Exercise 1.1. *For the following complex numbers $z_1, z_2 \in \mathbb{C}$, find their* canonical representation *and their representation in* polar form*.*

1. $z_1 = \left(1+\sqrt{3}i\right)^{20} \cdot \left(-\frac{1}{4}+\frac{\sqrt{3}}{4}i\right)^{15}$,

2. $z_2 = \dfrac{16+4i}{-5+3i}$.

Exercise 1.2. *Use* Gaussian elimination *to solve $A\mathbf{x} = \mathbf{b}$, where*

$$A = \begin{pmatrix} -5 & -4 & 8 & 14 \\ 17 & 19 & -5 & 19 \\ 15 & -6 & -20 & -30 \\ -2 & -10 & -14 & -46 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -5 \\ 209 \\ -235 \\ -214 \end{pmatrix}.$$

Moreover, determine a matrix C in reduced row echelon form*, which is equivalent to A. Determine then the* rank *of A.*

Exercise 2.1. *1. Consider the function* $f\colon \mathbb{R} \to \mathbb{R}$*, defined by* $f(x) = 3x - 2$*.*

   (a) *Sketch the function* f*.*

   (b) *Prove* f *is invertible.*

2. *Let* $a, b \in \mathbb{R}$*. Consider* $g\colon \mathbb{R} \to \mathbb{R}$*, defined by* $g(x) = ax + b$*. Use mathematical symbols to fill the gap, such that the following statement is true:*

   $$g \text{ is invertible, if and only if } a \text{ \_\_\_\_ }.$$

   Prove the statement.
   Hint: Restrict the variables, if needed.

Exercise 2.2. *Let* $\mathbf{v} = (v_i)_{i=1}^d \in \mathbb{C}^d$*. Then, we define* $\|\mathbf{v}\|_1$ *by the formula*

$$\|\mathbf{v}\|_1 := \sum_{i=1}^{d} |v_i|.$$

1. *Compute* $\|\mathbf{w}\|_1$ *for*

   $$\mathbf{w} = \begin{pmatrix} -3 - i \\ 3 + 2i \\ -2 \\ 3i \\ -2 + 4i \end{pmatrix} \in \mathbb{C}^5.$$

2. *Prove the following statements:*

   (a) $\forall \mathbf{v} \in \mathbb{C}^d : \ \|\mathbf{v}\|_1 \geq 0,$

   (b) $\forall \mu \in \mathbb{C} \ \forall \mathbf{v} \in \mathbb{C}^d : \ \|\mu \mathbf{v}\|_1 = |\mu| \, \|\mathbf{v}\|_1,$

   (c) $\forall \mathbf{v}, \mathbf{w} \in \mathbb{C}^d : \ \|\mathbf{v} + \mathbf{w}\|_1 \leq \|\mathbf{v}\|_1 + \|\mathbf{w}\|_1.$

Exercise 2.3. *Use "proof by induction" to prove the following statement:*

$$\sum_{\ell=1}^{n} \ell \cdot \ell! = (\ell + 1)! - 1$$

Exercise 2.4. *For the matrix $A \in \mathbb{R}^{2\times 2}$, given by*

$$A = \begin{pmatrix} 1 & 0 \\ -5 & 3 \end{pmatrix},$$

we define the function $p_A(\lambda) = \det(A - \lambda I_2)$. Find all values $\lambda \in \mathbb{R}$, such that $p_A(\lambda) = 0$.
Hint: Treat λ as an unknown constant and compute $A - \lambda I_2$. Then compute the determinant of this matrix. This will give you an expression, which only depends on λ.