31. Prove that
$$\sqrt{3a^2+ab}+\sqrt{3b^2+bc}+\sqrt{3c^2+ca}\le 2(a+b+c)$$
holds for all non-negative real numbers a,b,c.
Hint. Use the Cauchy-Schwarz inequality.

Solution:
Let $\mathbf{u}=\left(\sqrt{a},\sqrt{b},\sqrt{c}\right)$ and $\mathbf{v}=\left(\sqrt{3a+b},\sqrt{3b+c},\sqrt{3c+a}\right)$. The Cauchy-Schwarz inequality states that for all $\mathbf{u},\mathbf{v}\in\mathbb{C}^d$,
$$|\langle\mathbf{u},\mathbf{v}\rangle|\le\|\mathbf{u}\|_2\|\mathbf{v}\|_2.$$
Since $(u_i,v_i)_{1\le i\le 3}\in\mathbb{R}_{\ge 0}\times\mathbb{R}_{\ge 0}$, $\overline{u_i}=u_i$ and $\overline{v_i}=v_i$ for all $1\le i\le 3$. Therefore,
$$\begin{aligned}|\langle\mathbf{u},\mathbf{v}\rangle|=\left|\sum_{i=1}^{3}u_iv_i\right|&=\left|\sqrt{a}\left(\sqrt{3a+b}\right)+\sqrt{b}\left(\sqrt{3b+c}\right)+\sqrt{c}\left(\sqrt{3c+a}\right)\right|\\&=\sqrt{3a^2+ab}+\sqrt{3b^2+bc}+\sqrt{3c^2+ca}.\end{aligned}$$
Note that for any $r\in\mathbb{R}_{\ge 0}$, $|r|=r$. Consequently,
$$\|\mathbf{u}\|_2=\sqrt{\sum_{i=1}^{3}|u_i|^2}=\sqrt{\left|\sqrt{a}\right|^2+\left|\sqrt{b}\right|^2+\left|\sqrt{c}\right|^2}=\sqrt{a+b+c}.$$
Similarly,
$$\begin{aligned}\|\mathbf{v}\|_2=\sqrt{\sum_{i=1}^{3}|v_i|^2}&=\sqrt{\left|\sqrt{3a+b}\right|^2+\left|\sqrt{3b+c}\right|^2+\left|\sqrt{3c+a}\right|^2}\\&=\sqrt{(3a+b)+(3b+c)+(3c+a)}\\&=\sqrt{4\left(a+b+c\right)}=2\sqrt{a+b+c}.\end{aligned}$$
Finally, applying the Cauchy-Schwarz inequality, we obtain
$$\sqrt{3a^2+ab}+\sqrt{3b^2+bc}+\sqrt{3c^2+ca}\le 2\cdot\sqrt{a+b+c}\cdot\sqrt{a+b+c}=2\left(a+b+c\right).$$

$\square$

32. Let $A \in \mathbb{R}^{2\times 3}$, $B \in \mathbb{R}^{3\times 3}$ and $C \in \mathbb{R}^{3\times 2}$ be the matrices as follows:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -7 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -3 \\ -2 & 1 \\ 0 & 0 \end{pmatrix}$$

Which of the following expressions are well-defined? Compute the result if possible.

(a) $A \cdot B$ (b) $B \cdot A$ (c) $A \cdot (B \cdot C)$ (d) $C \cdot (B \cdot A)$ (e) $A \cdot (B + C)$ (f) $5 \cdot (A^\top + C)$

(g) $B^\top \cdot A^\top$.

Solution:

(a) $A \cdot B = \begin{pmatrix} 3 & 6 & 9 \\ -5 & -10 & -15 \end{pmatrix}$.

(b) Inner matrix dimensions do not agree.

(c) $A \cdot (B \cdot C) = \begin{pmatrix} 1 & 1 & 0 \\ -7 & -1 & 0 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ -6 & -2 \\ -9 & -3 \end{pmatrix} = \begin{pmatrix} -9 & -3 \\ 15 & 5 \end{pmatrix}$.

(d) Inner matrix dimensions do not agree.

(e) Problem with addition of matrices B and C as dimensions of the matrices B and C do not agree.

(f) $5 \cdot (A^\top + C) = 5 \cdot \left(\begin{pmatrix} 1 & -7 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -2 & 1 \\ 0 & 0 \end{pmatrix} \right) = 5 \cdot \begin{pmatrix} 2 & -10 \\ -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 10 & -50 \\ -5 & 10 \\ 0 & 0 \end{pmatrix}$.

(g) Following the definition of transposition of a matrix, we have

$$B^\top \cdot A^\top = \begin{pmatrix} 3 & -5 \\ 6 & -10 \\ 9 & -15 \end{pmatrix}.$$

$\square$

33. (a) Give an example to show that the matrix multiplication in $\mathbb{R}^{3\times 3}$ is not commutative.

(b) Prove that, for all $A, B \in \mathbb{R}^{m\times n}$ and any $\lambda \in \mathbb{R}$,

$$\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B.$$

(c) Prove that, for all $A \in \mathbb{R}^{m\times p}$ and $B, C \in \mathbb{R}^{p\times n}$,

$$A \cdot (B + C) = A \cdot B + A \cdot C.$$

(d) Prove that, for all $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times s}$,

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

(e) Let $I_n \in \mathbb{R}^{n\times n}$ and $I_m \in \mathbb{R}^{m\times m}$ be the identity matrices of order n and m respectively. Prove that for $A \in \mathbb{R}^{m\times n}$,

$$A \cdot I_n = A \text{ and } I_m \cdot A = A.$$

(f) Prove that for all $A, B \in \mathbb{R}^{m\times n}$

$$(A + B)^\top = A^\top + B^\top.$$

Solution:

(a) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$A \cdot B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $A \cdot B \neq B \cdot A$.

(b) Write $A = (a_{ij})_{i,j=1}^{m,n}$ and $B = (b_{ij})_{i,j=1}^{m,n}$. Then the ij entry of $\lambda \cdot (A + B)$ is

$$\lambda \cdot (a_{ij} + b_{ij}) = \lambda \cdot a_{ij} + \lambda \cdot b_{ij},$$

which is exactly the ij entry of $\lambda \cdot A + \lambda \cdot B$.

(c) Write $A = (a_{ij})_{i,j=1}^{m,p}$, $B = (b_{ij})_{i,j=1}^{p,n}$ and $C = (c_{ij})_{i,j=1}^{p,n}$. Then the ij entry of $A \cdot (B + C)$ is

$$\sum_{k=1}^{p} a_{ik} \cdot (b_{k,j} + c_{k,j}) = \sum_{k=1}^{p} (a_{ik} \cdot b_{kj} + a_{ik} \cdot c_{kj}) = \sum_{k=1}^{p} a_{ik} \cdot b_{kj} + \sum_{k=1}^{p} a_{ik} \cdot c_{kj},$$

which is exactly the ij entry of $A \cdot B + A \cdot C$.

(d) Write $A = (a_{ij})_{i,j=1}^{m,n}$, $B = (b_{ij})_{i,j=1}^{n,p}$ and $C = (c_{ij})_{i,j=1}^{p,s}$. The ij entry of $A \cdot B$ is $\sum_{k=1}^{n} a_{ik} b_{kj}$ and then the ij entry of $(A \cdot B) \cdot C$ is

$$((A \cdot B) \cdot C)_{ij} = \sum_{\ell=1}^{p} (A \cdot B)_{i\ell} \cdot (C)_{\ell j} = \sum_{\ell=1}^{p} \left(\sum_{k=1}^{n} a_{ik} \cdot b_{k\ell} \right) \cdot c_{\ell j} = \sum_{\ell=1}^{p} \sum_{k=1}^{n} a_{ik} \cdot b_{k\ell} \cdot c_{\ell j}.$$

On the other hand, the ij entry of $B \cdot C$ is $\sum_{\ell=1}^{p} b_{i\ell} \cdot c_{\ell j}$ and then the ij entry of $A \cdot (B \cdot C)$ is

$$(A \cdot (B \cdot C))_{ij} = \sum_{k=1}^{n} (A)_{ik} \cdot (B \cdot C)_{kj} = \sum_{k=1}^{n} a_{ik} \cdot \left(\sum_{\ell=1}^{p} b_{k\ell} \cdot c_{\ell j} \right) = \sum_{k=1}^{n} \sum_{\ell=1}^{p} a_{ik} \cdot b_{k\ell} \cdot c_{\ell j}.$$

Changing the order of double sum, we get

$$\sum_{\ell=1}^{p} \sum_{k=1}^{n} a_{ik} \cdot b_{k\ell} \cdot c_{\ell j} = \sum_{k=1}^{n} \sum_{\ell=1}^{p} a_{ik} \cdot b_{k\ell} \cdot c_{\ell j}.$$

This implies that the ij entry of $(A \cdot B) \cdot C$ and $A \cdot (B \cdot C)$ are equal. So we prove that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

(e) i. Note that

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Write $I_n = (b_{ij})_{i,j=1}^{n}$. Then the ij entry of the identity matrix I_n is

$$b_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

for all $i, j \in \{1, \dots, n\}$. Write $A = (a_{ij})_{i,j}^{m,n}$. By the definition of matrix multiplication, we get the ij entry of $A \cdot I_n$ is

$$\sum_{k=1}^{n} a_{ik} \cdot b_{kj} = a_{ij} \cdot b_{jj} + \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{ik} \cdot b_{kj} = a_{ij} \cdot 1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{ik} \cdot 0 = a_{ij}.$$

Hence $AI_n = A$.

ii. Similar as before, write $A = (a_{ij})_{i,j=1}^{m,n}$ and $I_m = (b_{i,j})_{i,j=1}^{m}$, where b_{ij} is defined as

$$b_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

for all $i, j \in \{1, \dots, m\}$. By the definition of matrix multiplication, we get the ij entry of $I_m \cdot A$ is

$$\sum_{k=1}^{m} b_{ik} \cdot a_{kj} = b_{ii} a_{ij} + \sum_{\substack{k=1 \\ k \neq i}}^{m} b_{ik} \cdot a_{kj} = 1 \cdot a_{ij} + \sum_{\substack{k=1 \\ k \neq i}}^{m} 0 \cdot a_{kj} = a_{ij}.$$

Hence $I_m A = A$.

(f) Write $A = (a_{ij})_{i,j}^{m,n}$ and $B = (b_{ij})_{i,j}^{m,n}$, and so $A+B = (a_{ij}+b_{ij})_{i,j}^{m,n} =: (c_{ij})_{i,j}^{m,n}$. By definition of transpose of a matrix, we see that

$$(A+B)^T = (c_{ji})_{i,j}^{m,n} = (a_{ji}+b_{ji})_{i,j}^{m,n} = (a_{ji})_{i,j}^{m,n} + (b_{ji})_{i,j}^{m,n} = A^T + B^T.$$

$\square$

34. Assume you have ordered 4 pizzas and 5 drinks, but you forgot the individual prices. You only know that you have paid total 50 EURO, and that a pizza was 8 EURO more expensive than a drink. How much is a pizza and how much is a drink?

Solution:
Let x_1 and x_2 be the price of a pizza and a drink, respectively. From the assumption on the overall cost, we know that $4x_1+5x_2 = 50$, and the second assumption reads $x_1 = x_2 + 8$. This can be written as the linear system

$$\begin{aligned} 4x_1 + 5x_2 &= 50, \\ x_1 - x_2 &= 8. \end{aligned}$$

By substituting $x_2 = x_1 - 8$ in the first equation, we see that a solution must satisfy $4x_1 + 5(x_1 - 8) = 50$, which simplifies to $9x_1 = 50 + 40 = 90$. So we obtain that the price of a pizza is $x_1 = 10$. From $x_2 = x_1 - 8$, we see that $x_2 = 2$. Hence the price of a pizza is 10 EURO and the price of a drink is 2 EURO.

Note. The linear system in matrix-vector form is

$$\begin{pmatrix} 4 & 5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 50 \\ 8 \end{pmatrix}.$$

Using the above argument, we see this system has a unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}.$$

Therefore the set of solutions in $\mathbb{R}^2$ is given by

$$L\left(\begin{pmatrix} 4 & 5 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 50 \\ 8 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 10 \\ 2 \end{pmatrix} \right\}.$$

$\square$

35. (a) Which of the following matrices are in row echelon form? Reduced row echelon form? For those matrices which are not in (reduced) row echelon form, explain the reason.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) For each of the following matrices, compute their reduced row echelon forms and ranks.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -2 & -4 \\ 2 & 4 & 1 & 2 \\ 1 & 3 & -3 & -3 \end{pmatrix}.$$

Solution:

(a) The matrices A and D are in row echelon form and the matrix A is in reduced row echelon form.

i. The matrix A is in reduced row echelon form.

ii. The matrix B is not in row echelon form, because the leading coefficient 3 in the second row has a non-zero entry below it.

iii. The matrix C is not in row echelon form, because it does not satisfy the required condition that all of the zero rows are at the bottom of the matrix.

iv. The matrix D is in row echelon form, but not in reduced row echelon form. Because its leading coefficient of the second row is 2 (not 1). Also the leading coefficient 2 in the second row has a non-zero entry above it.

(b) We reduce the matrices to reduced row echelon form using row operations.

i.

$$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} \xrightarrow[R_2 = R_2 - 2R_1]{} \underbrace{\begin{pmatrix} 1 & 5 \\ 0 & -7 \end{pmatrix}}_{\textbf{echelon form}}$$

$$\xrightarrow[R_2 = -\frac{1}{7}R_2]{} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \xrightarrow[R_1 = R_1 - 5R_2]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $\operatorname{rank}(A) = 2$.

ii.

$$\begin{pmatrix} 1 & 2 & -2 & -4 \\ 2 & 4 & 1 & 2 \\ 1 & 3 & -3 & -3 \end{pmatrix} \xrightarrow[R_3 = R_3 - R_1]{} \begin{pmatrix} 1 & 2 & -2 & -4 \\ 2 & 4 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \xrightarrow[R_2 = R_2 - 2R_1]{} \begin{pmatrix} 1 & 2 & -2 & -4 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \underbrace{\begin{pmatrix} 1 & 2 & -2 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & 10 \end{pmatrix}}_{\textbf{echelon form}} \xrightarrow[R_3 = \frac{1}{5}R_3]{} \begin{pmatrix} 1 & 2 & -2 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 = R_1 + 2R_3 \\ R_2 = R_2 + R_3}]{} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow[R_1 = R_1 - 2R_2]{} \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Thus, $\operatorname{rank}(B) = 3$.

$\square$

36. Find all solutions (x_1, x_2, x_3) in $\mathbb{R}^3$ of the following systems of linear equations.

(a)

$$\begin{aligned} x_1 \qquad - \ x_3 &= 2 \\ x_2 + 2x_3 &= 5 \\ x_1 + x_2 + \ x_3 &= 7. \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= -4 \\ 2x_1 + 4x_2 + x_3 &= 2 \\ x_2 - x_3 &= 1. \end{aligned}$$

Solution:

(a) Since adding the first equation to the second equation yields the third one, we only need to solve the first two equations:

$$\begin{aligned} x_1 \qquad - \ x_3 &= 2 \\ x_2 + 2x_3 &= 5 \end{aligned}$$

We can treat $x_3 \in \mathbb{R}$ as a free variable, and conclude that any point of the form $(2+\lambda, 5-2\lambda, \lambda)$ such that $\lambda \in \mathbb{R}$ is a solution to the system of equations. Therefore the linear system

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$$

has infinitely many solutions in $\mathbb{R}^3$ and we have

$$L\left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 2+\lambda \\ 5-2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$$

(b) First we subtract twice the first equation from the second equation to estimate the variable x_1 and we obtain $5x_3 = 10$. This implies that $x_3 = 2$. From the third equation $x_2 - x_3 = 1$, we see $x_2 = 1 + x_3 = 3$. From the first equation, we see $x_1 = -4 - 2x_2 + 2x_3 = -4 - 2 \cdot 3 + 2 \cdot 2 = -6$. Therefore the linear system

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$$

has a unique solution:

$$L\left(\begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \right) = \left\{ \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} \right\}.$$

$\square$