25. (a) Determine the set $\{x \in \mathbb{R} : 3|x+2| - 4x + 3 \leq 5|x-1|\}$. (Hint: Use case distinction.) Plot the solution set and the functions $f(x) = 3|x+2| - 4x + 3$ and $g(x) = 5|x-1|$. You may use any software of your choice for the plots.

(b) Let $x_i \in \mathbb{R}, i \in \{1, \dots, n\}$. Prove the generalized triangle inequality

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i| \, .$$

Hint: Use induction on the predicate $P(n) : |\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$.

Solution:

(a) We have to distinct the three cases $x < -2, -2 \leq x < 1$ and $x \geq 1$.

- Case $\underline{x < -2}$:

$$\begin{aligned} -3(x+2) - 4x + 3 &\leq -5(x-1) \\ -7x - 3 &\leq -5x + 5 \\ -2x &\leq 8 \\ x &\geq -4 \end{aligned}$$

  We get the solution set $L_1 = [-4, -2)$.

- Case $\underline{-2 \leq x < 1}$:

$$\begin{aligned} 3(x+2) - 4x + 3 &\leq -5(x-1) \\ -x + 9 &\leq -5x + 5 \\ 4x &\leq -4 \\ x &\leq -1 \end{aligned}$$

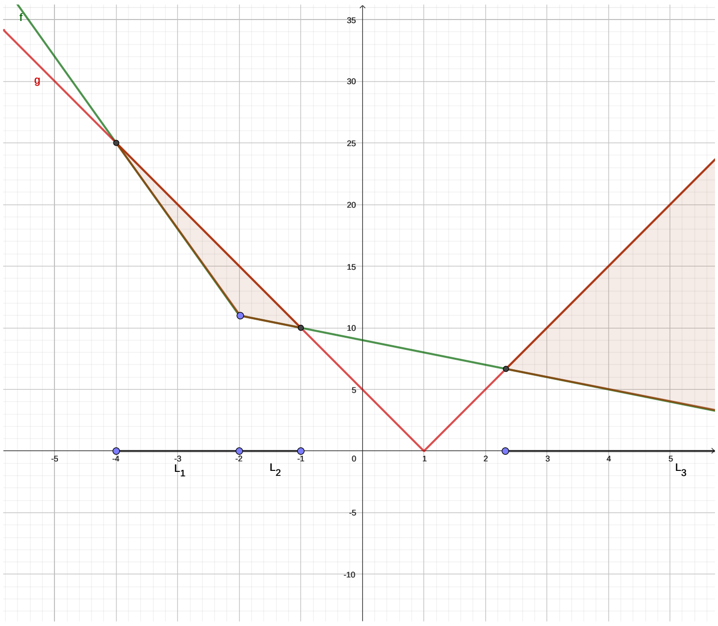
  We get the solution set $L_2 = [-2, -1]$.

- Case $\underline{x \geq 1}$:

$$\begin{aligned} 3(x+2) - 4x + 3 &\leq 5(x-1) \\ -x + 9 &\leq 5x - 5 \\ -6x &\leq -14 \\ x &\geq \frac{7}{3} \end{aligned}$$

  We get the solution set $L_3 = [\frac{7}{3}, \infty)$.

The inequality is fulfilled by all $x \in L_1 \cup L_2 \cup L_3 = [-4, -1] \cup [\frac{7}{3}, \infty)$.



(b) Proof by induction on the predicate $P(n)$: From the lecture we know that the assertion is true for $n = 1$ and $n = 2$. That is, if $n = 1$ we have

$$\left|\sum_{i=1}^{1} x_i\right| = |x_1| = \sum_{i=1}^{1} |x_i| \implies P(1) \text{ is true,}$$

and that for $n = 2$ we have

$$\left|\sum_{i=1}^{2} x_i\right| = |x_1 + x_2| \leq |x_1| + |x_1| = \sum_{i=1}^{2} |x_i| \implies P(2) \text{ is true.}$$

Let the assertion be true, that is, we assume that $P(n)$ is true for any $n \in \mathbb{N}$ (hence $|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$ is true for any $n \in \mathbb{N}$). Now we need to show that $P(n+1)$ is also true:

$$\begin{aligned} |x_1 + \ldots + x_n + x_{n+1}| &= |(x_1 + \ldots + x_n) + x_{n+1}| && (1) \\ &\leq |(x_1 + \ldots + x_n)| + |x_{n+1}| && (2) \\ &\leq (|x_1| + \ldots + |x_n|) + |x_{n+1}| && (3) \end{aligned}$$

where in (2) we used the fact that $P(2)$ is true, and in (3) we used the induction step.

$\square$

26. (a) Determine the following sets:
    - $\left\{x \in (4,\infty) : 4(\sqrt{3})^{2x-1}5^{4x+3} = 3\pi^{x-4}\right\}$.
    - $\{x \in (0,\infty) : \log_3 x = 5\}$.
    - $\{x \in (-1,\infty) : 2\ln(x+3) - 3\ln(x+2) + \ln(x+1) = 0\}$.

    (b) Compute

$$\left\{x \in [-\pi,\pi] : \cos x = \frac{\sqrt{3}}{2}\right\} \quad \text{and} \quad \left\{x \in [-\pi,\pi] : \sin^2 x = \frac{1}{2}\right\}$$

Solution:

(a)
- We have

$$\begin{aligned}
&4(\sqrt{3})^{2x-1}5^{4x+3} = 3\pi^{x-4}\\
\iff &\ln(4(\sqrt{3})^{2x-1}5^{4x+3}) = \ln(3\pi^{x-4})\\
\iff &\ln(4) + (2x-1)\ln(\sqrt{3}) + (4x+3)\ln(5) = \ln(3) + (x-4)\ln(\pi)\\
\iff &2x\ln(\sqrt{3}) + 4x\ln(5) - x\ln(\pi) = -\ln(4) + \ln(\sqrt{3}) - 3\ln(5) + \ln(3) - 4\ln(\pi)\\
\iff &x(2\ln(\sqrt{3}) + 4\ln(5) - \ln(\pi)) = -\ln(4) + \ln(\sqrt{3}) - 3\ln(5) + \ln(3) - 4\ln(\pi)\\
\iff &x(\ln((\sqrt{3})^2) + \ln(5^4) - \ln(\pi)) = -\ln(4) + \ln(\sqrt{3}) - \ln(5^3) + \ln(3) - \ln(\pi^4)\\
\iff &x\ln\left(\frac{3\cdot 5^4}{\pi}\right) = \ln\left(\frac{3\sqrt{3}}{5^3\pi^4\cdot 4}\right)
\end{aligned}$$

  Consequently we have

$$x = \frac{\ln\left(\frac{3\sqrt{3}}{500\pi^4}\right)}{\ln\left(\frac{1875}{\pi}\right)} = \ln_{\frac{1875}{\pi}}\left(\frac{3\sqrt{3}}{500\pi^4}\right) < 0.$$

  Then we conclude that $\left\{x \in (4,\infty) : 4(\sqrt{3})^{2x-1}5^{4x+3} = 3\pi^{x-4}\right\} = \emptyset$

-

$$\begin{aligned}
\{x \in (0,\infty) : \log_3 x = 5\} &= \left\{x \in (0,\infty) : \frac{\ln x}{\ln(3)} = 5\right\}\\
&= \{x \in (0,\infty) : \ln x = 5\ln(3)\}\\
&= \left\{x \in (0,\infty) : \ln x = \ln(3^5)\right\}
\end{aligned}$$

  Then we conclude that $\{x \in (0,\infty) : \log_3 x = 5\} = \{3^5\}$.

- Since $x > -1$ we have

$$\begin{aligned}
2\ln(x+3) - 3\ln(x+2) + \ln(x+1) &= \ln((x+3)^2) - \ln((x+2)^3) + \ln(x+1)\\
&= \ln((x+3)^2) + \ln\left(\frac{1}{(x+2)^3}\right) + \ln(x+1)\\
&= \ln\left(\frac{(x+3)^2(x+1)}{(x+2)^3}\right).
\end{aligned}$$

The fact that $\ln(1) = 0$, implies that $2\ln(x+3) - 3\ln(x+2) + \ln(x+1) = 0$ is the same as $\frac{(x+3)^2(x+1)}{(x+2)^3} = 1$. Therefore, we have

$$\frac{(x+3)^2(x+1)}{(x+2)^3} = 1 \iff (x+3)^2(x+1) = (x+2)^3$$
$$(x+3)^2(x+1) = (x+2)^3 \iff (x+3)^2(x+1) - (x+2)^3 = 0.$$

Furthermore, we have

$$(x+3)^2(x+1) - (x+2)^3 = (x^2+6x+9)(x+1) - (x^3+6x^2+12x+8)$$

and that

$$(x^2+6x+9)(x+1) - (x^3+6x^2+12x+8) = x^2+3x+1.$$

Then we have

$$2\ln(x+3) - 3\ln(x+2) + \ln(x+1) \iff x^2+3x+1 = 0.$$

That is the problem is reduced to solve a quadratic equation. The solutions of $x^2+3x+1=0$ are

$$x_1 = \frac{-3+\sqrt{5}}{2}, \quad x_2 = \frac{-3-\sqrt{5}}{2}$$

since $x_1 > -1$ and $x_2 \leq -1$ we conclude that

$$\{x \in (-1,\infty) : 2\ln(x+3) - 3\ln(x+2) + \ln(x+1) = 0\} = \{\frac{-3+\sqrt{5}}{2}\}.$$

(b)
- The equality $\cos(x) = \frac{\sqrt{3}}{2}$ implies that $\cos(x) = \cos(\frac{\pi}{6})$ or $\cos(x) = \cos(-\frac{\pi}{6})$, since $\cos(\theta) = \cos(-\theta)$. We discuss two cases:
  - **Case1:** For $\cos(x) = \cos(\frac{\pi}{6})$, the possible solutions for this equation are
  $$x = \frac{\pi}{6} + 2\pi k, \quad (k \in \mathbb{Z}).$$
    * For $k < 0$, we have $x_k \leq -\frac{11\pi}{6}$. Hence $x_k \notin [-\pi, +\pi]$.
    * For $k = 0$, we have $x_0 = \frac{\pi}{6} \in [-\pi, +\pi]$.
    * For $k \geq 1$, we have $x_k \geq \frac{13\pi}{6}$. Hence $x_k \notin [-\pi, +\pi]$.
  - **Case2:** For $\cos(x) = \cos(-\frac{\pi}{6})$, the possible solutions for this equation are
  $$x = -\frac{\pi}{6} + 2\pi k, \quad (k \in \mathbb{Z}).$$
  It follows that:
    * For $k < 0$, we have $x_k \leq -\frac{13\pi}{6}$. Hence $x_k \notin [-\pi, +\pi]$.
    * For $k = 0$, we have $x_0 = -\frac{\pi}{6} \in [-\pi, +\pi]$.
    * For $k \geq 1$, we have $x_k \geq \frac{11\pi}{6}$. Hence $x_k \notin [-\pi, +\pi]$.

  We then conclude that
  $$\left\{x \in [-\pi,\pi] : \cos x = \frac{\sqrt{3}}{2}\right\} = \{-\frac{\pi}{6}, \frac{\pi}{6}\}.$$

- The equality $\sin^2 x = \frac{1}{2}$ implies that $\sin x = \pm\frac{1}{\sqrt{2}}$. Then, we have two cases:
  Case1: $\sin x = \frac{1}{\sqrt{2}}$, then we get $\sin x = \sin(\frac{\pi}{4})$ or $\sin x = \sin(\pi - \frac{\pi}{4})$, since $\sin(\pi - \theta) = \sin(\theta)$.
  - For $\sin x = \sin(\frac{\pi}{4})$, it follows that
  $$x_k = \frac{\pi}{4} + 2\pi k, \quad (k \in \mathbb{Z}).$$
  Here, we notice that
    * For $k < 0$, we have $x_k \leq -\frac{7\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.
    * For $k = 0$, we have $x_0 = \frac{\pi}{4} \in [-\pi, +\pi]$.
    * For $k \geq 1$, we have $x \geq \frac{9\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.
  - For $\sin x = \sin(\pi - \frac{\pi}{4}) = \sin(\frac{3\pi}{4})$. It follows that
  $$x_k = \frac{3\pi}{4} + 2\pi k, \quad (k \in \mathbb{Z}).$$
  Again, we notice that:
    * For $k < 0$, we have $x_k \leq -\frac{5\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.
    * For $k = 0$, we have $x_0 = \frac{3\pi}{4} \in [-\pi, +\pi]$.
    * For $k \geq 1$, we have $x_k \geq \frac{11\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.

  Case2: $\sin x = -\frac{1}{\sqrt{2}}$, then we get $\sin x = \sin(-\frac{\pi}{4})$ or $\sin x = \sin(-\pi + \frac{\pi}{4})$, since $\sin(-\pi + \theta) = \sin(-\theta)$.
  - For $\sin x = \sin(-\frac{\pi}{4})$, it follows that
  $$x_k = -\frac{\pi}{4} + 2\pi k, \quad (k \in \mathbb{Z})$$
    * For $k < 0$, we have $x_k \leq -\frac{-9\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.
    * For $k = 0$, we have $x_0 = -\frac{\pi}{4} \in [-\pi, +\pi]$.
    * For $k \geq 1$, we have $x_k \geq \frac{7\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.
  - Now, we have $\sin x = \sin(-\pi + \frac{\pi}{4}) = \sin(-\frac{3\pi}{4})$. It follows that
  $$x_k = -\frac{3\pi}{4} + 2\pi k, \quad (k \in \mathbb{Z}).$$
  We notice that:
    * For $k < 0$, we get $x_k \leq -\frac{11\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.
    * For $k = 0$, we get $x_0 = -\frac{3\pi}{4} \in [-\pi, +\pi]$.
    * For $k \geq 1$, we get $x_k \geq \frac{5\pi}{4}$. Hence $x_k \notin [-\pi, +\pi]$.

  Finally, we conclude that
  $$\left\{x \in [-\pi, \pi] : \sin^2 x = \frac{1}{2}\right\} = \left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\right\}.$$

$\square$

27. Let $f : \mathbb{R} \to \mathbb{R}$ be the ReLU activation function defined as

$$f(x) := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Plot and describe in words, how the following transformations change the ReLU activation function f, that is, how the graph of the function $F(x)$ looks like compared to the graph of $f(x)$. You may use any software of your choice for the plots.

Let $k \in \{-1,\ 0.5,\ 2\}$

(a) $F(x) = f(x) + k$,

(b) $F(x) = f(x + k)$,

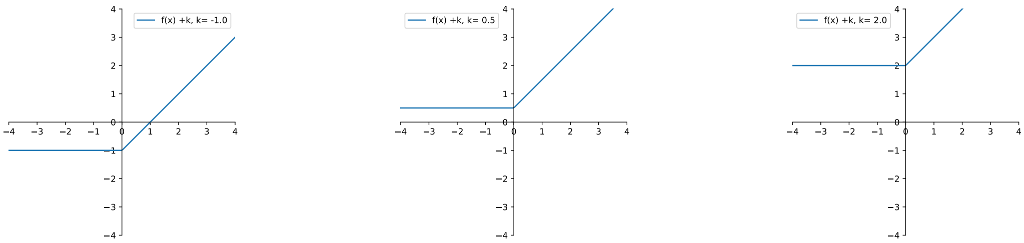
(c) $F(x) = k \cdot f(x)$,

(d) $F(x) = f(k \cdot x)$,

(e) $F(x) = |f(x)|$,

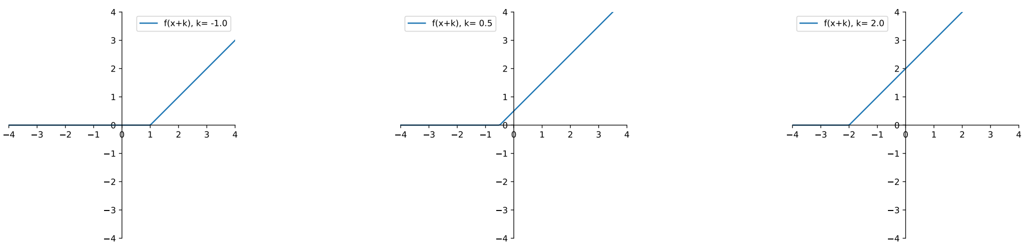
(f) $F(x) = f(|x|)$.

Solution:



(a) $F(x) = f(x) - 1$ (b) $F(x) = f(x) + 0.5$ (c) $F(x) = f(x) + 2$

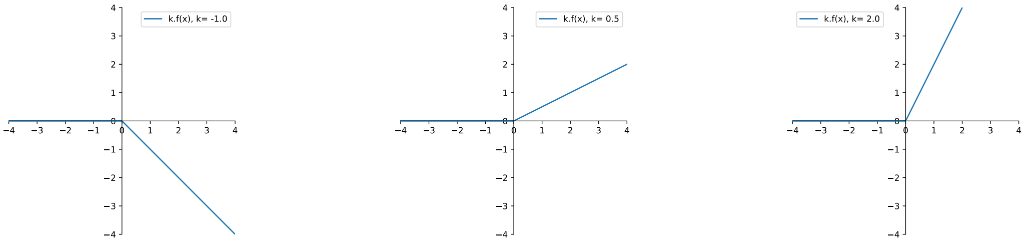
Figure 1: The graph of $f(x)$ is shifted in direction of the y-axis.



(a) $F(x) = f(x - 1)$ (b) $F(x) = f(x + 0.5)$ (c) $F(x) = f(x + 2)$

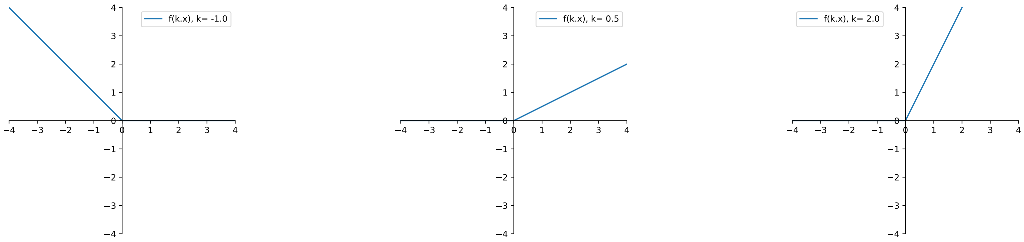
Figure 2: The graph of $f(x)$ is shifted in direction of the x-axis.

$\square$



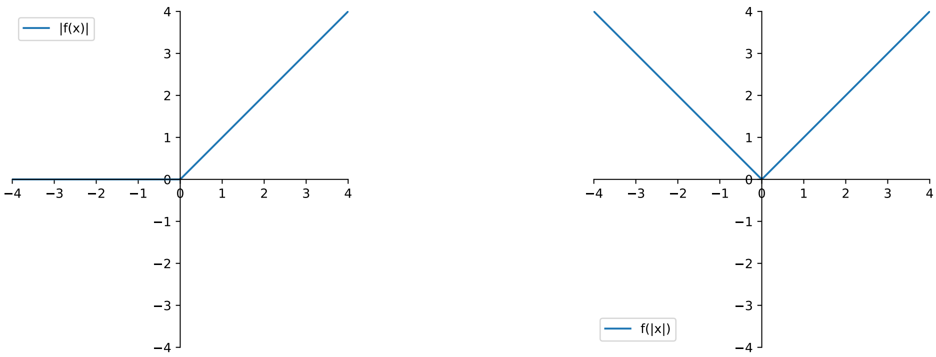
(a) $F(x) = -f(x)$ (b) $F(x) = \frac{1}{2}f(x)$ (c) $F(x) = 2f(x)$

Figure 3: The graph of $f(x)$ is stretched (i.e. gets steeper) ($|k| > 1$) or compressed (i.e. gets flatter) ($|k| < 1$) in direction of the y-axis; if $k < 0$ the graph is additionally mirrored with respect to the x-axis.



(a) $F(x) = f(-x)$ (b) $F(x) = f(\frac{x}{2})$ (c) $F(x) = f(2x)$

Figure 4: The graph of $f(x)$ is stretched ($|k| > 1$) or compressed ($|k| < 1$) in direction of the x-axis; if $k < 0$ the graph is additionally mirrored at the y-axis.



(a) $F(x) = |f(x)|$ (b) $F(x) = f(|x|)$

Figure 5: In (a), the graph of $f(x)$ with $f(x) > 0$ are mirrored at the x-axis (in this particular choice of function it was not clear since ReLU is always positive). In (b) the graph of $f(x)$ with $x < 0$ is mirrored at the y-axis.

28. Let $z_1 = \frac{\sqrt{3}}{2}\,\mathrm{i} + \frac{\sqrt{3}-\mathrm{i}}{(\sqrt{2}+\sqrt{2}\,\mathrm{i})^2}$ and $z_2 = 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$.

    (a) Determine the real and imaginary part of z_1 and z_2.

    (b) Determine the absolute value and the argument of z_1 and z_2.

    (c) Determine the absolute value and the argument of $z_1 z_2$. Furthermore, give the real and imaginary part of $z_1 z_2$.

Solution:

(a)

$$\begin{aligned} z_1 &= \frac{\sqrt{3}}{2}\,\mathrm{i} + \frac{\sqrt{3}-\mathrm{i}}{(\sqrt{2}+\sqrt{2}\,\mathrm{i})^2} \\ &= \frac{\sqrt{3}}{2}\,\mathrm{i} + \frac{\sqrt{3}-\mathrm{i}}{2+2\sqrt{2}\cdot\sqrt{2}\,\mathrm{i}+(\sqrt{2}\,\mathrm{i})^2} \\ &= \frac{\sqrt{3}}{2}\,\mathrm{i} + \frac{\sqrt{3}-\mathrm{i}}{2+4\,\mathrm{i}-2} = \frac{\sqrt{3}}{2}\,\mathrm{i} + \frac{\sqrt{3}-\mathrm{i}}{4\,\mathrm{i}} \end{aligned}$$

We recall that $\frac{1}{\mathrm{i}} = -\,\mathrm{i}$ (because $\frac{1}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}\cdot\mathrm{i}} = \frac{\mathrm{i}}{-1}$).

$$\begin{aligned} z_1 &= \frac{\sqrt{3}}{2}\,\mathrm{i} - \frac{\mathrm{i}\sqrt{3}-\mathrm{i}}{4} \\ &= \frac{\sqrt{3}}{2}\,\mathrm{i} - \frac{\mathrm{i}(\sqrt{3}-\mathrm{i})}{4} \\ &= \frac{\sqrt{3}}{2}\,\mathrm{i} - \frac{\mathrm{i}\sqrt{3}+1}{4} \\ &= -\frac{1}{4} + \frac{\sqrt{3}}{4}\,\mathrm{i} \end{aligned}$$

Then the real part of z_1: $Re(z_1) = -\frac{1}{4}$ and the imaginary part of z_1: $Im(z_1) = \frac{\sqrt{3}}{4}$.

$$\begin{aligned} z_2 &= 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) \\ &= 2\left(\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\right) \\ &= 1 + \sqrt{3}\,\mathrm{i} \end{aligned}$$

Then the real part of z_2: $Re(z_2) = 1$ and the imaginary part of z_2: $Im(z_2) = \sqrt{3}$.

(b) We first find the magnitude of $z_1 = -\frac{1}{4} + \frac{\sqrt{3}}{4}\,\mathrm{i}$.

$$\begin{aligned} |z_1| &= \sqrt{\left(-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} \\ &= \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \frac{1}{2}. \end{aligned}$$

Now we find the argument ϕ. Recall that $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\cos(x) \geq 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Note that $\cos(x) = -\cos(x+\pi)$ and $\sin(x) = -\sin(x+\pi)$. Hence $\tan(x+\pi) = \tan(x)$. Knowing that $a < 0$, we need to add π to $\arctan(b/a)$ to obtain the correct argument:

$$\phi_1 = \arctan\left(\frac{(\sqrt{3}/4)}{(-1/4)}\right) + \pi = \arctan(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

Therefore, $|z_1| = \frac{1}{2}$ and the argument of z_1 equals $\frac{2\pi}{3}$.
The absolute value of $z_2 = 1 + \sqrt{3}i$ can be determined as follows:

$$|z_2| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2.$$

We get the argument ϕ_2 of z_2 as follows:

$$\phi_2 = \arctan\left(\frac{\sqrt{3}}{1}\right) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}.$$

Therefore, $|z_2| = 2$ and the argument of z_2 equals $\frac{\pi}{3}$.

(c) Using Euler's formula, we can write

$$z_1 = r_1 e^{i\phi_1} = \frac{1}{2} e^{i\frac{2\pi}{3}}$$

and

$$z_2 = r_2 e^{i\phi_2} = 2e^{i\frac{\pi}{3}}.$$

Therefore,

$$z_1 z_2 = r_1 \cdot r_2 \cdot e^{i(\phi_1+\phi_2)} = \frac{1}{2} \cdot 2 \cdot e^{i\pi} = e^{i\pi} = -1.$$

Since $z_1 z_2 = e^{i\pi} = -1$ it follows that $|z_1 z_2| = 1$, moreover the fact that $z_1 z_2 = e^{i\pi}$ implies that the argument of $z_1 z_2$ equals to π. Furthermore, the $Re(z_1 z_2) = -1$ and $Im(z_1 z_2) = 0$.

$\square$

29. (a) Calculate $\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\,\mathrm{i}\right)^{12}$ using de Moivre's formula.

(b) How many real solutions do the following equations have? Write them down.

i. $x^2 = -1, \quad x \in \mathbb{R}$

ii. $x^3 = -1, \quad x \in \mathbb{R}$

(c) How many complex solutions do the following equations have? Write them down.

i. $x^2 = -1, \quad x \in \mathbb{C}$

ii. $x^3 = -1, \quad x \in \mathbb{C}$

(*Hint: For c) ii), use Euler's formula or/and de Moivre's formula.*)

Solution:

(a) We first write $z = (\frac{\sqrt{3}}{2} + \frac{1}{2}\,\mathrm{i})^{12}$ in polar coordinates. The magnitude is given by

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1.$$

The argument ϕ is given by

$$\phi = \arctan\left(\frac{(1/2)}{(\sqrt{3}/2)}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Using de Moivre's formula, we have,

$$\begin{aligned} z^{12} &= r^{12}e^{12\,\mathrm{i}\,\phi} \\ &= 1^{12}\left(\cos\left(12\frac{\pi}{6}\right) + i\sin\left(12\frac{\pi}{6}\right)\right) \\ &= (\cos(2\pi) + i\sin(2\pi)) \\ &= 1 \end{aligned}$$

(b) i. Since x^2 is always non-negative for real numbers, there is no solution to the equation $x^2 = -1$ over the real numbers.

ii. $x = -1$ is the only solution over the real numbers for the equation $x^3 = -1$.

(c) i. The polynomial equation $x^2 = -1$ has two solutions over the complex numbers.

$$x^2 = -1 \Longrightarrow x = -\,\mathrm{i} \text{ or } x = \mathrm{i}\,.$$

ii. The polynomial equation $x^3 = -1$ has three solutions over the complex numbers.

$$\begin{aligned} x^3 &= -1 \\ x^3 &= e^{ik\pi} = -1, \quad \text{for any odd } k \in \mathbb{Z} \\ x &= e^{ik\pi/3}. \end{aligned}$$

We note that in Euler's formula the argument should be bounded in $[0, 2\pi)$ hence $k\pi/3 \in [0, 2\pi)$ if $k \in \{1, 3, 5\}$. Therefore, $x \in \left\{e^{\mathrm{i}\,\pi/3}, \underbrace{e^{\mathrm{i}\,\pi}}_{-1}, e^{5\pi\,\mathrm{i}\,/3}\right\}$

are solutions to the equation $x^3 = -1$. Remark: the solution -1 which is the solution in $\mathbb{R}$ is also a solution of $\mathbb{C}$ because $\mathbb{R} \subset \mathbb{C}$.

$\square$

30. Let z and w be two non-zero complex numbers. Show that $|z+w| = |z| + |w|$ if and only if z and w have the same argument.
(Hint: Use Theorem 1.50 from the lecture notes and use polar coordinates for z and w.)

Solution:
Let $z = r_1 e^{i\phi_1}$ and $w = r_2 e^{i\phi_2}$, $r_1, r_2 \in \mathbb{R}^+$.

- " $\Rightarrow$ " Assume $|z+w| = |z| + |w|$. From Theorem 1.50, we obtain that $z\bar{w}$ is a real number and $z\bar{w} \geq 0$.

  $$z\bar{w} = \left(r_1 e^{\mathrm{i}\,\phi_1}\right)\left(r_2 e^{-\mathrm{i}\,\phi_2}\right) = r_1 r_2 e^{\mathrm{i}(\phi_1-\phi_2)} = r_1 r_2 \left(\cos\left(\phi_1-\phi_2\right) + \mathrm{i}\sin\left(\phi_1-\phi_2\right)\right).$$

  Since $z\bar{w}$ is real, we obtain

  $$\mathrm{i}\, r_1 r_2 \sin\left(\phi_1-\phi_2\right) = 0$$

  Thus

  $$\sin\left(\phi_1-\phi_2\right) = 0 \Longrightarrow \phi_1 - \phi_2 = k\pi, \quad k \in \mathbb{Z}.$$

  Since $\phi_1 - \phi_2 \in [0, 2\pi)$, we obtain that $k = 0$ or $k = 1$. For $k = 1, z\bar{w} = -r_1 r_2$. This contradicts our assumption that $z\bar{w} \geq 0$. Therefore $k = 0$ is the only possible case and thus $\phi_1 = \phi_2$.

- " $\Leftarrow$ " Assume $\phi_1 = \phi_2$. Then
  $|z+w| = |r_1 e^{\mathrm{i}\,\phi_1} + r_2 e^{\mathrm{i}\,\phi_2}| = |r_1 e^{\mathrm{i}\,\phi_1} + r_2 e^{\mathrm{i}\,\phi_1}|$
  $= |(r_1 + r_2) e^{\mathrm{i}\,\phi_1}| = |r_1 + r_2||e^{\mathrm{i}\,\phi_1}| = r_1 + r_2 = |z| + |w|$

$\square$