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7. Find the largest possible domain in R and the corresponding range.

a) $f(x) = (2-x)^4$

For domain, f(x) is defined for all x.

$\therefore x \in \mathbb{R}$

So, Domain $= (-\infty, \infty)$

Largest possible domain and range is ∞.

For range,

Let, $f(x) = y = (2-x)^4$

So, $y \geq 0$ [as ~~the~~ $(2-x)^4$ always gives positive result]

$\therefore$ Range $= [0, \infty)$

b) $g(x) = \frac{1}{1+\sqrt{x}}$

For domain,

$1+\sqrt{x} \geq 1$ [$\because$ Denominator cannot be 0]

or, $\sqrt{x} \geq 0$

$\therefore x \geq 0$

$\therefore$ Domain $= [0, \infty)$

For range: At $x = 0$, $g(x) = 1$

At $x \to \infty$, $g(x) \to 0$

$\therefore$ Range $= (0, 1]$

Largest possible range is 1.

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c) $h(x) = 3 - |x|$

For domain: $x \in \mathbb{R}$

$\therefore$ Domain $= (-\infty, \infty)$

For range,

Let, $h(x) = 3 - |x| = y$

where, $y \leq 3$

$\therefore$ Range $= (-\infty, 3]$

Largest possible domain is ∞.
and corresponding range is $-\infty$.

d) $u(x) = \frac{1}{\ }$

* d) $u(x) = \frac{1}{1-\sqrt{x}}$

For domain,

$\sqrt{x} \geq 0$ i.e $x \geq 0$

&

$\sqrt{x} \neq 1$ $[\because$ as $x \to 0$, $1-\sqrt{x} \to 0]$

$\therefore$ Domain $= [0, \infty) - 1$

$= [0, 1) \cup (1, \infty)$

For range,

$y = u(x) = \frac{1}{1-\sqrt{x}} \leq 1$

$\therefore$ Range $= (-\infty, 0) \cup (0, 1]$

$= (-\infty, 0) \cup (0, 1]$

10)

8.a) $f : D \to C$ defined as $x \to x^2$

i.e. Let $f(x) = x^2$ where, $x \in D \subset \mathbb{R}$.

To say it's injective, let $D = \{3, 4, 5\}$

then, $C = \{9, 16, 25\}$

every element in C is unique when input is $x_1 \neq x_2 \cdot (x_1, x_2 \in D)$

To say it's not injective:

$D = \{-2, 2\}$

$C = \{4, 4\} = \{4\}$

$\Rightarrow$ -2, & 2 are mapped to 4 which means it's not injective.

b) i) $f(S_i)$, for $i = 1, 2, 3$.

$\boxed{\begin{array}{l} S_1 = \{-1, 0, 1, 2, \pi\} \\ S_2 = \{x \in R : -3 \leq x \leq 4\} \\ S_3 = \{x \in R : x < 0 \vee x > 16\} \end{array}}$

$\Rightarrow \cancel{f(S_1) = f(1)}$ ~~$[1 \in i]$ &~~

$\cancel{= 1^2 = 1}$ ~~$1 \in S_1$~~

$\Rightarrow f(S_1) = \{1, 0, 1, 4, \pi^2\}$

$\Rightarrow f(S_2) = \{y \in R : 9 \leq y \leq 16\}$

$\Rightarrow f(S_3) = \{y \in R : y > 0\}$

c) ii) $T_1 = \{0, 1, 2, \pi\}$, $T_2 = \{x \in R : x \geq 0\}$, $T_3 = \{x \in R : x < 0\}$

$f^{-1}(T_1) = \{0, 1, \sqrt{2}, \sqrt{\pi}\}$

$f^{-1}(T_2) = \{y \in R : y \geq 0\}$

$f^{-1}(T_3) = \{\phi\}$

8b) Let, $C \in R$

Let $C = \{9, 4\} \subset R$

C has pre-image $\{3, 2\}$

Under this, the function f is surjective

Let, $C = \{-3, -4\}$

↳ This has no pre-image under function $f(x) = x^2$.

9.a) $f: R \to R$ with $f(x) = x+3$ and $g(x) = x-5$.

$(g \circ f)(x) = g(f(x)) = g(x+3) = (x+3) - 5 = x - 2$

$(f \circ g)(x) = f(g(x)) = f(x-5) = (x-5)+3 = x - 2$

b) $f: R \to R$ with $f(x) = 3x$ and $g(x) = -5x$

$(g \circ f)(x) = g(f(x)) = g(3x) = -5 \times (3x) = -15x$

$(f \circ g)(x) = f(g(x)) = f(-5x) = 3 \times (-5x) = -15x$

c) No, ~~it~~ the composite $(g \circ f)(x)$ and $(f \circ g)(x)$ are equal if both $f(x)$ and $g(x)$ are linear. But in general, it does not hold true for others.

For example; $f(x) = x^2$ $g(x) = x+3$

$f(g(x)) = f(x+3) = (x+3)^2$

$g(f(x)) = g(x^2) = x^2 + 3$

Here, $(f \circ g)(x) \neq (g \circ f)(x)$

Hence, the statement ~~is~~ can't be apply as a general.

10) a) $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^4$

Injective

Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$

$$x_1^4 = x_2^4$$

or, $x_1 = \pm x_2$ i.e. $x_1 \neq x_2$

$\therefore f(x_1) = f(x_2)$ does not imply $x_1 = x_2$. Hence, it's not injective.

Surjective

Let $y \in \mathbb{R}$, we show $\exists x \in \mathbb{R}: f(x) = y$

Here, $x = \sqrt[4]{y}$, then $f(x) = x^4 = (\sqrt[4]{y})^4 = \pm y \neq y$

Hence, it's neither surjective nor injective.

So, this function is not bijective either.

b) $g: \mathbb{R}_0^+ \to \mathbb{R}_0^-$ with $g(x) = -x^2$

injective

Let $x_1, x_2 \in \mathbb{R}_0^+$ such that $g(x_1) = g(x_2)$

if it's injective then $x_1 = x_2$.

Now,

$$g(x_1) = g(x_2)$$

or, $-(x_1)^2 = -(x_2)^2$

$\therefore x_1 = x_2$

Here, $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$. Hence, it's Bijective.

10c

Surjective

Let $y \in \mathbb{R}_0^-$, we show $\exists x \in \mathbb{R}_0^+ : g(x) = y$

We know, $\cancel{y = \sqrt{-x^2}}$

$$y = -x^2$$

$$\text{or, } \boxed{x = \sqrt{-y}} \quad (y \in \mathbb{R}_0^-)$$

then, $g(x) = -x^2 = -(\sqrt{-y})^2 = y$.

So, this function is surjective.

Inverse

$y = g(x) = -x^2$

or, $x^2 = -y$

or, $x = \sqrt{-y}$

Interchanging x and y.

$y = \sqrt{-x}$

$\therefore \boxed{g^{-1}(x) = \sqrt{-x}}$

As, it is both injective and surjective, the given function is bijective.

10.c) $h : R \setminus \{0\} \to R \setminus \{0\}$ with $h(x) = -x^{-1}$

$$h(x) = -x^{-1} = -\frac{1}{x}$$

Injective

Let $x_1, x_2 \in R \setminus \{0\}$ such that $h(x_1) = h(x_2)$

$$h(x_1) = h(x_2)$$

$$\text{or, } -\frac{1}{x_1} = -\frac{1}{x_2}$$

$\therefore x_1 = x_2$ implies the function is injective.

Surjective

Let $y \in \mathbb{R} \setminus \{0\}$, we show $\exists x \in \mathbb{R} \setminus \{0\} : h(x) = y$

We know, $y = -\frac{1}{x}$ or, $\boxed{x = -\frac{1}{y}}$

then, $h(x) = \frac{-1}{x} = -\frac{1}{\frac{-1}{y}} = y$

so, this is surjective.

Hence, the function is bijective

Inverse

$h(x) = y = -x^{-1} = -\frac{1}{x}$

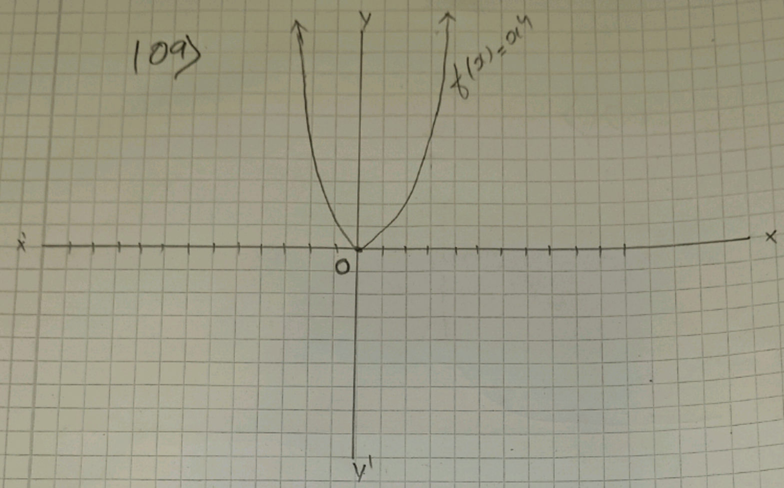
or, $x = -\frac{1}{y}$

Interchangin x and y.

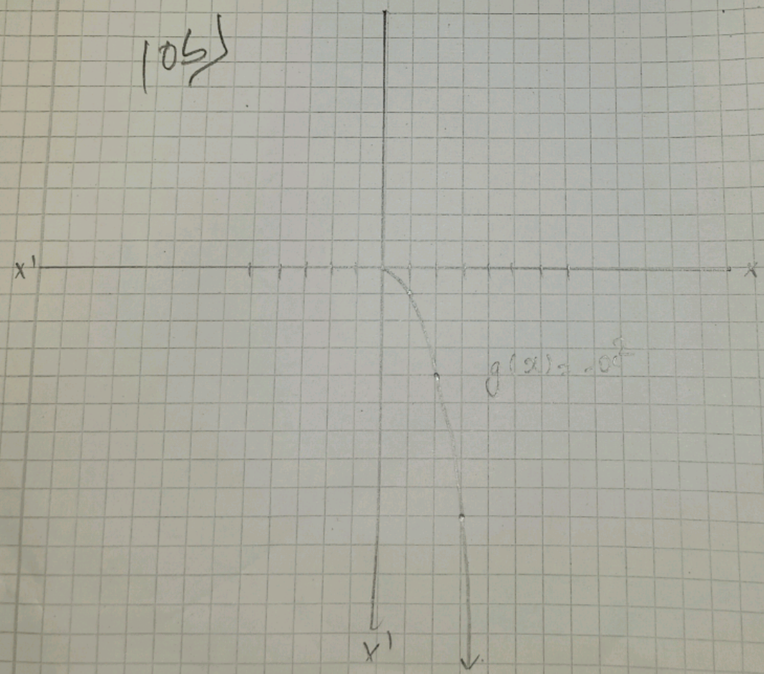
$y = -\frac{1}{x}$

$\therefore \boxed{h^{-1}(x) = \frac{-1}{x} = -x^{-1}}$

10a)

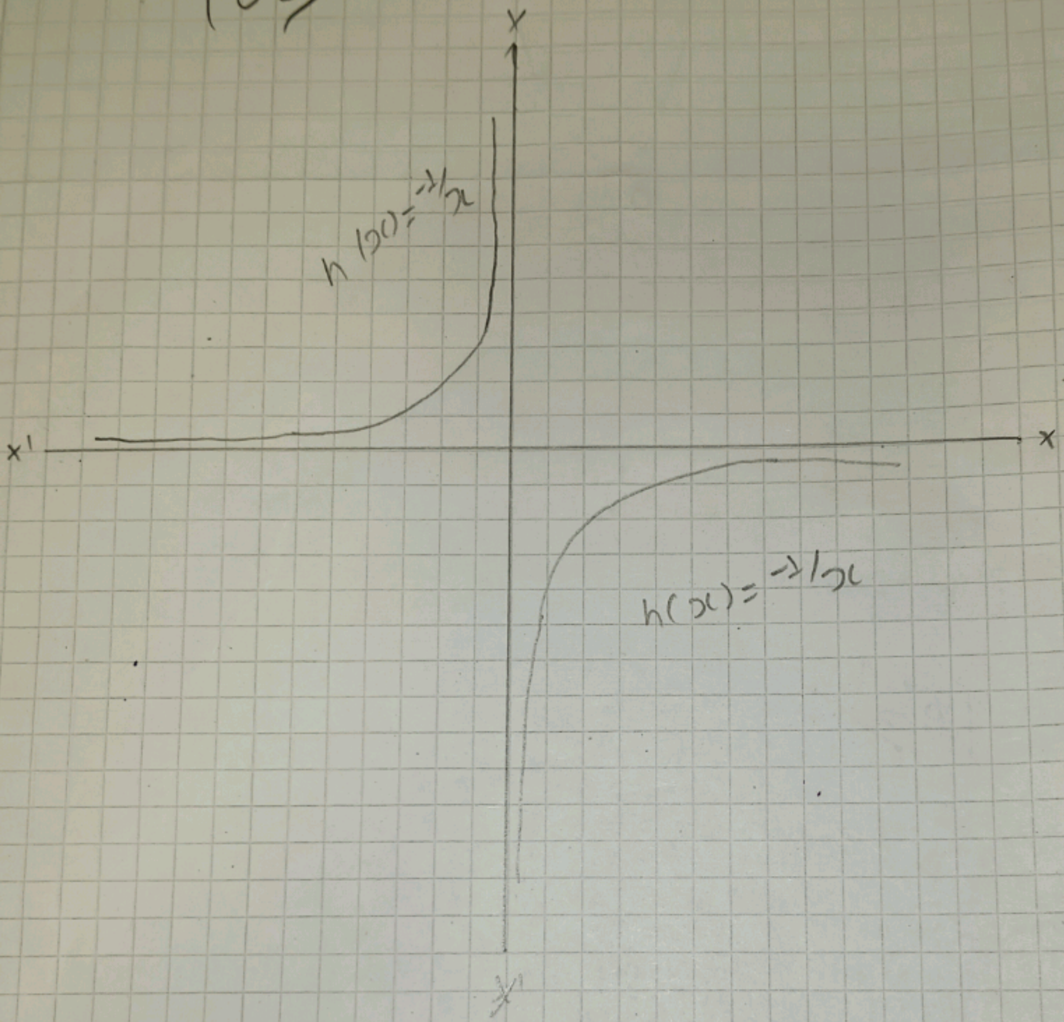


10b)



10c

10c)



11. a) For the divisibility relation has to be partial order, it has to be reflexive, anti-symmetric and transitive.

Let m and n be integers. we say that m divides n the $m|n$, if there exists $k \in N$ such that $\underline{mk = n}$.

$$\boxed{m | n} \longleftrightarrow \boxed{mk = n}$$

- **Reflexity**: to show: ~~m|n~~ $m|m$

$$m|m \longleftrightarrow mk = m$$
$$\cancel{or}\ k = 1 \in R$$

Hence, it is reflexive.

- ~~Anti-symmetric~~ : ~~$m|b \wedge b|m \Rightarrow a = b$~~

~~$a|b \wedge b|a \longleftrightarrow a\cdot$~~

- **Anti-symmetric**

$$m|n \wedge n|m \Rightarrow m = n \quad \text{(proving)}$$

$$m|n \wedge n|m \longleftrightarrow \boxed{m \cdot k = n} \wedge \boxed{n \cdot l = m} \quad (l \in R, k \in R)$$

$$\text{or } m \cdot k \cdot l = m$$

Case i: $m \neq 0$ ~~or, k.l~~

$k \cdot l = 1$

i.e $\boxed{k = 1 \wedge l = 1}$ $(k, l \in R)$

which means

$\boxed{m = n}$

Case II: $m = 0$

that means

$m \cdot k = n$

$\therefore n = 0$

$\therefore \boxed{m = n = 0}$

- Transitivity: Let q be another integer

~~$m|n \wedge n|q \Rightarrow$~~

To prove:
$$m|n \wedge n|q \Rightarrow m|q$$

Here,
$$m|n \wedge n|q \Leftrightarrow \boxed{m \cdot k = n} \wedge n \cdot l = q$$

or, $mk \cdot l = q$

or, $\boxed{m \cdot k \cdot l = q}$

Let, $k \cdot l$ is natural number p.

$$\boxed{\therefore m \cdot p = q \Leftrightarrow m|q}$$

As, divisibility relation is reflexive, anti-symmetric and transitive, it's partial order.

~~1+b)~~ ~~2|4|6~~

~~3|6~~

1(i·b) 1|2 2|4 3|6

1|3 2|6

1|4

1|5

1|6

These are the pairs.