43. Find an example for a sequence with the following properties, if possible:

(a) a bounded sequence that is divergent.

(b) a non-increasing bounded sequence that is divergent.

(c) a strictly increasing sequence converging to π.

(d) an unbounded null sequence.

Solution:

(a) e.g. $a_n = (-1)^n$

(b) this is not possible, as the monotonicity principle (Theorem 3.20) asserts that any monotonous bounded sequence is convergent.

(c) e.g. $a_n = \pi - \frac{1}{n}$

(d) this is not possible, as any convergent sequence is bounded (Theorem 3.6)

$\square$

44. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n := (-1)^n\left(\frac{n+\cos n\pi}{2n}\right)$ for $n \in \mathbb{N}$ is not convergent.

Solution:
Consider the expression for a_n:

$$a_n = (-1)^n\left(\frac{n+\cos(n\pi)}{2n}\right).$$

First, note that $\cos(n\pi)$ takes the values 1 for even values of n and -1 for odd values of n. So, we can express a_n as:

$$a_n = (-1)^n\left(\frac{n+(-1)^n}{2n}\right).$$

Now, let's consider the cases for even and odd n:

- For even n:

$$a_{2k} = (-1)^{2k}\left(\frac{2k+(-1)^{2k}}{4k}\right) = \frac{2k+1}{4k} = \frac{1}{2}+\frac{1}{4k}.$$

- For odd n:

$$a_{2k+1} = (-1)^{2k+1}\left(\frac{2k+1+(-1)^{2k+1}}{4k+2}\right) = -\frac{2k}{4k+2} = -\frac{1}{2}+\frac{1}{4k+2}.$$

By definition, if $(a_n)_{n\in\mathbb{N}}$ is a convergences to $a \in \mathbb{R}$, then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$ we have

$$|a_n - a| < \epsilon.$$

Assume that $\lim_{n\to\infty} a_n = a$, let $\epsilon = \frac{1}{4}$, we have

$$|a_n - a| < \frac{1}{4}.$$

For all $2k > N$ we have

$$|a_{2k} - a_{2k+1}| = |a_{2k} - a + a - a_{2k+1}| \le |a_{2k} - a| + |a - a_{2k+1}| \le \frac{1}{4} + \frac{1}{4} \le \frac{1}{2}.$$

On the other hand

$$\begin{aligned} |a_{2k} - a_{2k+1}| = \left|\frac{1}{2}+\frac{1}{4k}-\left(-\frac{1}{2}+\frac{1}{4k+2}\right)\right| &= \left|1+\frac{1}{4k}-\frac{1}{4k+2}\right| \\ &= \left|1+\frac{1}{2k(4k+2)}\right| \\ &\ge 1 > \frac{1}{2}. \end{aligned}$$

This is a contradiction. Therefore, the sequence $(a_n)_{n\in\mathbb{N}}$ is not convergent. $\square$

45. Give an example for non-constant sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ satisfying

(a) $\lim_{n\to\infty} a_n = 3, b_n \neq 0$ and $\lim_{n\to\infty}(a_n b_n^2) = 0$.

(b) $\lim_{n\to\infty} a_n = 5, |b_n| \neq |a_n|$ and $\lim_{n\to\infty} (a_n + (-1)^n b_n) = 0$.

Solution:

(a) $(b_n^2)_{n\in\mathbb{N}}$ has to converge to 0, e.g., $b_n = \frac{1}{\sqrt{n}}$. Then by Theorem 3.24 we have $\lim_{n\to\infty}(a_n b_n^2) = 0$. We can choose any sequence $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = 3$, e.g., $a_n = 3 + \frac{1}{n}$.

(b) $a_n = 5 + \frac{1}{n}, b_n = (-1)^n(-5 + \frac{1}{n}) \Rightarrow a_n + (-1)^n b_n = (-1)^{2n}\frac{2}{n}$. Then $\lim_{n\to\infty} (a_n + (-1)^n b_n) = \lim_{n\to\infty} \frac{2}{n} = 0$.

$\square$

46. Determine the limits of sequences

$$a_n = \sqrt{n^2+n} - n, \qquad b_n = \frac{n^2+n+1}{n^2+n\sin n+1},$$
$$c_n = \frac{n!(n+5)+2^n}{(n+1)!+3^n}, \qquad d_n = \sqrt[n]{3^n+4^n},$$
$$e_n = \frac{n^3-3n+7}{7n+1}, \qquad t_n = \sqrt{n^3-n^2+1} - n,$$

where $n \in \mathbb{N}$.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n}$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n}+1}+1} = \frac{1}{2}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{n^2\left(1+\frac{\sin n}{n}+\frac{1}{n^2}\right)} = 1$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{(n+1)!\left(\frac{n+5}{n+1}+\frac{2^n}{(n+1)!}\right)}{(n+1)!\left(1+\frac{3^n}{(n+1)!}\right)} = \frac{\lim_{n\to\infty}\left(\frac{n+5}{n+1}+\frac{2^n}{(n+1)!}\right)}{\lim_{n\to\infty}\left(1+\frac{3^n}{(n+1)!}\right)} = 1,$$

since $0 < \dfrac{2^n}{(n+1)!} < \left(\dfrac{2}{3}\right)^n$ with n large enough, $\lim\limits_{n\to\infty}\left(\dfrac{2}{3}\right)^n = 0$,

$$\Rightarrow \lim_{n\to\infty} \frac{2^n}{(n+1)!} = 0.$$

Similarly, $\lim\limits_{n\to\infty} \dfrac{3^n}{(n+1)!} = 0.$

$\lim\limits_{n\to\infty} d_n = 4$ (by using the example in slide 51.)

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} \frac{n^3\left(1-\frac{3}{n^2}+\frac{7}{n^3}\right)}{n^3\left(\frac{7}{n^2}+\frac{1}{n^3}\right)} = \frac{\lim_{n\to\infty}\left(1-\frac{3}{n^2}+\frac{7}{n^3}\right)}{\lim_{n\to\infty}\left(\frac{7}{n^2}+\frac{1}{n^3}\right)} = \infty$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{(\sqrt{n^3-n^2+1}-n)(\sqrt{n^3-n^2+1}+n)}{\sqrt{n^3-n^2+1}+n}$$
$$= \lim_{n\to\infty} \frac{n^3-2n^2+1}{\sqrt{n^3-n^2+1}+n} = \lim_{n\to\infty} \frac{n^3\left(1-\frac{2}{n}+\frac{1}{n^3}\right)}{n^3\left(\sqrt{\frac{1}{n^3}-\frac{2}{n^4}+\frac{1}{n^6}}+\frac{1}{n^2}\right)}$$
$$= \lim_{n\to\infty} n^{3/2} \frac{1-\frac{2}{n}+\frac{1}{n^3}}{\sqrt{1-\frac{1}{n}+\frac{1}{n^3}}+\frac{1}{n^{\frac{1}{2}}}} = \infty$$

$\square$

47. Let $q > 0$, determine the limit of sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$, defined by

$$\begin{aligned} a_n &= (q^n + 5^n + 11^n)^{\frac{1}{n}}, \\ b_n &= \sqrt[n]{q^n + n^2}, \\ c_n &= \frac{\sin(q^n\pi)}{n}, \quad \text{where } n \in \mathbb{N}. \end{aligned}$$

Solution:

- $\lim_{n\to\infty} a_n$
  + If $0 < q \le 11$, we have
    $$11 = \sqrt[n]{11^n} \le \sqrt[n]{q^n + 5^n + 11^n} \le \sqrt[n]{3.11^n}.$$
    Using Sandwich law and $\lim_{n\to\infty} \sqrt[n]{3.11^n} = 11$, we obtain
    $$\lim_{n\to\infty} a_n = 11.$$
  + If $q > 11$, we have
    $$q = \sqrt[n]{q^n} \le \sqrt[n]{q^n + 5^n + 11^n} \le \sqrt[n]{3q^n}.$$
    Using Sandwich law and $\lim_{n\to\infty} \sqrt[n]{3q^n} = q$, then
    $$\lim_{n\to\infty} a_n = q.$$
- $\lim_{n\to\infty} b_n$
  + If $0 < q < 1$, we have $\lim_{n\to\infty} q^n = 0$, then as n approaches infinity, the term n^2 dominates q^n. Therefore,
    $$\sqrt[n]{n^2} \le \sqrt[n]{q^n + n^2} \le \sqrt[n]{2n^2}.$$
    Since $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} \sqrt[n]{n^2} = 1$, then $\lim_{n\to\infty} b_n = 1$
  + If $q \ge 1$, then as n approaches infinity, the term q^n dominates n^2. Therefore,
    $$\sqrt[n]{q^n} \le \sqrt[n]{q^n + n^2} \le \sqrt[n]{2q^n}.$$
    Since $\lim_{n\to\infty} \sqrt[n]{q^n} = \lim_{n\to\infty} \sqrt[n]{2q^n} = q$, then $\lim_{n\to\infty} b_n = q$.
- For all $q > 0$ we have
  $$0 \le \left|\frac{\sin(q^n\pi)}{n}\right| \le \frac{1}{n}.$$
  Therefore, $\lim_{n\to\infty} c_n = 0$, by using Sandwich law, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

$\square$

48. Let $\lim_{n\to\infty} a_n = a$. We define $b_n = \frac{1}{n}\sum_{j=1}^{n} a_j$ for $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} b_n = a$$

Hint: Use the boundedness of sequence $(a_n)_{n\in\mathbb{N}}$ and $a = \frac{\sum_{i=1}^{n} a}{n}$ for $a \in \mathbb{C}$, $n \in \mathbb{N}$.

Solution:

Assume that $\lim_{n\to\infty} a_n = a$. Sequence $(a_n)_{n\in\mathbb{N}}$ is convergent and thus bounded. There exist $C > 0$ such that

$$|a| \le C, \qquad \text{and} \qquad |a_n| \le C \quad \text{for } n \in \mathbb{N}. \tag{1}$$

By the definition of limit there exists $K \in \mathbb{N}$ such that for $n > K$

$$|a_n - a| \le \frac{\varepsilon}{2} \tag{2}$$

Observe that

$$\begin{aligned}
|b_n - a| = \left|\frac{\sum_{j=1}^{n} a_j}{n} - a\right| &= \left|\frac{\sum_{j=1}^{n} a_j - \sum_{j=1}^{n} a}{n}\right| \\
&= \left|\frac{\sum_{j=1}^{n}(a_j - a)}{n}\right| \\
&\le \frac{\sum_{j=1}^{n} |a_j - a|}{n} \\
&= \frac{\sum_{j=1}^{K} |a_j - a|}{n} + \frac{\sum_{j=K+1}^{n} |a_j - a|}{n}
\end{aligned}$$

By (1) we have

$$\sum_{j=1}^{K} |a_j - a| \le \sum_{j=1}^{K} (|a_j| + |a|) \le \sum_{j=1}^{K} 2C = 2KC.$$

By (2) we have

$$\sum_{j=K+1}^{n} |a_j - a| \le \sum_{j=K+1}^{n} \frac{\varepsilon}{2} = \frac{(n-K)\varepsilon}{2}$$

Therefore for all $n > \frac{4KC}{\varepsilon}$ we get

$$|b_n - a| \le \frac{2KC}{n} + \frac{(n-K)\varepsilon}{2n} \le \frac{2KC\varepsilon}{4KC} + \frac{\varepsilon}{2} \le \varepsilon.$$

$\square$