31. Prove that
$$\sqrt{3a^2+ab}+\sqrt{3b^2+bc}+\sqrt{3c^2+ca}\le 2(a+b+c)$$
holds for all non-negative real numbers a, b, c.
Hint. Apply the Cauchy-Schwarz inequality.

32. Let $A \in \mathbb{R}^{2\times 3}$, $B \in \mathbb{R}^{3\times 3}$ and $C \in \mathbb{R}^{3\times 2}$ be the matrices as follows:
$$A=\begin{pmatrix}1 & 1 & 0\\ -7 & 1 & 0\end{pmatrix},\quad B=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix},\quad C=\begin{pmatrix}1 & -3\\ -2 & 1\\ 0 & 0\end{pmatrix}$$
Which of the following expressions are well-defined? Compute the result if possible.
$(a)\ A\cdot B \quad (b)\ B\cdot A \quad (c)\ A\cdot(B\cdot C) \quad (d)\ C\cdot(B\cdot A) \quad (e)\ A\cdot(B+C) \quad (f)\ 5\cdot(A^\top+C)$
$$(g)\ B^\top\cdot A^\top.$$

33. (a) Give an example to show that the matrix multiplication in $\mathbb{R}^{3\times 3}$ is not commutative.
(b) Prove that, for all $A, B \in \mathbb{R}^{m\times n}$ and any $\lambda \in \mathbb{R}$,
$$\lambda(A+B)=\lambda A+\lambda B.$$
(c) Prove that, for all $A \in \mathbb{R}^{m\times p}$ and $B, C \in \mathbb{R}^{p\times n}$,
$$A\cdot(B+C)=A\cdot B+A\cdot C.$$
(d) Prove that, for all $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times s}$,
$$(A\cdot B)\cdot C=A\cdot(B\cdot C).$$
(e) Let $I_n \in \mathbb{R}^{n\times n}$ and $I_m \in \mathbb{R}^{m\times m}$ be the identity matrices of order n and m respectively. Prove that for $A \in \mathbb{R}^{m\times n}$,
$$A\cdot I_n = A \text{ and } I_m\cdot A = A.$$
(f) Prove that for all $A, B \in \mathbb{R}^{m\times n}$
$$(A+B)^\top = A^\top + B^\top.$$

34. Assume you have ordered 4 pizzas and 5 drinks, but you forgot the individual prices. You only know that you have paid total 50 EURO, and that a pizza was 8 EURO more expensive than a drink. How much is a pizza and how much is a drink?

35. (a) Which of the following matrices are in row echelon form? Reduced row echelon form? For those matrices which are not in (reduced) row echelon form, explain the reason.
$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & 3\\ 0 & 1\end{pmatrix},\quad C=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 3\\ 0 & 0 & 0\end{pmatrix},\quad D=\begin{pmatrix}1 & 2 & 3 & 1\\ 0 & 0 & 2 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

(b) For each of the following matrices, compute their reduced row echelon forms and ranks.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -2 & -4 \\ 2 & 4 & 1 & 2 \\ 1 & 3 & -3 & -3 \end{pmatrix}.$$

36. Find all solutions (x_1, x_2, x_3) in $\mathbb{R}^3$ of the following systems of linear equations.

(a)

$$\begin{aligned} x_1 \qquad - \ x_3 &= 2 \\ x_2 + 2x_3 &= 5 \\ x_1 + x_2 + \ x_3 &= 7. \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= -4 \\ 2x_1 + 4x_2 + x_3 &= 2 \\ x_2 - x_3 &= 1. \end{aligned}$$