19. (a) Prove that if n is even, then n^2 is divisible by 4.
    (b) Prove that if n is odd, then $n^2 - 1$ is divisible by 8.

Solution:

(a) Recall that an *even number is an integer which is divisible by* 2. *An integer that is not an even number is an odd number*. (See Sheet 2, Exercise 11 for the divisibility relation)
Since n is even, then n is written as $n = 2m$ where $m \in \mathbb{Z}$. It follows that $n^2 = 4m^2$ where $m^2 \in \mathbb{N}$. This means there exists an integer $k = m^2$ such that $n^2 = 4k$. Therefore, we get $4|n^2$.

(b) Since n is odd, then n is written as $n = 2m + 1$ where $m \in \mathbb{Z}$. Then

$$n^2 - 1 = 4m^2 + 4m = 4m(m + 1).$$

Here, we discus two cases:

- if m is even, by the definition of even integer, m is divisible by 2. This is, there exists $q \in \mathbb{Z}$ such that $m = 2q$. It follows that $n^2-1 = 4m(m+1) = 8q(2q + 1)$, which is divisible by 8.
- If m is odd, then we have $m+1$ is even. Again, we get $m+1$ is divisible by 2. Then, similar to the first case, there exists $p \in \mathbb{Z}$ such that $m+1 = 2p$ and it follows that $n^2 - 1 = 4(2p - 1)(2p) = 8p(2p - 1)$. Therefore, we get $8|(n^2 - 1)$.

$\square$

20. Let A and B be non-empty subsets of $\mathbb{R}$, and let $A \subset B$. Prove that:

    (a) if $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$,

    (b) if $\inf A$ and $\inf B$ exist, then $\inf A \geq \inf B$.

    (c) Let C and D be non-empty subsets of $\mathbb{R}$, and let $x \leq y$, for all $x \in C$ and $y \in D$. Then $\sup C \leq \inf D$.

    Solution:
    In this exercise, we suppose that $\sup A, \sup B, \sup C, \inf A, \inf B, \inf D$ exist.

    (a) Since $A \subset B$, it follows that, for all $a \in A$, we have $a \in B$. By the definition of the supremum of B ($\sup B$ is an upper bound of B), we have $a \leq \sup B$, for all $a \in A$. Hence, $\sup B$ is an upper bound of A too. Again, by the definition of the supremum of A, we have $\sup A$ is the smallest upper bound among the upper bounds of A, it follows that $\sup A \leq \sup B$.

    (b) The proof for the infimum is similar. Since $A \subset B$, it follows that, for all $a \in A$, we have $a \in B$. By the definition of the infimum of B ($\inf B$ is a lower bound of B), we have $\inf B \leq a$, for all $a \in A$. Hence $\inf B$ is a lower bound of A too. By the definition of $\inf A$, we have $\inf A$ is the largest lower bound of A among the lower bounds of A. This means $\inf A \geq \inf B$.

    (c) Fix $y \in D$. Since $x \leq y$ for all $x \in C$, it follows that y is an upper bound of C. Again, by the definition of the supremum of C (which is $\sup C$ is the smallest among the upper bounds of C), this gives us $y \geq \sup C$. Now, for all $y \in D$ and $x \leq y$, we have $\sup C$ is a lower bound of D. We recall that $\inf D$ is the largest among the lower bounds of D. Then, we get $\sup C \leq \inf D$. This completes the proof.

$\square$

21. Prove by induction the following equality: for all $n \in \mathbb{N}$, we have

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \tag{1}$$

Solution:
Induction basis: For $n = 1$, we have

$$2 = \sum_{k=0}^{1} \binom{1}{k} = \binom{1}{0} + \binom{1}{1} = 2.$$

Where we use the fact that $\binom{1}{0} = \binom{1}{1} = 1$.
Induction step: We assume that

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

holds for some $n \in \mathbb{N}$, and we want to prove that it is true for $n+1$, i.e., we want to show that

$$2^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k}.$$

By using the binomial identity (Lemma 1.41), we can write the above binomial as follows:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Then, we have

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \sum_{k=0}^{n+1} \binom{n}{k} + \sum_{k=0}^{n+1} \binom{n}{k-1} \\
&= \sum_{k=0}^{n} \binom{n}{k} + \binom{n}{n+1} + \sum_{k=0}^{n+1} \binom{n}{k-1} \\
&= \sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n+1} \binom{n}{k-1}, \qquad (2)
\end{aligned}$$

where we used $\binom{n}{n+1} = 0$. Moreover, we have

$$\sum_{k=0}^{n+1} \binom{n}{k-1} = \binom{n}{-1} + \sum_{k=1}^{n+1} \binom{n}{k-1} = \sum_{k=1}^{n+1} \binom{n}{k-1},$$

where $\binom{n}{-1} = 0$.

Now we use index shift argument, recall that $\sum_{k=\ell}^{n} a_k = \sum_{k=\ell+m}^{n+m} a_{k-m}$ such that in our setting $a_k = \binom{n}{k-1}$, $\ell = 0$ and $m = 1$. Hence we get

$$\sum_{k=0}^{n+1} \binom{n}{k-1} = \sum_{k=1}^{n+1} \binom{n}{k-1} = \sum_{k=0}^{n} \binom{n}{k}. \tag{3}$$

From (2) and (3) we find that

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{n+1}{k} &= \sum_{k=0}^{n}\binom{n}{k} + \sum_{k=0}^{n+1}\binom{n}{k-1} \\ &= \sum_{k=0}^{n}\binom{n}{k} + \sum_{k=0}^{n}\binom{n}{k} = 2\sum_{k=0}^{n}\binom{n}{k} \\ &= 2\cdot 2^n = 2^{n+1},\end{aligned}$$

where we used the induction hypothesis. Therefore, we conclude that equality (1) is true for all $n \in \mathbb{N}$. $\square$

22. Let
$$A = \left\{ \frac{1}{n^2 - n - 3} \;:\; n \in \mathbb{N} \right\}.$$
Compute, if they exist, the following quantities:
$$\inf A, \quad \sup A, \quad \max A, \quad \min A.$$

Solution:
For $n \in \mathbb{N}$, we have
$$A = \left\{ -\frac{1}{3}, -1, \frac{1}{3}, \frac{1}{9}, \frac{1}{17}, \frac{1}{27}, \frac{1}{39}, \dots \right\}.$$
In other words, we have
$$\begin{aligned} A &= \left\{ -\frac{1}{3}, -1, \right\} \bigcup \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{17}, \frac{1}{27}, \frac{1}{39}, \dots \right\} \\ &= \left\{ -\frac{1}{3}, -1, \right\} \bigcup \left\{ \frac{1}{n^2 - n - 3} \;:\; n \in \{3, 4, 5, , \dots\} \right\} \\ &= A_1 \bigcup A_2 \end{aligned}$$
Indeed the sign of the elements in A_2 are characterized by $n^2 - n - 3$ which is equal to $(n - \frac{1-\sqrt{13}}{2})(n - \frac{1+\sqrt{13}}{2})$. Since $\frac{1-\sqrt{13}}{2} < 0$, then the only place where A_2 is negative is when $n \leq \lfloor \frac{1+\sqrt{13}}{2} \rfloor = 2$, that is $n = 1$ and $n = 2$ which are not part of A_2. Then we conclude that all the elements of A_2 are positive. Hence A_1 contains only negative numbers while A_2 contains only positive numbers. Then $\inf A$ and $\min A$ are the same as $\inf A_1 = -1$ and $\min A_1 = -1$ respectively. That is $\inf A = \inf A_1 = \min A_1 = \min A = -1$. Since A_2 contains only positive numbers then $\sup A = \sup A_2$ and $\max A = \max A_2$. First, we show that $P(n) = n^2 - n - 3$ is an increasing function for all $n \in \mathbb{N}$ such that $n \geq 3$. Which implies that $\frac{1}{n^2-n-3}$ is a decreasing function for all $n \in \mathbb{N}$ such that $n \geq 3$, and then it attends its maximum and supremum in the first position, that is, when $n = 3$. Indeed, to show that $P(n) = n^2 - n - 3$ is an increasing function we compute $P(n+1) - P(n)$ if its positive than $P(n)$ is an increasing function.

$$\begin{aligned} P(n+1) - P(n) &= (n+1)^2 - (n+1) - 3 - (n^2 - n - 3) \\ &= n^2 + 2n + 1 - n - 1 - 3 - n^2 + n + 3 \\ &= 2n \geq 0 \end{aligned}$$
the last inequality holds true since $n \geq 3$. Now we conclude that $P(n)$ is an increasing function, which implies that $\frac{1}{n^2-n-3}$ is a decreasing function for all $n \in \mathbb{N}$ such that $n \geq 3$. Hence $\max A = \max A_2 = \sup A_2 = \sup A = \frac{1}{P(3)} = \frac{1}{3}$.

Thus, we find that $\sup A = \max A = \frac{1}{3}$ and $\inf A = \min A = -1$. $\square$

23. Suppose that $n, k \in \mathbb{N}$.
Let B be the set of k-element subsets of $\{1, \dots, n\}$.
Let U be the set of $(k+1)$-element subsets of $\{1, \dots, n+1\}$ containing the element $n+1$.

(a) Determine the sets B and U and construct a bijection $B \longrightarrow U$ under the assumption that $n = 4$ and $k = 2$.

(b) Construct a bijection $B \longrightarrow U$ for all $n, k \in \mathbb{N}$.

(c) Express the cardinality $|U|$ in terms of n and k.

Solution:

(a) For $n = 4$ and $k = 2$, we define:
$B :=$ the set of 2-element subsets of the set $\{1, 2, 3, 4\}$,
$U :=$ the set of 3-element subsets of {1,2, 3, 4, 5} containing 5.

We can read the sets B and U in another way:

- B is the set formed out from $6 = \binom{4}{2}$ subsets, such that each subset contains two elements chosen of the set $\{1, 2, 3, 4\}$. Namely

$$B = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

- U is the set of subsets, such that each subset contains three elements chosen of $\{1, 2, 3, 4, 5\}$, where one of these elements should be number 5. This means: two elements are chosen from the set $\{1, 2, 3, 4\}$, and the third one should be number 5. Namely

$$U = \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}.$$

Notice that $|U| = 6$.

Now, we construct a function $f : B \longrightarrow U$, such that:
$f(\{1, 2\}) = \{1, 2, 5\}$, $f(\{1, 3\}) = \{1, 3, 5\}$, $f(\{1, 4\}) = \{1, 4, 5\}$, $f(\{2, 3\}) = \{2, 3, 5\}$, $f(\{2, 4\}) = \{2, 4, 5\}$, $f(\{3, 4\}) = \{3, 4, 5\}$.
From above, it is easy to see that f is bijective, and then $|B| = |U| = 6$, (Precisely what we have seen before).

(b)
- Let B be the set formed out from $\binom{n}{k}$ subsets, such that each subset contains k elements chosen of $\{1, \dots, n\}$. Namely

$$B = \{\{b_1, \dots, b_k\}; \quad b_1, \dots, b_k \in \{1, 2, \dots, n\}\}.$$

- Let U be the set of subsets, such that each subset contains $k + 1$ elements chosen of $\{1, \dots, n, n + 1\}$, where one of these elements should be the number $n + 1$. This means, k elements can be chosen from the set $\{1, 2, 3, \dots, n\}$ and the last element $k + 1$ should be the number $n + 1$. Namely,

$$U = \{\{b_1, \dots, b_k, n + 1\}; \quad b_1, \dots, b_k \in \{1, 2, \dots, n\}\}.$$

We construct the function $f : B \longrightarrow U$, such that: $f(\{b_1, \ldots, b_k\}) = \{b_1, \ldots, b_k, n+1\}$. One can check that every subset of B has an unique partner of U. It follows that f is a bijective.

(c) Since the function f is a bijective, then we have $|B| = |U| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

$\square$

24. Prove the following identity for all $r, m, n \in \mathbb{N}_0$ such that $r \leq m + n$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

Hint. Let A and B be sets such that $|A| = m$, $|B| = n$ and $|A \cup B| = m + n$. For all r-element subsets $U \subset A \cup B$ we have that $A \cap U$ is an k-element subset of A and $B \cap U$ is an $(r - k)$-element subset of B where $k = |A \cap U|$.

Solution:
This identity is called the Chu-Vandermonde identity. Its proof can be algebraic, geometrical, or Combinatorial. In this exercise, we show the combinatorial proof.

- Let A and B be sets such that $|A| = m$, $|B| = n$, and $|A \cup B| = m + n$.
- Let U be a subset consisting of r elements chosen from $A \cup B$. Namely $|U| = r$.

Then, there are $\binom{m+n}{r}$ ways to choose U from $A \cup B$.

Now, we notice that $U = (A \cap U) \cup (B \cap U)$. Hence $|U| = r = |A \cap U| + |B \cap U|$.

- Let $A \cap U$ be a subset from U consisting of k elements chosen from A, namely $|A \cap U| = k$. Here, we see that k can be $0, 1, 2, \ldots, r$.
- Let $B \cap U$ be a subset from U consisting of $r - k$ elements chosen from B, namely $|B \cap U| = r - k$. Again, we have $(r - k)$ can be $r, r-1, r-2, \ldots, 2, 1, 0$.

Then, there are $\binom{m}{k}$ ways to choose the subset $A \cap U \subset U$ of k elements from A, and $\binom{n}{r-k}$ ways to choose the subset $B \cap U \subset U$ of $(r - k)$ elements from B. Therefore, the sum over all possible values of k, of the number of subsets consisting of k elements from A and $(r - k)$ elements from B is same to the number of subsets consisting of r elements from $A \cup B$. That means:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

This completes the proof. $\square$