Exercise Sheet 3

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13. <u>Recall</u>

We call f invertible iff $f^{-1}: B \to A$ such that
$\forall a \in A: f^{-1}(f(a)) = a$

To show that there exists a unique $g: B \to A$ that satiesfies $g \circ f = ID_A$ and $f \circ g = ID_B$, we will prove:

1. There exists $g: B \to A$ such that $g \circ f = ID_A$ and $f \circ g = ID_B$
2. If $g \circ f = ID_A$ and $f \circ g = ID_B$ for some $g: B \to A$, then g is unique.

Now,
Let $b \in B$, and f is surjective as it is investible, then $\exists a \in A$ such that $f(a) = b$. Since, f is injective, a is unque.

We can define a function: $g: B \to A$ by $g(b) = a$ for $\forall b \in B$.

For $\forall a \in A$,

$$g \circ f: = g(f(a)) = g(b) = a \quad (\because f(a) = b)$$

$$\boxed{\therefore g \circ f = ID_A}$$

similarly,
for, $\forall b \in B$

$$f \circ g: f(g(b)) = f(a) = b \quad (\because g(b) = a)$$

$$\therefore \boxed{f \circ g = ID_B}$$

To prove g's uniqueness, ~~let~~

Lets assume, $h : B \to A$ such that $h \circ f = ID_A$ and $f \circ h = ID_B$. ~~and~~ And then prove $h = g$.

$\Rightarrow$ $\forall b \in B$, we have

$$\underline{g(b)} = (h \circ f)\, g(b) = h(f(g(b))) = h(f(a)) = \underline{\underline{h(b)}}$$

$$\therefore \boxed{g = h}$$

Therefore, there exists a unique $g : B \to A$ that satisfies $g \circ f = ID_A$ and $f \circ g = ID_B$.

14. Let $h : \mathbb{N} \to \mathbb{R}$

$$h(1) = 2$$
$$h(n+1) = \sqrt{3 + 3h(n)}$$

$\forall n \in \mathbb{N}$, we have
$$h(n+1) = \sqrt{3 + 3h(n)}$$

Induction basis:
$$h(1) = 2 < 4$$

For $n=1$, $h(1+1) = h(2) = \sqrt{3 + 3 \times h(1)} = \sqrt{3+3\times 2} = \sqrt{9} < 4$

suppose, it is true for $n = p$. $\Rightarrow \boxed{h(p) < 4}$ — (i)

Then, $h(p+1) = \sqrt{3 + 3h(p)} = \sqrt{3 + 3\times 4} = \sqrt{15} < 4$

$\Rightarrow \boxed{h(p+1) < 4}$

So, the assertion is true for $n = p+1$, whenever it is true for $n = p$.

So, by principle of mathematical induction,
$$\forall n \in \mathbb{N},\ h(n) < 4.$$

15. Let, $\forall n \in N \setminus \{1,2\}$: $\boxed{2n+1 < 2^n}$

when $n=3$,

or, $2\times3+1 < 2^3$

$\boxed{7 < 8}$, is True

suppose for $n=p$, the assertion is true.

$\therefore\ 2p+1 < 2^p$ —— (i)

Multiplying (i) on both side.

$2(2p+1) < 2\cdot2^p$

$\boxed{4p+2 < 2^{p+1}}$ —— (ii)

substituting, $p=p+1$ in L.H.S of (i)

$2(p+1)+1 < \cancel{2^{p+1}}$

$\boxed{2p+3 < \cancel{2^{p+1}}}$ —— (iii)

Comparing (ii) and (iii),

$\boxed{2p+3 < 4p+2}$

Hence, the assertion is true for $n=p+1$, whenever it is true for $n=p$.

So, by principle of mathematical induction: $\forall n \in N\{1,2\}$, $\boxed{2n+1 < 2^n}$

16. Let $F = (F; \{+, \cdot\})$ be a field.

a) ~~Since~~ Take $x \in F$, such that $x = x + x$
since F is a field, by additive identity $0 \in F$.
So
$$x + 0 = x = x + x$$
$$\Rightarrow x + 0 = x + x$$
$$\Rightarrow x = 0$$

b) Take $x \in F$, ~~such that~~.
since, $x = x + x \Rightarrow x = 0$
consider,
$$0 \cdot x = 0 \cdot (x + x) \quad \because 0 \in F$$
By distributive property.
$$0 \cdot x = 0 \cdot x + 0 \cdot x$$
$$\Rightarrow 0 \cdot x = 0$$
consider,
$$x \cdot 0 = (x + x) \cdot 0$$
By distributive property
$$x \cdot 0 = x \cdot 0 + x \cdot 0$$
$$\Rightarrow x \cdot 0 = 0$$
$$\therefore \boxed{x \cdot 0 = 0 \cdot x = 0}$$

(6.c) Let $x, y \in F$ such that $x \cdot y = 0$

$x \cdot y = 0$

or, $x \cdot y = x \cdot 0$ (By using (b)) — (i)

Adding $x \cdot y$ both side of (i)

$x \cdot y + x \cdot y = x \cdot 0 + x \cdot y$ — (ii)

By using distributive property,

$x \cdot (y + y) = x \cdot (0 + y)$

or, $y + y = 0 + y$

or, $y = y + y$

$\Rightarrow \boxed{y = 0}$ (From a)

~~Adding x.y on both side of (ii)~~

From (ii), By using distributive property.

$y \cdot (x + x) = (x + 0) \cdot y$

or $x + x = x + 0$

since, 0 is the additive identity

$x + x = x$

$\Rightarrow \boxed{x = 0}$ $\boxed{\text{by (a)}}$

So Hence, $x = 0$ or $y = 0$, proved

18.

18. Let $A \subset \mathbb{R}$ and $A \neq \phi$, let $k \in \mathbb{R}$,

$B := \{-a \mid a \in A\}$. — (i)

let $\sup A = k$

By definition,

$\Rightarrow k \geq a \quad (\forall a \in A)$

Let's multiply by (-1)

$-k \leq -a \quad (\forall a \in A)$

$\Rightarrow -k = \inf B \quad$ (From i)

proved