JOHANNES KEPLER
UNIVERSITY LINZ

Mathematics for AI 1

3. Sequences and Series

JOHANNES KEPLER
UNIVERSITY LINZ
Altenbergerstraße 69
4040 Linz, Austria
jku.at

3.1. Convergence of a sequences

Definition of a sequence

Definition

- Let $M \neq \emptyset$ be an arbitrary set.
- Let $I \subset \mathbb{Z}$ be an infinite set.

A sequence in M is a function $a : I \to M$. Usually we use the following notation $a_n := a(n)$. We write the sequence as $(a_n)_{n \in I}$.

- Range of a sequence $(a_n)_{n \in I}$ is the set $\{a_n : n \in I\}$.
- We call the set I the index set of the sequence.

In the special cases $M = \mathbb{R}$ or $M = \mathbb{C}$ we say that $(a_n)_{n \in I}$ is a **real-valued** or **complex-valued** sequence, respectively.

We can define a sequence in multiple ways

- Explicit formula:

$$a_n = 2^n, \quad b_n = 1 + \frac{\sin(n)}{n}, \quad c_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We can define a sequence in multiple ways

- Explicit formula:

$$a_n = 2^n, \quad b_n = 1 + \frac{\sin(n)}{n}, \quad c_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

- Recursion. We give starting value(s) of the sequence and a rule for how to calculate a new term using previous terms.

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad n \in \mathbb{N} \text{ and } n \geq 2.$$

  This is the Lucas sequence. See also Fibonacci numbers in lecture notes for a similar example.

Exercise

Show that $c_n = L_n$ for $n \in \mathbb{N}$.

We can define a sequence in multiple ways

- Explicit formula:

$$a_n = 2^n, \quad b_n = 1 + \frac{\sin(n)}{n}, \quad c_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

- Recursion. We give starting value(s) of the sequence and a rule for how to calculate a new term using previous terms.

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad n \in \mathbb{N} \text{ and } n \geq 2.$$

  This is the Lucas sequence. See also Fibonacci numbers in lecture notes for a similar example.

Exercise

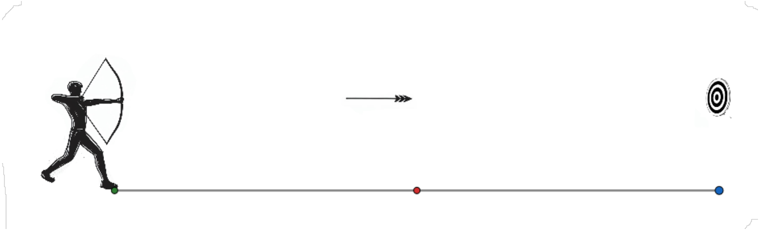
Show that $c_n = L_n$ for $n \in \mathbb{N}$.

- Description. Value of d_n is the largest number of regions into which n straight lines can divide the plane.

Zeno's Paradox

"That which is in locomotion must arrive at the half-way stage before it arrives at the goal" *Aristotle, Physics VI:9, 239b10*

Zeno's Paradox



Where is the paradox?

- Achilles shoots an arrow at a d meter distant target.
- Arrow flies through air in the direction of the target. At a certain moment in time (t_1 seconds) arrow passes half of the distance between Achilles and the target. We put $d_1 = \frac{d}{2}$.

Zeno's Paradox

Where is the paradox?

- Let $d_2 = \frac{d_1}{2} = \frac{d}{4}$. After t_2 seconds arrow is exactly at the distance d_2 to the target.

Zeno's Paradox

Where is the paradox?

- Finally we put $d_n = \frac{d_{n-1}}{2} = 2^{-n}d$. After t_n seconds arrow is exactly at the distance d_n to the target.

Zeno's Paradox

Where is the paradox?

- Ancient Greek's saw a paradox here. Before arrow hits the target infinitely many steps have to occur. Hence, the arrow can not reach its target. In real life we know that if we shoot arrow with enough force, it will reach the target, let say at the time T. For large n, the value of t_n is getting closer to T and the distance to the target is close to zero. In mathematics we deal with this phenomena using the limit of a sequence. There are infinitely many steps but

$$\lim_{n\to\infty} d_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} t_n = T.$$

How do we define the limit of a sequence?

Definition (Convergent sequence)

Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued sequence and $a \in \mathbb{C}$. We say that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a if and only if

$$\forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} :\ n \geq n_0 \implies |a_n - a| < \varepsilon.$$

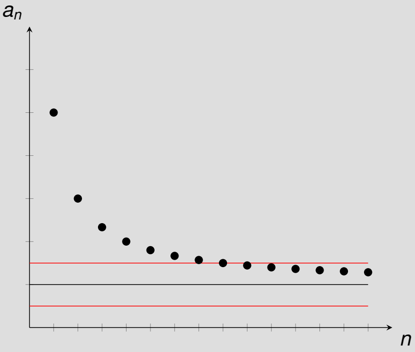
We call a the limit of the sequence and write

$$a = \lim_{n\to\infty} a_n.$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent if there exists some $a \in \mathbb{C}$ such that $a_n \to a$. Otherwise, the sequence $(a_n)_{n\in\mathbb{N}}$ is called divergent.

Visualisation

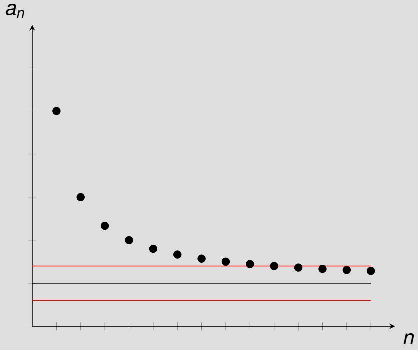
$$a_n = 1 + \frac{4}{n}, \quad \varepsilon = 0.5, \quad \lim_{n\to\infty} a_n = 1.$$



Visualisation

$$a_n = 1 + \frac{4}{n}, \quad \varepsilon = 0.4, \quad \lim_{n \to \infty} a_n = 1.$$

No matter how small $\varepsilon > 0$ we choose there are only finitely many $n \in \mathbb{N}$ such that $a_n \notin (1 - \varepsilon, 1 + \varepsilon)$.



Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued sequence such that $\lim\limits_{n\to\infty} a_n = 0$. Then we call $(a_n)_{n\in\mathbb{N}}$ a null sequence.

Lemma

We fix $\alpha < 0$. For $n \in \mathbb{N}$ we define $b_n = n^\alpha$. Sequence b_n is a null-sequence.

Proof.

Fix arbitrary $\varepsilon > 0$. By Archimedean property there exists $N \in \mathbb{N}$ such that $N > \varepsilon^{\frac{1}{\alpha}}$. Function $f(x) = x^\alpha$ is a decreasing function. Therefore for $n > N$ we have

$$|b_n| = |n^\alpha| \leq |N^\alpha| \leq |\varepsilon^{\frac{1}{\alpha}}|^\alpha = \varepsilon^{\frac{\alpha}{\alpha}} = \varepsilon.$$

Since ε was arbitrary by the definition $\lim\limits_{n\to\infty} a_n = 0$.

Proposition

Let $\lambda \in \mathbb{C}$. If $(a_n)_{n\in\mathbb{N}}$ is a null sequence then $(\lambda a_n)_{n\to\mathbb{N}}$ is a null sequence.

Proof. We assume $\lambda \neq 0$, the case $\lambda = 0$ is trivial. We fix $\varepsilon > 0$. Since a_n is a null sequence there exists $N(\varepsilon) \in \mathbb{N}$ such that for $n > N(\varepsilon)$ we have $|a_n| \leq \varepsilon|\lambda|^{-1}$. Therefore for $n > N(\varepsilon)$ we have

$$|b_n| = |\lambda||a_n| \leq |\lambda|\frac{\varepsilon}{|\lambda|} = \varepsilon. \qquad \square$$

Remark

Let $a \in \mathbb{C}$. Note that $\lim\limits_{n\to\infty} a_n = a \iff b_n = a - a_n$ is a null seq.

Example

Let $a_n = 1 + \frac{4}{n}$. Since $b_n = a_n - 1 = \frac{4}{n}$ is a null sequence $\lim\limits_{n\to\infty} a_n = 1$

Example - divergent sequence

Sequence $\xi_n = (-1)^n$ is divergent. We assume opposite $\lim_{n\to\infty} \xi_n = a$. There exists $N \in \mathbb{N}$ such that for $n > N$ we have $|a_n - a| \leq \frac{1}{2}$. Therefore for $n > N$ we have

$$|1 - a| = |a_{2n} - a| \leq \frac{1}{2}.$$

By triangle inequality

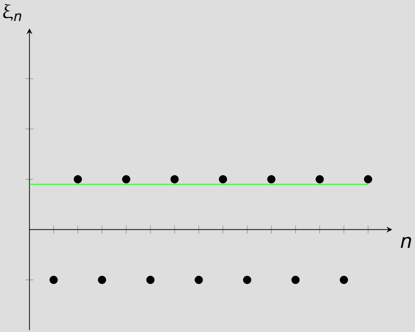
$$2 = |a_{2n+1} - a_{2n}| = |a_{2n+1} - a + a - a_{2n}| \leq |a_{2n+1} - a| + |a_{2n} - a|.$$

Thus

$$\frac{1}{2} \geq |a_{2n+1} - a| \geq 2 - |a_{2n} - a| \geq 2 - \frac{1}{2} \geq \frac{3}{2}. \quad \text{(Contradiction)}$$

Visualisation

In graph the hypothetical value of a is denoted by the green line and $\xi_n = (-1)^n$. We summarise last slide in simple words: if a is close to 1, then it is far from (-1).



Example

Let $a_n = \frac{n^2-1}{n^2+1}$, $n \in \mathbb{N}$. Proof by definition that $\lim\limits_{n\to\infty} a_n = 1$.

Proof. We fix $\varepsilon > 0$. We want to prove there is $N \in \mathbb{N}$ such that for $n > N$ we have

$$|1 - a_n| \leq \varepsilon.$$

We calculate

$$\left|1 - \frac{n^2-1}{n^2+1}\right| = \frac{2}{n^2+1} \leq \frac{2}{n^2}.$$

It is enough to have

$$\frac{2}{n^2} \leq \varepsilon.$$

It is equivalent to

$$n \geq \sqrt{\frac{2}{\varepsilon}}.$$

Example

Let $a_n = \frac{n^2-1}{n^2+1}$, $n \in \mathbb{N}$. Proof by definition that $\lim_{n\to\infty} a_n = 1$.

Proof(part II). By Archimedean property there exists $N \in \mathbb{N}$ such that

$$N \geq \sqrt{\frac{2}{\varepsilon}}.$$

For every $n > N$ we have

$$|1 - a_n| \leq \varepsilon.$$

$\square$

Exercise

Prove by the definition that the sequence $(b_n)_{n\in\mathbb{N}}$ given by the formula

$$b_n = \frac{n}{2^{2n} + 9}.$$

is a null sequence. **Hint.** Use Exercise 15 and $2^{2n} = 2^n \cdot 2^n$.

Exercise

Let $k \in \mathbb{N}$. We define a sequence $(a_n)_{n\in\mathbb{N}}$ by the following formula

$$a_n = \begin{cases} (-2)^n & \text{for } n \in \mathbb{N},\ n < k, \\ 2 + \frac{1}{n^3} & \text{for } n \in \mathbb{N},\ n \geq k. \end{cases}$$

Does the sequence (a_n) converge or diverge? If it converges determine the limit?

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a complex valued sequence. We call the sequence bounded (by C) if and only if

$$\exists\, C > 0 \;: \forall\, n \in \mathbb{N}, |a_n| \leq C.$$

A real-valued sequence $(a_n)_{n\in\mathbb{N}}$ is bounded from above if and only if

$$\exists\, C \in \mathbb{R} : \forall\, n \in \mathbb{N}, a_n \leq C,$$

and bounded from below if and only if

$$\exists\, C \in \mathbb{R} : \forall\, n \in \mathbb{N}, a_n \geq C.$$

Remark

This notion is connected to the boundedness of a set. Definitions above could be restated in terms of the range of a sequence.

Example

Sequence $a_n = (-1)^n, n \in \mathbb{N}$ is bounded by 1.

Example

Fix $\theta \in [0, 2\pi]$. Sequence $(e^{\theta ni})_{n\in\mathbb{N}}$ is bounded by 1.

Example

Sequence $(2^n)_{n\in\mathbb{N}}$ is unbounded but it is bounded from below by 0.

Example

If $(d_n)_{n\in\mathbb{N}}$ is bounded by $D > 0$ and $(f_n)_{n\in\mathbb{N}}$ is bounded by $F > 0$ then $d_n + f_n$, $d_n \cdot f_n$ are bounded by $D + F$ and $D \cdot F$ respectively.

Theorem

If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $(a_n)_{n\in\mathbb{N}}$ is bounded.

Proof.

Let $a = \lim_{n\to\infty} a_n$. We choose your favorite ε. In my case $\varepsilon = 42$. By the definition of convergence, there exists N such that, for all $n \geq N$,

$$|a - a_n| < 42.$$

It follows from the triangle inequality that

$$|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| < 42 + |a|$$

holds for all $n \geq N$. Therefore for all $n \in \mathbb{N}$,

$$|a_n| \leq \max\{42 + |a|, |a_1|, |a_2|, \ldots, |a_{N-1}|\}. \qquad \square$$

Example

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = \frac{n}{\log_2(n)}$. This sequence is unbounded (check a_{2^n}). From the last slide we know that this sequence is divergent.

Exercise

Check if the following sequences are bounded and compute the limit if possible.

$$(a) \quad a_n = 1 - \frac{(-2)^{n+7}}{2^{n+5}} \qquad (d) \quad d_n = \frac{7n^4 + 4n + 3n}{5n^4 + 5}$$

$$(b) \quad b_n = \sin(n)\frac{(n-5)}{\sqrt[3]{n^4+1}}$$

$$(c) \quad c_n = \frac{n^3+5}{(1+2i)^n}$$

Exercise

Let $\lim_{n\to\infty} a_n = a$. We define $b_n = \frac{1}{n}\sum_{j=1}^{n} a_j$. Prove that

$$\lim_{n\to\infty} b_n = a$$

Hint. Use the boundedness of the sequence (a_n).

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a real-valued sequence. The sequence $(a_n)_{n\in\mathbb{N}}$ tends to $+\infty$ if and only if

$$\forall\, C \in \mathbb{R}, \exists\, n_0 \in \mathbb{N} : \forall\, n \geq n_0, a_n \geq C.$$

In this case, we write $\lim\limits_{n\to\infty} a_n = \infty$ and call ∞ the improper limit of $(a_n)_{n\in\mathbb{N}}$.

The sequence $(a_n)_{n\in\mathbb{N}}$ tends to $-\infty$ if and only if

$$\forall\, C \in \mathbb{R}, \exists\, n_0 \in \mathbb{N} : \forall\, n \geq n_0, a_n \leq C.$$

In this case, we write $\lim\limits_{n\to\infty} a_n = -\infty$ or and call $-\infty$ the improper limit of $(a_n)_{n\in\mathbb{N}}$.

If the sequence $(a_n)_{n\in\mathbb{N}}$ tends to ∞ or $-\infty$, it is called definitely divergent.

Example

The function $\ln(x)$ is an increasing function and therefore for $n > 3^k$ we have

$$\ln(n) > \ln(3^k) = k\ln(3).$$

For any $C \in \mathbb{R}$ there exists $k(C) > \frac{C}{\ln(3)}$ (Archimedean property). Thus for $n > 3^{k(C)}$

$$\ln(n) > C$$

We have proven $\lim\limits_{n\to\infty} \ln(n) = +\infty$.

Exercise

Let $\lim_{n\to\infty} a_n = +\infty$ and $a_n \neq 0$ for $n \in \mathbb{N}$. We put $b_n = -a_n$, $c_n = \frac{1}{a_n}$. Prove that

$$\lim_{n\to\infty} b_n = -\infty, \qquad \lim_{n\to\infty} c_n = 0.$$

Exercise

Show that $\lim_{n\to\infty} \frac{n!}{2^n} = +\infty$. **Hint.** Show that $\frac{n!}{2^n} \geq \frac{n}{4}$.

3.2. Calculation rules for limits

Theorem (Algebraic Properties of Limits)

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent complex-valued sequences and let $\lambda \in \mathbb{C}$. Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then, we have

(i) $\lim_{n\to\infty}(a_n + b_n) = a + b$,

(ii) $\lim_{n\to\infty}(\lambda \cdot a_n) = \lambda \cdot a$,

(iii) $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$,

(iv) *if $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Proof of i). $\lim_{n\to\infty}(a_n + b_n) = a + b$,
Let $\varepsilon > 0$ be arbitrary. By the definition of convergence, there exist $M, N \in \mathbb{N}$ such that

$$n \geq M \implies |a_n - a| < \frac{\varepsilon}{2} \quad \text{and} \quad n \geq N \implies |b_n - b| < \frac{\varepsilon}{2}.$$

In particular, for all $n \geq \max\{M, N\}$, we have $|a_n - a|, |b_n - b| < \frac{\varepsilon}{2}$.
Then, by the triangle inequality

$$|(a_n+b_n)-(a+b)| = |(a_n - a)+(b_n - b)| \leq |a_n - a|+|b_n - b| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

holds for all $n \geq \max\{M, N\}$.
For proofs of ii),iv) see lecture notes

Proof of iii). $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$. We assume $a \neq 0$, case $a = 0$ is left as an exercise.

Since (b_n) is a convergent sequence, it is a bounded sequence i.e. there is some $C > 0$ such that $|b_n| \leq C$ holds for all $n \in \mathbb{N}$.

Since (a_n) and (b_n) are convergent there exists $N \in \mathbb{N}$ (see proof of i)) such that

$$n \geq N \implies |a_n - a| < \frac{\varepsilon}{2C} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{2|a|}. \tag{1}$$

Then, by the triangle inequality,

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab|.$$

By boundedness of the sequence (b_n) and (1)

$$|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b| < C|a_n - a| + |a||b_n - b|.$$

Thus

$$|a_n b_n - ab| < C\frac{\varepsilon}{2C} + |a|\frac{\varepsilon}{2|a|} = \varepsilon.$$

Remark

Fix $k \in \mathbb{N}$. Let $(a_{s,n})_{n\in\mathbb{N}}$ be complex valued, convergent sequences for $s \in \{1, ..., k\}$. The following identities hold

$$\lim_{n\to\infty} \sum_{s=1}^{k} a_{s,n} = \sum_{s=1}^{k} \lim_{n\to\infty} a_{s,n}$$

and

$$\lim_{n\to\infty} \prod_{s=1}^{k} a_{s,n} = \prod_{s=1}^{k} \lim_{n\to\infty} a_{s,n}$$

Proof. By induction, left as an exercise.

Example

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = \frac{10n^3 - 42n + 57}{4n^3 + 6n^2 - 13n + n^{5/3}}.$$

We will use the rules of calculation we have proven to calculate this limit. First we factor out the highest power of n from the nominator and the denominator, in both cases it is n^3.

$$a_n = \frac{n^3(10 - \frac{42}{n^2} + \frac{57}{n^3})}{n^3(4 + \frac{6}{n} - \frac{13}{n^2} + \frac{1}{n^{4/3}})} = \frac{10 - \frac{42}{n^2} + \frac{57}{n^3}}{4 + \frac{6}{n} - \frac{13}{n^2} + \frac{1}{n^{4/3}}}.$$

We put

$$b_n = 10 - \frac{42}{n^2} + \frac{57}{n^3}, \quad c_n = 4 + \frac{6}{n} - \frac{13}{n^2} + \frac{1}{n^{4/3}}.$$

Example

By that i) and ii) from the Theorem on the slide 29 we have

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty} 10 - \frac{42}{n^2} + \frac{57}{n^3} \\ &= \lim_{n\to\infty} 10 - \lim_{n\to\infty} \frac{42}{n^2} + \lim_{n\to\infty} \frac{57}{n^3} \\ &= 10 - 0 + 0 = 10,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} c_n &= \lim_{n\to\infty} 4 + \frac{6}{n} - \frac{13}{n^2} + \frac{1}{n^{4/3}} \\ &= \lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{6}{n} - \lim_{n\to\infty} \frac{13}{n^2} + \lim_{n\to\infty} \frac{1}{n^{4/3}} \\ &= 4 + 0 - 0 + 0 = 4.\end{aligned}$$

Example-part II

Using from the same Theorem as before but point (iv) gives us

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{b_n}{c_n} = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} c_n} = \frac{10}{4}.$$

Bernoulli's Inequality

For $x \geq 0$ and $n \in \mathbb{N}$ we have $(1 + x)^n \geq 1 + nx$. *Proof.* Since $x \geq 0$ we have

$$(1 + x)^n = 1 + \binom{n}{1} x + \sum_{j=2}^{n} \binom{n}{j} x^j \geq 1 + nx.$$

*Remark.*Using induction we can prove the above inequality for $x > -1$.

Lemma

Let $z \in \mathbb{C}$ with $|z| < 1$. Then

$$\lim_{n \to \infty} z^n = 0.$$

Proof.

Let

$$x = \frac{1}{|z|} - 1 > 0.$$

Observe that x is a positive real number. Bernoulli's Inequality implies that $(1+x)^n \geq 1 + nx$ holds for all $n \in \mathbb{N}$. Therefore,

$$|z^n| = |z|^n = \left(\frac{1}{1+x}\right)^n = \frac{1}{(1+x)^n} \leq \frac{1}{1+nx} < \frac{1}{nx}.$$

Fix $\varepsilon > 0$. By Archimedean property there exists $N \in \mathbb{N}$ such that $N > \frac{1}{x\varepsilon}$. For $n > N$ we have

$$|z^n| < \frac{1}{nx} < \frac{1}{x} x\varepsilon = \varepsilon.$$

Since ε was arbitrary we have $\lim\limits_{n\to\infty} z^n = 0$ □

Example - Zeno's Paradox

For simplicity we assume that the arrow travels at a constant speed v. Then the distance travelled at the time t is expressed by the formula

$$S(t) = v \cdot t.$$

Recall that at the time t_n we were at the distance

$$d_n = 2^{-n} d$$

from the target. Hence

$$d - d_n = S(t_n) = v \cdot t_n.$$

Morever

$$d = S(T) = v \cdot T,$$

where at the time T we hit the target.

Example - Zeno's Paradox II

Hence

$$t_n = \frac{d - d_n}{v}.$$

Thus

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{d - d_n}{v} = \frac{d}{v} \lim_{n\to\infty} (1 - 2^{-n}) = \frac{d}{v} = T$$

and

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} 2^{-n} d = 0.$$

Exercise

Let $z \in \mathbb{C}$ with $|z| > 1$. Show that the sequence $(z^n)_{n\in\mathbb{N}}$ is divergent. For what $z \in \mathbb{C}$ with $|z| = 1$ is the sequence $(z^n)_{n\in\mathbb{N}}$ convergent?

Lemma

Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued null sequence and let $c, C > 0$ be arbitrary positive real numbers. Suppose that the sequence $(b_n)_{n\in\mathbb{N}}$ satisfies

$$|b_n| \leq C|a_n|^c$$

for all but finitely many $n \in \mathbb{N}$. Then $(b_n)_{n\in\mathbb{N}}$ is a null sequence.

Proof.

This is left as an exercise. $\square$

Example

We will use last slide's Lemma to show $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Observe that for $0 < k \leq \frac{n}{2}$

$$\frac{k}{n} \leq \frac{n}{2n} = \frac{1}{2}$$

and for $\frac{n}{2} < k < n$

$$\frac{k}{n} \leq 1.$$

Since there are more than $\frac{n-1}{2}$ elements in $\{k : 0 < k \leq \frac{n}{2}\}$ we get

$$\frac{n!}{n^n} = \prod_{k=1}^{n} \frac{k}{n} \leq \prod_{0<k\leq\frac{n}{2}} \frac{k}{n} \prod_{\frac{n}{2}<k<n} \frac{k}{n} \leq \prod_{0<k\leq\frac{n}{2}} \frac{k}{n} \leq \left(\frac{1}{2}\right)^{\frac{n-1}{2}} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{n}$$

The sequence $(2^{-n/2})$ is a null sequence. By Lemma from the last slide $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Lemma

Let $z \in \mathbb{C}$, $|z| > 1$ and $k \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{n^k}{z^n} = 0$.

Proof- Case k=1.

Since $|z| > 1$ then $|z|^{\frac{1}{2}} > 1$. We put $x = |z|^{\frac{1}{2}} - 1$. By Bernoulli's Inequality

$$\left|\frac{n}{z^n}\right| = \frac{n}{|z|^n} = \frac{n}{|z|^{\frac{n}{2}}}\frac{1}{|z|^{\frac{n}{2}}} = \frac{n}{(1+x)^n}\frac{1}{|z|^{\frac{n}{2}}} \leq \frac{n}{1+nx}\frac{1}{|z|^{\frac{n}{2}}} \leq \frac{1}{x}\frac{1}{|z|^{\frac{n}{2}}}$$

Since $\frac{1}{|z|^{\frac{n}{2}}}$ is a null sequence then $\lim_{n\to\infty} \frac{n}{|z|^n} = 0$.

Proof- Case k>1.

Observe that

$$\left|\frac{n^k}{z^n}\right| \leq \frac{n^k}{|z|^n}.$$

It is enough to show $\frac{n^k}{|z|^n}$ is a null sequence. We put $y = |z|^{\frac{1}{k}} > 1$. We use Remark on the slide 32 and the case "$k = 1$" on the last slide

$$\lim_{n\to\infty} \frac{n^k}{|z|^n} = \lim_{n\to\infty} \prod_{j=1}^{k} \frac{n}{|z|^{\frac{n}{k}}} = \prod_{j=1}^{k} \lim_{n\to\infty} \frac{n}{|z|^{\frac{n}{k}}} = \left(\lim_{n\to\infty} \frac{n}{y^n}\right)^k = 0.$$

Lemma

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Proof.

Define $x_n = \sqrt[2n]{n} - 1 > 0$ for all $n \in \mathbb{N}$. We will show that $(x_n)_{n\in\mathbb{N}}$ is a null sequence. Using algebraic properties of limits we show that

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (x_n + 1)^2 = (\lim_{n\to\infty} (x_n + 1))^2 = (0 + 1)^2 = 1.$$

To show that $(x_n)_{n\in\mathbb{N}}$ is a null sequence, we use Bernoulli's inequality.

$$n = (1 + x_n)^{2n} = ((1 + x_n)^n)^2 \geq (1 + nx_n)^2.$$

Proof.

A rearrangement of this inequality gives

$$0 < x_n \leq \frac{\sqrt{n}-1}{n}.$$

We calculate that $\lim_{n\to\infty} \frac{\sqrt{n}-1}{n} = 0$. By Lemma from the slide 41 (x_n) is a null sequence. $\square$

Remark

For any $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \sqrt[n]{n^k} = \prod_{j=1}^{k} \lim_{n\to\infty} \sqrt[n]{n} = 1^k = 1$$

Theorem (Sandwich Rule)

Let $(a_n)_{n\in\mathbb{N}}$ *and* $(c_n)_{n\in\mathbb{N}}$ *be convergent real-valued sequences. Suppose that* $(b_n)_{n\in\mathbb{N}}$ *is a real-valued sequence and that there exists* $N \in \mathbb{N}$ *such that*

$$a_n \leq b_n \leq c_n \quad \forall n \geq N.$$

Suppose also that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then $(b_n)_{n\in\mathbb{N}}$ *is convergent and*

$$\lim_{n\to\infty} b_n = L.$$

Proof.

Observe that

$$a_n - L \leq b_n - L \leq c_n - L \quad \forall n \geq N.$$

Hence

$$|b_n - L| \leq \max\{|a_n - L|, |c_n - L|\} \leq |a_n - L| + |c_n - L| \quad \forall n \geq N.$$

The sequence $(|a_n - L| + |c_n - L|)$ is a null sequence. Therefore $(|b_n - L|)$ is a null sequence (slide 41) and $\lim_{n\to\infty} b_n = L$. □

Lemma

Let a be a positive real number. Then

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

Proof

For $a \geq 1$, observe that, for all $n \geq a$

$$1 \leq \sqrt[n]{a} \leq \sqrt[n]{n}.$$

We have bounded the sequence $(\sqrt[n]{a})_{n\in\mathbb{N}}$ from above and below by two sequences which both converge to 1 (by Lemma on the slide 45). From the Sandwich Rule

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

Proof.

For $0 < a < 1$, let $(b_n)_{n\in\mathbb{N}}$ be the sequence given by

$$b_n = \sqrt[n]{\frac{1}{a}}.$$

Since $1/a > 1$, it follows from what we have just proven that $\lim_{n\to\infty} b_n = 1 > 0$. From the algebraic properties of limits we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n} = 1$$

□

Example

We give another application of the Sandwich Rule:
Let $x > y > 0$ be fixed real numbers and consider the sequence $(b_n)_{n\in\mathbb{N}}$ where

$$b_n = \sqrt[n]{x^n + y^n}.$$

We will show that

$$\lim_{n\to\infty} b_n = x.$$

Note that

$$x = \sqrt[n]{x^n} \leq \sqrt[n]{x^n + y^n} \leq \sqrt[n]{2x^n} = \sqrt[n]{2}\sqrt[n]{x^n} = \sqrt[n]{2} \cdot x.$$

Apply the Sandwich Rule with $a_n = x$ for all n and $c_n = \sqrt[n]{2} \cdot x$.

Example-part II

We know from the slide 49 that $\lim_{n\to\infty} \sqrt[n]{2} = 1$. From algebraic properties of limits we get $\lim_{n\to\infty} c_n = x$. The Sandwich Rule then implies that

$$\lim_{n\to\infty} b_n = x.$$

Exercise

Determine limits of sequences

$$a_n = \sqrt[n]{4^n + 3^{2n} + 2^n}, \quad b_n = \sqrt[n]{3^{-n} + 2^{-n} + 4^{-n}},$$
$$c_n = \sqrt[n]{3^{-n} + 2^{-n} + n}, \quad d_n = \sqrt[n^2]{3^n + 4^n}.$$

Exercise

Determine limits of sequences

$$a_n = \frac{n^2 + 5}{n^3 + 11n + 2}, \quad b_n = \frac{n^2 + n + 1}{n^2 + n\sin(n) + 1}$$
$$c_n = \frac{n!(n+5) + 2^n}{(n+1)! + 3^n}, \quad d_n = \frac{4^n + 3^n + 1}{2^{2n}\sqrt[n]{n} + 2 \cdot 3^n}.$$

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be real-valued sequences and let* $b, \lambda \in \mathbb{R}$*. Then*

(i) $\lim_{n\to+\infty} a_n = +\infty$ *and* $\lim_{n\to+\infty} b_n = +\infty \implies \lim_{n\to+\infty}(a_n + b_n) = +\infty$,

(ii) $\lim_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} b_n = +\infty \implies \lim_{n\to\infty}(a_n b_n) = +\infty$,

(iii) $\lim_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} b_n = b \implies \lim_{n\to\infty}(a_n + b_n) = +\infty$,

(iv) $\lim_{n\to\infty} a_n = +\infty \implies \lim_{n\to\infty} \frac{\lambda}{a_n} = 0$,

(v) $\lim_{n\to\infty} a_n = +\infty$ *and* $\lambda > 0 \implies \lim_{n\to\infty} \lambda a_n = +\infty$,

(vi) $\lim_{n\to\infty} a_n = +\infty$ *and* $\lambda < 0 \implies \lim_{n\to\infty} \lambda a_n = -\infty$.

Proof. This is left as an exercise.

Sandwich rule for definitely divergent sequences.

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be a real valued sequences. Assume there exists $N \in \mathbb{N}$ such that for $n > N$

$$b_n > a_n$$

If $\lim\limits_{n\to\infty} a_n = +\infty$ then $\lim\limits_{n\to\infty} b_n = +\infty$.

Proof. Left as an exercise.

Exercise - Indeterminate forms- limits of type: $\frac{\infty}{\infty}$

Determine limits of sequences:

$$a_n = \frac{n^2+5n-7}{n^2+5n+1}, \quad b_n = \frac{n^2-3n+1}{n^3+11n^2-4n+1},$$
$$c_n = \frac{n^3-3n+1}{7n+1}, \quad d_n = \frac{n^3-3n+1}{1-n}.$$

Exercise - Indeterminate forms- limits of type: $\infty - \infty$

Determine limits of sequences:

$$a_n = \sqrt{n^2 + n} - n,$$

$$b_n = \sqrt{n^4 - 3n + 1} - n^2,$$

$$c_n = \sqrt{n^3 - n^2 + 1} - n.$$

3.3. Monotone sequences

Definition

A real-valued sequence $(a_n)_{n\in\mathbb{N}}$ is called

- increasing if and only if
$$\forall n \in \mathbb{N},\ \ a_{n+1} > a_n,$$
- non-decreasing if and only if
$$\forall n \in \mathbb{N},\ \ a_{n+1} \geq a_n,$$
- decreasing if and only if
$$\forall n \in \mathbb{N},\ \ a_{n+1} < a_n,$$
- non-increasing if and only if
$$\forall n \in \mathbb{N},\ \ a_{n+1} \leq a_n.$$

Definition

Moreover, we say that a sequence is monotone if it is non-increasing or non-decreasing, and strictly monotone if it is either increasing or decreasing.

Remark

Note that, since the definition of monotonicity requires a notion of order, we do not have an analogue of the definition above for complex-valued sequences.

Examples

Many of the sequences that we have discussed so far are monotone. For example:

- $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is decreasing (and thus also strictly monotone).
- Any sequence of the form $(n^k)_{n\in\mathbb{N}}$ with $k > 0$ is increasing.
- If $k < 0$ then the sequence $(n^k)_{n\in\mathbb{N}}$ is decreasing.
- The sequence $(n^0)_{n\in\mathbb{N}}$ is constant (and so both non-increasing and non-decreasing).
- The sequence $((-1)^n)_{n\in\mathbb{N}}$ is not monotone.

Exercise

Let $a_1 = 1$ and $a_{n+1} = \frac{a_n+5}{2}$. Show that the sequence is bounded by 5. Prove that the sequence (a_n) is increasing.

Quotient criterion

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with positive values i.e. $a_n > 0$ for any $n \in \mathbb{N}$.

- The sequence is increasing if and only if

$$\frac{a_{n+1}}{a_n} > 1$$

  holds for all $n \in \mathbb{N}$.
- The sequence is decreasing if and only if

$$\frac{a_{n+1}}{a_n} < 1$$

  holds for all $n \in \mathbb{N}$.

Lemma

The sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n := \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$$

is non-decreasing.

Proof

We consider quotients of successive terms. We will show

$$\frac{a_{n+1}}{a_n} \geq 1$$

holds for all $n \in \mathbb{N}$.

Proof

Observe that

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^{n}} = \left(\frac{(n+2)n}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n}$$

$$= \left(\frac{(n+1)^2-1}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n}$$

$$= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n}.$$

Proof.

An application of Bernoulli's Inequality with $x = -\frac{1}{(n+1)^2} \geq -1$ yields

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n} \geq \left(1 - (n+1)\frac{1}{(n+1)^2}\right) \frac{n+1}{n} = 1.$$

$\square$

Lemma

The sequence $(b_n)_{n\in\mathbb{N}}$ *given by*

$$b_n := \left(1 + \frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1}$$

is non-increasing.

Proof.

This is left as an exercise. $\square$

Theorem (Monotonicity Principle)

(i) *If* $(a_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence which is bounded above, then*

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\} \in \mathbb{R}.$$

(ii) *If* $(a_n)_{n\in\mathbb{N}}$ *is non-increasing sequence which is bounded below, then*

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\} \in \mathbb{R}.$$

(iii) *If* $(a_n)_{n\in\mathbb{N}}$ *is a monotone sequence. Then*

$$(a_n)_{n\in\mathbb{N}} \text{ is convergent} \iff (a_n)_{n\in\mathbb{N}} \text{ is bounded.} \tag{2}$$

Notation

We use the notation $\sup(a_n)$ as a shorthand for $\sup\{a_n : n \in \mathbb{N}\}$, and similarly $\inf(a_n) = \inf\{a_n : n \in \mathbb{N}\}$.

Proof

i)

Suppose that $(a_n)_{n\in\mathbb{N}}$ is non-decreasing sequence which is bounded. We put $g = \sup(a_n)$. By the definition of the supremum the number $g-\frac{1}{n}$ is not an upper bound of the range of the sequence $(a_n)_{n\in\mathbb{N}}$. Hence there exists a_m such that

$$g-\frac{1}{n} \leq a_m \leq g$$

However, since $(a_n)_{n\in\mathbb{N}}$ is non-decreasing, it follows that for all $k \geq m$

$$g-\frac{1}{n} \leq a_m \leq a_k \leq g.$$

Proof

Hence

$$-\frac{1}{n} \le a_k - g \le 0 \quad \Rightarrow |a_k - g| \le \frac{1}{n}$$

For $\varepsilon > 0$ we choose $n > \varepsilon^{-1}$ and we get for $k > m$

$$|a_k - g| < \varepsilon$$

ii)

Let $(a_n)_{n\in\mathbb{N}}$ be non-increasing and bounded sequence. We put $b_n = -a_n$ for $n \in \mathbb{N}$. Th sequence $(b_n)_{n\in\mathbb{N}}$ non-decreasing and bounded. It follows from i) that $(b_n)_{n\in\mathbb{N}}$ has a limit equal to $\sup(b_n) \in \mathbb{R}$. Observe that by algebraic properties of limits and properties of supremum we get

$$\lim_{n\to\infty} a_n = -\lim_{n\to\infty} b_n = -\sup(b_n) = -\sup(-a_n) = -(-\inf(a_n)) = \inf(a_n)$$

Proof

iii) If monotone sequence is bounded it is convergent by i) and ii). On the other hand we know that every convergent sequence is bounded.

Exercise

Let $(a_n)_{n\in\mathbb{N}}$ be a non-decreasing real-valued sequence which is unbounded. Prove that

$$\lim_{n\to\infty} a_n = \infty.$$

Exercise

Prove that $b_n = \binom{2n}{n}$ is an increasing sequence.

Exercise

Let $(x_n)_{n\in\mathbb{N}}$ be given by the formula $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \in \mathbb{N}$ and $x_1 = 1$. Prove that

- $(x_n)_{n\in\mathbb{N}}$ has positive values,
- $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence,
- $(x_n)_{n\in\mathbb{N}}$ is unbounded.

Lemma

The sequence $(a_n)_{n\in\mathbb{N}}$ *given by*

$$a_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$$

is convergent.

Remark

We define this limit to be Euler's number, denoted e. That is,

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sup \left(1 + \frac{1}{n}\right)^n .$$

Proof.

We have already proven that this sequence is monotone. To prove this lemma, we will show that $(a_n)_{n\in\mathbb{N}}$ is bounded. We are then done, by Monotonicity Principle .

Since $(a_n)_{n\in\mathbb{N}}$ is non-decreasing, it follows from slide 62 that, for all $n \in \mathbb{N}$,

$$a_n \geq a_1 = 2.$$

It remains to establish an upper bound for a_n. For this, we recall the related sequence $(b_n)_{n\in\mathbb{N}}$, given by

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1} = a_n \left(1 + \frac{1}{n}\right) > a_n.$$

We also know that $(b_n)_{n\in\mathbb{N}}$ is non-increasing, which implies that, for all $n \in \mathbb{N}$,

$$a_n \leq b_n \leq b_1 = 4.$$

Fact

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real valued sequences such that $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} a_n b_n = g \in \mathbb{R}$. Then

$$\lim_{n\to\infty}(1 + a_n)^{b_n} = e^g$$

Proof - simple case

Let $a_n = b_n^{-1}$ and $a_n > 0$. Observe that $a_n \cdot b_n = 1$. For every n there exists $m_n \in \mathbb{N}$ such that

$$\frac{1}{m_n + 1} \leq a_n \leq \frac{1}{m_n}.$$

Thus

$$1 + \frac{1}{m_n + 1} \leq 1 + a_n \leq 1 + \frac{1}{m_n}$$

Proof - simple case

We know that $b_n > 0$ for $n \in \mathbb{N}$. Therefore function $f(x) = x^{b_n}$ is increasing for $x > 0$. Therefore

$$(1 + \frac{1}{m_n + 1})^{b_n} \leq (1 + a_n)^{b_n} \leq (1 + \frac{1}{m_n + 1})^{b_n}.$$

Recall that $m_n \leq b_n \leq m_n + 1$. Therefore

$$1 < (1 + \frac{1}{m_n + 1})^{m_n} \leq (1 + \frac{1}{m_n})^{b_n}$$

and

$$1 < (1 + \frac{1}{m_n})^{b_n} \leq (1 + \frac{1}{m_n})^{m_n+1}.$$

Proof - simple case

Observe that

$$\lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = \lim_{n\to\infty}(1+\frac{1}{n+1})^{n} = e$$

Since $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} m_n = +\infty$. We fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for $n > N$ we have

$$e - \varepsilon \leq (1+\frac{1}{m_n+1})^{m_n} \quad \text{and} \quad (1+\frac{1}{m_n})^{m_n+1} \leq e + \varepsilon$$

Thus

$$e - \varepsilon \leq (1+a_n)^{\frac{1}{a_n}} \leq e + \varepsilon$$

and

$$\lim_{n\to\infty}(1+a_n)^{\frac{1}{a_n}} = e.$$

Exercise

Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence such that $\lim\limits_{n\to\infty} na_n = 0$. Prove that

$$\lim_{n\to\infty}(1+a_n)^n = 1.$$

Hint. Use Bernoulli's inequality and Sandwich rule.

Exercise - Indeterminate forms- limits of type: 1^∞

Determine limits of sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$,$(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ where

$$a_n = \left(1+\frac{1}{5n+1}\right)^{6n+1}, \quad b_n = \left(1+\frac{15n+1}{7n^3+11n+5}\right)^{3n^2+1},$$
$$c_n = \left(\frac{2n^2-n}{2n^2+13}\right)^{n}, \quad d_n = \left(\frac{n^2+1}{n^2}\right)^{n}, \quad \text{for } n\in\mathbb{N}.$$

Example- Babylonian square-root algorithm

We define $a_1 = 2$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

where $n \in \mathbb{N}$. We will show by induction that $a_n > \sqrt{2}$. First we check the base of induction $a_1 = 2 > \sqrt{2}$. We assume that $a_n > \sqrt{2} > 0$ for fixed $n \in \mathbb{N}$. We will prove that $a_{n+1} > \sqrt{2}$. By definition this inequality is equivalent to

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) > \sqrt{2}$$

We multiply both sides by $2a_n > 0$ and we get

$$a_n^2 + \left(\sqrt{2}\right)^2 > 2\sqrt{2}a_n \Leftrightarrow (a_n - \sqrt{2})^2 > 0.$$

Example- Babylonian square-root algorithm

Therefore $a_{n+1} > \sqrt{2}$. Now we will show that $(a_n)_{n\in\mathbb{N}}$ is a monotone sequence

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) - a_n = \frac{1}{2}\left(\frac{2}{a_n} - a_n\right)$$
$$= \frac{2 - a_n^2}{2a_n} < \frac{2 - \left(\sqrt{2}\right)^2}{2a_n} = 0$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is decreasing. Since $(a_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below then the sequence is bounded and convergent.

Example- Babylonian square-root algorithm

We use the recursion to find limit $g = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

$$g = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}(g + \frac{2}{g})$$

Simple algebraic manipulation give us

$$g^2 = 2$$

But we know that $g > 0$ (it is a limit of numbers greater than $\sqrt{2}$).
Therefore

$$\lim_{n\to\infty} a_n = \sqrt{2}$$

3.4. Subsequences

Definition

Let $(n_1, n_2, n_3, \dots)$ be an (infinite) increasing sequence of natural numbers and let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Then, we call

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{n_1}, a_{n_2}, \dots)$$

a subsequence of $(a_n)_{n\in\mathbb{N}}$.

Example

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = (-1)^n.$$

Two notable subsequences are given by taking the odd and even terms of the sequences. That is, we can consider $(n_1, n_2, \ldots) = (1, 3, \ldots)$ and $(n_1, n_2, \ldots) = (2, 4, \ldots)$. These subsequences are convergent (in fact, they are constant) with limit (-1) and 1 respectively.

Exercise

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to} a_n = a$. Show that any subsequence $(a_{n_k})_{k\in\mathbb{N}}$ satisfies

$$\lim_{k\to\infty} a_{n_k} = a.$$

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued sequence. We call $a \in \mathbb{C}$ an accumulation point of $(a_n)_{n\in\mathbb{N}}$ if there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with

$$\lim_{k\to\infty} a_{n_k} = a.$$

Example

The accumulation points of the sequence $((-1)^n)_{n\in\mathbb{N}}$ are -1 and 1. We can also consider some more complicated examples.

Example

Consider the sequence defined by

$$a_n = \begin{cases} 1 & \text{if } n \text{ is not prime} \\ \frac{1}{n} & \text{if } n \text{ is prime} \end{cases} .$$

The accumulation points of this sequence are 0 and 1. The subsequence $(a_n)_{n\in\mathbb{P}}$, where $\mathbb{P}$ denotes the set of all primes, converges to 0 and the subsequence $(a_n)_{n\notin\mathbb{P}}$ is constant, always taking value 1.

Exercise

Determine all the accumulation points for the following sequences.

1. $a_n := i^n$
2. $b_n := (-1)^n + \frac{n^2+1}{n^2}$
3. $c_n := \sin\left(n\frac{\pi}{6}\right)$

Theorem (Bolzano-Weierstrass Theorem)

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded real-valued sequence. Then $(a_n)_{n\in\mathbb{N}}$ has at least one convergent subsequence.

Proof

Let $I_0 = [\inf(a_n), \sup(a_n)]$. We divide I_0 into two equal intervals left I_0^- and right I_0^+. One of sets

$$\{n : a_n \in I_0^-\} \text{ or } \{n : a_n \in I_0^+\}$$

contains infinitely many elements. Let's say I_0^-. We define $I_1 = I_0^-$. Similarly we define I_2 for I_1. Inductively we define sequence of intervals $(I_k)_{k\in\mathbb{N}}$.

Proof

We define

$$n_1 = \min\{n : a_n \in I_1\}$$

and inductively

$$n_k = \min\{n : a_n \in I_k \text{ and } n > n_{k-1}\}.$$

We put b_k as a left end of I_k. It is a bounded monotone sequence. Hence (b_k) converges. We put

$$\lim_{n\to\infty} b_k = a.$$

Proof.

For $k \in \mathbb{N}$ we have both a_k and a in I_k. Therefore

$$|a_{n_k} - a| \leq |I_N| = \frac{\sup(a_n) - \inf(a_n)}{2^k}.$$

On right hand side we have null sequence. Thus

$$\lim_{n \to \infty} a_{n_k} = a.$$

$\square$

Example

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = \frac{n \cdot \cos(3n^2 - 5)}{n + 1}.$$

It is not easy to see a pattern in this sequence, with the values of a_n jumping around fairly randomly, somewhere in the range $(-1, 1)$. However

$$|a_n| \leq 1.$$

So the Bolzano-Weierstrass Theorem tells us that there exists a convergent subsequence.

Example

Note that the converse of the Bolzano-Weierstrass Theorem does not hold, i.e. not every sequence with a convergent subsequence is bounded. One may consider, for example, the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = \begin{cases} n^3 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$