrt{2}+\sqrt{2}i)^2}$

$= \frac{\sqrt{3}}{2} i + \frac{\sqrt{3} - i}{\cancel{2} + 4i - \cancel{2}}$

$= \frac{\sqrt{3}}{2} i + \frac{\sqrt{3} - i}{4i}$

$= \frac{\sqrt{3}}{2} i + \frac{\sqrt{3} - i}{4i} \times \frac{i}{i}$

$= \frac{\sqrt{3}}{2} i + \frac{(\sqrt{3} - i)i}{-4}$

$= \frac{\sqrt{3}}{2} i - \frac{\sqrt{3}}{4} i - \frac{1}{4}$

$= -\frac{1}{4} + \frac{\sqrt{3}}{4} i = \frac{1}{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \cancel{\sqrt{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}$

$\Rightarrow \frac{1}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

a) Real part of $z_1 = -\frac{1}{4}$ Imaginary part of $z_1 = \frac{\sqrt{3}}{4}$

Real part of $z_2 = 1$ Imaginary part of $z_2 = \sqrt{3}$

b) Absolute value of z_1 is $\frac{1}{2}$ ← comparing with $z = r(\cos\theta + i\sin\theta)$

Absolute value of z_2 is 2.

$\sqrt{a^2+b^2}$

29b) i. $x^2 = -1$, $x \in R$

$x^2 + 1 = 0$

comparing with $ax^2+bx+c=0$

$a=1$, $b=0$, $c=1$

For the solutions to be real,

$\boxed{b^2-4ac \geq 0}$

$0^2 - 4 \times 1 \times 1$

$= -4 < 0$

Hence, it does not have real solutions.

ii. $x^3 = -1$, $x \in R$

$x^3 = (-1)^3$

$\therefore x = -1 \in R$

or

$x^3 + 1^3 = 0$

$\underbrace{(x+1)}_{\text{Real}} \underbrace{(x^2-x+1)}_{\text{Imaginary}} = 0$

Hence, it has one solution.

29.c) i) $x^2 = -1$

$0, x^2 + 1 = 0$

$x = \frac{-0 \pm \sqrt{0-4 \cdot 1 \cdot 1}}{2 \cdot 1}$

$x = \frac{\cancel{2}\sqrt{-1}}{\cancel{2}} = \pm\sqrt{-1} = \pm i$

Hence, it has two solutions

(c.i) $x^3 = -1$, $x \in \mathbb{C}$

or, $x^3 = e^{i\pi}$ (Euler's formula)

or, $x = \sqrt[3]{e^{i\pi}}$

or, $x = e^{\frac{i\pi}{3}}$

or, $x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\therefore \boxed{x = \frac{1}{2} + \frac{\sqrt{3}}{2} i}$

30. Let z and ω be two non-zero complex numbers.

Let, $z = r_1 \cdot e^{i\theta_1}$

$\omega = r_2 e^{i\theta_2}$

By theorem 1.50,

$|z+\omega| = |z| + |\omega| \Leftrightarrow z\bar{\omega}$ is a real number and $z\bar{\omega} \geq 0$

$\therefore z\bar{\omega} = (r_1 \cdot e^{i\theta_1}) \cdot (\overline{r_2 e^{i\theta_2}})$ — (i)

$\overline{r_2 e^{i\theta_2}} = r_2 \overline{(\cos\theta_2 + i\sin\theta_2)}$

$= r_2 (\cos\theta_2 - i\sin\theta_2)$ {r_2 is real}