37. Let $M := \mathbb{R}^{m\times n}$ and suppose that $R \subset M \times M$ is the relation such that $(A, B) \in R$ if and only if the matrices A and B are related by a sequence of row operations. Show that R is an equivalence relation.

    Solution:

    (a) **Reflexive**. For any matrix M in $\mathbb{R}^n$, the pair (M, M) belongs to R trivially, as any matrix is trivially related to itself by a sequence of zero row operations.

    (b) **Symmetric**. For any matrices A and B in $\mathbb{R}^n$, if (A, B) belongs to R, this implies that A can be transformed into B through a series of row operations. Since each row operation has an inverse, applying these inverse operations in reverse order to B yields A. Thus, (B, A) also belongs to R.

    (c) **Transitive**. Consider matrices A, B, and C in $\mathbb{R}^n$. If (A, B) is in R, A is related to B by a series of row operations. Similarly, if (B, C) is in R, B is related to C by another series of row operations. Concatenating these two sequences (first applying the operations that transform A to B, and then those transforming B to C), leads to a direct transformation from A to C, making (A, C) a member of R as well.

    $\square$

38. Determine the solution set $L(A, \mathbf{b})$ where,

$$A = \begin{pmatrix} 2 & 4 & 7 & 2 & -1 & -4 \\ -1 & 9 & 9 & -3 & -6 & -3 \\ -1 & 2 & 8 & -2 & 7 & 1 \\ 3 & -6 & 4 & 4 & 9 & 1 \\ -3 & 4 & -8 & -6 & -5 & -3 \\ -3 & -5 & 0 & -3 & 8 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -6 \\ -3 \\ 1 \\ 3 \\ -1 \\ -3 \end{pmatrix}.$$

Solution:

- **Note**: In this exercise $\mathbf{b}$ is not the correct vector. Instead, the correct $\mathbf{b}$ is

$$\mathbf{b} = \begin{pmatrix} 2 \\ -6 \\ -8 \\ 4 \\ -6 \\ 10 \end{pmatrix}.$$

Then the system $A\mathbf{x} = \mathbf{b}$, has the unique solution

$$\mathbf{x} = \begin{pmatrix} -6 \\ -3 \\ 1 \\ 3 \\ -1 \\ -3 \end{pmatrix}.$$

which mistakenly was given as $\mathbf{b}$.
In the following solution will use the correct $\mathbf{b}$. This does not change the row operations we are going to use or the order in which we use them. It only changes the final solution $\mathbf{x}$ (the last row in the augmented matrix).

So starting from the augmented matrix we get:

$$\left(\begin{array}{cccccc|c} 2 & 4 & 7 & 2 & -1 & -4 & 2 \\ -1 & 9 & 9 & -3 & -6 & -3 & -6 \\ -1 & 2 & 8 & -2 & 7 & 1 & -8 \\ 3 & -6 & 4 & 4 & 9 & 1 & 4 \\ -3 & 4 & -8 & -6 & -5 & -3 & -6 \\ -3 & -5 & 0 & -3 & 8 & 2 & 10 \end{array}\right)$$

$$\xrightarrow[R_1=(1/2)R_1]{} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ -1 & 9 & 9 & -3 & -6 & -3 & -6 \\ -1 & 2 & 8 & -2 & 7 & 1 & -8 \\ 3 & -6 & 4 & 4 & 9 & 1 & 4 \\ -3 & 4 & -8 & -6 & -5 & -3 & -6 \\ -3 & -5 & 0 & -3 & 8 & 2 & 10 \end{array}\right)$$

$$\xrightarrow[\substack{R_4=R_4-3R_1\\R_5=R_5+3R_1\\R_6=R_6+3R_1}]{\substack{R_2=R_2+1R_1\\R_3=R_3+1R_1}} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 11 & 25/2 & -2 & -13/2 & -5 & -5 \\ 0 & 4 & 23/2 & -1 & 13/2 & -1 & -7 \\ 0 & -12 & -13/2 & 1 & 21/2 & 7 & 1 \\ 0 & 10 & 5/2 & -3 & -13/2 & -9 & -3 \\ 0 & 1 & 21/2 & 0 & 13/2 & -4 & 13 \end{array}\right)$$

$$\xrightarrow[R_2=(1/11)R_2]{} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 25/22 & -2/11 & -13/22 & -5/11 & -5/11 \\ 0 & 4 & 23/2 & -1 & 13/2 & -1 & -7 \\ 0 & -12 & -13/2 & 1 & 21/2 & 7 & 1 \\ 0 & 10 & 5/2 & -3 & -13/2 & -9 & -3 \\ 0 & 1 & 21/2 & 0 & 13/2 & -4 & 13 \end{array}\right)$$

$$\xrightarrow[\substack{R_5=R_5-10R_2\\R_6=R_6-1R_2}]{\substack{R_3=R_3-4R_2\\R_4=R_4+12R_2}} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 25/22 & -2/11 & -13/22 & -5/11 & -5/11 \\ 0 & 0 & 153/22 & -3/11 & 195/22 & 9/11 & -57/11 \\ 0 & 0 & 157/22 & -13/11 & 75/22 & 17/11 & -49/11 \\ 0 & 0 & -195/22 & -13/11 & -13/22 & -49/11 & 17/11 \\ 0 & 0 & 103/11 & 2/11 & 78/11 & -39/11 & 148/11 \end{array}\right)$$

$$\xrightarrow[R_3=(22/153)R_3]{} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 25/22 & -2/11 & -13/22 & -5/11 & -5/11 \\ 0 & 0 & 1 & -2/51 & 65/51 & 2/17 & -38/51 \\ 0 & 0 & 157/22 & -13/11 & 75/22 & 17/11 & -49/11 \\ 0 & 0 & -195/22 & -13/11 & -13/22 & -49/11 & 17/11 \\ 0 & 0 & 103/11 & 2/11 & 78/11 & -39/11 & 148/11 \end{array}\right)$$

$$\xrightarrow[\substack{R_5=R_5+195/22R_3\\R_6=R_6-103/11R_3}]{R_4=R_4-157/22R_3} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 25/22 & -2/11 & -13/22 & -5/11 & -5/11 \\ 0 & 0 & 1 & -2/51 & 65/51 & 2/17 & -38/51 \\ 0 & 0 & 0 & -46/51 & -290/51 & 12/17 & 44/51 \\ 0 & 0 & 0 & -26/17 & 182/17 & -58/17 & -86/17 \\ 0 & 0 & 0 & 28/51 & -247/51 & -79/17 & 1042/51 \end{array}\right)$$

$$\xrightarrow[R_4\leftrightarrow R_5]{} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 25/22 & -2/11 & -13/22 & -5/11 & -5/11 \\ 0 & 0 & 1 & -2/51 & 65/51 & 2/17 & -38/51 \\ 0 & 0 & 0 & -26/17 & 182/17 & -58/17 & -86/17 \\ 0 & 0 & 0 & -46/51 & -290/51 & 12/17 & 44/51 \\ 0 & 0 & 0 & 28/51 & -247/51 & -79/17 & 1042/51 \end{array}\right)$$

$$\xrightarrow[R_4=(-17/26)R_4]{} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 25/22 & -2/11 & -13/22 & -5/11 & -5/11 \\ 0 & 0 & 1 & -2/51 & 65/51 & 2/17 & -38/51 \\ 0 & 0 & 0 & 1 & -7 & 29/13 & 43/13 \\ 0 & 0 & 0 & -46/51 & -290/51 & 12/17 & 44/51 \\ 0 & 0 & 0 & 28/51 & -247/51 & -79/17 & 1042/51 \end{array}\right)$$

$$\xrightarrow[\substack{R_5=R_5+46/51R_4\\R_6=R_6-28/51R_4}]{}\left(\begin{array}{cccccc|c}1&2&7/2&1&-1/2&-2&1\\0&1&25/22&-2/11&-13/22&-5/11&-5/11\\0&0&1&-2/51&65/51&2/17&-38/51\\0&0&0&1&-7&29/13&43/13\\0&0&0&0&-12&106/39&50/13\\0&0&0&0&-1&-229/39&242/13\end{array}\right)$$

$$\xrightarrow[R_5\leftrightarrow R_6]{}\left(\begin{array}{cccccc|c}1&2&7/2&1&-1/2&-2&1\\0&1&25/22&-2/11&-13/22&-5/11&-5/11\\0&0&1&-2/51&65/51&2/17&-38/51\\0&0&0&1&-7&29/13&43/13\\0&0&0&0&-1&-229/39&242/13\\0&0&0&0&-12&106/39&50/13\end{array}\right)$$

$$\xrightarrow[R_5=(-1)R_5]{}\left(\begin{array}{cccccc|c}1&2&7/2&1&-1/2&-2&1\\0&1&25/22&-2/11&-13/22&-5/11&-5/11\\0&0&1&-2/51&65/51&2/17&-38/51\\0&0&0&1&-7&29/13&43/13\\0&0&0&0&1&229/39&-242/13\\0&0&0&0&-12&106/39&50/13\end{array}\right)$$

$$\xrightarrow[R_6=R_6+12R_5]{}\left(\begin{array}{cccccc|c}1&2&7/2&1&-1/2&-2&1\\0&1&25/22&-2/11&-13/22&-5/11&-5/11\\0&0&1&-2/51&65/51&2/17&-38/51\\0&0&0&1&-7&29/13&43/13\\0&0&0&0&1&229/39&-242/13\\0&0&0&0&0&2854/39&-2854/13\end{array}\right)$$

$$\xrightarrow[R_6=(39/2854)R_6]{}\left(\begin{array}{cccccc|c}1&2&7/2&1&-1/2&-2&1\\0&1&25/22&-2/11&-13/22&-5/11&-5/11\\0&0&1&-2/51&65/51&2/17&-38/51\\0&0&0&1&-7&29/13&43/13\\0&0&0&0&1&229/39&-242/13\\0&0&0&0&0&1&-3\end{array}\right)$$

$$\xrightarrow[\substack{R_3=R_3-2/17R_6\\R_2=R_2+5/11R_6\\R_1=R_1+2R_6}]{\substack{R_5=R_5-229/39R_6\\R_4=R_4-29/13R_6}}\left(\begin{array}{cccccc|c}1&2&7/2&1&-1/2&0&-5\\0&1&25/22&-2/11&-13/22&0&-20/11\\0&0&1&-2/51&65/51&0&-20/51\\0&0&0&1&-7&0&10\\0&0&0&0&1&0&-1\\0&0&0&0&0&1&-3\end{array}\right)$$

$$\xrightarrow[\substack{R_2=R_2+13/22R_5\\R_1=R_1+1/2R_5}]{\substack{R_4=R_4+7R_5\\R_3=R_3-65/51R_5}}\left(\begin{array}{cccccc|c}1&2&7/2&1&0&0&-11/2\\0&1&25/22&-2/11&0&0&-53/22\\0&0&1&-2/51&0&0&15/17\\0&0&0&1&0&0&3\\0&0&0&0&1&0&-1\\0&0&0&0&0&1&-3\end{array}\right)$$

$$\xrightarrow[\substack{R_2=R_2+2/11R_4\\R_1=R_1-1R_4}]{R_3=R_3+2/51R_4} \left(\begin{array}{cccccc|c} 1 & 2 & 7/2 & 0 & 0 & 0 & -17/2 \\ 0 & 1 & 25/22 & 0 & 0 & 0 & -41/22 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

$$\xrightarrow[\substack{R_2=R_2-25/22R_3\\R_1=R_1-7/2R_3}]{} \left(\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 0 & 0 & -12 \\ 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

$$\xrightarrow[R_1=R_1-2R_2]{} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

$\square$

39. (a) Show that elementary matrices $E \in \mathbb{R}^{n\times n}$ are invertible and that the inverse E^{-1} is also an elementary matrix.

(b) For each of the following matrices, determine its inverse and clearly describe the row operation used to obtain it, starting from the identity matrix:

$$E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Solution:

(a) Elementary matrices $E \in \mathbb{R}^{n\times n}$ are defined as matrices that can be obtained from the identity matrix by performing a single row operation. When we multiply a matrix A by an elementary matrix E from the left ($E \cdot A$), we are effectively applying the corresponding row operation of E to A.

Now, note that the effect of a single row operation can be reversed by another single row operation. Specifically:

- Swapping rows i and j can be reversed by swapping the same rows again (the inverse of $R_i \leftrightarrow R_j$ is $R_i \leftrightarrow R_j$).
- Multiplying a row by a nonzero scalar λ can be reversed by multiplying the same row by $\frac{1}{\lambda}$ (the inverse of $R_i = \lambda R_i$ is $R_i = \frac{1}{\lambda} R_i$).
- Adding λ times row j to row i can be reversed by adding $-\lambda$ times row j from row i (the inverse of $R_i = R_i + \lambda R_j$ is $R_i = R_i - \lambda R_j$).

Therefore, for each elementary matrix E corresponding to a row operation, there exists another elementary matrix F, constructed as described above, corresponding to the reverse row operation.

Multiplying E by F results in the identity matrix. Since E is derived from the identity matrix by a row operation, and the multiplication by F (from the left) reverses this row operation, the product must be the identity matrix. This is also true if we multiply F by E (from the left), confirming that E is invertible and $F = E^{-1}$ is also an elementary matrix.

(b)
- E_1 is a 2×2 matrix corresponding to the row operation $R_1 = R_1 + R_2$. Therefore, its inverse would be the elementary matrix corresponding to the row operation $R_1 = R_1 - R_2$:

$$E_1^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

- E_2 is a 3×3 matrix corresponding to the row operation $R_2 = (-2)R_2$. Therefore, its inverse would be the elementary matrix corresponding to the row operation $R_2 = -\frac{1}{2} R_2$:

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- E_3 is a 3×3 matrix corresponding to the row operation $R_3 = R_3 + 2R_1$. Therefore, its inverse would be the elementary matrix corresponding to the row operation $R_3 = R_3 - 2R_1$:

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

□

40. In this exercise, you are required to prove, through a two-part process, that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that $T = T_A$.

    (a) First, prove the direct statement: if such a matrix A exists, then T is a linear transformation.

    (b) Then, prove the converse statement: if T is a linear transformation, then such a matrix A exists.

    Hint: For the second part, it may be helpful to use the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.

    Solution:

    Consider the function $T : \mathbb{R}^n \to \mathbb{R}^m$.

    (a) Suppose there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that $T = T_A$ (i.e., $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$). Then, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any scalar $\lambda \in \mathbb{R}$, we have:

    - $T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$.
    - $T(\lambda\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda T(\mathbf{x})$.

    Therefore, T is a linear transformation.

    (b) Conversely, if T is a linear transformation, we aim to demonstrate the existence of a matrix $A \in \mathbb{R}^{m\times n}$ such that $T = T_A$.

    Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ with $T(\mathbf{x}) = \mathbf{y}$. Using the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, namely $\{e_1, e_2, \ldots, e_n\}$ and $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_m\}$, respectively, we can express $\mathbf{x}$ and $\mathbf{y}$ as

    $$\mathbf{x} = (x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i \quad \text{and} \quad \mathbf{y} = (y_1, y_2, \ldots, y_m) = \sum_{j=1}^{m} y_j \tilde{e}_j.$$

    The linearity of T implies:

    $$T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T(e_i).$$

    Given that $T(e_i) \in \mathbb{R}^m$ for each i, these vectors can be represented in terms of the basis vectors $\tilde{e}_j$. Suppose

    $$T(e_i) = \sum_{j=1}^{m} a_{ji}\tilde{e}_j$$

    for some $a_{ji} \in \mathbb{R}$.

    Then,

    $$T(\mathbf{x}) = \sum_{i=1}^{n}\left(x_i \sum_{j=1}^{m} a_{ji}\tilde{e}_j\right) = \sum_{j=1}^{m} y_j \tilde{e}_j.$$

    Upon expanding, we find

    $$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_i a_{ji}\tilde{e}_j\right) = \sum_{j=1}^{m} y_j \tilde{e}_j.$$

Since any vector in $\mathbb{R}^m$ can be uniquely represented in terms of the basis vectors $\tilde{e}_j$, it follows that

$$\sum_{i=1}^{n} a_{ji} x_i = y_j \quad \text{for each } j = 1, \ldots, m.$$

This establishes the existence of a matrix $A \in \mathbb{R}^{m \times n}$ with entries $A_{ji} = a_{ji}$ such that for every $\mathbf{x}$, $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$. Consequently, $T = T_A$.

□

41. Compute the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 6 & 0 \\ 4 & 11 & 8 \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & \cdots & 0 & d_n \\ 0 & 0 & \cdots & d_{n-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & d_2 & \cdots & 0 & 0 \\ d_1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Notice that C is a 4×4 matrix, with C_1 and C_2 being 2×2 matrices with known determinants, $\det C_1$ and $\det C_2$, and O is the zero 2×2 matrix. Also, D is a $n \times n$ matrix where $d_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$.

Solution:
Matrix A is just a simple case of a 2×2 matrix and its determinant is given by

$$\det A = \det \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix} = 1 \cdot 6 - 2 \cdot 4 = -2$$

Matrix B is a 3×3 matrix and its determinant can be calculated either using Sarrus rule, or by expansion along any row or column. For instance, along the first row:

$$\begin{aligned} \det B = \det \begin{pmatrix} 1 & 3 & -2 \\ 2 & 6 & 0 \\ 4 & 11 & 8 \end{pmatrix} &= 1 \det \begin{pmatrix} 6 & 0 \\ 11 & 8 \end{pmatrix} - 3 \det \begin{pmatrix} 2 & 0 \\ 4 & 8 \end{pmatrix} + (-2) \det \begin{pmatrix} 2 & 6 \\ 4 & 11 \end{pmatrix} \\ &= 1 \cdot (6 \cdot 8) - 3 \cdot (2 \cdot 8) - 2 \cdot (2 \cdot 11 - 4 \cdot 6) \\ &= 48 - 48 - 2(22 - 24) = 4 \end{aligned}$$

For matrix C, let

$$C_1 = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \qquad \text{and} \qquad C_2 = \begin{pmatrix} c_5 & c_6 \\ c_7 & c_8 \end{pmatrix}$$

from which

$$C = \begin{pmatrix} c_1 & c_2 & 0 & 0 \\ c_3 & c_4 & 0 & 0 \\ 0 & 0 & c_5 & c_6 \\ 0 & 0 & c_7 & c_8 \end{pmatrix}$$

for $c_1, c_2, \ldots, c_8 \in \mathbb{R}$. Now by expansion along the first row of C, we get

$$\begin{aligned} \det C &= c_1 \det \begin{pmatrix} c_4 & 0 & 0 \\ 0 & c_5 & c_6 \\ 0 & c_7 & c_8 \end{pmatrix} - c_2 \det \begin{pmatrix} c_3 & 0 & 0 \\ 0 & c_5 & c_6 \\ 0 & c_7 & c_8 \end{pmatrix} \\ &= c_1 c_4 \det \begin{pmatrix} c_5 & c_6 \\ c_7 & c_8 \end{pmatrix} - c_2 c_3 \det \begin{pmatrix} c_5 & c_6 \\ c_7 & c_8 \end{pmatrix} \end{aligned}$$

Now, $\det\begin{pmatrix} c_5 & c_6 \\ c_7 & c_8 \end{pmatrix}$ is just $\det C_2$ and taking out as a common factor we get

$$\det C = (c_1 c_4 - c_2 c_3) \det C_2$$

The term inside the parenthesis is just the determinant of C_1, which leads to

$$\det C = \det C_1 \cdot \det C_2$$

For matrix D, we can also try to use expansion along the first row:

$$\det D = (-1)^{1+n} d_n \det \begin{pmatrix} 0 & 0 & \cdots & d_{n-1} \\ \vdots & \vdots & \iddots & \vdots \\ 0 & d_2 & \cdots & 0 \\ d_1 & 0 & \cdots & 0 \end{pmatrix}$$

We see that only one term of this expansion survives, since there is only one non zero element in the first row of D. Now we can repeat the same procedure for the determinant of the $(n-1) \times (n-1)$ and we get

$$\det D = (-1)^{1+n} d_n \cdot (-1)^{1+(n-1)} d_{n-1} \det \begin{pmatrix} 0 & 0 & \cdots & d_{n-2} \\ \vdots & \vdots & \iddots & \vdots \\ 0 & d_2 & \cdots & 0 \\ d_1 & 0 & \cdots & 0 \end{pmatrix}$$

Repeating this iteratevly untile we then get

$$\begin{aligned} \det D &= (-1)^{1+n} d_n \cdot (-1)^{1+(n-1)} d_{n-1} \cdots (-1)^{1+2} d_2 \det \begin{pmatrix} d_1 \end{pmatrix} \\ &= \prod_{k=2}^{n} (-1)^{1+k} d_k \ \cdot \ d_1 \\ &= \prod_{k=2}^{n} (-1)^{1+k} \prod_{k=2}^{n} d_k \ \cdot \ d_1 \\ &= (-1)^{\sum_{k=2}^{n}(1+k)} \prod_{k=1}^{n} d_k \end{aligned}$$

Now,

$$\begin{aligned} \sum_{k=2}^{n}(k+1) = \sum_{k=1}^{n-1}(k+2) &= \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 2 \\ &= \frac{(n-1)n}{2} + 2(n-1) \\ &= \frac{n(n+3)}{2} - 2 \end{aligned}$$

So

$$\begin{aligned} \det D &= (-1)^{\frac{n(n+3)}{2} - 2} \prod_{k=1}^{n} d_k \\ &= (-1)^{\frac{n(n+3)}{2}} \prod_{k=1}^{n} d_k \end{aligned}$$

which hold for any $n \in \mathbb{N}$

This results become more clear if we separately look at even and odd cases for n

- For $n = 2m$ with $m \in \mathbb{N}$, (that is, for n even), we get

$$\det D = (-1)^{\frac{n(n+3)}{2}} \prod_{k=1}^{n} d_k = (-1)^{m(2m+3)} \prod_{k=1}^{n} d_k$$

  Now $2m + 3$ is always odd, so the exponent $m(2m + 3)$ is even if and only if m is even. Therefor we can write

$$\det D = (-1)^m \prod_{k=1}^{n} d_k$$

- For $n = 2m + 1$ with $m \in \mathbb{N}$, (that is, for n odd), we get

$$\det D = (-1)^{\frac{n(n+3)}{2}} \prod_{k=1}^{n} d_k = (-1)^{\frac{(2m+1)(2m+4)}{2}} \prod_{k=1}^{n} d_k = (-1)^{(2m+1)(m+2)} \prod_{k=1}^{n} d_k$$

  Now $2m + 1$ is always odd, so the exponent $(2m + 1)(m + 2)$ is even if and only if m is even. Therefor we can again write

$$\det D = (-1)^m \prod_{k=1}^{n} d_k$$

$\square$

42. Compute the inverse of

 (a) $A = \begin{pmatrix} -9 & 7 & 3 \\ -13 & 9 & 4 \\ -3 & 2 & 1 \end{pmatrix}$

 (b) $\mathrm{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and $d \in \mathbb{R}$. Determine the condition(s) that a, b, c and d must satisfy for matrix B to be invertible.

Solution:

(a) Using Gaussian elimination we get

$$\left(\begin{array}{ccc|ccc} -9 & 7 & 3 & 1 & 0 & 0 \\ -13 & 9 & 4 & 0 & 1 & 0 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1=R_1-3R_3]{} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -3 \\ -13 & 9 & 4 & 0 & 1 & 0 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1\leftrightarrow R_2]{} \left(\begin{array}{ccc|ccc} -13 & 9 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1=R_1-9R_2]{} \left(\begin{array}{ccc|ccc} -13 & 0 & 4 & -9 & 1 & 27 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3=R_3-2R_2]{} \left(\begin{array}{ccc|ccc} -13 & 0 & 4 & -9 & 1 & 27 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ -3 & 0 & 1 & -2 & 0 & 7 \end{array}\right)$$

$$\xrightarrow[R_1=R_1-4R_3]{} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ -3 & 0 & 1 & -2 & 0 & 7 \end{array}\right)$$

$$\xrightarrow[R_1=(-1)R_1]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ -3 & 0 & 1 & -2 & 0 & 7 \end{array}\right)$$

$$\xrightarrow[R_3=R_3+3R_1]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & -3 & 10 \end{array}\right)$$

It follows that

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 10 \end{pmatrix}$$

(b) If B is invertible then there must exist a matrix $C \in \mathbb{R}^{2\times 2}$

$$C = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

such that $BC = CB = I_2$, or more explicitly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrix multiplication then gives

$$\begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from which the following system of equations is derived.

$$ax + bz = 1 \tag{1}$$
$$cx + dz = 0 \tag{2}$$
$$ay + bw = 0 \tag{3}$$
$$cy + dw = 1 \tag{4}$$

Multiplying eq.(1) by d and eq.(2) by $-b$ gives

$$\begin{aligned} dax + dbz &= d \\ -bcx - bdz &= 0 \end{aligned}$$

Adding these equations together leads to

$$(ad - bc)x = d$$

which admits the solution

$$x = \frac{d}{ad - bc}$$

if and only if $ad - bc \neq 0$.
Similarly, multiplying eq.(1) by $-c$ and eq.(2) by a gives

$$\begin{aligned} -cax - cbz &= -c \\ acx + adz &= 0 \end{aligned}$$

Adding these equations together leads to

$$(ad - bc)z = -c$$

which admits the solution

$$z = \frac{-c}{ad - bc}$$

if and only if $ad - bc \neq 0$
We do the same for eq.(3) and eq.(4). Multiplying eq.(3) by $-c$ and eq.(4) by a gives

$$\begin{aligned} -cay - cbw &= 0 \\ acy + adw &= a \end{aligned}$$

Adding these equations together leads to

$$(ad - bc)w = a$$

which admits the solution

$$w = \frac{a}{ad - bc}$$

if and only if $ad - bc \neq 0$. Similarly, multiplying eq.(3) by d and eq.(4) by $-b$ we eventually find y

$$y = \frac{-b}{ad - bc}.$$

Having found all x, y, z, and w, we conclude that the matrix C exists and is given by

$$C = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for $ad - bc \neq 0$.

Notice that $ad - bc = \det(B)$ by the definition of the determinant of a 2×2 matrix. We conclude that this is the condition that must be fulfilled in order for C to exist and B to be invertible.

$\square$