37. Let $M := \mathbb{R}^{m\times n}$ and suppose that $R \subset M \times M$ is the relation such that $(A, B) \in R$ if and only if the matrices A and B are related by a sequence of row operations. Show that R is an equivalence relation.

38. Determine the solution set $L(A, \mathbf{b})$ where,

$$A = \begin{pmatrix} 2 & 4 & 7 & 2 & -1 & -4 \\ -1 & 9 & 9 & -3 & -6 & -3 \\ -1 & 2 & 8 & -2 & 7 & 1 \\ 3 & -6 & 4 & 4 & 9 & 1 \\ -3 & 4 & -8 & -6 & -5 & -3 \\ -3 & -5 & 0 & -3 & 8 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -6 \\ -3 \\ 1 \\ 3 \\ -1 \\ -3 \end{pmatrix}.$$

39. (a) Show that elementary matrices $E \in \mathbb{R}^{n\times n}$ are invertible and that the inverse E^{-1} is also an elementary matrix.

    (b) For each of the following matrices, determine its inverse and clearly describe the row operation used to obtain it, starting from the identity matrix:

$$E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

40. In this exercise, you are required to prove, through a two-part process, that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that $T = T_A$.

    (a) First, prove the direct statement: if such a matrix A exists, then T is a linear transformation.

    (b) Then, prove the converse statement: if T is a linear transformation, then such a matrix A exists.

    Hint: For the second part, it may be helpful to use the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.

41. Compute the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 6 & 0 \\ 4 & 11 & 8 \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & \cdots & 0 & d_n \\ 0 & 0 & \cdots & d_{n-1} & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & d_2 & \cdots & 0 & 0 \\ d_1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

    Notice that C is a 4×4 matrix, with C_1 and C_2 being 2×2 matrices with known determinants, $\det C_1$ and $\det C_2$, and O is the zero 2×2 matrix. Also, D is a $n \times n$ matrix where $d_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$.

42. Compute the inverse of

    (a) $A = \begin{pmatrix} -9 & 7 & 3 \\ -13 & 9 & 4 \\ -3 & 2 & 1 \end{pmatrix}$

    (b) $\mathrm{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and $d \in \mathbb{R}$. Determine the condition(s) that a, b, c and d must satisfy for matrix B to be invertible.