19. (a) Prove that if n is even, then n^2 is divisible by 4.
    (b) Prove that if n is odd, then $n^2 - 1$ is divisible by 8.

20. Let A and B be non-empty subsets of $\mathbb{R}$, and let $A \subset B$. Prove that:
    (a) if $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$,
    (b) if $\inf A$ and $\inf B$ exist, then $\inf A \geq \inf B$.
    (c) Let C and D be non-empty subsets of $\mathbb{R}$, and let $x \leq y$, for all $x \in C$ and $y \in D$. Then $\sup C \leq \inf D$.

21. Prove by induction the following equality: for all $n \in \mathbb{N}$, we have

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

22. Let

$$A = \left\{ \frac{1}{n^2 - n - 3} \;:\; n \in \mathbb{N} \right\}.$$

Compute, if they exist, the following quantities:

$$\inf A, \quad \sup A, \quad \max A, \quad \min A.$$

23. Suppose that $n, k \in \mathbb{N}$.
Let B be the set of k-element subsets of $\{1, \ldots, n\}$.
Let U be the set of $(k+1)$-element subsets of $\{1, \ldots, n+1\}$ containing the element $n+1$.
    (a) Determine the sets B and U and construct a bijection $B \longrightarrow U$ under the assumption that $n = 4$ and $k = 2$.
    (b) Construct a bijection $B \longrightarrow U$ for all $n, k \in \mathbb{N}$.
    (c) Express the cardinality $|U|$ in terms of n and k.

24. Prove the following identity for all $r, m, n \in \mathbb{N}_0$ such that $r \leq m + n$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

Hint. Let A and B be sets such that $|A| = m$, $|B| = n$ and $|A \cup B| = m + n$. For all r-element subsets $U \subset A \cup B$ we have that $A \cap U$ is an k-element subset of A and $B \cap U$ is an $(r-k)$-element subset of B where $k = |A \cap U|$.