1. Compute the finite sums below. Give the computational steps, not only the results. Apply the learned rules for Σ whenever appropriate.

$$A = \sum_{j=5}^{10}(3j),$$

$$B = \sum_{i=1}^{50}(i^3 - (i-1)^3),$$

C is the sum of the first 110 numbers in the list $3, 7, 11, 15, 19, 23, \ldots$.

Solution:

(a) $\sum_{j=5}^{10}(3j) = 3\sum_{j=5}^{10} j = 3 \cdot (5+6+7+8+9+10) = 3 \cdot 45 = 135,$

(b) $\sum_{i=1}^{50}(i^3-(i-1)^3) = \sum_{i=1}^{50} i^3 - \sum_{i=1}^{50}(i-1)^3 = \left(\sum_{i=1}^{49} i^3 + 50^3\right) - \sum_{i=0}^{49} i^3 = \left(\sum_{i=1}^{49} i^3 + 50^3\right) - \left(0^3 + \sum_{i=1}^{49} i^3\right) = 50^3 + \left(\sum_{i=1}^{49} i^3 - \sum_{i=1}^{49} i^3\right) = 50^3 + 0 = 50^3,$

(c) $C = \sum_{k=1}^{110}(3+4(k-1)) = \sum_{k=1}^{110}(3) + 4 \cdot \sum_{k=1}^{109} k =_{\text{(Gauss)}} 330 + 4 \cdot \frac{109 \cdot 110}{2} = 24310.$

$\square$

2. Compute the finite products below. Give the computational steps, not only the results. Apply the learned rules for Π whenever appropriate.

$$A = \prod_{i=1}^{5}(2i),$$
$$B = \prod_{i=3}^{6}(i-2)!,$$
$$C = \prod_{i=1}^{3}\prod_{k=2}^{4}(2i+k).$$

Solution:

(a) $\prod_{i=1}^{5}(2i) = 2^5 \prod_{i=1}^{5}(i) = 32 \cdot 5! = 32 \cdot 120 = 3840,$

(b) $\prod_{i=3}^{6}(i-2)! = \prod_{i=1}^{4}(i)! = 2! \cdot 3! \cdot 4! = 2 \cdot 6 \cdot 24 = 288,$

(c) $= \prod_{i=1}^{3}\prod_{k=2}^{4}(2i+k) = \left(\prod_{k=2}^{4}(2+k)\right)\left(\prod_{k=2}^{4}(4+k)\right)\left(\prod_{k=2}^{4}(6+k)\right) = (4 \cdot 5 \cdot 6) \cdot (6 \cdot 7 \cdot 8) \cdot (8 \cdot 9 \cdot 10) = 29030400.$

$\square$

3. Let $n \in \mathbb{N}$. Try to give the following expressions in a simple closed form. Perform some computations for particular, small values of n. Give the computational steps, not only the results.

$$A = \sum_{k=1}^{n} k \cdot k! \ (\textit{Hint: } A = \sum_{k=1}^{n} (k+1-1) \cdot k! = \cdots = (n+1)! - 1\,),$$

$$B = \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right), \ (\textit{Hint: common denominator for the factors})$$

$$C = \sum_{k=1}^{n} \frac{1}{k(k+1)} \ (\textit{Hint: use that the term } \tfrac{1}{k(k+1)} \textit{ can be rewritten as } \tfrac{\alpha}{k} + \tfrac{\beta}{k+1}$$
for some α and β in $\mathbb{R}$),

Solution:

(a) $A = \sum_{k=1}^{n} (k+1-1) \cdot k! = \sum_{k=1}^{n} (k+1)! - k! = \sum_{k=1}^{n} (k+1)! - \sum_{k=1}^{n} k! = \sum_{k=2}^{n+1} k! - \sum_{k=1}^{n} k! = (n+1)! + \left(\sum_{k=2}^{n} k! - \sum_{k=2}^{n} k!\right) - 1! = (n+1)! - 1,$

(b) $B = \prod_{i=2}^{n} \frac{i-1}{i} = \frac{\prod_{i=2}^{n} i - 1}{\prod_{i=2}^{n} i} = \frac{\prod_{i=1}^{n-1} i}{n!} = \frac{(n-1)!}{n!} = \frac{1}{n},$

(c) $C =^{(*)} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 + \left(\sum_{k=2}^{n} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k}\right) - \frac{1}{n+1} = \frac{n}{n+1}.$

(*) Note that α and β can be determined e.g. by coefficient comparison: $\frac{\alpha}{k} + \frac{\beta}{k+1} = \frac{(\alpha+\beta) \cdot k + (\alpha) \cdot 1}{k(k+1)} = \frac{(0) \cdot k + (1) \cdot 1}{k(k+1)} = \frac{1}{k(k+1)}$ if and only if $\alpha = 1$ and $\beta = -1$.

$\square$

4. Think about how the following expressions can be written explicitly and try to calculate the sum of the product and the product of the sum:

   a) $\displaystyle\sum_{i=1}^{2}\prod_{k=1}^{3}\frac{i}{k}$,

   b) $\displaystyle\prod_{k=1}^{3}\sum_{i=1}^{2}\frac{i}{k}$,

   c) Are points a) and b) sufficient to say that, in general, in sum and products together, the order of the symbols sum and product is important and should not be changed?

Solution:

(a) $\displaystyle\sum_{i=1}^{2}\prod_{k=1}^{3}\frac{i}{k}=\prod_{k=1}^{3}\frac{1}{k}+\prod_{k=1}^{3}\frac{2}{k}=\left(1\cdot\frac{1}{2}\cdot\frac{1}{3}\right)+\left(2\cdot 1\cdot\frac{2}{3}\right)=\frac{3}{2}$,

(b) $\displaystyle\prod_{k=1}^{3}\sum_{i=1}^{2}\frac{i}{k}=\sum_{i=1}^{2}\frac{i}{1}\cdot\sum_{i=1}^{2}\frac{i}{2}\cdot\sum_{i=1}^{2}\frac{i}{3}=(1+2)\cdot\left(1+\frac{1}{2}\right)\cdot\left(\frac{1}{3}+\frac{2}{3}\right)=3\cdot\frac{3}{2}=\frac{9}{2}$,

(c) yes, in general, if we interchange the order of sum and product in the above expressions, then the resulting numbers are different (contrary to the finite double products, (see Exercise 2c), where the order of multiplication does not matter).

$\square$

5. Consider the following table.

x_i	-6	-3	-1	1	3	4
y_i	5	4	2	1	1	0

(a) Compute $\mu_x := \frac{1}{6}\sum_{\ell=1}^{6} x_\ell$ and $\mu_y := \frac{1}{6}\sum_{\ell=1}^{6} y_\ell$.

(b) Compute $\sigma_x^2 := \frac{1}{6}\sum_{\ell=1}^{6}(x_\ell - \mu_x)^2$ and $\sigma_{xy} := \frac{1}{6}\sum_{\ell=1}^{6}(x_\ell - \mu_x)(y_\ell - \mu_y)$.

(c) Determine $k := \frac{\sigma_{xy}}{\sigma_x^2}$.

(d) Solve for d: $\mu_y = k\mu_x + d$.

(e) Plot the points (x_i, y_i) and $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = kx + d$.

(f) Compute $f(-2)$ and $f(2)$.

Solution:

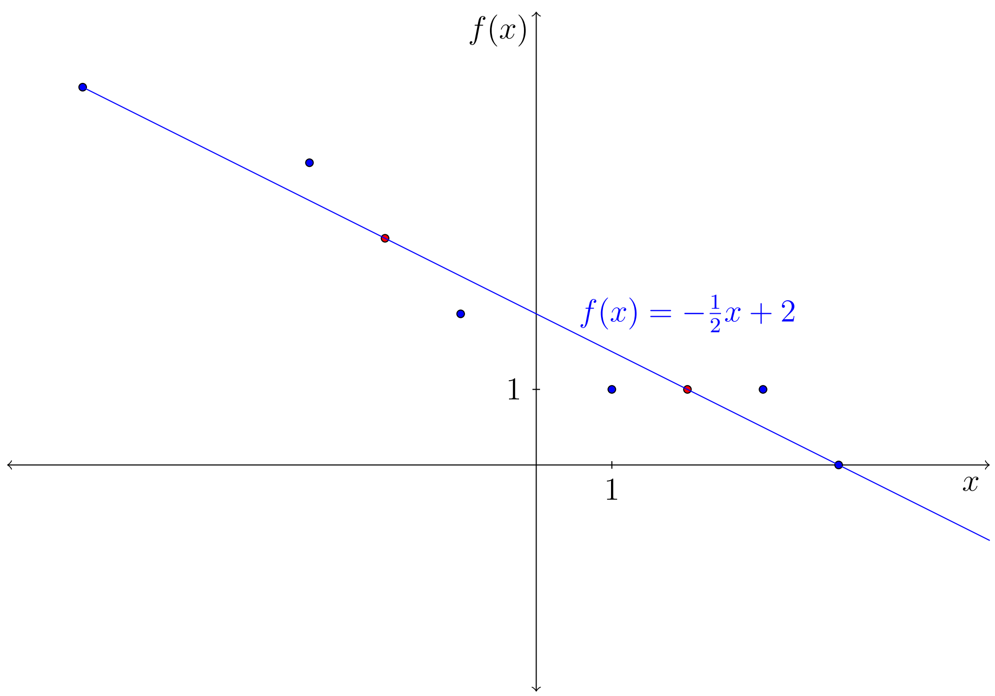
(a) $\mu_x = -\frac{2}{6}$, $\mu_y = \frac{13}{6}$.

(b) $\sigma_x^2 = \frac{107}{9}$, $\sigma_{xy} = -\frac{107}{18}$.

(c) $k = -\frac{1}{2}$.

(d) $d = \frac{13}{6} - \left(-\frac{1}{2} \cdot \left(-\frac{2}{6}\right)\right) = \frac{13}{6} - \frac{1}{6} = 2$.

(e) A plot looks like this:



(f) $f(-2) = 3$ and $f(2) = 1$.

Remark: This is simple linear regression, i.e. the line $f(x)$ is the best linear predictor given the data x_i, y_i. □

6. Let $n \in \mathbb{N}$, and $x_0 < x_1 < x_2 < \cdots < x_n \in \mathbb{R}$ a sorted list of $n+1$ different real numbers. For $i = 0, 1, 2, \ldots, n$, we define the polynomials

$$\ell_i(x) := \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,.$$

This notation for the lower and upper index of the product corresponds to the index set

$$\{0, 1, 2, \ldots, n\} \setminus \{i\} = \{0, 1, 2, \ldots, i-1, i+1, \ldots, n\}.$$

Furthermore, for $y_0, y_1, y_2, \ldots, y_n \in \mathbb{R}$, another list of $n+1$ real numbers (not necessarily sorted and different from each other), we define

$$p(x) := \sum_{i=0}^{n} y_i \cdot \ell_i(x)\,.$$

For this exercise, let $n = 3$ and take the following table:

i	0	1	2	3
x_i	-1	0	2	3
y_i	3	-3	6	5

(a) Compute the four polynomial functions $\ell_0(x), \ell_1(x), \ell_2(x), \ell_3(x)$.

(b) Plot the graph of $\ell_i(x)$ for $i = 0, \ldots, 3$ in the interval $[-1.5, 3.5] \subset \mathbb{R}$.

- Look at the zeros of the graphs. What do you observe?
- Also look at the points $(x_i, \ell_i(x_i)) \in \mathbb{R}^2$ for $i = 0, \ldots, 3$. What do you observe?

Can you justify both observations by considering the definition of $\ell_i(x)$?
(For plotting/visualizing, use a tool of your choice!)
Hint: a zero of $\ell_i(x)$ *is a value* $\overline{x} \in \mathbb{R}$*, such that* $\ell_i(\overline{x}) = 0$.

(c) Compute $p(x)$ and plot the graph of $p(x)$ in the interval $[-1.5, 3.5] \subset \mathbb{R}$. In this plot, highlight the points $(x_i, y_i) \in \mathbb{R}^2$ for $i = 0, \ldots, n$.
(For plotting/visualizing, use a tool of your choice!)

(d) Compute $p(1)$.

Solution:

(a) The polynomials are given by

$$\ell_0(x) = -\tfrac{1}{12}(x-0)(x-2)(x-3) = \tfrac{1}{12}\Big(-x^3 + 5x^2 - 6x\Big),$$
$$\ell_1(x) = \tfrac{1}{6}(x+1)(x-2)(x-3) = \tfrac{1}{6}\Big(x^3 - 4x^2 + x + 6\Big),$$
$$\ell_2(x) = -\tfrac{1}{6}(x+1)(x-0)(x-3) = \tfrac{1}{6}\Big(-x^3 + 2x^2 + 3x\Big),$$
$$\ell_3(x) = \tfrac{1}{12}(x+1)(x-0)(x-2) = \tfrac{1}{12}\Big(x^3 - x^2 - 2x\Big),$$

Evaluating $\ell_i(x_j)$ always results in 0, for $i, j = 0, \ldots, 3$ and $i \neq j$. This is because one of the factors of every polynomial $\ell_i(x)$ will always be

$$\frac{x_j - x_j}{x_i - x_j} = \frac{0}{x_i - x_j} = 0.$$

On the other hand,

$$\ell_i(x_i) = \prod_{\substack{j=0 \\ j \neq i}}^{3} \frac{x_i - x_j}{x_i - x_j} = 1 \cdot 1 \cdot 1 = 1.$$

(b) $p(x) = \frac{1}{6}\left(-8x^3 + 29x^2 + x - 18\right)$.

(c) We have $p(1) = \frac{2}{3}$.

Plots etc: `https://www.geogebra.org/calculator/yqvkdg5d` □