M. Ullrich, J.-M. Holzinger et al.

Final exam:
Mathematics for Artificial Intelligence 1

Duration: 180 minutes
Maximum of 50 points

Rules:

- **Put your name and matriculation nr. on every sheet!**
- Use a new sheet for every exercise!
- Electronic devices are not allowed!
- Do not use pencil or a red pen.
- The whole procedure of solution is required.
- All results must be simplified as far as possible.

Good luck!

EXERCISE 1: **Induction** (4 points)

Prove by induction that

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for all $n \in \mathbb{N}$.

Solution: For $n = 1$ we have $1 = 1$. If $n \geq 1$ we get

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\ &= \frac{n^4+2n^3+n^2}{4} + (n^3+3n^2+3n+1) = \frac{n^4+6n^3+13n^2+12n+4}{4} \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2\end{aligned}$$

EXERCISE 2: **Inequalities** (4 points)

Determine the set of all real numbers $x \in \mathbb{R}$ with

$$\Big||x-2|-2\Big| < 1.$$

Solution: [Points: 1P for every case]
$L = (-1, 1) \cup (3, 5)$

EXERCISE 3: **Complex numbers** (4 points)

Write $z_1 := \dfrac{6+2\mathrm{i}}{2+4\mathrm{i}}$, $z_2 := (1+\mathrm{i})^6$ and $z_3 := \mathrm{i}^5 \cdot e^{\mathrm{i}\frac{5\pi}{2}}$ in canonical and polar form.

Solution: [Points: 1P for every number; form and overall appearance (1P)]

$$\begin{aligned}z_1 =& 1-\mathrm{i} = \sqrt{2}\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}, \\ z_2 =& (\sqrt{2}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}})^6 = 8\mathrm{e}^{\frac{6\pi}{4}\mathrm{i}} = -8, \\ z_3 =& \mathrm{i}^5 \cdot e^{\frac{5\pi}{2}\mathrm{i}} = \mathrm{i}^6 = -1 = \mathrm{e}^{\pi\mathrm{i}}.\end{aligned}$$

EXERCISE 4: **Inequalities with complex numbers** (4 points)

Determine all complex numbers $z = x + iy \in \mathbb{C}$ with

$$|\overline{z} + 2i| \leq |2 - z|$$

and make a sketch of this set in the complex plane.

Solution: $y \geq x$

EXERCISE 5: **Determinants** (3 points)

Calculate the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 24 \end{pmatrix}.$$

Solution: $\det(A) = 5$

Exercise 6: **Linear systems** (5 points)

For $t \in \mathbb{R}$ consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \\ 1 & 2 & t \end{pmatrix}.$$

For which $t \in \mathbb{R}$ has the linear system $Ax = b$ with $b = (2, 0, 2)^T$ a solution $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$? Determine the set of solutions $L(A, b)$.

Solution: [Points: uniquely solvable for $t \neq 3$ (1p); not or not uniquely solvable for $t = 3$ (1p); solution for $t \neq 3$ (1p); solution for $t = 3$ (2p)]
By Gaussian elimination we obtain that the system is equivalent to the system

$$\begin{aligned} x_1 - x_3 &= 4 \\ 2x_2 + 4x_3 &= -2 \\ (t-3)x_3 &= 0. \end{aligned}$$

Hence, for $t \neq 3$ the solution is $x = (4, -1, 0)^T$, and hence $L(A, b) = \{(4, -1, 0)^T\}$.
For $t = 3$, we have $L(A, b) = \{(s + 4, -1 - 2s, s) \colon s \in \mathbb{R}\}$.

Exercise 7: **Sequences** (3+3 points)

Verify if the limits of the following sequences exist, determine the limits, if possible, and otherwise compute the lim inf and lim sup of

a) $a_n := \sqrt[n]{n \cdot 2^{3n} + 3^{2n}}$.

b) $c_n := \sqrt[n]{1 - (-1)^n}$.

Solution: [Points per task: argument (1P); correct calc. (1P); form (1P)]
a) 9
b) n.e., $\liminf(c_n) = 0$, $\limsup(c_n) = 1$

Exercise 8: **Series I** (3+3 points)

Are the following series convergent or divergent?

$$\text{a) } \sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}) \qquad \text{b) } \sum_{n=1}^{\infty} \binom{n+2}{3} n^{-4}$$

Solution: [Points per task: argument (1P); correct calc. (1P); form (1P)]
a) We have

$$\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

which is monotone decreasing null sequence. Therefore, by the Leibnitz criterion the series $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$ is convergent.
b) divergent (comparison to harmonic series)

Exercise 9: **Series II** (4 points)

For which $x \in \mathbb{R}$ does the series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} \left(\frac{10 - x^2}{6} \right)^k$$

converge?

Solution: [non-boundary points (2P); boundary points (1P); form (1P)]
We have

$$\limsup_{k \to \infty} \sqrt[k]{\frac{k}{k+1} \left| \frac{|10 - x^2|}{6} \right|^k} = \frac{|10 - x^2|}{6}.$$

(Alternatively: $\sum_{k=1}^{\infty} \frac{k}{k+1} y^k$ has radius of convergence 1.)

Since $|10 - x^2| < 6$ for $x \in (-4, -2) \cup (2, 4)$, we have convergence for those x.
For $|x| \in \{2, 4\}$ the sum is divergent since the sequences

$$\left(\frac{k}{k+1} \right)_{k \in \mathbb{N}} \quad \text{and} \quad \left((-1)^k \frac{k}{k+1} \right)_{\in \mathbb{N}}$$

do not tend to 0.

Exercise 10: (10 points)

Write your answers into the corresponding fields of the tabular. All variables are assumed to be real numbers. Every wrong answer will deduct two points. You do not have to hand in your calculations.

1. a)	1. b)	2. a) $b =$	2. b) $x =$
3. a) $x \in$	3. b) $x \in$	4. a) $x \in$	4. b) $x \in$
5. a) $x =$	5. b) $x =$		

1. Compute the following sums.

$$\text{a)} \sum_{t=0}^{2}(t + 2^{3t}) \qquad \text{b)} \sum_{n=1}^{4}\sum_{j=1}^{n} j$$

   Solution:

   a) 1 + (1+8)+ (2+64) = 76

   b) 1+3+6+10 = 20

2. Solve for the given variable. The expression should contain no brackets and at most a single fraction.

$$\text{a)}\quad W = \tfrac{7}{b-k} \quad \text{for } b \qquad \text{b)}\quad 2b = \tfrac{b}{x} - \tfrac{1}{b} \quad \text{for } x$$

   What do the other parameters need to fulfill for your calculation?

   Solution:

   a) $b = \frac{7}{W} + k \quad (k \neq b, W \neq 0)$

   b) $x = \frac{b}{2b^2+1} \quad (b \neq 0)$

3. Solve the following equations.

$$\text{a)}\quad x^2 - x + 3 = 6 + x \qquad \text{b)}\quad \frac{x}{x-2} - \frac{x}{x+2} = \frac{28}{x^2-4}$$

   Solution: a) $x \in \{-1, 3\}$; b) $x = 7$

4. Determine the set of solutions of the following inequalities.

$$\text{a)}\quad (x+2)(x-42) > 0 \qquad \text{b)}\quad \tfrac{2x}{x-2} \geq 3$$

   Solution: a) $x \in (-2, 42)$; b) $x \in (2, 6]$

5. Solve the following equations.

$$\text{a)}\quad \log_2(1 + 7x) = 3 \qquad \text{b)}\quad \mathrm{e}^{2(x-1)} = \tfrac{\mathrm{e}^{x+1}}{\mathrm{e}^2}$$

   Solution: a) $x = 1$; b) $x = 1$