Mathematics for AI 1

3. Sequences and Series

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3.5 Cauchy criterion

Definition - Cauchy sequences

A complex-valued sequence $(a_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} : m, n \geq N \implies |a_n - a_m| < \varepsilon.$$

Example

We define a sequence $(a_n)_{n\in\mathbb{N}}$ by $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We choose $N > \frac{2}{\varepsilon}$ and for n, m greater than N by the triangle inequality,

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \varepsilon.$$

Example

The sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = \sin\left(n\frac{\pi}{2}\right)$$

is a not Cauchy sequence. Suppose, for the sake of contradiction, that it is Cauchy. Then, for all $\varepsilon > 0$, there is a number $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ holds for all $m, n \geq N$. But, if we set $\varepsilon = \frac{1}{2}$, we obtain a contradiction. Indeed, no matter which value we choose for $k \in \mathbb{N}$, we have

$$|a_{2k} - a_{2k+1}| = 1 > \frac{1}{2}.$$

Exercise

Prove by definition of a Cauchy sequence that the sequence:

$$a_n = \frac{n^3}{n^3 + n}, \qquad n \in \mathbb{N},$$

is a Cauchy sequence.

Exercise

Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence and let $\lambda > 0$. Suppose that sequence $(b_n)_{n\in\mathbb{N}}$ satisfies following inequality

$$|b_n - b_m| < \lambda|a_n - a_m| \quad \text{for} \quad m, n \in \mathbb{N}.$$

Prove that $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Exercise

Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ is such that

$$|a_n - a_m| \leq \frac{1}{n^2 + m^2}$$

for all $m, n \in \mathbb{N}$.

a) Prove that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

b) Prove that $(a_n)_{n\in\mathbb{N}}$ is a constant sequence.

Exercise

Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence such that $a_n \in \mathbb{N}$ for $n \in \mathbb{N}$. Prove there are $N, m \in \mathbb{N}$ satisfying

$$a_{N+n} = m, \quad \forall n \in \mathbb{N}.$$

Remark

We had two examples of sequences: the convergent one is a Cauchy sequence, divergent one is not. It turns out it is not a coincidence.

Theorem (Cauchy criterion)

Let $(a_n)_{n\in\mathbb{N}}$ be a real-valued sequence. Then

$$(a_n)_{n\in\mathbb{N}} \text{ is convergent } \iff (a_n)_{n\in\mathbb{N}} \text{ is Cauchy.}$$

Remark

Note that the complex-valued sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then $(\mathrm{Re}(a_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(a_n))_{n\in\mathbb{N}}$ are Cauchy sequences. By the theorem from the last slide there are real numbers x and y such that

$$\lim_{n\to N} \mathrm{Re}(a_n) = x \quad \text{and} \quad \lim_{n\to N} \mathrm{Im}(a_n) = y.$$

It is easy to check that $\lim_{n\to N} a_n = x + iy$. Therefore theorem from the last slide also holds for complex valued sequences.

Lemma

Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued sequence which is Cauchy. Then $(a_n)_{n\in\mathbb{N}}$ is bounded.

Proof.

Apply the definition of a Cauchy sequence with your favourite $\varepsilon = \frac{e}{\pi}$. It follows that there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

$$|a_n - a_m| < \frac{e}{\pi}.$$

Also, by the triangle inequality,

$$|a_n| = |(a_n - a_N) + a_N| \leq |a_n - a_N| + |a_N| < \frac{e}{\pi} + |a_N|.$$

Proof.

This gives a bound for all $n \geq N$. It then follows that, for all $n \in \mathbb{N}$, $|a_n| \leq C$, where

$$C = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, \frac{e}{\pi} + |a_N|\}.$$

$\square$

Proof of the Cauchy Criterion

First, we show that

$$(a_n)_{n\in\mathbb{N}} \text{ is convergent } \implies (a_n)_{n\in\mathbb{N}} \text{ is Cauchy.}$$

Suppose that $(a_n)_{n\in\mathbb{N}}$ is convergent with $a_n \to a$. Let $\varepsilon > 0$ be arbitrary. By the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|a_m - a| < \frac{\varepsilon}{2}, \quad \text{and} \quad |a_n - a| < \frac{\varepsilon}{2}.$$

It follows from the triangle inequality that, for all $m, n \geq N$,

$$|a_m - a_n| = |(a_m - a) + (a - a_n)| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Proof of the Cauchy Criterion

$$(a_n)_{n\in\mathbb{N}} \text{ is Cauchy } \implies (a_n)_{n\in\mathbb{N}} \text{ is convergent .}$$

Suppose that $(a_n)_{n\in\mathbb{N}}$ is Cauchy. Lemma on slide 2 implies that $(a_n)_{n\in\mathbb{N}}$ is bounded. By the Bolzano-Weierstrass Theorem, $(a_n)_{n\in\mathbb{N}}$ has at least one convergent subsequence.
Let $(a_{n_k})_{n\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} a_{n_k} = a.$$

Proof of the Cauchy Criterion

Let $\varepsilon > 0$ be arbitrary. Since $(a_{n_k})_{n\in\mathbb{N}}$ tends to a, it follows from the definition of convergence that there is some $K \in \mathbb{N}$ such that, for all $k \geq K$,

$$|a_{n_k} - a| < \frac{\varepsilon}{2}.$$

Also, since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, it follows that there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

$$|a_n - a_m| < \frac{\varepsilon}{2}.$$

Let $k \in \mathbb{N}$ be any integer such that both $k \geq K$ and $n_k \geq N$ hold. Then, by the triangle inequality, we have that for all $n \geq N$

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Example

Assume that complex valued sequence $(a_n)_{n\in\mathbb{N}}$ satisfies

$$|a_{n+1} - a_{n+2}| \leq \frac{1}{2}|a_n - a_{n+1}|.$$

Sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof

By induction we prove that there exists $C > 0$ such that

$$|a_n - a_{n+1}| \leq \frac{C}{2^n}. \tag{1}$$

For $C = 2|a_1 - a_2|$ obviously we have

$$|a_1 - a_2| \leq \frac{C}{2}.$$

We assume that (1) holds for n and we prove that

$$|a_{n+1} - a_{n+2}| \leq \frac{1}{2}|a_n - a_{n+1}| \leq \frac{1}{2}\frac{C}{2^n} = \frac{C}{2^{n+1}}.$$

Proof

For natural numbers $n > m$ by triangle inequality we have

$$|a_m - a_n| \leq \sum_{k=m}^{n-1} |a_k - a_{k+1}| \leq \sum_{k=m}^{n-1} \frac{C}{2^k} = \frac{C}{2^m} \sum_{k=0}^{n-m-1} \frac{1}{2^k}$$
$$= \frac{C}{2^m}(2 - \frac{1}{2^{n-m}}) \leq \frac{C}{2^{m-1}}.$$

We fix $\varepsilon > 0$. Observe that for m large enough we obtain (2^{-m+1} is a null sequence):

$$|a_m - a_n| \leq \varepsilon.$$

Example

Let $(a_n)_{n\in\mathbb{N}}$ be a recursively defined sequence with $a_1 = 1$ and

$$a_{n+1} = \begin{cases} a_n + \frac{1}{2^n} & \text{if } n \text{ is prime,} \\ a_n - \frac{1}{2^n} & \text{if } n \text{ is not prime.} \end{cases}$$

Behaviour of the above sequence is hard to predict since it is connected to properties of prime numbers. However

$$|a_{n+2} - a_{n+1}| = \frac{1}{2^{n+1}} = \frac{1}{2}|a_{n+1} - a_n|.$$

From last slides we know that the sequence $(a_n)_{n\in\mathbb{N}}$ converges.