13. Let A and B be two sets and let $f\colon A \to B$ be an invertible function. Show that there exists a unique $g\colon B \to A$ that satisfies

$$g \circ f = \mathrm{ID}_A \quad \text{and} \quad f \circ g = \mathrm{ID}_B.$$

14. Let $h\colon \mathbb{N} \to \mathbb{R}$ be defined by

$$h(1) = 2$$
$$h(n+1) = \sqrt{3 + 3h(n)}.$$

Prove by induction that $\forall n \in \mathbb{N}\colon h(n) < 4$.

15. Prove by induction that for each $n \in \mathbb{N} \setminus \{1, 2\}$ we have $2n + 1 < 2^n$.

16. Let $\mathbb{F} = (F; \{+, \cdot\})$ be a field (cf. Definition 1.22). Prove that

(a) $\forall x \in F\colon x = x + x \Rightarrow x = 0$.

(b) $\forall x \in F\colon x \cdot 0 = 0 \cdot x = 0$.

(c) $\forall x, y \in F\colon x \cdot y = 0 \Rightarrow x = 0$ or $y = 0$.

Note that you are allowed to use (16a) to prove (16b), and that your are allowed to use (16a) and (16b) to prove (16c).

17. Let $A := \mathbb{R} \times \mathbb{R}$ and let $\star\colon A^2 \to A$ be defined as follows: For all $((x, y), (z, w)) \in A^2$

$$\star((x, y), (z, w)) = (xz - yw, xw + yz).$$

(a) Prove that $\star$ is associative and commutative;

(b) prove that for each $(x, y) \in A$ we have $\star((x, y), (1, 0)) = (x, y)$;

(c) prove that for each $(a, b) \in A \setminus \{(0, 0)\}$ there exists $(x, y) \in A$ such that

$$\star((a, b), (x, y)) = (1, 0).$$

18. Let $A \subset \mathbb{R}$ with $A \neq \emptyset$, let $k \in \mathbb{R}$, let $B := \{-a \mid a \in A\}$, and let us assume that $\sup A = k$. Prove that $\inf B = -k$.