Sheet 9

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49) $(a_n)_{n\in N}$; $a_1 = 1$, $a_{n+1} = \frac{a_n + 5}{2}$

For. $n = 1$

$$a_{1+1} = \frac{a_1 + 5}{2}$$

$$a_2 = \frac{1+5}{2} = 3 \leq 5$$

Asum, $a_k \leq 5$ for some k

$$a_{k+1} = \frac{a_k + 5}{2} \leq \frac{5+5}{2} = 5$$

By induction, a_n bounded by 5 for all $n \in N$.
For increasing sequence,

$$a_{n+1} - a_n = \frac{a_n + 5}{2} - a_n = \frac{5 - a_n}{2} > 0 \quad (\because a_n \leq 5)$$

$\therefore$ Sequence is increasing
The limit exists and is equal to 5.

$$\lim_{n\to\infty} a_n = \lim$$

50) Let $B_1 = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$

$B_{n+1} = B_1 \cdot B_n \, ; \quad n \in N$

~~det~~ Then, for $n=1$,

$$B_2 = B_1 \cdot B_1 = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 21 & 30 \\ 6 & 9 \end{bmatrix}.$$

$$B_3 = B_1 \cdot B_2 = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 21 & 30 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 114 & 165 \\ 33 & 48 \end{bmatrix}$$

$$\det(B_1) = 4 \times 2 - 5 \times 1 = 3$$

$$\det(B_2) = 21 \times 9 - 30 \times 6 = 9$$

$$\det(B_3) = \cancel{120 \times 30 - 180 \times 2} \; 114 \times 48 - 165 \times 33 = 27$$

$$\therefore \det(B_n) = 3^n$$

$$\lim_{n \to \infty} \frac{3^n}{5 + a^2 \cdot b^n} = \frac{1}{9}$$

Divide by 3^n on denominator and nominator,

$$\lim_{n \to \infty} \frac{1}{\frac{5}{3^n} + a^2 \cdot \left(\frac{b}{3}\right)^n} = \frac{1}{9}$$

Then, $\lim_{n \to \infty} \frac{5}{3^n} + a^2 \cdot \left(\frac{b}{3}\right)^n = 9$

$$\text{or } \lim_{n \to \infty} a^2 \cdot \left(\frac{b}{3}\right)^n = 9$$

$$\therefore a = b = 3,$$

53. $a_n = \left(1 + \frac{1}{5n+1}\right)^{6n+1}$

i) Let $b_n = \frac{1}{5n+1}$, then $\lim_{n\to\infty} b_n = 0$

Let $c_n = 6n+1$

~~Then, $b_n \cdot c_n =$~~

Then, $\lim_{n\to\infty} b_n \cdot c_n = \frac{1}{5n+1} \cdot 6n+1 = \frac{6}{5}$

$\therefore \boxed{\lim_{n\to\infty} a_n = e^{\frac{6}{5}}}$ $\left(\because \lim_{n\to\infty} b_n = 0\right)$

similarly,

ii) $b_n = \left(1 + \frac{15n+1}{7n^3+11n+5}\right)^{3n^2+1}$

Let, $a_n = \frac{15n+1}{7n^3+11n+5}$

$\lim_{n\to\infty} a_n = 0$

Let, $c_n = 3n^2+1$

Then, $\lim_{n\to\infty} a_n \cdot c_n = \lim_{n\to\infty} \frac{15n+1}{7n^3+11n+5} \cdot (3n^2+1)$

$= \lim_{n\to\infty} \frac{45n^3+3n^2+15n+1}{7n^3+11n+5} = \frac{45}{7}$

$\therefore \boxed{\lim_{n\to\infty} b_n = e^{\frac{45}{7}}}$

iii) $c_n = \left(\frac{2n^2-n}{2n^2+13}\right)^n = \left(1 + \frac{-(n+13)}{2n^2+13}\right)^n$

~~$\lim_{n\to\infty}$~~ Let $a_n = \frac{-(n+13)}{2n^2+13}$

$\rightarrow \lim_{n\to\infty} a_n = 0$

Let, $b_n = n$

Then, $\lim_{n\to\infty} a_n \cdot b_n = \frac{-(n+13)}{2n^2+13} \cdot n = -\frac{1}{2}$

$\therefore \boxed{\lim_{n\to\infty} c_n = e^{-\frac{1}{2}}}$

iv) $d_n = \left(\frac{n^2+1}{n^2}\right)^n = \left(1 + \frac{1}{n^2}\right)^n$

Let $a_n = \frac{1}{n^2}$; $\lim_{n\to\infty} a_n = 0$

Let $b_n = n$;

$\lim_{n\to\infty} a_n \cdot b_n = \left(\frac{1}{n^2} \cdot n\right) = \left(\frac{1}{n}\right) = 0$

$\therefore \boxed{\lim_{n\to\infty} d_n = e^0 = 1}$

54. $a_n = \left(\sqrt{n^2+4} - n\right) \prod_{k=1}^{n} \left(1 + \frac{1}{k+1}\right)$

$$= \left(\sqrt{n^2+4} - n\right) \cdot \prod_{k=1}^{n} \left(\frac{k+2}{k+1}\right)$$

$$= \left(\sqrt{n^2+4} - n\right) \cdot \left(\frac{\cancel{3}}{2} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdot \frac{\cancel{6}}{\cancel{5}} \cdots \frac{\cancel{n+1}}{\cancel{n}} \cdot \frac{n+2}{\cancel{n+1}}\right)$$

$$a_n = \left(\sqrt{n^2+4} - n\right) \cdot \left(\frac{n+2}{2}\right)$$

$$\therefore \boxed{\lim_{n \to \infty} a_n = \infty}$$