55. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that for every $n \in \mathbb{N}$, one has $|a_n - a_{n+1}| \le \frac{1}{n}$. Is such a sequence always convergent?

56. Find a closed formula for
$$\sum_{k=1}^{n} \frac{1}{(4k+3)(4k+7)}$$
in terms of $n \in \mathbb{N}$, and use it to compute $\sum_{k=1}^{\infty} \frac{1}{(4k+3)(4k+7)}$.

57. (a) Prove that $\sum_{k=1}^{n} \frac{1}{k^3} \le 2 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Is the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ convergent?

    (b) Prove that $\sum_{k=1}^{n} \frac{1}{k^{1/3}} \ge \frac{3}{2}(n+1)^{2/3} - \frac{3}{2}$ for all $n \in \mathbb{N}$. Is the series $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ convergent?

    Hint: Use induction on n to prove the inequality in either part.

58. Let q be a real number with $|q| < 1$. Find a closed formula for $\sum_{k=1}^{n} (k+3)q^k$ in terms of n and q, and use it to compute $\sum_{k=1}^{\infty} (k+3)q^k$ in terms of q. *Hint:* $\sum_{k=1}^{n} kq^k = \sum_{j=1}^{n} \sum_{\ell=j}^{n} q^\ell$.

59. Let σ be a permutation of the set $\{0,1,2,3,4,5\}$. We define the function $\tau : \mathbb{N}_0 \to \mathbb{N}_0$, depending on σ, such that $\tau(6k+r) = 6k + \sigma(r)$ for all $k \in \mathbb{N}_0$ and all $r \in \{0,1,2,3,4,5\}$. Assume that $\sum_{n=0}^{\infty} a_n$ is a convergent series. Prove that $\sum_{n=0}^{\infty} a_{\tau(n)}$ is also convergent, with $\sum_{n=0}^{\infty} a_{\tau(n)} = \sum_{n=0}^{\infty} a_n$.

60. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely convergent. *Hint:* To prove convergence, you can proceed in the following steps, letting $S_N := \sum_{n=1}^{N} \frac{(-1)^n}{n}$ denote the N-th partial sum of the series.

    (a) The sequence $(S_{2M})_{M\in\mathbb{N}}$ is strictly decreasing and bounded from below by $S_1 = -1$. Hence, $\lim_{M\to\infty} S_{2M} =: \lambda$ exists.

    (b) Observe that $S_{2M} = S_{2M-1} + \frac{1}{2M}$, and conclude that $\lim_{M\to\infty} S_{2M-1}$ exists as well and is equal to λ.

    (c) From (a) and (b), conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent (with value λ).