49. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_1 = 1$, $a_{n+1} = \frac{a_n+5}{2}$ is bounded by 5. Then, prove that it is increasing. Provide a justification for the existence of the limit of the sequence $(a_n)_{n\in\mathbb{N}}$, and then calculate it.

Solution:
First, we want to show that the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded by 5. We will prove by induction that, for all $n \in \mathbb{N}$, we have $a_n \leq 5$.
Induction basis: The statement is clearly true for $n = 1$.
Induction step: Suppose that $a_k \leq 5$ for some $k \in \mathbb{N}$. Then

$$a_{k+1} = \frac{a_k + 5}{2} \leq \frac{5+5}{2} = \frac{10}{2} = 5.$$

It follows that $a_n \leq 5$ for all $n \in \mathbb{N}$.
Now, we want to prove that the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing. We will show by induction that $a_n < a_{n+1}$, for all $n \in \mathbb{N}$.
Induction basis: For $n = 1$, we have

$$a_2 = \frac{a_1 + 5}{2} = \frac{1+5}{2} = 3 > a_1 = 1.$$

Then, the statement is true for $n = 1$.
Induction step: We assume $a_k < a_{k+1}$ holds for some $k \in \mathbb{N}$. Then,

$$a_k + 5 < a_{k+1} + 5,$$

and

$$\frac{a_k + 5}{2} < \frac{a_{k+1} + 5}{2}.$$

Therefore, we find that

$$a_{k+1} < a_{k+2}.$$

It follows by induction that the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing.

From the above and the Monotonicity Principle Theorem, we deduce that there is $\ell \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = \ell$. It follows that any subsequence $(a_{n_k})_{k\in\mathbb{N}}$ satisfies

$$\lim_{k\to\infty} a_{n_k} = \ell.$$

Taking limits on both sides of the equation

$$a_{k+1} = \frac{a_k + 5}{2},$$

we obtain

$$\ell = \frac{\ell + 5}{2},$$

and therefore, $\ell = 5$. □

50. Let $B_1 = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$, and let $B_{n+1} = B_1 B_n$ for $n \in \mathbb{N}$. Find pairs of real numbers (a, b) such that:

$$\lim_{n\to\infty} \frac{\det B_n}{5 + a^2 b^n} = \frac{1}{9}.$$

Solution:
Notice that $\det B_1 = 8-5 = 3$, and that $B_2 = B_1 B_1$. Then $\det B_2 = (\det B_1)(\det B_1) = 3^2$. We need to calculate $\det B_n$. By induction, we prove that $\det B_n = 3^n$ for all $n \in \mathbb{N}$.
Induction basis: For $n = 1$, we have seen that $\det B_2 = (\det B_1)(\det B_1) = 3^2$.
Induction step: We assume that $\det B_k = 3^k$ holds for some $k \in \mathbb{N}$, we want to show that $\det B_{k+1} = 3^{k+1}$. From the inductive hypothesis, we have

$$\det B_{k+1} = \det(B_1 B_k) = (\det B_1)(\det B_k) = 3^{k+1}.$$

Therefore $\det B_n = 3^n$ for all $n \in \mathbb{N}$. Thus, we have

$$\lim_{n\to\infty} \frac{3^n}{5 + a^2 b^n} = \frac{1}{9}.$$

Now, we discuss the following cases for a and b:

- If $a = 0$ then
$$\lim_{n\to\infty} \frac{3^n}{5} = +\infty.$$
- If $a \neq 0$ and $|b| > 3$ then
$$\lim_{n\to\infty} \frac{3^n}{5 + a^2 b^n} = \lim_{n\to\infty} \frac{\left(\frac{3}{b}\right)^n}{\frac{5}{b^n} + a^2} = 0,$$
where we used the fact that if $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$ (where $z = \frac{3}{b}$ in our exercise).
- If $a \neq 0$ and $|b| < 3$ then
$$\lim_{n\to\infty} \frac{3^n}{5 + a^2 b^n} = \lim_{n\to\infty} \frac{\left(\frac{3}{b}\right)^n}{\frac{5}{b^n} + a^2} = +\infty,$$
where we used the fact that if $|z| > 1$, then $\lim_{n\to\infty} z^n = +\infty$.
- If $a \neq 0$ and $b = -3$ then the limit doesn't exists. To show that we discuss two cases if n is even and if n is odd. In case n is even, we write $n = 2k$, and in case n is odd, we write $n = 2k - 1$ for $k \in \mathbb{N}$. Then, we have
$$\lim_{k\to\infty} \frac{3^{2k}}{5 + a^2(-3)^{2k}} = \lim_{k\to\infty} \frac{3^{2k}}{(-3)^{2k}} \frac{1}{5(-3)^{-2k} + a^2} = \frac{1}{a^2}$$
and
$$\lim_{k\to\infty} \frac{3^{2k-1}}{5 + a^2(-3)^{2k-1}} = \lim_{k\to\infty} \frac{3^{2k-1}}{(-3)^{2k-1}} \frac{1}{5(-3)^{-2k+1} + a^2} = -\frac{1}{a^2}.$$

- If $a \neq 0$ and $b = 3$ then

$$\lim_{n\to\infty} \frac{3^n}{5 + a^2 3^n} = \lim_{n\to\infty} \frac{1}{\frac{5}{3^n} + a^2} = \frac{1}{a^2}.$$

  Taking $a = 3$ or $a = -3$, we find that

$$\lim_{n\to\infty} \frac{3^n}{5 + 9 \cdot 3^n} = \frac{1}{9}.$$

Therefore, $(a, b) \in \{(-3, 3), (3, 3)\}$. $\square$

51. Determine all the accumulation points for the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ defined by

- $a_n := i^n$,
- $b_n := (-1)^n + \frac{n^2+1}{n^2}$,
- $c_n := \sin\left(n\frac{\pi}{6}\right)$.

Solution:

- For the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = i^n$. We notice that the sequence takes the values, $i, -1, -i, 1, i, -1, -i, \ldots$ for $n = 1, 2, 3, 4, 5, 6, 7 \cdots$. Thus, we have four notable subsequences $(a_{4k-3})_{k\in\mathbb{N}}$, $(a_{4k-2})_{k\in\mathbb{N}}$, $(a_{4k-1})_{k\in\mathbb{N}}$, and $(a_{4k})_{k\in\mathbb{N}}$ are convergent to $i, -1, -i$ and 1 respectively. Therefore, the accumulation points for the sequence $(a_n)_{n\in\mathbb{N}}$ are $i, -1, -i, 1$.
- For the sequence $(b_n)_{n\in\mathbb{N}}$ defined by $b_n = (-1)^n + \frac{n^2+1}{n^2}$. We notice that $(-1)^n$ can be 1 when n is even, and -1 when n is odd. On the other hand, $\frac{n^2+1}{n^2}$ can be written as $\frac{n^2+1}{n^2} = 1 + \frac{1}{n^2}$ and then $\lim\limits_{n\to\infty} \frac{n^2+1}{n^2} = 1$. Therefore, we have two notable subsequences $(a_{2k})_{k\in\mathbb{N}}$ and $(a_{2k-1})_{k\in\mathbb{N}}$ convergent with limit 2 and 0 respectively. We conclude that the accumulation points for the sequence $(b_n)_{n\in\mathbb{N}}$ are 2 and 0.
- For the sequence $(c_n)_{n\in\mathbb{N}}$ defined by $c_n = \sin\left(n\frac{\pi}{6}\right)$. We know that the sine function is periodic, its period is 2π. Thus we have $c_{n+12} = c_n$. The set of accumulation points is equal to the set $\{c_n : n \in \{1, 2, \ldots, 12\}\}$. Then, the accumulation points are $0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}$, and -1.

$\square$

52. Let $p > 1$, and let $a_n = \sqrt{n^p + n + 1} - \sqrt{n^p - n + 1}$. Determine for which $p > 1$ the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded.

Solution:
First, We can simplify a_n by multiplying it by its conjugate.

$$\begin{aligned}
a_n &= \frac{\left(\sqrt{n^p+n+1} - \sqrt{n^p-n+1}\right)\left(\sqrt{n^p+n+1} + \sqrt{n^p-n+1}\right)}{\sqrt{n^p+n+1} + \sqrt{n^p-n+1}} \\
&= \frac{(n^p+n+1) - (n^p-n+1)}{\sqrt{n^p+n+1} + \sqrt{n^p-n+1}} \\
&= \frac{2n}{\sqrt{n^p+n+1} + \sqrt{n^p-n+1}} \\
&= \frac{2n}{n^{\frac{p}{2}}\left(\sqrt{1+\frac{1}{n^{p-1}}+\frac{1}{n^p}} + \sqrt{1+\frac{1}{n^{p-1}}+\frac{1}{n^p}}\right)} \\
&= \frac{2}{n^{\frac{p}{2}-1}\left(\sqrt{1+\frac{1}{n^{p-1}}+\frac{1}{n^p}} + \sqrt{1+\frac{1}{n^{p-1}}+\frac{1}{n^p}}\right)}
\end{aligned}$$

Here we discuss three cases:

- If $p > 2$, then we have $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$, and $\lim\limits_{n\to\infty} \frac{1}{n^{p-1}} = 0$. It follows that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{2n^{\frac{p}{2}-1}} = 0,$$
because $\frac{p}{2} - 1 > 0$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ is a null sequence. Thus $(a_n)_{n\in\mathbb{N}}$ converges and therefore it is bounded.
- If $1 < p < 2$, we have $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$, and $\lim\limits_{n\to\infty} \frac{1}{n^{p-1}} = 0$, but
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\frac{p}{2}-1}} = \infty,$$
because $\frac{p}{2} - 1 < 0$. Therefore, the sequence $(a_n)_{n\in\mathbb{N}}$ is divergent and then it is not bounded.
- If $p = 2$, we have
$$\begin{aligned}
a_n &= \frac{\left(\sqrt{n^2+n+1} - \sqrt{n^2-n+1}\right)\left(\sqrt{n^2+n+1} + \sqrt{n^2-n+1}\right)}{\sqrt{n^2+n+1} + \sqrt{n^2-n+1}} \\
&= \frac{(n^2+n+1) - (n^2-n+1)}{\sqrt{n^2+n+1} + \sqrt{n^2-n+1}} \\
&= \frac{2n}{\sqrt{n^2+n+1} + \sqrt{n^2-n+1}} \\
&= \frac{2}{\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1+\frac{1}{n}+\frac{1}{n^2}}\right)}.
\end{aligned}$$
Thus, we have
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1+\frac{1}{n}+\frac{1}{n^2}}\right)} = 1.$$
Thus $(a_n)_{n\in\mathbb{N}}$ converges and therefore it is bounded.

From above, we conclude that the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded for $p \geq 2$. □

53. Determine limits of sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ defined by

$$a_n = \left(1 + \frac{1}{5n+1}\right)^{6n+1}, \quad b_n = \left(1 + \frac{15n+1}{7n^3+11n+5}\right)^{3n^2+1},$$
$$c_n = \left(\frac{2n^2-n}{2n^2+13}\right)^n, \quad d_n = \left(\frac{n^2+1}{n^2}\right)^n.$$

Solution:
We have studied in the lecture (Slides: sections 3.1-3,4) that:
Fact: Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be real valued sequences such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\lim\limits_{n\to\infty} \alpha_n\beta_n = g \in \mathbb{R}$. Then

$$\lim_{n\to\infty} (1+\alpha_n)^{\beta_n} = e^g.$$

In this exercise, we use this fact to determine the limits of the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$.

- For the sequence $(a_n)_{n\in\mathbb{N}}$: we take $\alpha_n = \frac{1}{5n+1}$ and $\beta_n = 6n+1$, then, we find that
$$\lim_{n\to\infty} \frac{1}{5n+1} = 0, \qquad \lim_{n\to+\infty} \frac{6n+1}{5n+1} = \frac{6}{5}.$$
It follows that $\lim\limits_{n\to\infty} a_n = e^{\frac{6}{5}}$.
- For the sequence $(b_n)_{n\in\mathbb{N}}$: we take $\alpha_n = \frac{15n+1}{7n^3+11n+5}$ and $\beta_n = 3n^2+1$. Then, we have
$$\lim_{n\to\infty} \frac{15n+1}{7n^3+11n+5} = 0, \qquad \lim_{n\to+\infty} \frac{(15n+1)(3n^2+1)}{7n^3+11n+5} = \frac{45}{7}.$$
It follows that $\lim\limits_{n\to\infty} b_n = e^{\frac{45}{7}}$.
- For the sequence $(c_n)_{n\in\mathbb{N}}$: we write c_n as follows
$$\left(\frac{2n^2-n}{2n^2+13}\right)^n = \left(\frac{2n^2+13-13-n}{2n^2+13}\right)^n = \left(1 - \frac{n+13}{2n^2+13}\right)^n$$
We take $\alpha_n = \frac{-(n+13)}{2n^2+13}$ and $\beta_n = n$. Then, we have
$$\lim_{n\to\infty} \frac{-(n+13)}{2n^2+13} = 0, \qquad \lim_{n\to+\infty} \frac{-n(n+13)}{2n^2+13} = \frac{-1}{2}.$$
It follows that $\lim\limits_{n\to\infty} c_n = e^{-\frac{1}{2}}$.
- For the sequence $(d_n)_{n\in\mathbb{N}}$: we notice that
$$\left(\frac{n^2+1}{n^2}\right)^n = \left(1 + \frac{1}{n^2}\right)^n$$
We take $\alpha_n = \frac{1}{n^2}$ and $\beta_n = n$. Then, we have
$$\lim_{n\to\infty} \frac{1}{n^2} = 0, \qquad \lim_{n\to+\infty} \frac{n}{n^2} = 0.$$
It follows that $\lim\limits_{n\to\infty} d_n = 1$.

□

54. Determine the limit of the sequence $(a_n)_{n\in\mathbb{N}}$ defined by

$$a_n = (\sqrt{n^2+4} - n)\prod_{k=1}^{n}\left(1 + \frac{1}{k+1}\right),$$

where $n \in \mathbb{N}$.

Solution:
First, we simplify a_n as follows

$$\sqrt{n^2+4} - n = \frac{\left(\sqrt{n^2+4} - n\right)\left(\sqrt{n^2+4} + n\right)}{\sqrt{n^2+4}+n} = \frac{n^2+4-n^2}{\sqrt{n^2+4}+n} = \frac{4}{n + n\sqrt{1+\frac{4}{n^2}}},$$

and

$$\prod_{k=1}^{n}\left(1 + \frac{1}{k+1}\right) = \prod_{k=1}^{n}\frac{k+2}{k+1} = \frac{2}{2}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n+2}{n+1} = \frac{n+2}{2}.$$

Then, we write

$$a_n = (\sqrt{n^2+4} - n)\prod_{k=1}^{n}\left(1 + \frac{1}{k+1}\right) = \frac{2n+4}{n + n\sqrt{1+\frac{4}{n^2}}}$$

It follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2n+4}{n + n\sqrt{1+\frac{4}{n^2}}}\right) = 1.$$

□