JOHANNES KEPLER
UNIVERSITY LINZ

Mathematics for AI I

Jan-Michael Holzinger

JOHANNES KEPLER
UNIVERSITY LINZ
Altenbergerstraße 69
4040 Linz, Austria
jku.at

Sigma- and Pi-Notation for Sums and Products

Indices

Indices are used in mathematics for various purposes, specifying the elements in a tuple (vector, array) is just one (but an important one).

Example

- A vector a of length 7 may be given as $a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$.
- If $b = (5, -3)$, then $b_1 = 5$ and $b_2 = -3$.
- If $c = (1, 2, 3, 4)$, then $c_1 = 1$, $c_2 = 2$, $c_3 = 3$ and $c_4 = 4$.

Example

- $\forall i \in \{1, 2, 3, 4\} : c_i = i$.
- For $i = 1, \ldots, 4 : c_i = i$.

Indices

Being able to work with indices is also a core feature in many programming languages. E.g.

```
for i in range(0,4):
    c[i] = i
```

(In most programming languages, the first index of a list/array/tuple with n entries is `0` and the last index is `n-1`.)

Indices

Indices allow us to write complex formulae/expressions in a compact way.

Example

Let $A_1, A_2, \ldots, A_n$ be non-empty sets. The Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n := \{(a_1, a_2, \ldots, a_n) : \; a_i \in A_i \;\; \forall i = 1, 2, \ldots, n\}.$$

Example

Let $V_1, V_2, \ldots, V_n$ be sets, s.t.

$$V_i \cap V_j = \emptyset \text{ for all } i, j \in \{1, 2, \ldots, n\} \text{ with } i \neq j.$$

Indices

Until now, we only used indices that were taken from a subset of the natural numbers $\mathbb{N}_0$. We might however also use different index sets.

Example

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, and any $\alpha \in \mathbb{R}$, we define

$$L_\alpha(f) = \{x \in \mathbb{R} : f(x) = \alpha\}.$$

Summation

Very common task in programming: given a list/array/tuple of numbers, we have to sum (some of) them up.

```python
s = 0
for i in range(0,len(c)):
    s = s + c[i]
```

A corresponding mathematical expression is $s = c_1 + c_2 + \cdots + c_n$.

Summation

We already know, a problem of the “dot” notation is possible ambiguity, but in many cases it is not feasible to write down every summand explicitly. A solution is the so-called “Sigma”-notation.

Σ is the upper case Greek letter “S” - “S” stands for sum.

Example

- $\sum_{i=1}^{n} c_i = c_1 + c_2 + \cdots + c_n.$
- $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15.$

Summation

How to use the Sigma-notation? $\sum_{i=\ell}^{u} a_i$

i ... (summation) index,

ℓ ... lower bound (start value),

u ... upper bound (end value),

a_i ... summation term.

Summation

How to use the Sigma-notation? $\sum_{i=\ell}^{u} a_i$

1 Set res = 0 and $i = \ell$.

2 While $i \leq u$:

  2.a Replace every occurrence of i in the term a_i by the current value of i.

  2.b Add the resulting term to res: res $\leftarrow$ res + a_i.

  2.c Increment i by 1: $i \leftarrow i + 1$.

3 The result is the value of res.

Summation

Example

We compute $\sum_{i=2}^{5} 2i$.

We notice: The sum index is i, the lower bound is 2, the upper bound is 5 and the summation term is $2i$.

i	$i \leq 5$?	res (old)	a_i	res (new)	$i+1$
2	Yes.	0	$2 \cdot 2 = 4$	$0 + 4 = 4$	3
3	Yes.	4	$2 \cdot 3 = 6$	$4 + 6 = 10$	4
4	Yes.	10	$2 \cdot 4 = 8$	$10 + 8 = 18$	5
5	Yes.	18	$2 \cdot 5 = 10$	$18 + 10 = 28$	6
6	No.	28	-	-	-

Summation

Example

$$\sum_{i=6}^{10} \frac{i}{2} + 1.$$

The sum index is i. The lower bound is 6, the upper bound is 10. The summation term is $\frac{i}{2} + 1$.

Written down explicitly, the sum is $(\frac{6}{2} + 1) + (\frac{7}{2} + 1) + (\frac{8}{2} + 1) + (\frac{9}{2} + 1) + (\frac{10}{2} + 1) = 25$.

Summation

Example

$$\sum_{j=1}^{3} 2 \cdot j + j^2.$$

The sum index is j. The lower bound is 1, the upper bound is 3. The summation term is $2 \cdot j + j^2$.

Written down explicitly, the sum is $(2{\cdot}1{+}1^2){+}(2{\cdot}2{+}2^2){+}(2{\cdot}3{+}3^2) = 26$.

Summation

Example

Attention: $\sum_{i=0}^{2} 1 + j$.

The sum index is **i**. The lower bound is 0, the upper bound is 2. The summation term is 1 + **j**.

As i does not occur in the summation term: written down explicitly, the sum is $(1 + j) + (1 + j) + (1 + j) = 3 + 3j$.

Summation

Similar notation, given an index set A: $\sum_{i \in A} a_i$.

Example

Let $A := \{2, 4, 8, 16\}$, then

$$\sum_{i \in A} \frac{1}{i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

Summation - further properties

- empty sum,
- change of variable,
- distributive law,
- sums with same limits,
- decomposition,
- index reflection,
- index shift.

Empty sum

Example

$$\sum_{i=2}^{1} i^2 = 0.$$

Remark

If the lower summation index is bigger than the upper index, then the sum is 0*. I.e., if* $\ell > u$*, then* $\sum_{i=\ell}^{u} a_i = 0$.

(Other authors may define the case $u < \ell$ differently. Be careful when you refer to other literature.)

Change of variable

Example

$$a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{i=1}^{5} a_i.$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{j=1}^{5} a_j.$$

Remark

If we replace the summation index and all its occurrences in the summation term by another index, then the resulting sums are the same. $\displaystyle\sum_{i=\ell}^{u} a_i = \sum_{j=\ell}^{u} a_j$.

Distributive law

Example

(1) $$\sum_{i=1}^{5} c \cdot a_i = c \cdot a_1 + c \cdot a_2 + c \cdot a_3 + c \cdot a_4 + c \cdot a_5 \,.$$

(2) $$c \cdot \sum_{i=1}^{5} a_i = c \cdot (a_1 + a_2 + a_3 + a_4 + a_5) \,.$$

Remark

By ***distributivity*** *of multiplication over addition, to multiply a sum by a factor, each summand is multiplied by the factor and the resulting products are added. I.e.,* $\forall c \in \mathbb{R},\; c \cdot \sum_{i=\ell}^{u} a_i = \sum_{i=\ell}^{u} c \cdot a_i \,.$

Sums with same limits

Example

$$\left(\sum_{i=1}^{5} a_i\right) + \left(\sum_{i=1}^{5} b_i\right) = (a_1 + a_2 + \cdots + a_5) + (b_1 + b_2 + \cdots + b_5)$$
$$= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_5 + b_5)$$
$$= \sum_{i=1}^{5} a_i + b_i.$$

Sums with same limits

Remark

*Using **commutativity** and **associativity** of summation, we are able to "merge" or "split" sums (if lower and upper index of summation fit).* $\sum_{i=\ell}^{u} a_i + \sum_{i=\ell}^{u} b_i = \sum_{i=\ell}^{u}(a_i + b_i)$.

Decomposition

Example

$$\left(\sum_{i=1}^{2} a_i\right) + \left(\sum_{i=3}^{5} a_i\right) = (a_1 + a_2) + (a_3 + a_4 + a_5)$$
$$= a_1 + a_2 + a_3 + a_4 + a_5$$
$$= \sum_{i=1}^{5} a_i.$$

Remark ($\ell \leq m \leq u$)

*Using **associativity**, we may compute intermediate results (or "glue" together certain sums).* $\sum_{i=\ell}^{m} a_i + \sum_{i=m+1}^{u} a_i = \sum_{i=\ell}^{u} a_i.$

Index reflection

Example

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5$$
$$= a_5 + a_4 + a_3 + a_2 + a_1$$
$$= \sum_{i=1}^{5} a_{5-i+1}.$$

Remark

*Using **commutativity**, the result is the same, if we sum from lower to upper limit or vice versa.* $\displaystyle\sum_{i=\ell}^{u} a_i = \sum_{i=\ell}^{u} a_{u-i+\ell}.$

Index shift

Example

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5$$

$$\sum_{i=3}^{7} a_{i-2} = a_{(3-2)} + a_{(4-2)} + a_{(5-2)} + a_{(6-2)} + a_{(7-2)}.$$

Remark

If we change the summation bounds, we have to change the summation term accordingly. $\sum_{i=\ell}^{u} a_i = \sum_{i=\ell+m}^{u+m} a_{i-m}$.

Double summation

How can we compute the following sum? $\displaystyle\sum_{i=\ell}^{u}\sum_{j=m}^{v} a_{i,j}$.

Remark

$$\begin{aligned}\sum_{i=\ell}^{u}\sum_{j=m}^{v} a_{i,j} &= \sum_{j=m}^{v} a_{\ell,j} + \sum_{j=m}^{v} a_{\ell+1,j} + \cdots + \sum_{j=m}^{v} a_{u,j}\\ &= (a_{\ell,m} + a_{\ell,m+1} + \cdots + a_{\ell,v})\\ &+ (a_{\ell+1,m} + a_{\ell+1,m+1} + \cdots + a_{\ell+1,v})\\ &+ \dots\\ &+ (a_{u,m} + a_{u,m+1} + \cdots + a_{u,v}).\end{aligned}$$

Products

Similar to the Sigma-notation for sums, we use Pi-Notation for products.

Π is the upper case Greek letter “P” - “P” stands for product.

Example

- $\displaystyle\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n.$
- $\displaystyle\prod_{i=1}^{5} i = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120\ (= 5!).$

Products

How to use the Pi-notation? $\prod_{i=\ell}^{u} a_i$

i ... (multiplication) index,

ℓ ... lower bound (start value),

u ... upper bound (end value),

a_i ... multiplication term.

Products

How to use the Pi-notation? $\prod_{i=\ell}^{u} a_i$

1 Set res = 1 and $i = \ell$.

2 While $i \leq u$:

  2.a Replace every occurrence of i in the term a_i by the current value of i.

  2.b Multiply the resulting term to res: res $\leftarrow$ res $\cdot\, a_i$.

  2.c Increment i by 1: $i \leftarrow i + 1$.

3 The result is the value of res.

Products - properties

Similar as for summation, e.g.

$$\forall c \in \mathbb{R}, \; \prod_{i=\ell}^{u} c \cdot a_i = c^{u-\ell+1} \cdot \prod_{i=\ell}^{u} a_i \,.$$

Example

$$\prod_{i=2}^{5}(2 \cdot i) = (2 \cdot 2) \cdot (2 \cdot 3) \cdot (2 \cdot 4) \cdot (2 \cdot 5) = 2^4 \prod_{i=2}^{5} i \;\; (= 1920).$$

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