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Mathematics for AI 1

3. Sequences and Series

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3.6 Introduction to series

Definition - Series

Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued sequence and

$$s_n = \sum_{k=1}^{n} a_k.$$

We call s_n the **n**th partial sum of the series

$$\sum_{k=1}^{\infty} a_k.$$

Definition - Series

If the sequence $(s_n)_{n\in\mathbb{N}}$ converges with $\lim_{n\to\infty} s_n = s$ then we say that the series converges. We call s the sum of the series, and write

$$\sum_{k=1}^{\infty} a_k = s.$$

If a series is not convergent, then it is divergent.
If $\lim_{n\to\infty} s_n = \pm\infty$ then we write

$$\sum_{k=1}^{\infty} a_k = \pm\infty$$

and say that the series is definitely divergent.

Remark

Note that “series" is just another word for an infinite sum of elements of a sequence. Moreover, the notation $\sum_{k=1}^{\infty} a_k$ should be understood as a formal symbol for the limit of the corresponding sequence $(s_n)_{n\in\mathbb{N}}$: it might be a number or $\pm\infty$, but it might also not exist.

We will also sometimes consider series which do not start with index 1, i.e. series of the form $\sum_{k=k_0}^{\infty} a_k$ for some $k_0 \in \mathbb{Z}$. The most common variant we consider is with $k_0 = 0$.

Remark

It should also be noted that, if we consider an arbitrary series $\sum_{k=1}^{\infty} a_k$, we should assume that the terms a_k are complex numbers unless stated otherwise. There will be some instances later where we make the additional restriction that $a_k \in \mathbb{R}$.

Lemma (Geometric series)

Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have that

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad \text{and} \quad \sum_{k=1}^{\infty} q^k = \frac{q}{1-q}. \tag{1}$$

Moreover, we have

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}. \tag{2}$$

Proof

First we work with finite sums. Consider following equations:

$$
\begin{aligned}
(1-q)\sum_{k=0}^{n} q^k &= (1-q)(1+q+q^2+\cdots+q^n)\\
&= (1-q)+(q-q^2)+\cdots+(q^{n-1}-q^n)+(q^n-q^{n+1})\\
&= 1-\cancel{q}+\cancel{q}-q^2+q^2-q^3+\cdots+q^{n-1}-q^n+q^n-q^{n+1}\\
&= 1-\cancel{q}+\cancel{q}-\cancel{q^2}+\cancel{q^2}-\cancel{q^3}+\cdots\cancel{q^{n-1}}-\cancel{q^n}+\cancel{q^n}-q^{n+1}\\
&= 1-q^{n+1}.
\end{aligned}
$$

Therefore

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}.$$

Proof

Just for practice we write this argument only using properties of the sum notation.

$$
\begin{aligned}
(1-q)\sum_{k=0}^{n} q^k &= \sum_{k=0}^{n} q^k - q\sum_{k=0}^{n} q^k \\
&= \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1} \\
&= \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k \\
&= 1 + \sum_{k=1}^{n} q^k - (q^{n+1} + \sum_{k=1}^{n} q^k) \\
&= 1 - q^{n+1}.
\end{aligned}
$$

Proof

To prove the first part of the Lemma, we make use of some facts about convergence of sequences:

$$\sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} \sum_{k=0}^{n} q^k = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} \cdot \lim_{n\to\infty} (1-q^{n+1}) = \frac{1}{1-q}.$$

and for the second sum

$$\sum_{k=1}^{\infty} q^k = \lim_{n\to\infty} \sum_{k=1}^{n} q^k = \lim_{n\to\infty} q \sum_{k=0}^{n-1} q^k = q \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{q}{1-q}.$$

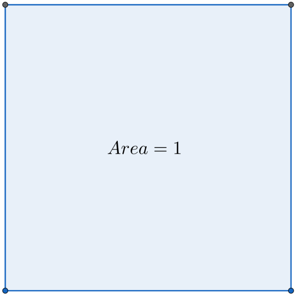
Remark

In the proof above, we considered two long sums and observed that almost all of the terms cancelled out. Sums of this form are called telescoping sums.

Example

we set $q = \frac{1}{2}$ in Lemma from the slide 6, it follows that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2, \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$



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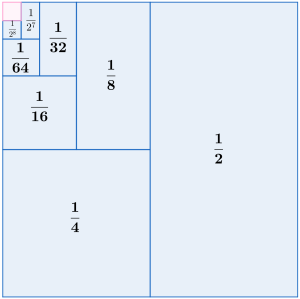
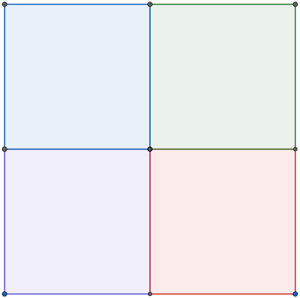


Example

We set $q = \frac{1}{4}$ in Lemma from slide 6, it follows that

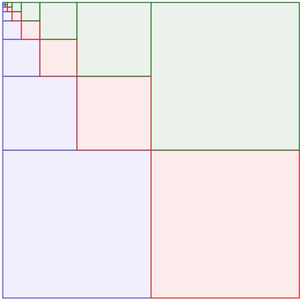
$$\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3}, \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}.$$



Example - Visualization

Observe that

$$3\sum_{k=1}^{\infty}\frac{1}{4^k} = \textit{Area} + \textit{Area} + \textit{Area} = 1$$



Exercise

Let q be a real number with $|q| < 1$. Find a closed formula for

$$\sum_{k=1}^{n} (k+3)q^k$$

in terms of n and q, and use it to compute

$$\sum_{k=1}^{\infty} (k+3)q^k$$

in terms of q. Hint. $\sum_{k=1}^{n} kq^k = \sum_{k=1}^{n} \sum_{j=k}^{n} q^j$.

Lemma (Harmonic series - IMPORTANT EXAMPLE !!!)

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \frac{1}{n}$. Then, the corresponding series satisfies

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Remark

In the above example, we can see that $\lim_{n\to\infty} a_n = 0$ is not sufficient to ensure the convergence of a sequence.

Proof

Observe that $s_n = \sum_{k=1}^{n} \frac{1}{k}$ is an increasing sequence. Thus, either it converges or diverges to $+\infty$. It is enough to study the convergence of a subsequence $(s_{2^n-1})_{n\in\mathbb{N}}$. We prove an auxiliary inequality

$$\sum_{k=n}^{2n-1} \frac{1}{k} \geq \sum_{k=n}^{2n-1} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}.$$

Therefore, we get

$$s_{2^{n+1}-1} = s_{2^n-1} + \sum_{k=2^n}^{2^{n+1}-1} \geq s_{2^n-1} + \frac{1}{2}.$$

Note that $s_{2-1} = 1 > \frac{1}{2}$. By induction we prove that $s_{2^n-1} > \frac{n}{2}$.

Proof

By the definition of the improper limit

$$\lim_{n\to\infty} s_{2^n-1} = +\infty.$$

Since $(s_n)_{n\in\mathbb{N}}$ is a monotone sequence,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{2^n-1} = +\infty.$$

Remark

The series $\sum_{k=1}^{\infty} \frac{1}{n}$ is called the harmonic series. By contrast with Slide 15, the series $\sum_{k=1}^{\infty} n^{-\alpha}$ converges for any $\alpha > 1$, and so the harmonic series is something of a critical example at which there is a change of behavior.

Example

In this example, we discuss how the aforementioned telescoping trick can sometimes be a powerful tool for obtaining the precise value of apparently complicated series. We will prove that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

We do not even know that this series is convergent yet. However, we first make the helpful observation that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}. \tag{3}$$

Example

It therefore follows that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 - \frac{1}{n+1}.$$

Thus, we get

$$\lim_{n \longrightarrow \infty} s_n = \lim_{n \longrightarrow \infty} 1 - \frac{1}{n+1} = 1$$

Remark

In the example above we have used the method of partial fraction decomposition to write a fraction with a product in the denominator as a sum of two fractions with more simple denominators.

Exercise

Find a closed formula for

$$\sum_{k=1}^{n} \frac{1}{(4k+3)(4k+7)}$$

in terms of $n \in \mathbb{N}$, and use it to compute

$$\sum_{k=1}^{\infty} \frac{1}{(4k+3)(4k+7)}.$$

Remark

However, it is worth noting that computing the sum of a series precisely is rarely an easy task. Even for a closely related example,

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

determining the value of the sum is considerably more challenging. We will use more sophisticated mathematics later in this program to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. For many other sums, there is just no closed expression.

Example

We will show that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent. The sequence $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is an increasing sequence. We will prove by induction that

$$s_n \leq 2 - \frac{1}{n} \tag{4}$$

for $n \in \mathbb{N}$. Clearly for $n = 1$ we have $s_1 = 1 = 1 - \frac{1}{1}$. We assume that (4) holds true for $n \in \mathbb{N}$. We have

$$s_{n+1} = s_n + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$

Example

We only need to show that

$$-\frac{1}{n}+\frac{1}{(n+1)^2}\leq -\frac{1}{n+1}.$$

We multiply both sides by $n(n+1)^2$ and obtain

$$-(n+1)^2+n\leq -n(n+1).$$

Observe that

$$n(n+1)+n=n(n+2)=(n+1)^2-1\leq (n+1)^2.$$

The sequence s_n is bounded and monotone. Therefore sequence s_n is convergent. By definition series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent.

Calculation rules and basic properties of series

Theorem

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent series and let $c \in \mathbb{C}$. Then we have

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

and

$$\sum_{k=1}^{\infty} c \cdot a_k = c \cdot \sum_{k=1}^{\infty} a_k.$$

Proof

We put $s_n = \sum_{k=1}^n a_k$, $\sigma_n = \sum_{k=1}^n b_k$, $\tau_n = \sum_{k=1}^n (a_k + b_k)$, $\theta_n = \sum_{k=1}^n c \cdot a_k$. Clearly $\tau_n = s_n + \sigma_n$ and $\theta_n = cs_n$. By the algebraic properties of limits we have

$$\sum_{k=1}^{\infty}(a_k + b_k) = \lim_{n\to\infty} \tau_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} \sigma_n = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

and

$$\sum_{k=1}^{\infty} ca_k = \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} cs_n = c\sum_{k=1}^{\infty} a_k$$

Example

Recall from the previous slides that

$$\sum_{k=1}^{\infty} 2^{-k} = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

It therefore follows from the slide 25 that

$$\sum_{k=1}^{\infty} \frac{\pi k(k+1) + 2^k}{k(k+1)2^k} = \sum_{k=1}^{\infty} \left(\frac{\pi}{2^k} + \frac{1}{k(k+1)} \right)$$

$$= \pi \sum_{k=1}^{\infty} 2^{-k} + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \pi + 1.$$

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a real-valued sequence such that $a_n \geq 0$ for all $n \in \mathbb{N}$. Let $s_n = \sum_{k=1}^{n} a_k$. Then

$$(s_n)_{n\in\mathbb{N}} \text{ is bounded } \iff \sum_{k=1}^{\infty} a_k \text{ converges.}$$

Proof.

Since $a_k \geq 0$ for all $k \in \mathbb{N}$, it follows that the sequence $(s_n)_{n\in\mathbb{N}}$ is non-decreasing. In particular, this sequence is monotone. Recall that for monotone sequence we have

$$(s_n)_{n\in\mathbb{N}} \text{ is bounded } \iff (s_n)_{n\in\mathbb{N}} \text{ is convergent .}$$

This proves the theorem. $\square$

Example

We can use the previous slide to show that the series $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges. We just need to show that the sequence $s_n = \sum_{k=1}^{n} \frac{1}{k!}$ is bounded. To see this, first observe that $k! \geq 2^{k-1}$ holds for all $k \in \mathbb{N}$. From this and the formula for the sum of the geometric series it follows that

$$\sum_{k=1}^{n} \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}} = \sum_{k=0}^{n-1} \frac{1}{2^k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges, it immediately follows that

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}$$

and so $\sum_{k=0}^{\infty} \frac{1}{k!}$ also converges.

Example

Moreover we will prove that

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

The binomial theorem yields

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k}$$

$$= \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{n-k-1} \frac{n-j}{n} \leq \sum_{k=0}^{n} \frac{1}{k!} \leq \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Hence,

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Example

On the other hand for fixed $m \in \mathbb{N}$ and $n > m$ by similar arguments

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{n-k-1} \frac{n-j}{n} \geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{j=0}^{n-k-1} \frac{n-j}{n}.\end{aligned}$$

Example

Hence,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq \lim_{n\to\infty}\sum_{k=0}^{m}\frac{1}{k!}\prod_{j=0}^{n-k-1}\frac{n-j}{n}$$
$$= \sum_{k=0}^{m}\frac{1}{k!}\prod_{j=0}^{n-k-1}\lim_{n\to\infty}\frac{n-j}{n} = \sum_{k=0}^{m}\frac{1}{k!}.$$

Here it is crucial that m is a fixed number!!! We obtain for $m \in \mathbb{N}$ the following inequality

$$\sum_{k=0}^{m}\frac{1}{k!} \leq e \leq \sum_{k=0}^{\infty}\frac{1}{k!}.$$

By sandwich rule $e = \sum_{k=0}^{\infty}\frac{1}{k!}$.

Theorem (Cauchy criterion for series)

Let $\sum_{k=1}^{\infty} a_k$ be a series. Then $\sum_{k=1}^{\infty} a_k$ is convergent if and only if

$$\forall \epsilon > 0, \exists\, n_0 \in \mathbb{N} \,:\, \forall m > n > n_0, \left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.$$

Proof of the Cauchy Criterion

By definition, the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges. By the Cauchy criterion for sequences, this sequence converges if and only if it is Cauchy.

But what does it mean for the sequence of partial sums to be Cauchy? By definition, this means that

$$\forall \epsilon > 0, \exists\, n_0 \in \mathbb{N} : \forall m > n > n_0, |s_m - s_n| < \epsilon.$$

Observe that $s_m - s_n = \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k = \sum_{n+1}^{m} a_k$. Therefore, we get

$$\forall \epsilon > 0, \exists\, n_0 \in \mathbb{N} : \forall m > n > n_0, \left| \sum_{n+1}^{m} a_k \right| < \epsilon.$$

Example

During the last lecture, we proved:

$$\sum_{k=n}^{2n-1} \frac{1}{k} \geq \sum_{k=n}^{2n-1} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}.$$

It follows that the harmonic series does not satisfy the Cauchy criterion.

Corollary

Let $(a_n)_{n\in\mathbb{N}}$ *be a sequence and suppose that the series* $\sum_{k=1}^{\infty} a_k$ *converges. Then*

$$\lim_{n\to\infty} a_n = 0.$$

In other words, a series can only be convergent if the corresponding sequence is a null sequence.

Proof.

Suppose that the series $\sum_{k=1}^{\infty} a_k$ converges. Then it follows from Cauchy criterion (setting $m = n + 1$ in the statement on slide 33) that

$$\forall \epsilon > 0,\ \exists\, n_0 \in \mathbb{N}\ :\ \forall n > n_0,\ |a_{n+1}| < \epsilon.$$

This implies that $\lim_{n\to\infty} a_n = 0$. □

Example

We can immediately use the previous corollary to see that, for any divergent sequence, or any convergent sequence which is not a null sequence, the corresponding series is not convergent. In particular, the series

$$\sum_{k=1}^{\infty} (-1)^k$$

is divergent. Also, for any $m \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \sqrt[k]{k^m}$$

is divergent. For $\lambda \geq 0$ the series

$$\sum_{k=1}^{\infty} n^{\lambda}$$

Remark

An important remark is that the converse of Corollary on the slide 36 does not hold. For instance, as we have shown earlier, the harmonic series is definitely divergent with

$$\sum_{k=1}^{\infty} \frac{1}{n} = \infty,$$

although the sequence of its summands $(\frac{1}{n})_{n\in\mathbb{N}}$ is a null sequence.

Definition

Absolute convergence of a series Let $(a_n)_{n\in\mathbb{N}}$ be a complex-valued sequence with the property that there exists $C \in \mathbb{R}$ such that

$$\sum_{k=1}^{n} |a_k| \leq C, \qquad \forall n \in \mathbb{N}.$$

Then we say that the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Remark

Note that, by the slide 28, the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if and only if the series $\sum_{k=1}^{\infty} |a_k|$ is convergent (which we could have used as an equivalent definition). It is therefore perfectly reasonable to write $\sum_{k=1}^{\infty} |a_k| < \infty$ as a shorthand for the series $\sum_{k=1}^{\infty} a_k$ being absolutely convergent.

Lemma

The geometric series $\sum_{k=1}^{\infty} q^k$, *with* $|q| < 1$, *is absolutely convergent.*

Proof.

We have

$$\sum_{k=1}^{\infty} |q^k| = \sum_{k=1}^{\infty} |q|^k = \frac{|q|}{1 - |q|},$$

where we have just applied the formula for the sum of a geometric series, with $|q|$ in the role of q.

$\square$

Example

The series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

is called the alternating harmonic series. This series is not absolutely convergent, since

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

is the harmonic series, which is divergent. However, it turns out that the alternating harmonic series is convergent.

Exercise

Prove that the alternating harmonic series is convergent.

Theorem

If a series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent then it is also convergent.

Proof

Let $\sum_{k=1}^{\infty} a^k$ be an absolutely convergent series and let $\epsilon > 0$ be arbitrary. We apply Cauchy criterion to the series $\sum_{k=1}^{\infty} |a_k|$ there exists $n_0 \in \mathbb{N}$ such that, for all $m > n \geq n_0$,

$$\sum_{k=n+1}^{m} |a_k| = \left| \sum_{k=n+1}^{m} |a_k| \right| < \epsilon.$$

Proof.

By the triangle inequality

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| = \left|\sum_{k=n+1}^{m} |a_k|\right| < \epsilon.$$

It then follows from a second application of Cauchy criterion that $\sum_{k=1}^{\infty} a_k$ is convergent. $\square$