55. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that for every $n \in \mathbb{N}$, one has $|a_n - a_{n+1}| \leq \frac{1}{n}$. Is such a sequence always convergent?

Solution:
No, such a sequence is not always convergent. For a counter-example, let $a_n := \sum_{k=1}^{n} \frac{1}{k}$, so that $(a_n)_{n\in\mathbb{N}}$ is the sequence of partial sums of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Then

$$|a_n - a_{n+1}| = \left|\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n+1} \frac{1}{k}\right| = \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} \leq \frac{1}{n},$$

but by Lemma 3.30 from the Lecture Notes, we have

$$\lim_{n\to\infty} a_n = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so $(a_n)_{n\in\mathbb{N}}$ is *not* convergent. □

56. Find a closed formula for
$$\sum_{k=1}^{n} \frac{1}{(4k+3)(4k+7)}$$
in terms of $n \in \mathbb{N}$, and use it to compute $\sum_{k=1}^{\infty} \frac{1}{(4k+3)(4k+7)}$.

Solution:

We start by determining the partial fraction decomposition of $\frac{1}{(4k+3)(4k+7)}$. There are $A, B \in \mathbb{R}$ such that
$$\frac{1}{(4k+3)(4k+7)} = \frac{A}{4k+3} + \frac{B}{4k+7} = \frac{A(4k+7)+B(4k+3)}{(4k+3)(4k+7)},$$
for all $k \in \mathbb{N}$, and we conclude that
$$1 = A(4k+7) + B(4k+3) = 4(A+B)k + 7A + 3B$$
for all $k \in \mathbb{N}$, which leads us to the linear equation system
$$\begin{aligned} A + B &= 0 \\ 7A + 3B &= 1. \end{aligned}$$
This system simplifies to $A = \frac{1}{4}, B = -\frac{1}{4}$. Therefore,
$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{(4k+3)(4k+7)} = \sum_{k=1}^{n} \left(\frac{1}{4} \cdot \frac{1}{4k+3} - \frac{1}{4} \cdot \frac{1}{4k+7} \right) \\ =&\frac{1}{4} \cdot \sum_{k=1}^{n} \left(\frac{1}{4k+3} - \frac{1}{4k+7} \right) = \frac{1}{4} \sum_{k=1}^{n} \left(\frac{1}{4k+3} - \frac{1}{4(k+1)+3} \right) \\ =&\frac{1}{4} \left(\frac{1}{7} - \frac{1}{4n+7} \right), \end{aligned}$$
where the last equality uses that the sum on its left-hand side is telescopic. Finally, we conclude that
$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{(4k+3)(4k+7)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(4k+3)(4k+7)} = \lim_{n\to\infty} \frac{1}{4} \left(\frac{1}{7} - \frac{1}{4n+7} \right) \\ =&\frac{1}{4} \left(\frac{1}{7} - 0 \right) = \frac{1}{28}. \end{aligned}$$

□

57. (a) Prove that $\sum_{k=1}^{n} \frac{1}{k^3} \leq 2 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Is the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ convergent?

    (b) Prove that $\sum_{k=1}^{n} \frac{1}{k^{1/3}} \geq \frac{3}{2}(n+1)^{2/3} - \frac{3}{2}$ for all $n \in \mathbb{N}$. Is the series $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ convergent?

Hint: Use induction on n to prove the inequality in either part.

Solution:

(a) As suggested in the hint, we prove the inequality by induction on n. For $n = 1$, the left-hand side of the inequality is $\sum_{k=1}^{1} \frac{1}{k^3} = \frac{1}{1^3} = 1$, and the right-hand side is $2 - \frac{1}{1^2} = 1$ as well, so the inequality holds for $n = 1$. Now assume that it holds for n; we aim to infer that it holds for $n+1$. And, indeed,

$$\sum_{k=1}^{n+1} \frac{1}{k^3} = \sum_{k=1}^{n} \frac{1}{k^3} + \frac{1}{(n+1)^3} \leq 2 - \frac{1}{n^2} + \frac{1}{(n+1)^3},$$

so it suffices to prove that for all $n \in \mathbb{N}$, one has

$$2 - \frac{1}{n^2} + \frac{1}{(n+1)^3} \leq 2 - \frac{1}{(n+1)^2},$$

which is equivalent to

$$\frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} \leq \frac{1}{n^2},$$

and further (via multiplying both sides by $n^2(n+1)^3$) to

$$n^2(n+1) + n^2 \leq (n+1)^3,$$

which simplifies to the obviously true inequality

$$n^3 + 2n^2 \leq n^3 + 3n^2 + 3n + 1.$$

This concludes the inductive proof of the inequality. Finally, we note that the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is, indeed, convergent, because its sequence of partial sums is strictly increasing and (by the inequality we just proved) bounded from above by 2.

(b) Again, we prove the inequality by induction on n. For $n = 1$, the left-hand side of the inequality is $\sum_{k=1}^{1} \frac{1}{k^{1/3}} = \frac{1}{1^{1/3}} = 1$, while the right-hand side is $\frac{3}{2} \cdot 2^{2/3} - \frac{3}{2}$. The inequality is thus equivalent to $\frac{5}{2} = 1 + \frac{3}{2} \geq \frac{3}{2} \cdot 2^{2/3}$, and further (through multiplying both sides by 2, then raising both sides to the third power) to $125 = 5^3 \geq 3^3 \cdot 2^2 = 108$, which is true. Now assume that the inequality holds for n, and aim to prove that it holds for $n+1$. Due to

$$\sum_{k=1}^{n+1} \frac{1}{k^{1/3}} = \sum_{k=1}^{n} \frac{1}{k^{1/3}} + \frac{1}{(n+1)^{1/3}} \geq \frac{3}{2}(n+1)^{2/3} - \frac{3}{2} + \frac{1}{(n+1)^{1/3}},$$

it suffices to prove that

$$\frac{3}{2}(n+1)^{2/3} - \frac{3}{2} + \frac{1}{(n+1)^{1/3}} \geq \frac{3}{2}(n+2)^{2/3} - \frac{3}{2}.$$

Through adding $\frac{3}{2}$ to both sides, then multiplying both sides by $(n+1)^{1/3}$, we see that this is equivalent to

$$\frac{3}{2}(n+1)+1 \geq \frac{3}{2}(n+2)^{2/3}(n+1)^{1/3}.$$

Furthermore, through multiplying both sides by 2, then raising both sides to the third power, we find that it is equivalent to

$$(3n+5)^3 \geq 27(n+2)^2(n+1),$$

which can be simplified to the obviously true inequality

$$27n^3 + 135n^2 + 225n + 125 \geq 27n^3 + 135n^2 + 216n + 108.$$

This concludes the inductive proof of the inequality. Moreover, because $\lim_{n\to\infty}\left(\frac{3}{2}(n+1)^{2/3} - \frac{3}{2}\right) = \infty$, we infer that $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^{1/3}} = \infty$. In particular, the series $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is *not* convergent.

$\square$

58. Let q be a real number with $|q| < 1$. Find a closed formula for $\sum_{k=1}^{n} (k+3)q^k$ in terms of n and q, and use it to compute $\sum_{k=1}^{\infty} (k+3)q^k$ in terms of q. *Hint:* $\sum_{k=1}^{n} kq^k = \sum_{j=1}^{n} \sum_{\ell=j}^{n} q^\ell$.

Solution:

Let us first try and find a closed formula for $\sum_{k=1}^{n} kq^k$, following the hint. Throughout the solution of this exercise, we make use of the geometric series formula,

$$\sum_{j=0}^{m} q^j = \frac{q^{m+1}-1}{q-1}.$$

Now,

$$\begin{aligned}
\sum_{k=1}^{n} kq^k &= \sum_{j=1}^{n}\sum_{\ell=j}^{n} q^\ell = \sum_{j=1}^{n} q^j \sum_{\ell=j}^{n} q^{\ell-j} = \sum_{j=1}^{n} q^j \sum_{\ell=0}^{n-j} q^\ell = \sum_{j=1}^{n} q^j \frac{q^{n-j+1}-1}{q-1} \\
&= \frac{1}{q-1}\sum_{j=1}^{n}\left(q^{n+1}-q^j\right) = \frac{1}{q-1}\left(\sum_{j=1}^{n} q^{n+1} - \sum_{j=1}^{n} q^j\right) \\
&= \frac{1}{q-1}\left(nq^{n+1} - q\sum_{j=0}^{n-1} q^j\right) = \frac{1}{q-1}\left(nq^{n+1} - q\cdot\frac{q^n-1}{q-1}\right) \\
&= \frac{1}{(q-1)^2}(nq^{n+1}(q-1) - q(q^n-1)) = \frac{1}{(q-1)^2}(nq^{n+2} - (n+1)q^{n+1} + q).
\end{aligned}$$

Using this closed formula for $\sum_{k=1}^{n} kq^k$, we can find one for $\sum_{k=1}^{n} (k+3)q^k$. Namely,

$$\begin{aligned}
\sum_{k=1}^{n} (k+3)q^k &= \sum_{k=1}^{n} kq^k + \sum_{k=1}^{n} 3q^k = \frac{1}{(q-1)^2}(nq^{n+2} - (n+1)q^{n+1} + q) + 3q\cdot\frac{q^n-1}{q-1} \\
&= \frac{1}{(q-1)^2}(nq^{n+2} - (n+1)q^{n+1} + q + 3q(q^n-1)(q-1)) \\
&= \frac{1}{(q-1)^2}(nq^{n+2} - (n+1)q^{n+1} + q + 3q^{n+2} - 3q^{n+1} - 3q^2 + 3q) \\
&= \frac{1}{(q-1)^2}((n+3)q^{n+2} - (n+4)q^{n+1} - 3q^2 + 4q).
\end{aligned}$$

Finally, we observe that

$$\begin{aligned}
\sum_{k=1}^{\infty} (k+3)q^k &= \lim_{n\to\infty}\sum_{k=1}^{n} (k+3)q^k \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{(q-1)^2}((n+3)q^{n+2} - (n+4)q^{n+1} - 3q^2 + 4q) \\
&= \frac{1}{(q-1)^2}(0 - 0 - 3q^2 + 4q) = \frac{q(4-3q)}{(q-1)^2}.
\end{aligned}$$

$\square$

59. Let σ be a permutation of the set $\{0,1,2,3,4,5\}$. We define the function $\tau : \mathbb{N}_0 \to \mathbb{N}_0$, depending on σ, such that $\tau(6k+r) = 6k+\sigma(r)$ for all $k \in \mathbb{N}_0$ and all $r \in \{0,1,2,3,4,5\}$. Assume that $\sum_{n=0}^{\infty} a_n$ is a convergent series. Prove that $\sum_{n=0}^{\infty} a_{\tau(n)}$ is also convergent, with $\sum_{n=0}^{\infty} a_{\tau(n)} = \sum_{n=0}^{\infty} a_n$.

Solution:
For $N \in \mathbb{N}_0 \cup \{-1\}$, let $S_N := \sum_{n=0}^{N} a_n$ and $S'_N := \sum_{n=0}^{N} a_{\tau(n)}$ be the N-th partial sum of $\sum_{n=0}^{\infty} a_n$ and of $\sum_{n=0}^{\infty} a_{\tau(n)}$, respectively (note that $S_{-1} = S'_{-1} = 0$, because each of them is an empty sum). Moreover, for notational simplicity, let $\lambda := \sum_{n=0}^{\infty} a_n$. We need to prove that the sequence $(S'_N)_{N\in\mathbb{N}_0}$ converges to λ. To that end, we first prove that for each $r \in \{0,1,2,3,4,5\}$, the subsequence $(S'_{6k+r})_{k\in\mathbb{N}_0}$ converges to λ.

So, let $r \in \{0,1,2,3,4,5\}$ be fixed. Observe that for each $k \in \mathbb{N}_0$, we have

$$\begin{aligned} S'_{6k-1} &= \sum_{n=0}^{6k-1} a_{\tau(n)} = \sum_{t=0}^{k-1}\sum_{m=0}^{5} a_{\tau(6t+m)} = \sum_{t=0}^{k-1}\sum_{m=0}^{5} a_{6t+\sigma(m)} = \sum_{t=0}^{k-1}\sum_{m=0}^{5} a_{6t+m} = \sum_{n=0}^{6k-1} a_n \\ &= S_{6k-1}. \end{aligned}$$

We can use this to write, for each $k \in \mathbb{N}_0$,

$$\begin{aligned} S'_{6k+r} &= \sum_{n=0}^{6k+r} a_{\tau(n)} = \sum_{n=0}^{6k-1} a_{\tau(n)} + \sum_{n=6k}^{6k+r} a_{\tau(n)} = S'_{6k-1} + \sum_{m=0}^{r} a_{\tau(6k+m)} \\ &= S_{6k-1} + \sum_{m=0}^{r} a_{6k+\sigma(m)}. \end{aligned}$$

Through this last formula, the sequence $(S'_{6k+r})_{k\in\mathbb{N}_0}$ is written as the sum of the convergent sequence $(S_{6k-1})_{k\in\mathbb{N}_0}$, with limit λ, and the $r+1$ null sequences $(a_{6k+\sigma(m)})_{k\in\mathbb{N}_0}$ for $m = 0,1,\ldots,r$. The computation rules for limits therefore imply that $(S'_{6k+r})_{k\in\mathbb{N}_0}$ is convergent, with

$$\lim_{k\to\infty} S'_{6k+r} = \lim_{k\to\infty} S_{6k-1} + \sum_{m=0}^{r} \lim_{k\to\infty} a_{6k+\sigma(m)} = \lambda + \sum_{m=0}^{r} 0 = \lambda,$$

as we wanted to show.

Now we conclude that $(S'_N)_{N\in\mathbb{N}_0}$ is convergent with limit λ. Indeed, let $\epsilon > 0$ be arbitrary but fixed. For each $r \in \{0,1,2,3,4,5\}$, we just saw that the sequence $(S'_{6k+r})_{k\in\mathbb{N}_0}$ is convergent with limit λ, so there exists a $K_r = K_r(\epsilon) \in \mathbb{N}_0$ such that if $k \geq K_r$, then $|S'_{6k+r} - \lambda| < \epsilon$. Let $\mathcal{N} := \max\{6K_r + r : 0 \leq r \leq 5\}$. Then, if $N \in \mathbb{N}_0$ satisfies $N \geq \mathcal{N}$, we have the following, letting $r = r(N) \in \{0,1,2,3,4,5\}$ be the remainder upon dividing N by 6:

$$\frac{N-r}{6} \geq K_r, \text{ and thus } |S'_N - \lambda| = |S'_{6\cdot\frac{N-r}{6}+r}| < \epsilon.$$

Because $\epsilon > 0$ was arbitrary, this shows that $\lim_{N\to\infty} S'_N = \lambda$, as we needed to prove. $\square$

60. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely convergent. *Hint:* To prove convergence, you can proceed in the following steps, letting $S_N := \sum_{n=1}^{N} \frac{(-1)^n}{n}$ denote the N-th partial sum of the series.

    (a) The sequence $(S_{2M})_{M\in\mathbb{N}}$ is strictly decreasing and bounded from below by $S_1 = -1$. Hence, $\lim_{M\to\infty} S_{2M} =: \lambda$ exists.

    (b) Observe that $S_{2M} = S_{2M-1} + \frac{1}{2M}$, and conclude that $\lim_{M\to\infty} S_{2M-1}$ exists as well and is equal to λ.

    (c) From (a) and (b), conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent (with value λ).

Solution:
This series is not absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which is divergent (see Lemma 3.30 in the Lecture Notes). For the argument that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent, we follow the hint, and for notational simplicity, we set $b_n := \frac{1}{n}$. Please observe that $(b_n)_{n\in\mathbb{N}}$ is a positive, strictly decreasing null sequence (we will make use of these properties throughout the argument below).

(a) For each $M \in \mathbb{N}$, we have

$$\begin{aligned} S_{2(M+1)} &= \sum_{n=1}^{2M+2} (-1)^n b_n = \sum_{n=1}^{2M} (-1)^n b_n + (-1)^{2M+1} b_{2M+1} + (-1)^{2M+2} b_{2M+2} \\ &= S_{2M} - b_{2M+1} + b_{2M+2} < S_{2M}, \end{aligned}$$

which proves that $(S_{2M})_{M\in\mathbb{N}}$ is strictly decreasing. Moreover, for each $M \in \mathbb{N}$,

$$\begin{aligned} S_{2M} &= \sum_{n=1}^{2M} (-1)^n b_n = (-1)^1 b_1 + (-1)^2 b_2 + \sum_{n=3}^{2M} (-1)^n b_n \\ &= -b_1 + b_2 + \sum_{m=2}^{M} (-b_{2m-1} + b_{2m}) \geq -b_1 = S_1, \end{aligned}$$

proving that $(S_{2M})_{M\in\mathbb{N}}$ is bounded from below by S_1. This concludes part (a) of the hint.

(b) Indeed, for each $M \in \mathbb{N}$, we have

$$S_{2M} = \sum_{n=1}^{2M} (-1)^n b_n = \sum_{n=1}^{2M-1} (-1)^n b_n + (-1)^{2M} b_{2M} = S_{2M-1} + b_{2M} = S_{2M-1} + \frac{1}{2M}.$$

In particular, we have

$$S_{2M-1} = S_{2M} - b_{2M} = S_{2M} - \frac{1}{2M},$$

and the computation rules for limits imply that $(S_{2M-1})_{M\in\mathbb{N}}$ is convergent, with limit

$$\lim_{M\to\infty} S_{2M-1} = \lim_{M\to\infty} S_{2M} - \lim_{M\to\infty} \frac{1}{2M} = \lambda - 0 = \lambda,$$

as asserted in part (b) of the hint.

(c) Let $\epsilon > 0$ be arbitrary but fixed. From part (a), we know that there exists an $\mathcal{N}_{\text{even}} = \mathcal{N}_{\text{even}}(\epsilon) \in \mathbb{N}$ such that if $N \in \mathbb{N}$ is even with $N \geq \mathcal{N}_{\text{even}}$, then $|S_N - \lambda| < \epsilon$. Moreover, from part (b), we know that there exists an $\mathcal{N}_{\text{odd}} = \mathcal{N}_{\text{odd}}(\epsilon) \in \mathbb{N}$ such that if $N \in \mathbb{N}$ is odd with $N \geq \mathcal{N}_{\text{odd}}$, then $|S_N - \lambda| < \epsilon$. Let $\mathcal{N} := \max(\mathcal{N}_{\text{even}}, \mathcal{N}_{\text{odd}})$. Then, if $N \in \mathbb{N}$ satisfies $N \geq \mathcal{N}$, we have $|S_N - \lambda| < \epsilon$, because N is even or odd, and $N \geq \mathcal{N}_{\text{even}}$ and $N \geq \mathcal{N}_{\text{odd}}$ both hold. Because $\epsilon > 0$ was arbitrary, we conclude that $(S_N)_{N\in\mathbb{N}}$ is convergent, with $\lim_{N\to\infty} S_N = \lambda$, as we needed to show.

Remark: This convergence argument works for any series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$ where $(b_n)_{n\in\mathbb{N}}$ is a positive, strictly decreasing null sequence. That is, any such series is convergent. $\square$