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19) a) $n =$ even (if)

To show, n^2 is divisible by 4.

By definition, $n = 2k$ where $k \in \mathbb{N}$

Squaring both sides,

$$n^2 = (2k)^2$$

$$\text{or, } n^2 = 4k^2$$
$$= 2 \cdot 2k^2 \Rightarrow n^2 \text{ is even.}$$

b) When, $n = 2k+1$ whe $k \in \mathbb{N}$

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$\therefore n^2 - 1 = 4k^2 + 4k + 1 - 1$$
$$= 4k^2 + 4k$$
$$= 4(k^2 + k)$$

For $k = 1$, $n^2 - 1 = \underline{4 \cdot 2 = 8}$, is the least number than can be divided by 8.

Similarly, for $k = 2$, $n^2 - 1 = 24$.

~~Being the~~ $(n^2 - 1)$ being multiple of 4 and true for $k = 1$ and 2. proves it can be divided by 8 ~~who~~ for $k \in \mathbb{N}$.

Q No 20

Given

A and $B = \mathbb{R}$

$A \subset B$

(a)

If $\sup(A)$ and $\sup(B)$ exist

prove $\sup(A) \leq \sup(B)$

Since $A \subset B$, any upper bound for B is also an upper bound for A

$\therefore$ $\sup B$ is an upper Bound of A

$\sup A$ is the least upper Bound for A, Since $\sup B$ is an upper bound for A.

$$\sup A \leq \sup B$$

(b)

If $\inf A$ and $\inf B$ exists, then $\inf A \geq \inf B$

Since $A \subset B$, any upper bound for B is also an upper Bound for A.

$\therefore$ $\inf B$ is a lower bound for A

By definition $\inf A$ is greatest lower bound for A. Since $\inf B$ is a lower bound for A, we have $\inf A \geq \inf B$.

$\not{d}$ ϵ

(c)

If C and D are non empty subsets of $\mathbb{R}$
~~and $x \le y$ $\forall x$~~
Let consider that for every x in C and every y in D, $x \le y$

$$\forall x \in C \wedge y \in D \implies \sup C \le \inf D.$$

Let prove by contradiction,
Suppose $\sup C > \inf D$

Since $\sup C$ is and upper bound for C, it is greater than or equal to every element of C.

Similarly, $\inf D$ is a lower bound for D, it is less than or equal to every element of D.

Now, consider

$$\varepsilon = \frac{\sup C - \inf D}{2}$$

Since $\sup C > \inf D$, $\varepsilon > 0$.
By Defination of Supremum
$\exists x \in C$ such that $\sup C - \varepsilon < x \le \sup C$
$\exists y \in D$ such that $\inf D \le y < \inf D + \varepsilon$

This lead to contradiction because $x \le \sup C$ and $y \geq \inf D$, but $x > y$ which contradiction the assumption $x \leq y$

$\therefore$ we can conclude $\sup C \leq \inf D$.

QN22

Given
set $A = \left\{ \frac{1}{n^2 - n - 3} : n \in \mathbb{N} \right\}$.

$n \in \mathbb{N}$.

To compute, the infimimum and supremum. when n approches infinity.
as n becomes larger, the term $n^2 - n - 3$ becomes dominated by n^2

$\therefore \frac{1}{n^2 - n - 3}$ approches 0 as n approaches infinity.

Since sequence approaches 0 but never actually reaches, both the infimimum and supremum are 0.

max A and min A does not exist. Since sequence approaches zero but does not reach a specific minimum value.

Q23

Given

Suppose $n, k \in N$

$B \Rightarrow$ set of k elements subsets of $\{1, n\}$

$U \Rightarrow$ set $k+1$ element subsets. $\{1 .... n+1\}$

a

Given

$n = 4$

$k = 2$

for $B = \{\{1,2\}, \{1,3\}, \{1,4\}), \{2,3) \{2,4) \{3,4\}\}$

for $U = (k+1)$ elements subsets containing $n+1$.

$U = \{(1,2,5), (1,3,5), (1,4,5), (2,3,5), (2,4,5)\} (3,4,5)\}$

$(1,2) \rightarrow (1,2,5)$

$(1,3) \rightarrow (1,3,5)$

$(1,4) \rightarrow (1,4,5)$

$(2,3) \rightarrow (2,3,5)$

$(2,4) \rightarrow (2,4,5)$

$(3,4) \rightarrow (3,4,5)$

(b)

$f: B \to U$ $\forall$ $n, k \in N$

generalize the contruction

$B = (a, b)$

$U = (a, b, n+1)$

So the bijection is $f: B \to U$ defined by

$f(\{a, b\}) = \{a, b, n+1\}$

c

Cardinality $|U|$ in terms of n and k

$\{1, 2, 3 \dots n+1\} \Rightarrow (1, 2, 3 \dots n\} + n+1$

~~$\therefore |U| =$ the~~

$\therefore |U| = \binom{n}{k}$, the ~~biomial~~ binomial coefficient

"n choose k"

Q24

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Let A be a set of size m
Let B be a set of size n

Let AUB be union of sets A and B

$|A \cup B| = m+n$

Consider set $U \Rightarrow r$ element subsets of AUB

We want to count the number of ways to choose k element from set A and remaining r-k elements from set B

Let k be the size of the intersection of U with A. Then, the remaining r-k elements from set B.

Let k be the size of the intersection of U with A. Then remaining r-k elements of U must come from B.

Now, the total number of ways to choose k elements from set A is $\binom{m}{k}$.

Combinations of m things taken k at a time.

Total number of ways to choose r-k elements from set B is

$$\binom{n}{r-k}$$

(Combination of n things taken r-k at a time)

Since U is an r element subset, we need to consider all possible values of k from 0

$$\therefore \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

21) $\forall n \in \mathbb{N}$, we have $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Proof by induction:

$\Rightarrow$ Ind^n basis, for $n=1$ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\sum_{k=0}^{1} \binom{1}{k} = \binom{1}{0} + \binom{1}{1}$$

$$= \frac{1!}{0!(1-0)!} + \frac{1!}{1!(1-1)!}$$

$$= 1 + 1$$

$$= 2 = 2^1$$

$\Rightarrow$ Ind^n step: Assume $2^n = \sum_{k=0}^{n} \binom{n}{k}$, ~~we~~ is true,

we show $2^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k}$

$$2^{n+1} \Rightarrow 2 \cdot 2^n = 2 \cdot \sum \binom{n}{k} \quad -(i)$$

$$\text{or, } 2 \cdot 2^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)\cdot(n-k)!} + \frac{n!}{k(k-1)!\,(n-k)!}$$

$$= \frac{\cancel{n!\cdot k(k-1)!}\ \cancel{n}\ n!\cdot k + n!\cdot(n-k+1)}{k(k-1)!\,(n-k+1)\cdot(n-k)!}$$

$$= \frac{n!\,(\cancel{k}+n-k+1)}{k(k-1)!\,(n-k+1)\cdot(n-k)!}$$

$$= \frac{(n+1)\cdot n!}{k(k-1)!\,(n-k+1)\cdot(n-k)!}$$

$$= \frac{(n+1)!}{k!\cdot(n-k+1)!}$$

Now, $\sum_{k=0}^{n} 2\binom{n}{k}$

$$= \sum_{k=0}^{n} 2 \cdot \left(\binom{n}{k-1} + \binom{n}{k}\right)$$

$$= \sum_{k=0}^{n} 2\binom{n}{k-1} + \sum_{k=0}^{n} 2\binom{n}{k}$$

$$= \sum_{k=1}^{n+1} 2\binom{n}{k-1} + \sum_{k=0}^{n} 2\binom{n}{k}$$

$$= \left(\sum_{k=1}^{n} 2\binom{n}{k-1} + 2\binom{n}{0}\right) + \sum_{k=0}^{n} 2\binom{n}{k}$$

$$= 2 \cdot 1 + \sum_{k=1}^{n} 2\binom{n}{k-1} + \sum_{k=0}^{n} 2\binom{n}{k}$$

$$= 2 + \sum_{k=0}^{n} 2\binom{n}{k} = 2 + \sum_{k=1}^{n} 2\binom{n}{k-1} + \sum_{k=0}^{n} 2\binom{n}{k}$$

$$= 2 + \left(\sum_{k=0}^{n} 2 \cdot \binom{n}{k} \cdot 2\right) - 2 \cdot \left(1 + \sum_{k=0}^{n} \binom{n}{k}\right)$$

$$= 2 \cdot \left(1 + \sum_{k=0}^{n} \binom{n}{k}\right) = 2 \cdot \sum_{k=0}^{n} \binom{n}{k} \quad \square$$