61. Study the convergence of these series:

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{(k+1)(k+2)(k+3)}}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{((k+1)(k+2)(k+3))^{\frac{1}{4}}}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+k}{k^5+1}$

   Solution:

   (a) $(k+1)(k+2)(k+3) > k^3$ then $\displaystyle\frac{1}{\sqrt{(k+1)(k+2)(k+3)}} < \frac{1}{k^{\frac{3}{2}}}$.

   The serie $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ is convergent then $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{(k+1)(k+2)(k+3)}}$ converges.

   (b) $\displaystyle\frac{1}{((k+1)(k+2)(k+3))^{\frac{1}{4}}} > \frac{1}{(3+k)^{\frac{3}{4}}} > \frac{1}{3+k}$

   but $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3+k} = \sum_{k=4}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} - 1 - \frac{1}{2} - \frac{1}{3}$, and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ (harmonic serie) is divergent. Then, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{((k+1)(k+2)(k+3))^{\frac{1}{4}}}$ is divergent.

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+k}{k^5+1} = \sum_{k=1}^{\infty} \underbrace{\frac{k}{k^5+1}}_{<\frac{1}{k^4}} + \sum_{k=1}^{\infty} \underbrace{\frac{k^3}{k^5+1}}_{<\frac{1}{k^2}}$ thus $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+k}{k^5+1}$ is the sum of two convergent series and then it is convergent.

   $\square$

62. Study the convergence of these series:

(a) $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k^2}\right)^{k^3}$

(b) $\sum_{k=1}^{\infty} \frac{3^k}{k!}$

(c) $\sum_{k=1}^{\infty} \frac{k + k^k}{k^{2k}}$

(d) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

Solution:

(a) We will use the root test. Let's $a_k = \left(1 - \frac{1}{k^2}\right)^{k^3}$, $k \geq 1$.

Then $|a_k|^{\frac{1}{k}} = \left(\left(1 - \frac{1}{k^2}\right)^{k^3}\right)^{\frac{1}{k}} = \left(1 - \frac{1}{k^2}\right)^{k^2}$ and $\lim_{k\to\infty} \left(1 - \frac{1}{k^2}\right)^{k^2} = e^{-1} < 1$,

then $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k^2}\right)^{k^3}$ is convergent.

(b) We will use the ratio test. Let's $a_k = \frac{3^k}{k!}$.

Then $\frac{a_{k+1}}{a_k} = \frac{3^{k+1}}{(k+1)!}\frac{k!}{3^k} = \frac{3}{k+1} \leq \frac{3}{4} < 1$ if $k \geq 3$ then $\sum_{k=1}^{\infty} \frac{3^k}{k!}$ is convergent.

(c) $\sum_{k=1}^{\infty} \frac{k + k^k}{k^{2k}} = \sum_{k=1}^{\infty} \underbrace{\frac{k}{k^{2k}}}_{\frac{1}{k^{2k-1}}} \sum_{k=1}^{\infty} \underbrace{\frac{k^k}{k^{2k}}}_{=\frac{1}{k^k}}$, then $\sum_{k=1}^{\infty} \frac{k + k^k}{k^{2k}}$ is the sum of two convergent series and then converges.

(d) We will use the ratio test. Let's $a_k = \frac{k!}{k^k}$.

Then $\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1}}\frac{k^k}{k!} = \frac{k^k(k+1)}{(k+1)^{k+1}} = \frac{k^k}{(k+1)^k} = \left(\frac{k}{k+1}\right)^k = \left(\frac{1}{1+\frac{1}{k}}\right)^k$

We know that $\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e > 1$, so $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = e^{-1} < 1$ and $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ is convergent.

$\square$

63. Study the values of α for which $\sum_{k=2}^{\infty} \frac{1}{k \ln^{\alpha}(k)}$ converges.

Hint: you can try to use the condensation test.

Solution:
Let's $a_k = \frac{1}{k \ln^{\alpha}(k)}$. We use the condensation test, so we study the convergence of $\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{2^n}{2^n \ln^{\alpha}(2^n)}$.

Then, $\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{1}{\ln^{\alpha}(2^n)} = \sum_{k=1}^{\infty} \frac{1}{(n \ln(2))^{\alpha}} = \sum_{k=1}^{\infty} \frac{1}{n^{\alpha} \ln(2)^{\alpha}}$.

Then, $\sum_{n=2}^{\infty} 2^n a_{2^n}$ (and thus $\sum_{k=1}^{\infty} a_k$) converges if and only if $\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}}$. We know $\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}}$ converges if and only if $\alpha > 1$, so $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k \ln^{\alpha}(k)}$ converges if and only if $\alpha > 1$. $\square$

64. For each serie, find the values $b \in \mathbb{R}$ such as the following series converge:

 (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(b^2+2b)^k}{k^2}$

 (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^b}$

 Solution:

 (a) For the sake of simplicity, the notation $A = b^2 + 2b$ is used, so we study the convergence of the serie $\displaystyle\sum_{k=1}^{\infty} \frac{A^k}{k^2}$. We study the cases $|A| \leq 1$ and $|A| > 1$:

 - If $|A| > 1$, then the term $a_k = \dfrac{A^k}{k^2}$ does not converge to 0, so the serie $\displaystyle\sum_{k=1}^{\infty} \frac{A^k}{k^2}$ is divergent.
 - If $|A| \leq 1$, then $|a_k| < \dfrac{1}{k^2}$, and then $\displaystyle\sum_{k=1}^{\infty} \frac{A^k}{k^2}$ is absolutely convergent.

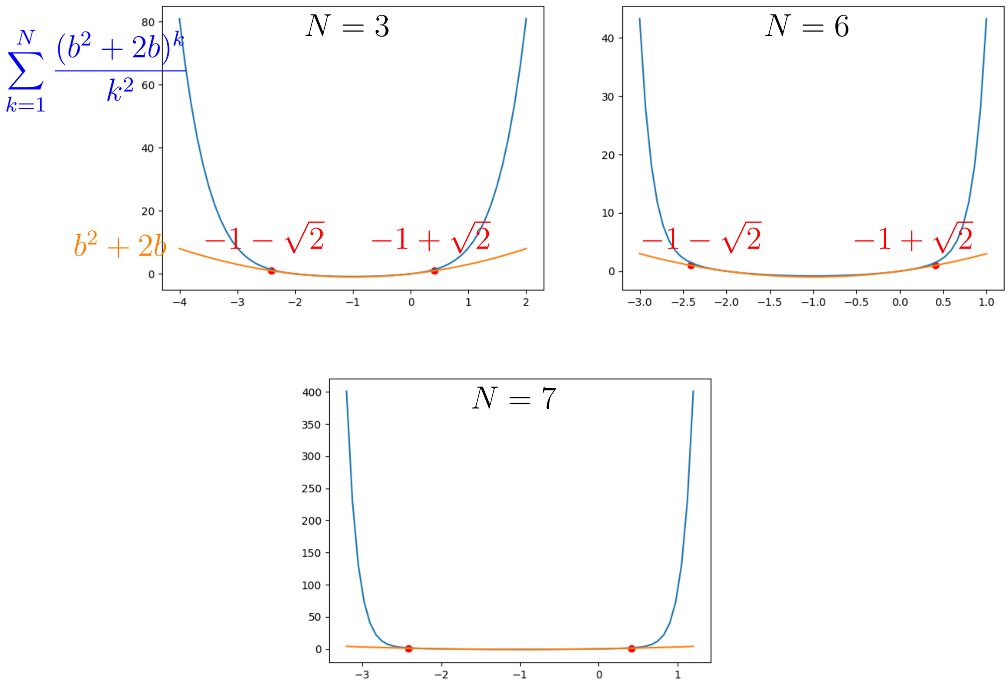
 Finallly, to determine for which values of b the serie $\displaystyle\sum_{k=1}^{\infty} \frac{(b^2+2b)^k}{k^2}$ is convergent, we have to solve $|b^2 + 2b| = 1$, in other terms, we have to solve:

 - $b^2 + 2b = 1$, which has $-1 + \sqrt{2}$ and $-1 - \sqrt{2}$ as solutions.
 - $b^2 + 2b = -1$, i.e. $(b+1)^2 = 0$ which has obviously -1 as unique solution.

 We deduce the parabola $b^2 + 2b$ is lower or equal than 1 if $b \in [-1-\sqrt{2}, -1+\sqrt{2}]$ and always greater or equal than -1 (equal to -1 when $b = -1$).

 To conclude, $\displaystyle\sum_{k=1}^{\infty} \frac{(b^2+2b)^k}{k^2}$ converges when $b \in [-1-\sqrt{2}, -1+\sqrt{2}]$.

 We can see what happens with partial sums depending on b, $f(b) = \displaystyle\sum_{k=1} \frac{(b^2+2b)^k}{k^2}$:



(b) Let's $a_k = \dfrac{1}{k^b}$. For $b = 0$, $(\lim_{k\to\infty})a_k = 1$. For $b < 0$, $\lim_{k\to\infty} a_k = +\infty$. Then, for $b \leq 0$, $\lim_{k\to\infty} a_k \neq 0$, so $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^b}$ is divergent.

We want to use the Leibniz rule for $b > 0$. Let's $a_k = \dfrac{1}{k^b}$, and its associated function $f(x) = \dfrac{1}{x^b} = x^{-b}, x \in \mathbb{R}^+$. According to the lecture notes page 48, f and then $(a_k)_{k\in\mathbb{N}}$ is decreasing only if $b > 0$. Moreover, with $b > 0$, $\lim_{k\to\infty} a_k = 0$. So we can apply the Leibniz rule, and then $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^b}$ is convergent if and only if $b > 0$.

$\square$

65. Let $(F_n)_{n\in\mathbb{N}}$ be a Fibonacci sequence. Show that $\sum_{k=1}^{\infty} \frac{1}{F_k}$ is convergent.

Solution:
A Fibonacci sequence can be defined by $F_{k+2} = F_{k+1} + F_k$, $k \geq 0$, $F_1 = 1$, $F_0 = 0$.
We want to use the ratio test. Let's $a_k = \frac{1}{F_k}$, for sake of simplicity, we will study the ratio $\frac{a_k}{a_{k+1}}$ instead of $\frac{a_{k+1}}{a_k}$, so we want to prove that $\exists C > 1, \forall k \in \mathbb{N}, \frac{a_k}{a_{k+1}} \geq C$.
A possibility would be to use the explicit formula of F_k, $F_k = \frac{1}{\sqrt{5}}\left(\phi^k + (1-\phi)^k\right)$, ϕ is the golden ratio: $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. Then, we can prove that $\lim_{k\to+\infty} \frac{a_k}{a_{k+1}} = \lim_{k\to+\infty} \frac{F_{k+1}}{F_k} = \phi > 1$. However, the explicit formula has not to be known in this class.
Let's try with the implicit form. For $k \geq 1$, $\frac{a_k}{a_{k+1}} = \frac{F_{k+1}}{F_k} = \frac{F_k + F_{k-1}}{F_k} = 1 + \frac{F_{k-1}}{F_k}$.
Now, we have to prove that $\exists C' > 0, \forall k \geq 1, \frac{F_{k-1}}{F_k} \geq C'$. It is easy to see (for example by induction) that $(F_k)_{k\in\mathbb{N}}$ is a non-decreasing sequence with positive terms. Then, $F_k = F_{k-1} + \underbrace{F_{k-2}}_{\leq F_{k-1}} \leq 2F_{k-1}$, so $\frac{F_{k-1}}{F_k} \geq \frac{1}{2}$ and then $\frac{a_k}{a_{k+1}} \geq \frac{3}{2} > 1$.

Thanks to the ratio test we conclude that $\sum_{k=1}^{\infty} \frac{1}{F_k}$ is convergent.

Remark: If we assume that $\frac{F_{k+1}}{F_k}$ *has a limit* $l \in \mathbb{R}$*, then* $\lim_{k\to+\infty} \frac{F_{k+1}}{F_k} = \lim_{k\to+\infty} 1 + \frac{F_{k-1}}{F_k}$ *and* $l = 1 + \frac{1}{l}$*, which is actually the polynomial* $l^2 - l - 1 = 0$*. The solutions are* $l = \phi$ *and* $l = 1 - \phi$*, and only* $\phi > 1$ *is relevant because* $l > 0$*. However, to use this argument, you have to show before that a such limit* l *exists...* $\square$

66. Let $(u_n)_{n\in\mathbb{N}}$ be a positive sequence such as the serie $\sum_{n=1}^{\infty} u_n$ is convergent. We want to study the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{u_n}}{n}$.

   (a) Prove that $\sum_{n=1}^{m} \frac{\sqrt{u_n}}{n} \leq \left(\sum_{n=1}^{m} u_n\right)^{\frac{1}{2}} \cdot \left(\sum_{n=1}^{m} \frac{1}{n^2}\right)^{\frac{1}{2}}$.
   Hint: Use a famous inequality you learnt at the beginning of the semester.

   (b) Conclude about the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{u_n}}{n}$.

Solution:

(a) We remind that for two vectors of same length, $\boldsymbol{a} = (a_1, ..., a_N)$ and $\boldsymbol{b} = (b_1, ..., b_N)$, the Cauchy-Schwarz inequality can be applied:

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle \leq \|\boldsymbol{a}\|_2 \cdot \|\boldsymbol{b}\|_2$$

$$\sum_{n=1}^{N} a_n b_n \leq \left(\sum_{n=1}^{N} a_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} a_n^2\right)^{\frac{1}{2}}$$

If we choose $a_n = \sqrt{u_n}$, $b_n = \frac{1}{n}$, then we get:

$$\sum_{n=1}^{N} a_n b_n \leq \left(\sum_{n=1}^{N} u_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{\frac{1}{2}}$$

(b) We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, let's denote its limite as $l = \left(\frac{\pi^2}{6}\right)$, and the statement asserts that $\sum_{n=1}^{\infty} u_n$ is convergent, let's denote its limit as l'. Then:

$$\forall N \in \mathbb{N} : \left(\sum_{n=1}^{N} \frac{\sqrt{u_n}}{n}\right)^2 \leq \left(\sum_{n=1}^{N} u_n\right) \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)$$

it follows that

$$\lim_{N\to+\infty} \left(\sum_{n=1}^{N} \frac{\sqrt{u_n}}{n}\right)^2 \leq \underbrace{\left(\lim_{N\to+\infty} \sum_{n=1}^{N} u_n\right)}_{l'} \underbrace{\left(\lim_{N\to+\infty} \sum_{n=1}^{N} \frac{1}{n^2}\right)}_{l}$$

therefore we get

$$\sum_{n=1}^{\infty} \frac{\sqrt{u_n}}{n} \leq \sqrt{ll'} \in \mathbb{R}^+$$

Then $\sum_{n=1}^{\infty} \frac{\sqrt{u_n}}{n}$ is convergent.

$\square$