This exercise sheet focuses on working with definitions, which might be new to you. Please study all the slides from Chapter 1 (in Moodle) and the definitions on this exercise sheet.

Read the following definition:

Definition. *A* two-place **predicate** $P(\cdot_1, \cdot_2)$ *is a linguistic expression with* two *blanks and* two *corresponding classes* U_1, U_2 *of objects, s.t. inserting an object* x *from class* U_1 *and an object* y *from class* U_2 *yields the proposition* $P(x, y)$. *Via the* **universal quantifier** $\forall$ *and the* **existential quantifier** $\exists$ *one builds the following propositions*

$$\begin{array}{ll} \forall x \in U_1\ \forall y \in U_2:\ P(x,y), & \forall y \in U_2\ \forall x \in U_1:\ P(x,y), \\ \forall x \in U_1\ \exists y \in U_2:\ P(x,y), & \exists y \in U_2\ \forall x \in U_1:\ P(x,y), \\ \exists x \in U_1\ \forall y \in U_2:\ P(x,y), & \forall y \in U_2\ \exists x \in U_1:\ P(x,y), \\ \exists x \in U_1\ \exists y \in U_2:\ P(x,y), & \exists y \in U_2\ \exists x \in U_1:\ P(x,y), \end{array}$$

whose truth values are defined by

$$\begin{aligned} |\forall x \in U_1\ \forall y \in U_2:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{for all } x \in U_1 \textit{ it holds that } (\forall y \in U_2:\ P(x,y)) \textit{ is true}, \\ |\forall y \in U_2\ \forall x \in U_1:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{for all } y \in U_2 \textit{ it holds that } (\forall x \in U_1:\ P(x,y)) \textit{ is true}, \\ |\forall x \in U_1\ \exists y \in U_2:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{for all } x \in U_1 \textit{ it holds that } (\exists y \in U_2:\ P(x,y)) \textit{ is true}, \\ |\exists y \in U_2\ \forall x \in U_1:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{there exists } y \in U_2,\ \textit{s.t. } (\forall x \in U_1:\ P(x,y)) \textit{ is true}, \\ |\exists x \in U_1\ \forall y \in U_2:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{there exists } x \in U_1,\ \textit{s.t. } (\forall y \in U_2:\ P(x,y)) \textit{ is true}, \\ |\forall y \in U_2\ \exists x \in U_1:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{for all } y \in U_2 \textit{ it holds that } (\exists x \in U_1:\ P(x,y)) \textit{ is true}, \\ |\exists x \in U_1\ \exists y \in U_2:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{there exists } x \in U_1,\ \textit{s.t. } (\exists y \in U_2:\ P(x,y)) \textit{ is true}, \\ |\exists y \in U_2\ \exists x \in U_1:\ P(x,y)| = \mathbb{T} \ &:\Leftrightarrow \ \textit{there exists } y \in U_2,\ \textit{s.t. } (\exists x \in U_1:\ P(x,y)) \textit{ is true}. \end{aligned}$$

1. Let $P(x, y)$ be the predicate $x \geq y$ for $x \in \mathbb{N}$ and $y \in \mathbb{N}$. Find the truth value of the following statements. What do you observe?

   (a) $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N}:\ P(x,y)$, $\quad \forall y \in \mathbb{N}\ \forall x \in \mathbb{N}:\ P(x,y)$.

   (b) $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}:\ P(x,y)$, $\quad \exists y \in \mathbb{N}\ \forall x \in \mathbb{N}:\ P(x,y)$.

   (c) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}:\ P(x,y)$, $\quad \forall y \in \mathbb{N}\ \exists x \in \mathbb{N}:\ P(x,y)$.

   (d) $\exists x \in \mathbb{N}\ \exists y \in \mathbb{N}:\ P(x,y)$, $\quad \exists y \in \mathbb{N}\ \exists x \in \mathbb{N}:\ P(x,y)$.

   Solution:
   We always either give a counterexample or a logical argument:

   (a) **false**, let $x = 1$, then e.g. for $y = 2$ we have $1 < 2$.
   Also **false**, if $y = 2$, then e.g. for $x = 1$, we have $1 < 2$.

   (b) **true**, let x, be any natural number, then we have $x \geq x$, hence if we choose $y = x$, we have $x \geq y$.
   true, let $y = 1$, then for all natural numbers x it holds $x \geq 1$.

(c) **false**, assume there is such a number x. We set y to be the natural number $x+1$, thus we get $x \geq x+1$, which is a contradiction.
true, let y be any natural number, and let x be the natural number $y+1$. We then have $y+1 \geq y$, which is a tautology.

(d) **true**, let $x = 5$, $y = 2$, then $5 \geq 2$.
true, let $y = 2$, $x = 5$, then $5 \geq 2$.

We may observe: changing the order of quantification of variables does not seem to matter, if the quantors are the same, i.e. both universal quantors $\forall$ or both existential quantors $\exists$. Changing the order of quantification might matter, if the quantors are different. $\square$

2. Let $P(x, y)$ be a two-place predicate with $x \in U_1$, $y \in U_2$. Negate the following statements.

   (a) $\forall x \in U_1\ \forall y \in U_2 :\ P(x, y)$,

   (b) $\forall x \in U_1\ \exists y \in U_2 :\ P(x, y)$,

   (c) $\exists x \in U_1\ \forall y \in U_2 :\ P(x, y)$,

   (d) $\exists x \in U_1\ \exists y \in U_2 :\ P(x, y)$,

   Hint: Use DeMorgan's laws.

   Solution:

   (a) $\neg(\forall x \in U_1\ \forall y \in \mathbb{N} :\ P(x, y)) \equiv \exists x \in U_1\ \exists y \in U_2 :\ \neg P(x, y)$,

   (b) $\neg(\forall x \in U_1\ \exists y \in \mathbb{N} :\ P(x, y)) \equiv \exists x \in U_1\ \forall y \in U_2 :\ \neg P(x, y)$,

   (c) $\neg(\exists x \in U_1\ \forall y \in \mathbb{N} :\ P(x, y)) \equiv \forall x \in U_1\ \exists y \in U_2 :\ \neg P(x, y)$,

   (d) $\neg(\exists x \in U_1\ \exists y \in \mathbb{N} :\ P(x, y)) \equiv \forall x \in U_1\ \forall y \in U_2 :\ \neg P(x, y)$.

$\square$

Read the following definition:

> **Definition.** *For a given set A, the* **power set** *of A, denoted by $\mathcal{P}(A)$ is the set of all subsets of A, i.e.*
> $$\mathcal{P}(A) := \{B : B \subset A\}.$$

3. Using the definition above, solve the following tasks:

   (a) Let $A_1 := \{\text{car}\}$. Write down explicitly $\mathcal{P}(A_1)$.

   (b) Let $A_2 := \{a, b\}$. Write down explicitly $\mathcal{P}(A_2)$.

   (c) Let $A_3 := \{n \in \mathbb{N} : \; 2 \leq n \leq 4\}$. Write down explicitly $\mathcal{P}(A_3)$.

   (d) Write down explicitly $\mathcal{P}(\emptyset)$,

   Solution:

   (a) The subsets of A_1 are: $\emptyset$ and $\{\text{car}\} = A_1$. Hence $\mathcal{P}(A_1) = \{\emptyset, A_1\}$.

   (b) The subsets of A_2 are: $\emptyset, \{a\}, \{b\}$ and $\{a, b\} = A_2$. Hence $\mathcal{P}(A_2) = \{\emptyset, \{a\}, \{b\}, A_2\}$.

   (c) The subsets of A_3 are: $\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$ and $\{2, 3, 4\}$. Hence $\mathcal{P}(A_2) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, A_3\}$.

   (d) The only subset of $\emptyset$ is $\emptyset$, thus $\mathcal{P}(\emptyset) = \{\emptyset\}$.

$\square$

4. Let A, B be any sets.

   (a) Prove or disprove: $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (b) Prove or disprove: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   Hint 1: Try some examples first.
   Hint 2: Two sets M, N are equal, if $M \subset N$ and $N \subset M$.
   Solution:

   (a) The first statement is true. As $\mathcal{P}(A \cap B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$ are both sets, we show $\mathcal{P}(A \cap B) \subset \mathcal{P}(A) \cap \mathcal{P}(B)$ and we show $\mathcal{P}(A) \cap \mathcal{P}(B) \subset \mathcal{P}(A \cap B)$.

   "$\subset$":
   If $C \in \mathcal{P}(A \cap B)$, it is a subset of $A \cap B$, thus it contains only elements, which are contained in A and in B. Hence $C \subset A$ and $C \subset B$. Thus, $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$. Finally, $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   "$\supset$":
   If $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$, it is a subset of A and a subset of B. Thus it contains only elements, which are contained in A and in B. Hence $C \subset A \cap B$. Thus, $C \in \mathcal{P}(A \cap B)$.

   (b) This statement is false, consider for example $A = \{a\}$ and $B = \{b, c\}$. Then

   $$\mathcal{P}(A) = \{\emptyset, \{a\}\} \text{ and } \mathcal{P}(B) = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}.$$

   Furthermore, $A \cup B = \{a, b, c\}$ and

   $$\mathcal{P}(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, , \{b, c\}, \{a, b, c\}\},$$

   but

   $$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}.$$

   $\square$

Read the following definitions:

> **Definition.** *A* **relation** *R beween two sets M and N is a subset of the cartesian product of M and N, i.e.*
> $$R \subset M \times N.$$
> *If $(x, y) \in R$, we also write xRy. If $M = N$, we call $R \subset M^2$ a relation* **on** *M.*

> **Definition.** *Let $R \subset M \times N$ be a releation. R is called*
>
> - **serial** *(or left-total), if for all $x \in M$ exists a $y \in N$, s.t. xRy, i.e. $(x, y) \in R$.*
> - **functional** *(or right-unique), if for all $x \in M$ and $y_1, y_2 \in N$ it holds that xRy_1 and xRy_2 implies $y_1 = y_2$.*

5. Let $M := \{1, 2, 3\}$ and $N := \{a, b, c\}$.

   (a) Determine $M \times N$.

   (b) Determine $M \times M =: M^2$.

   (c) Write down the definition of *serial* and *functional* using only mathematical symbols. (Quantors, logical connectives, etc.)

   Solution:

   (a) $M \times N = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$.

   (b) $M^2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

   (c)
   - serial: $\forall x \in M\ \exists y \in N : xRy$.
   - functional: $\forall x \in M\ \forall y_1, y_2 \in N : xRy_1 \wedge xRy_2 \Rightarrow y_1 = y_2$.

□

6. Let $M := \{1, 2, 3\}$ and $N := \{a, b, c\}$.

   (a) Give an example of a relation $R \subset M \times N$ that is

      i. serial, but not functional,

      ii. functional, but not serial,

      iii. functional and serial,

      iv. neither functional nor serial.

   (b) Visualize your solutions from (a), e.g. using a table.

Solution:
(More solutions possible)

(a) i. $R := \{(1, a), (1, b), (2, a), (3, a)\}$,

   ii. $R := \{(1, a)\}$,

   iii. $R := \{(1, a), (2, a), (3, a)\}$,

   iv. $R := \{(1, a), (1, b)\}$.

(b) (Depending on the prior solution)

i.

	1	2	3
a	•	•	•
b	•		
c			

ii.

	1	2	3
a	•		
b			
c			

iii.

	1	2	3
a	•	•	•
b			
c			

iv.

	1	2	3
a	•		
b	•		
c			

$\square$

The focus of this exercise sheet is using indices and the Σ- and Π-notation for sums and products respectively.

7. For a given n-tuple $x = (x_1, x_2, \dots, x_n)$, we define

$$\mu := \frac{1}{n}\sum_{i=1}^{n} x_i$$

and

$$\sigma^2 := \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2.$$

(a) Let $x = (x_1, x_2)$. Write down μ and σ^2, i.e. do not use dot notation or Σ-notation.

(b) Let $x = (x_1, x_2, x_3)$. Write down μ and σ^2, i.e. do not use dot notation or Σ-notation.

(c) Consider the following table.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
56	64	72	52	58	60	76	74	62	66

Compute μ and σ^2.

(d) Consider the following table.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
48	64	80	40	52	56	88	84	60	68

Compute μ and σ^2.

Solution:

(a) We have

$$\mu = \frac{1}{2}\sum_{i=1}^{2} x_i = \frac{x_1 + x_2}{2}$$

and

$$\begin{aligned}\sigma^2 &= \frac{1}{2}\sum_{i=1}^{2}(x_i - \mu)^2 \\ &= \frac{(x_1 - \mu)^2 + (x_2 - \mu)^2}{2} \\ &= \frac{(x_1 - \frac{x_1+x_2}{2})^2 + (x_2 - \frac{x_1+x_2}{2})^2}{2} \\ &= \frac{1}{4}(x_1 - x_2)^2.\end{aligned}$$

(Verify this computation yourself!)

(b) We have

$$\mu = \frac{1}{3}\sum_{i=1}^{3} x_i = \frac{x_1 + x_2 + x_3}{3}$$

and

$$\begin{aligned}
\sigma^2 &= \frac{1}{3}\sum_{i=1}^{3}(x_i - \mu)^2 \\
&= \frac{(x_1-\mu)^2 + (x_2-\mu)^2 + (x_3-\mu)^2}{3} \\
&= \frac{(x_1 - \frac{x_1+x_2+x_3}{3})^2 + (x_2 - \frac{x_1+x_2+x_3}{3})^2 + (x_3 - \frac{x_1+x_2+x_3}{3})^2}{3} \\
&= \frac{2}{9}(x_1^2 - x_1x_2 - x_1x_3 + x_2^2 - x_2x_3 + x_3^2).
\end{aligned}$$

(Verify this computation yourself!)

(c) We have $\mu = \frac{56+64+72+52+58+60+76+74+62+66}{10} = 64$ and

$$\begin{aligned}
\sigma^2 &= \frac{1}{10}\sum_{i=1}^{10}(x_i - 64)^2 \\
&= \frac{1}{10}\Big((-8)^2 + 0^2 + 8^2 + (-12)^2 + (-6)^2 + (-4)^2 + 12^2 + 10^2 + (-2)^2 + 2^2\Big) \\
&= 57.6
\end{aligned}$$

(d) We have $\mu = 64$ and $\sigma^2 = 230.4$

□

8. Show the following equalities by using dot Notation. **Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

   (a) $\sum_{\ell=1}^{u} a_\ell = \sum_{\ell=1}^{u} a_{u-\ell+1} = \sum_{\ell=0}^{u-1} a_{u-\ell}$.

   (b) $\left(\prod_{k=1}^{n} a_k\right) \cdot \left(\prod_{\ell=n+1}^{u} a_\ell\right) = \prod_{j=1}^{u} a_j$.

   Solution:

   (a) We have

$$\begin{aligned}
\sum_{\ell=1}^{u} a_\ell &= a_1 + a_2 + a_3 + \cdots + a_{u-1} + a_u \\
&= a_u + a_{u-1} + \cdots + a_3 + a_2 + a_1 \\
&= a_u + a_{u-1} + \cdots + a_{u-(u-3)} + a_{u-(u-2)} + a_{u-(u-1)} \\
&= a_{u-1+1} + a_{u-1-1+1} + \cdots + a_{u-(u-3)-1+1} + a_{u-(u-2)-1+1} + a_{u-(u-1)-1+1} \\
&= a_{u-(1)+1} + a_{u-(2)+1} + \cdots + a_{u-(u-2)+1} + a_{u-(u-1)+1} + a_{u-(u)+1} \\
&= \sum_{\ell=1}^{u} a_{u-\ell+1} \\
&= a_{u-(1)+1} + a_{u-(2)+1} + \cdots + a_{u-(u-2)+1} + a_{u-(u-1)+1} + a_{u-(u)+1} \\
&= a_{u-0} + a_{u-1} + \cdots + a_{u-(u-3)} + a_{u-(u-2)} + a_{u-(u-1)} \\
&= \sum_{\ell=0}^{u-1} a_{u-\ell}.
\end{aligned}$$

   (b) As

$$\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_{n-1} \cdot a_n,$$

and

$$\prod_{\ell=n+1}^{u} a_\ell = a_{n+1} \cdot a_{n+2} \cdot a_{n+3} \cdot \cdots \cdot a_{u-1} \cdot a_u,$$

we have

$$\begin{aligned}
\left(\prod_{k=1}^{n} a_k\right) \cdot \left(\prod_{\ell=n+1}^{u} a_\ell\right) &= (a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_{n-1} \cdot a_n) \cdot (a_{n+1} \cdot a_{n+2} \cdot a_{n+3} \cdot \cdots \cdot a_{u-1} \cdot a_u) \\
&= a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_{n-1} \cdot a_n \cdot a_{n+1} \cdot a_{n+2} \cdot a_{n+3} \cdot \cdots \cdot a_{u-1} \cdot a_u \\
&= a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_u \\
&= \prod_{j=1}^{u} a_j.
\end{aligned}$$

$\square$

9. Find the errors in the following computations, i.e. check for every equality "=" if it is always true. Show that you really found an error, by providing an example, for which the equality does not hold. *(Also find an example, for which the equality does hold.)*

   (a)
$$\begin{aligned}\sum_{\ell=1}^{n} a_\ell &= \sum_{\ell=1}^{m} a_\ell + \sum_{\ell=m+1}^{n} a_\ell \\ &= a_1 + \left(\sum_{\ell=1}^{m-1} a_\ell\right) + a_m + \left(\sum_{\ell=m+1}^{n-1} a_\ell\right) + a_n \\ &= (a_1 + a_m + a_n) + \sum_{\ell=1}^{n} a_\ell.\end{aligned}$$

   (b)
$$\begin{aligned}\sum_{\ell=1}^{m+u} b_\ell &= \sum_{j=0}^{m+u-1} b_{j+1} \\ &= \sum_{\ell=0}^{m+u-1} b_{m-\ell+u} \\ &= \sum_{\ell=1}^{m} b_{m-\ell+1} + \sum_{\ell=1}^{u-1} b_{u+1} \\ &= \sum_{\ell=1}^{m} b_\ell + \sum_{\ell=1}^{u} b_u.\end{aligned}$$

   (c) For constant $c \in \mathbb{R}$:
$$\begin{aligned}\sum_{j=0}^{m-1} c \cdot c_j &= \sum_{j=1}^{m} c \cdot c_j \\ &= c \cdot c_m \cdot \sum_{j=1}^{m-1} c_j \\ &= c \cdot \sum_{j=0}^{m-1} c_j.\end{aligned}$$

Solution:

(a) The first equality gives a possible "index out of bounds" error, if $m > n$, i.e. a_m might not even be defined.
In general,
$$\sum_{\ell=1}^{m} a_\ell + \sum_{\ell=m+1}^{n} a_\ell \neq a_1 + \left(\sum_{\ell=1}^{m-1} a_\ell\right) + a_m + \left(\sum_{\ell=m+1}^{n-1} a_\ell\right) + a_n,$$

as

$$\sum_{\ell=1}^{m} a_\ell = \left(\sum_{\ell=1}^{m-1} a_\ell\right) + a_m \text{ and } \sum_{\ell=m+1}^{n} a_\ell = \left(\sum_{\ell=m+1}^{n-1} a_\ell\right) + a_n,$$

the above equality holds iff $a_1 = 0$.
Furthermore,

$$a_1 + \left(\sum_{\ell=1}^{m-1} a_\ell\right) + a_m + \left(\sum_{\ell=m+1}^{n-1} a_\ell\right) + a_n \neq (a_1 + a_m + a_n) + \sum_{\ell=1}^{n} a_\ell,$$

as in general

$$\left(\sum_{\ell=1}^{m-1} a_\ell\right) + \left(\sum_{\ell=m+1}^{n-1} a_\ell\right) \neq \sum_{\ell=1}^{n} a_\ell.$$

We see this, if we consider for example $n = 4, a = (1, 2, 3, 4)$ and $m = 3$. Then $m + 1 = 4$, $n - 1 = 3$. Thus

$$\left(\sum_{\ell=1}^{m-1} a_\ell\right) + \left(\sum_{\ell=m+1}^{n-1} a_\ell\right) = \left(\sum_{\ell=1}^{2} a_\ell\right) + \underbrace{\left(\sum_{\ell=4}^{3} a_\ell\right)}_{=0} = 1 + 2 = 3,$$

but

$$\sum_{\ell=1}^{4} 3 = 10.$$

(b) First of all,

$$\sum_{\ell=0}^{m+u-1} b_{m-\ell+u} \neq \sum_{\ell=1}^{m} b_{m-\ell+1} + \sum_{\ell=1}^{u-1} b_{u+1},$$

in the second sum the term b_{u+1} appears, if $m = 0$ it might happen that the value is not defined. (In programming: array index out of bounds error.)
Moreover,

$$\sum_{\ell=1}^{m} b_{m-\ell+1} + \sum_{\ell=1}^{u-1} b_{u+1} \neq \sum_{\ell=1}^{m} b_\ell + \sum_{\ell=1}^{u} b_u,$$

as

$$\sum_{\ell=1}^{m} b_{m-\ell+1} = \sum_{\ell=1}^{m} b_\ell,$$

but in general

$$(u-1) \cdot b_{u+1} = \sum_{\ell=1}^{u-1} b_{u+1} \neq \sum_{\ell=1}^{u} b_u = u \cdot (b_u).$$

E.g. let $b = (7, 5, 25)$ and $u = 1$. Then $0 \cdot 25 = 0 \neq 5 = 1 \cdot 5$.

(c) Already the first step is erroneous:

$$\sum_{j=0}^{m-1} c \cdot c_j \neq \sum_{j=1}^{m} c \cdot c_j,$$

let $c = 1, m = 2$ and $c_i = i$ for $i = 0, 1, 2$. Then we get $0 + 1 = 1 \neq 3 = 1 + 2$.

Also the next step is not correct,

$$\sum_{j=1}^{m} c \cdot c_j \neq c \cdot c_m \cdot \sum_{j=1}^{m-1} c_j,$$

the same values as above yield $3 = 2$.
Using the same numbers again, we also find the error in the last step.

$$c \cdot c_m \cdot \sum_{j=1}^{m-1} c_j \neq c \cdot \sum_{j=0}^{m-1} c_j.$$

Thus in this computation, every step was incorrect, but the “overall result” is correct.

$\square$

10. Do the following statements hold in general? Justify your answer, i.e. give either a proof using dot-notation or a counterexample.

    (a) $\sum_{i=1}^{m}\sum_{\ell=1}^{n} a_{i,\ell} = \sum_{\ell=1}^{n}\sum_{i=1}^{m} a_{i,\ell}$.

    (b) $\sum_{i=1}^{m}\prod_{\ell=1}^{n} b_{i,\ell} = \prod_{\ell=1}^{n}\sum_{i=1}^{m} b_{i,\ell}$.

    (c) $\prod_{i=1}^{m}\prod_{\ell=1}^{n} c_{i,\ell} = \prod_{\ell=1}^{n}\prod_{i=1}^{m} c_{i,\ell}$.

    Solution:

    (a) Holds.

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{\ell=1}^{n} a_{i,\ell} &= \sum_{i=1}^{m}(a_{i,1} + a_{i,2} + \cdots + a_{i,n}) \\
&= (a_{1,1} + a_{1,2} + \cdots + a_{1,n}) + (a_{2,1} + a_{2,2} + \cdots + a_{2,n}) + \ldots \\
&\quad + (a_{m,1} + a_{m,2} + \cdots + a_{m,n}) \\
&= (a_{1,1} + a_{2,1} + \cdots + a_{m,1}) + (a_{1,2} + a_{2,2} + \cdots + a_{m,2}) + \ldots \\
&\quad + (a_{1,n} + a_{2,n} + \cdots + a_{m,n}) \\
&= \sum_{\ell=1}^{n}(a_{1,\ell} + a_{2,\ell} + \cdots + a_{m,\ell}) \\
&= \sum_{\ell=1}^{n}\sum_{i=1}^{m} a_{i,\ell}.
\end{aligned}$$

    (b) Does not hold. Let $m = n = 2$ and $b_{1,1} = 1, b_{1,2} = 2, b_{2,1} = 3, b_{2,2} = 4$. Then

$$\sum_{i=1}^{2}\prod_{\ell=1}^{2} b_{i,\ell} = (1 \cdot 2) + (3 \cdot 4) = 14,$$

    but

$$\prod_{\ell=1}^{2}\sum_{i=1}^{2} b_{i,\ell} = (1 + 3) \cdot (2 + 4) = 24.$$

    (c) Holds, similar to (a).

□

11. Take the following table:

i	0	1	2	3	4
x_i	-5	-3	0	3	4
y_i	3	-3	6	5	7

For $i = 0, \dots, n$, we define

$$\ell_i(x) := \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

This notation for the lower and upper index of the product corresponds to the index set

$$\{0, 1, 2, \dots, n\} \setminus \{i\} = \{0, 1, 2, \dots, i-1, i+1, \dots, n\}.$$

Furthermore, let

$$p(x) := \sum_{i=0}^{n} y_i \cdot \ell_i(x).$$

(a) Compute the five polynomial functions $\ell_0(x), \ell_1(x), \dots, \ell_4(x)$.

(b) Plot the graph of $\ell_i(x)$ for $i = 0, \dots, 4$ in the interval $[-6, 5] \subset \mathbb{R}$.

- Look at the zeros of the graphs. What do you observe?
- Also look at the points $(x_i, \ell_i(x_i)) \in \mathbb{R}^2$ for $i = 0, \dots, 4$. What do you observe?

Can you justify both observations? (For plotting/visualizing, use a tool of your choice!)

Hint: a zero of $\ell_i(x)$ is a value $\overline{x} \in \mathbb{R}$, such that $\ell_i(\overline{x}) = 0$.

(c) Compute $p(x)$ for the case $n = 4$ and plot the graph of $p(x)$ in the interval $[-6, 5] \subset \mathbb{R}$. In this plot, highlight the points $(x_i, y_i) \in \mathbb{R}^2$ for $i = 0, \dots, n$. (For plotting/visualizing, use a tool of your choice!)

(d) Compute $p(-1)$ and $p(2)$.

Solution:

(a) The polynomials are given by

$$\ell_0(x) = \tfrac{1}{720}\Big(x^4 - 4x^3 - 9x^2 + 36x\Big),$$
$$\ell_1(x) = \tfrac{1}{252}\Big(-x^4 + 2x^3 + 23x^2 - 60x\Big),$$
$$\ell_2(x) = \tfrac{1}{180}\Big(x^4 + x^3 - 29x^2 - 9x + 180\Big),$$
$$\ell_3(x) = \tfrac{1}{144}\Big(-x^4 - 4x^3 + 17x^2 + 60x\Big),$$
$$\ell_4(x) = \tfrac{1}{252}\Big(x^4 + 5x^3 - 9x^2 - 45x\Big).$$

Evaluating $\ell_i(x_j)$ always results in 0, for $i, j = 0, \ldots, 4$ and $i \neq j$. This is because one of the factors of every polynomial $\ell_i(x)$ will always be

$$\frac{x_j - x_j}{x_i - x_j} = \frac{0}{x_i - x_j} = 0.$$

On the other hand,

$$\ell_i(x_i) = \prod_{\substack{j=0 \\ j \neq i}}^{4} \frac{x_i - x_j}{x_i - x_j} = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

(b) $p(x) = \frac{1}{2520}\left(107x^4 - 18x^3 - 2363x^2 + 3522x + 15120\right)$.

(c) We have $p(-1) = \frac{26}{7}$ and $p(2) = \frac{17}{3}$.

Plots etc: `https://www.geogebra.org/calculator/tgcewfmb` $\square$

Definition. *Let $R \subset M \times N$ be a relation. R is called*

- **surjective** *(or right-total), if for all $y \in N$ exists a $x \in M$, s.t. xRy, i.e. $(x, y) \in R$.*
- **injective** *(or left-unique), if for all $y \in N$ and $x_1, x_2 \in M$ it holds that x_1Ry and x_2Ry implies $x_1 = x_2$.*

12. Let $M := \{1, 2, 3\}$ and $N := \{a, b, c\}$.

    (a) Give an example of a relation $R \subset M \times N$ that is

    i. surjective, but not injective,
    ii. injective, but not surjective,
    iii. injective and surjective,
    iv. neither injective nor surjective.

    (b) Visualize your solutions from (a), e.g. using a table. Compare

    - the definition of injective to the definition of functional,
    - the definition of surjective to the definition of serial.

    (c) Give two different examples of relations $R_1 \subset M \times N$ and $R_2 \subset M \times N$, s.t. R_1 and R_2 both are serial, functional, injective and surjective.

Solution:
(More solutions possible)

(a) i. $R := \{(1, a), (1, b), (2, a), (3, c)\}$,
ii. $R := \{(1, a), (2, c)\}$,
iii. $R := \{(1, a), (1, b), (1, c)\}$,
iv. $R := \{(1, a), (2, b)\}$.

(b) (Depending on the prior solution)

i.

	1	2	3
a	•	•	
b	•		
c			•

ii.

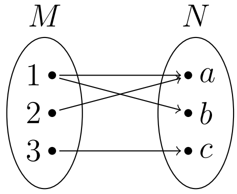
	1	2	3
a	•		
b			
c		•	

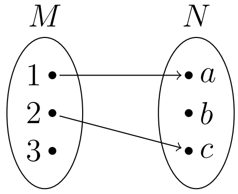
iii.

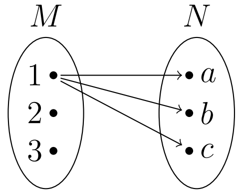
	1	2	3
a	•		
b	•		
c	•		

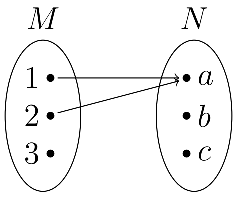
iv.

	1	2	3
a	•	•	
b			
c			

i. 

ii. 

iii. 

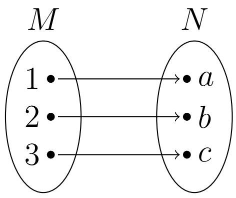
iv. 

What we might conjecture after looking at the visualization and the formulae is that a relation $R \subset M \times N$ is injective, iff the relation $\overline{R} \subset N \times M$ with $\overline{R} := \{(n, m) : \ (m, n) \in R\}$ is functional; and that a relation $R \subset M \times N$ is surjective, iff the relation $\overline{R} \subset N \times M$ with $\overline{R} := \{(n, m) : \ (m, n) \in R\}$ is serial.

(c) For example $R_1 = \{(1, a), (2, b), (3, c)\}$ and $R_1 = \{(1, c), (2, b), (3, a)\}$. This can be visualized as

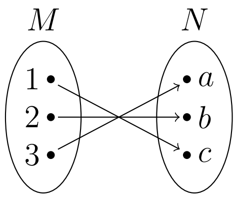
R_1:

	1	2	3
a	•		
b		•	
c			•



R_2:

	1	2	3
a			•
b		•	
c	•		



□

13. Consider the following table.

x_i	-6	-3	-1	1	3	4
y_i	5	4	2	1	1	0

(a) Compute $\mu_x := \frac{1}{6}\sum_{\ell=1}^{6} x_\ell$ and $\mu_y := \frac{1}{6}\sum_{\ell=1}^{6} y_\ell$.

(b) Compute $\sigma_x^2 := \frac{1}{6}\sum_{\ell=1}^{6}(x_\ell - \mu_x)^2$ and $\sigma_{xy} := \frac{1}{6}\sum_{\ell=1}^{6}(x_\ell - \mu_x)(y_\ell - \mu_y)$.

(c) Determine $k := \frac{\sigma_{xy}}{\sigma_x^2}$.

(d) Solve for d: $\mu_y = k\mu_x + d$.

(e) Plot the points (x_i, y_i) and $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = kx + d$.

(f) Compute $f(-2)$ and $f(2)$.

Solution:

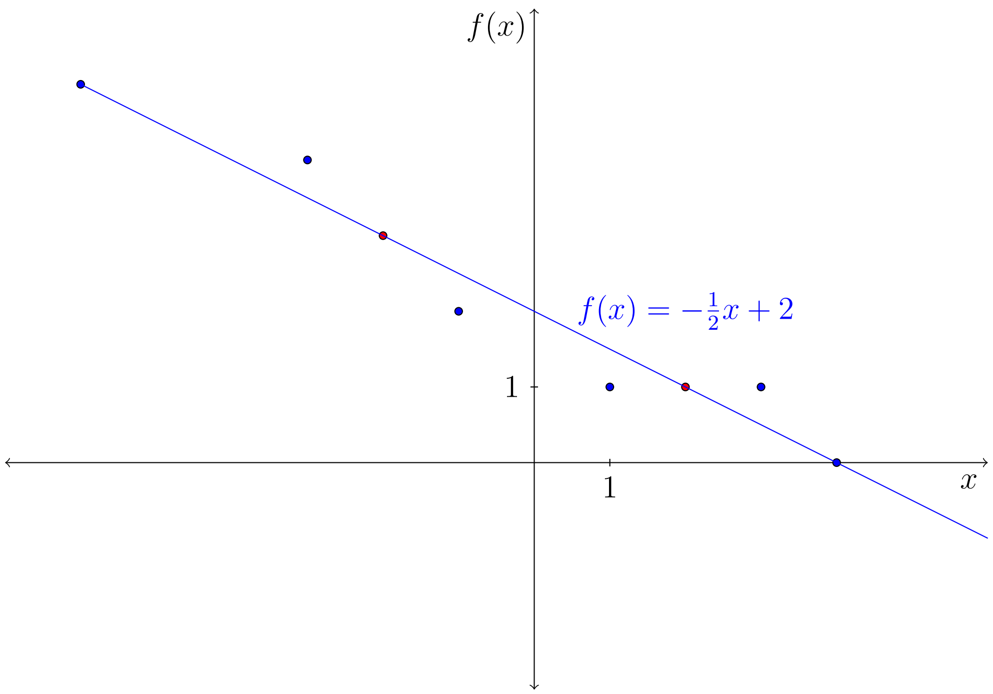
(a) $\mu_x = -\frac{2}{6}$, $\mu_y = \frac{13}{6}$.

(b) $\sigma_x^2 = \frac{107}{9}$, $\sigma_{xy} = -\frac{107}{18}$.

(c) $k = -\frac{1}{2}$.

(d) $d = \frac{13}{6} - \left(-\frac{1}{2} \cdot \left(-\frac{2}{6}\right)\right) = \frac{13}{6} - \frac{1}{6} = 2$.

(e) A plot looks like this:



(f) $f(-2) = 3$ and $f(2) = 1$.

Remark: This is simple linear regression, i.e. the line $f(x)$ is the best linear predictor given the data x_i, y_i. □

14. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x + 1$. Show
    i. f is injective,
    ii. f is surjective,
    iii. f is bijective.

    (b) For which values of $a \in \mathbb{R}$ is $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := ax + 1$ injective? For which values of $a \in \mathbb{R}$ is $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := ax + 1$ surjective?

Solution:

(a) i. We show f is injective. For this, we assume $f(x_1) = f(x_2)$ for arbitrary $x_1, x_2 \in \mathbb{R}$. We show $x_1 = x_2$. By definition of f, $f(x_1) = 2x_1 + 1$ and $f(x_2) = 2x_2 + 1$. Thus $2x_1 + 1 = 2x_2 + 1$, hence $2x_1 = 2x_2$. Therefore $x_1 = x_2$.

ii. We show f is surjective. For this we let $y \in \mathbb{R}$ be arbitrary but fixed. We show there exists $x \in \mathbb{R} : f(x) = y$. We claim $x = \frac{y-1}{2}$ has the desired property. First of all, if $y \in \mathbb{R}$, then also $\frac{y-1}{2} \in \mathbb{R}$. Moreover, $f(x) = 2\left(\frac{y-1}{2}\right) + 1 = (y-1) + 1 = y$.

iii. As f is injective and surjective, f is bijective.

(b) We claim: for $a \neq 0$, we have f is bijective and for $a = 0$ f is neither injective nor surjective?

- For the first claim, we show for $a \neq 0$, f is injective and f is surjective. We assume $f(x_1) = f(x_2)$ for arbitrary $x_1, x_2 \in \mathbb{R}$. We show $x_1 = x_2$. By definition of f, we have $f(x_1) = ax_1 + 1$ and $f(x_2) = ax_2 + 1$. Thus $ax_1 + 1 = ax_2 + 1$, hence $ax_1 = ax_2$. Therefore, as $a \neq 0$, $x_1 = x_2$.
  Moreover, let $y \in \mathbb{R}$ be arbitrary but fixed. We show there exists $x \in \mathbb{R} : f(x) = y$. We claim $x = \frac{y-1}{a}$ has the desired property. First of all, $\frac{y-1}{a}$ is well defined, as $a \neq 0$. Furthermore, if $y \in \mathbb{R}$, then also $\frac{y-1}{a} \in \mathbb{R}$. Moreover, $f(x) = a\left(\frac{y-1}{a}\right) + 1 = (y-1) + 1 = y$.
- For the second claim, we see that $2 \neq 1$, but $f(1) = 0 \cdot 1 + 1 = 1 = 0 \cdot 2 + 1 = f(2)$. Thus f is not injective.
  Moreover, assume for contradiction that f is surjective, then there exists x, s.t. $f(x) = 2$. Take such a x, then $2 = 0 \cdot x + 1 = 1$. This is a contradiction, hence f is not surjective.

$\square$

15. Let $f\colon D \to C$, defined as $x \mapsto x^2$, with $D, C \subset \mathbb{R}$.

 (a) Give an example for $D, C \subset \mathbb{R}$, such that f is injective. Find another example for $D, C \subset \mathbb{R}$, such that f is not injective.

 (b) Give an examples for $D, C \subset \mathbb{R}$, such that f is surjective. Find another example for $D, C \subset \mathbb{R}$, such that f is not surjective.

 (c) Consider $D = \mathbb{R}$ and $C = \mathbb{R}$.

  i. For $S_1 := \{-1, 0, 1, 2, \pi\}$, $S_2 := \{x \in \mathbb{R} :\ -3 \leq x \leq 4\}$ and $S_3 := \{x \in \mathbb{R} :\ x < 0 \vee x > 16\}$, determine the image of S_i under f, i.e. compute $f(S_i)$, for $i = 1, 2, 3$.

  ii. For $T_1 := \{0, 1, 2, \pi\}$, $T_2 := \{x \in \mathbb{R} :\ x \geq 0\}$ and $T_3 := \{x \in \mathbb{R} :\ x < 0\}$, determine the pre-image of T_i under f, i.e. compute $f^{-1}(T_i)$, for $i = 1, 2, 3$.

Solution:

(a) Let $D = \mathbb{R}_0^+ := \{x \in \mathbb{R} :\ x \geq 0\}$ and $C = \mathbb{R}$. Then f is injective: Assume for $x_1, x_2 \in D$ we have that $f(x_1) = f(x_2)$. This means $x_1^2 = x_2^2$. But then $x_1 = x_2$ (we only allow positive values).

If $D = \{-1, 1\}$ and $C = \mathbb{R}$, then $-1 \neq 1$, but $f(-1) = f(1) = 1$.

(b) Let $D = \mathbb{R}$ and $C = \mathbb{R}_0^+$. Let $y \in C$ be arbitrary but fixed. Then $x := \sqrt{y}$ is well defined (as $y \geq 0$) and $f(x) = (\sqrt{y})^2 = y$.

If $D = \mathbb{R}$ and $C = \mathbb{R}$, then there is no $x \in D$, s.t. $f(x) = x^2 = -1$.

(c) i. We have

$$
\begin{aligned}
f(S_1) &= \{f(x) \in C :\ x \in S_1\} = \{(-1)^2, 0^2, 2^2, \pi^2\} = \{1, 0, 4, \pi^2\},\\
f(S_2) &= \{f(x) \in C :\ x \in S_2\} = \{y \in C :\ 0 \leq y \leq 16\},\\
f(S_3) &= \{f(x) \in C :\ x \in S_3\} = \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}.
\end{aligned}
$$

ii. We have

$$
\begin{aligned}
f^{-1}(T_1) &= \{x \in D :\ f(x) \in T_1\} = \{-\sqrt{\pi}, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \sqrt{\pi}\},\\
f^{-1}(T_2) &= \{x \in D :\ f(x) \in T_2\} = \mathbb{R},\\
f^{-1}(T_3) &= \{x \in D :\ f(x) \in T_3\} = \emptyset.
\end{aligned}
$$

□

16. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

   (a) Let $f: \mathbb{R} \to \mathbb{R}$, with $f(x) = x + 2$ and $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x - 5$. Determine $(g \circ f)(x)$ and $(f \circ g)(x)$.

   (b) Let $f: \mathbb{R} \to \mathbb{R}$, with $f(x) = 2x$ and $g: \mathbb{R} \to \mathbb{R}$, $g(x) = -5x$. Determine $(g \circ f)(x)$ and $(f \circ g)(x)$.

   (c) Is function composition commutative, i.e. does $(g \circ f)(x) = (f \circ g)(x)$ hold in general? Justify your answer!

Solution:

(a) We have

$$(g \circ f)(x) = g(f(x)) = g(x + 2) = (x + 2) - 5 = x - 3,$$

and

$$(f \circ g)(x) = f(g(x)) = f(x - 5) = (x - 5) + 2 = x - 3.$$

(b) We have

$$(g \circ f)(x) = g(f(x)) = g(2x) = -5(2x) = -10x,$$

and

$$(f \circ g)(x) = f(g(x)) = f(-5x) = 2(-5x) = -10x.$$

(c) No, let for example $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$ and $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto x - 5$. Then

$$(g \circ f)(x) = g(f(x)) = g(2x) = (2x) - 5 = 2x - 5,$$

but

$$(f \circ g)(x) = f(g(x)) = f(x - 5) = 2(x - 5) = 2x - 10.$$

$\square$

17. Prove the following statements.

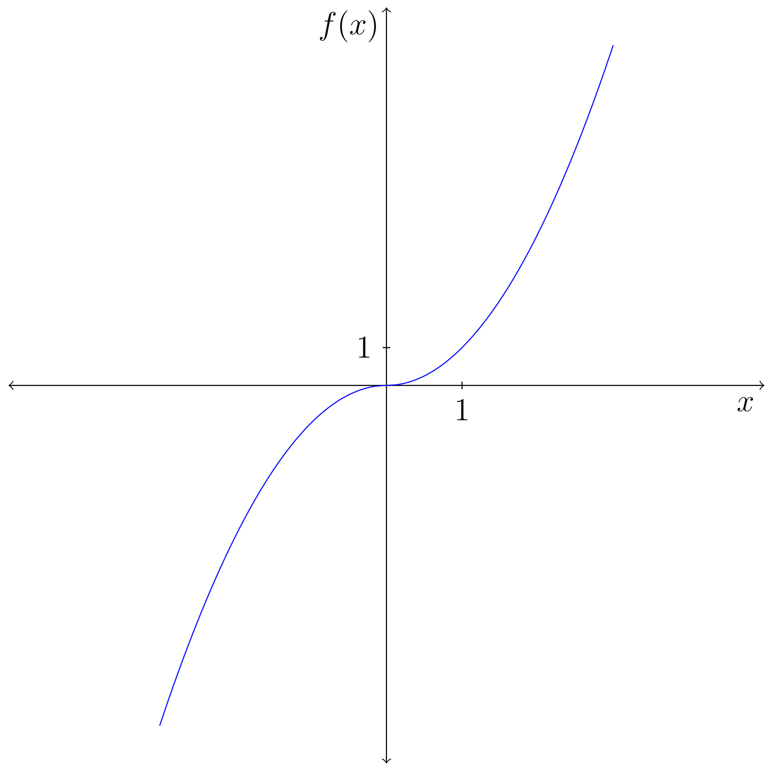
    (a) If $f\colon M \to N$ and $g\colon N \to O$ are both surjective, then also $(g \circ f)\colon M \to O$ is surjective.
    Hint 1: As in the lecture, transform $(g \circ f)$ into different notation.
    Hint 2: Use that g is surjective, then use f is surjective.

    (b) The function
    $$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} x^2 & x \geq 0, \\ -x^2 & x < 0. \end{cases}$$
    is invertible. Visualize f.
    Hint 1: It may be hard to directly find the inverse function. Look for an equivalent statement in the lecture slides.
    Hint 2: Use case distinction, i.e. split your proof into the parts $x \geq 0$ and $x < 0$.

Solution:

(a) We have to prove that $\forall y \in O\ \exists x \in M : (g \circ f)(x) = y$. Let $y \in O$ be arbitrary but fixed. As g is surjective, we have $\exists z \in N : g(z) = y$. Take such a z. As f is surjective, we know $\exists x \in M : f(x) = z$. For such an x it holds that $g(f(x)) = g(z) = y$.

(b) First we visualize f.



In order to show f is invertible, we show f is bijective. We first show f is injective, then we show f is surjective.

For proving injectivity, let $x_1, x_2 \in \mathbb{R}$ be a.b.f., s.t. $f(x_1) = f(x_2)$. This means either $f(x_1) \geq 0$ and $f(x_2) \geq 0$ or $f(x_1) < 0$ and $f(x_2) < 0$. In the first case, we have $x_1^2 = x_2^2$ and both $x_1 \geq 0$ and $x_2 \geq 0$. Thus, in this case $x_1 = x_2$. In the second case, we have $-x_1^2 = -x_2^2$ and both $x_1 < 0$ and $x_2 < 0$. Thus $x_1 = x_2$, hence f is injective

For proving surjectivity, let $y \in \mathbb{R}$ be a.b.f. If $y \geq 0$, we set $x := \sqrt{(y)} \geq 0$ and get $f(x) = x^2 = \sqrt{y}^2 = y$. If $y < 0$, then $-y > 0$. We set $x := -\sqrt{-y} < 0$ and have $f(x) = -x^2 = -\sqrt{-y}^2 = -(-y) = y$. Thus f is surjective.

Hence f is bijective, therefore invertible.

$\square$

18. Let $\equiv_3 \subset \mathbb{N}_0^2$ be defined as $x \equiv_3 y$ if x and y have the same remainder when divided by 3.

    (a) Show $\equiv_3$ is an equivalence relation on $\mathbb{N}_0$ by showing

    i. Show $\equiv_3$ is reflexive,

    ii. Show $\equiv_3$ is symmetric,

    iii. Show $\equiv_3$ is transitive.

    (b) Write down explicitly the equivalence class of 7, denoted by [7], and the equivalence class of 3001, denoted by [3001].

    (c) Determine $\mathbb{N}_0/\equiv_3$.

Solution:

(a) We show $\equiv_3$ is an equivalence relation on $\mathbb{N}_0$.

i. In order to show $\equiv_3$ is reflexive, we let $x \in \mathbb{N}_0$ be a.b.f. Then, clearly x has the same remainder when divided by 3 as x. Thus $x \equiv_3 x$ for all $x \in \mathbb{N}_0$.

ii. In order to show $\equiv_3$ is symmetric, we let $x, y \in \mathbb{N}_0$ be a.b.f., such that $x \equiv_3 y$. Thus, x has the same remainder when divided by 3 as y, but then also y has the same remaineder when divided by 3 as x. Hence, $y \equiv_3 x$. In total $x \equiv_3 y \Rightarrow y \equiv_3 x$ for all $x, y \in \mathbb{N}_0$.

iii. In order to show $\equiv_3$ is transitive, we let $x, y, z \in \mathbb{N}_0$ be a.b.f., such that $x \equiv_3 y$ and $y \equiv_3 z$. Thus, x has the same remainder when divided by 3 as y and y has the same remainder when divided by 3 as z, but then x also has the same reminder when divided by 3 as z. Hence, $x \equiv_3 z$. In total, $(x \equiv_3 y \wedge y \equiv_3 z) \Rightarrow x \equiv_3 z$ for all $x, y, z \in \mathbb{N}_0$.

(b) $[7] = \{x \in \mathbb{N}_0 : \ x \equiv_3 7\} = \{1, 4, 7, 10, 13, 16, \dots\} = \{n \in \mathbb{N}_0 : \ n = 3k + 1 \text{ for some } k \in \mathbb{N}_0\} = [3001]$.

(c) $\mathbb{N}_0/\equiv_3 = \{[0], [1], [2]\}$.

□

Read the following definition:

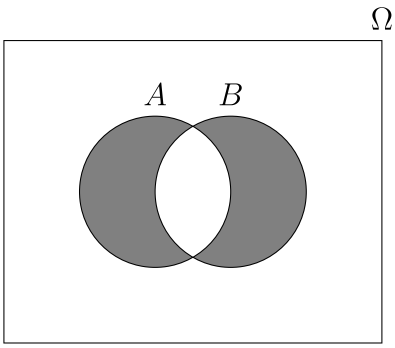
Definition. *For sets A and B and underlying set Ω, the **symmetric difference** of A and B , denoted by $A \triangle B$, is the set containing those elements in either A or B, but not in both A and B.*
$A \triangle B := \{x \in \Omega : \ (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$

19. (a) Find the symmetric difference $A \triangle B$ of $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4\}$.
    (b) Find the symmetric difference of the set of computer science majors at JKU and the set of mathematics majors at JKU.
    (c) Draw a Venn diagram for the symmetric difference of the sets A and B.
    (d) Express $A \triangle B$ by using the set operations $\cap, \cup, \setminus$.

Solution:

(a) $A \triangle B = \{2, 5, 7, 4\}$.

(b) Let A be the set of computer science majors at JKU and
B is the set of mathematics majors at JKU.
$\Rightarrow$ $A \triangle B$ is the set of all computer science majors and mathematics majors, but no majors that are both computer science majors and mathematics majors at JKU.

(c) 

(d) $A \triangle B = (A \setminus B) \cup (B \setminus A)$
or $A \triangle B = (A \cup B) \setminus (A \cap B)$.

$\square$

20. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

(a) Compute $\sum_{k=1}^{9} \frac{1}{k(k+1)}$.

(b) Show that $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is $(n+1)$-tuple of real numbers. This type of sum is called **telescoping**.

(c) Compute $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Hint: Using the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ *and telescoping.*

Solution:

(a) $\sum_{k=1}^{9} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \frac{1}{8.9} + \frac{1}{9.10} = \frac{9}{10}$

(b)

$$\begin{aligned}
\sum_{j=1}^{n}(a_j - a_{j-1}) &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) \\
&= a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + \ldots + a_n - a_{n-1} \\
&= -a_0 + a_1 - a_1 + a_2 - a_2 + a_3 + \ldots - a_{n-1} + a_n \\
&= a_n - a_0.
\end{aligned}$$

(c) Applying $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, we obtain

$$\begin{aligned}
&\frac{1}{1.2} = 1 - \frac{1}{2} \\
&\frac{1}{2.3} = \frac{1}{2} - \frac{1}{3} \\
&\ldots\ldots\ldots \\
&\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.
\end{aligned}$$

Therefore, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$.

□

Read the following definition:

> **Definition.** *The* ***factorial*** *of a non-negative integer* n*, denoted by* $n!$*, is the product of all positive integers less than or equal to* n*:*
>
> $$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$
>
> *We specify that* $0! = 1$.

21. Let n be non-negative integer.

    (a) Express $n!$ using product notation.

    (b) Compute $\sum_{j=0}^{4} j!$.

    (c) Compute $\prod_{j=0}^{4} j!$.

    Solution:

    (a) $n! = \prod_{j=1}^{n} j$.

    (b) $\sum_{j=0}^{4} j! = 0! + 1! + 2! + 3! + 4! = 1 + 1 + (2\cdot 1) + (3\cdot 2\cdot 1) + (4\cdot 3\cdot 2\cdot 1) = 34$.

    (c) $\prod_{j=0}^{4} j! = (0!)\cdot(1!)\cdot(2!)\cdot(3!)\cdot(4!) = 1\cdot 1\cdot(2\cdot 1)\cdot(3\cdot 2\cdot 1)\cdot(4\cdot 3\cdot 2\cdot 1) = 288$.

$\square$

22. Let us denote elements of $\mathbb{R}^2$ by $x = (x_1, x_2)$. Let $\preceq$ be a relation on $\mathbb{R}^2$, defined as $x \preceq y$, if $x_1 < y_1$ or $(x_1 = y_1 \wedge x_2 \leq y_2)$.

    (a) Give examples of three different tuples $x, y, z \in \mathbb{R}^2$, s.t. $x \preceq y$ and $x_1 = z_1$, $x \preceq z$.

    (b) Show $\preceq$ is reflexive.

    (c) Show $\preceq$ is antisymmetric.

    (d) Show $\preceq$ is transitive.
    Hint: case distinction. If $x \preceq y$, then $x_1 < y_1$; or $x_1 = y_1$ and $x_2 \leq y_2$.

Solution:

(a) Let $x = (-3, 2), y = (1, 1)$ and $z = (-3, 5)$. Then $x \preceq y$ and $x \preceq z$.

(b) In order to show $\preceq$ is reflexive, we let $x \in \mathbb{R}^2$ be a.b.f. Then $x_1 = x_1$ and $x_2 = x_2$, thus $x_2 \leq x_2$, hence $x \preceq x$.

(c) In order to show $\preceq$ is antisymmetric, we let $x, y \in \mathbb{R}^2$ be a.b.f., such that $x \preceq y$ and $y \preceq x$. We show, then $x = y$. On the one hand, if $x_1 > y_1$, then $x \not\preceq y$. On the other hand, if $y_1 > x_1$, then $y \not\preceq x$. Thus $x_1 = y_1$. Hence $x_2 \leq y_2$ and $y_2 \leq x_2$, thus $x_2 = y_2$. Therefore $x = y$.

(d) In order to show $\preceq$ is transitive, we let $x, y, z \in \mathbb{R}^2$ be a.b.f., such that $x \preceq y$ and $y \preceq z$. We show $x \preceq z$. From $x \preceq y$, we get $x_1 < y_1$; or $x_1 = y_1$ and $x_2 \leq y_2$.

    Case $x_1 < y_1$. From $y \preceq z$, we know $y_1 < z_1$ or $y_1 = z_1$. In both cases, we have $x_1 < z_1$, thus $x \preceq z$.

    Case $x_1 = y_2$ and $x_2 \leq y_2$. As $y \preceq z$, we get $x_1 = y_1 < z_1$; or $x_1 = y_1 = z_1$ and $x_2 \leq y_2 \leq z_2$. In both cases, we have $x \preceq z$.

$\square$

23. Determine the supremum and infimum of the following sets. Check if these are attained, i.e., whether the maximum or minimum exists.

 (a) $A_1 = \mathbb{R} \cap (0,1)^c$
 (b) $A_2 = [0, M) \cap (0,1)^c$ with $M > 1$.
 (c) $A_3 = [-M, 0] \cap (0,1)$ with $M > 1$.
 (d) $A_4 = \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$
 (e) $A_5 = \{-n + \frac{1}{m} : n, m \in \mathbb{N}\}$

Solution:

 (a) $A_1 = (0,1)^c = (-\infty, 0] \cup [1, \infty)$ and thus $\inf A_1 = -\infty$, $\sup A_1 = \infty$ and neither the maximum nor minimum exist.
 (b) $A_2 = \{0\} \cup [1, M)$ and thus $\inf A_2 = \min A_2 = 0$, $\sup A_2 = M$ and the maximum does not exist.
 (c) $A_3 = \emptyset$ and thus none of the quantities exist. However, it can make sense to define $\sup \emptyset = -\infty$ since every number is an upper bound and the least number is $-\infty$. Similarly $\inf \emptyset = \infty$.
 (d) We have $\inf A_4 = 0$ and $\sup A_4 = \infty$ and are not attained. Both the infimum and the supremum are given by the Archimedic principle: $\forall \varepsilon > 0 \exists n \in \mathbb{N} \colon \frac{1}{n} < \varepsilon$
 (e) By the Archimedic principle and choosing $m = 1$, we see the set is unbounded from below implying $\inf A_5 = -\infty$ and the minimum does not exist. If $n = 1$ and $m = 1$ we have $-1 + 1 = 0 \in A_5$ and no number in A_5 can be larger since increasing n or m decreases the number $-n + 1/m$. Therefore, $\sup A_5 = \max A_5 = 0$. The monotonicity in both coordinates basically follows again by the Archimedic principle.

□

24. A set $A \subset \mathbb{R}$ of real numbers is bounded. Prove that the supremum or infimum of A is unique if it exists.

Solution:
Suppose that M, M' are suprema of A. Then $M \leq M'$ since M' is an upper bound of A and M is a least upper bound; similarly, $M? \leq M$, so $M = M'$.
If m, m' are infima of A, then $m \geq m'$ since m' is a lower bound of A and m is a greatest lower bound; similarly, $m' \geq m$, so $m = m'$. $\square$

Exercise A *(Optional). This exercise does not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solution.*

Let Ω be the underlying set, and $A, B \subset \Omega$. By A^C and B^C we denote their complements in Ω. Prove or disprove the following equality:

$$(A \cap B)^C = A^C \cup B^C.$$

Solution:
Recall, that for any $D \subset \Omega$, we have that $D^C = \{x \in \Omega : \neg(x \in D)\}$.

$$\begin{aligned}
(A \cap B)^C &= (\{x \in \Omega : \ x \in A\} \cap \{x \in \Omega : \ x \in B\})^C \\
&= \{x \in \Omega : \ x \in A \cap B\}^C \\
&= \{x \in \Omega : \ \neg(x \in A \cap B)\} \\
&= \{x \in \Omega : \ (x \notin A \cap B)\} \\
&= \{x \in \Omega : \ (x \notin A) \vee (x \notin B)\} \\
&= \{x \in \Omega : \ \neg(x \in A) \vee \neg(x \in B)\} \\
&= \left\{x \in \Omega : \ (x \in A^C) \vee (x \in B^C)\right\} \\
&= \left\{x \in \Omega : \ x \in A^C\right\} \cup \left\{x \in \Omega : \ x \in B^C\right\} \\
&= A^C \cup B^C.
\end{aligned}$$

(This can also be shown, analogously to the proof in the lecture, by proving "$\subset$" and "$\supset$") $\square$

Exercise B *(Optional). This exercise does not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solution.*

Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. If $f(x) = 4x$, $g(x) = 2x - 1$ and $h(x) = x^2$, show that $(f \circ g) \circ h = f \circ (g \circ h)$.

Solution:
Firstly, we compute $(f \circ g) \circ h$. We have

$$\begin{aligned} &(f \circ g)(x) = f(g(x)) = f(2x-1) = 8x - 4, \\ \Rightarrow &\left((f \circ g) \circ h\right)(x) = (f \circ g)(h(x)) = (f \circ g)(x^2) = 8x^2 - 4. \end{aligned} \tag{1}$$

Secondly, we compute $f \circ (g \circ h)$. We have

$$\begin{aligned} &(g \circ h)(x) = g(h(x)) = g(x^2) = 2x^2 - 1, \\ \Rightarrow &\left(f \circ (g \circ h)\right)(x) = f\left((g \circ h)(x)\right) = f(2x^2 - 1) = 4(2x^2 - 1) = 8x^2 - 4. \end{aligned} \tag{2}$$

From (1) and (2), we obtain $(f \circ g) \circ h = f \circ (g \circ h)$. $\square$

Read the following definitions:

Definition. *The **floor function** assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.*

Definition. *The **ceiling function** assigns to the real number x the smallest integer number that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.*

Exercise C *(Optional). This exercise does not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solution.*

(a) Find these values: $\lfloor 2.25 \rfloor$, $\lceil 2.25 \rceil$, $\lfloor 3 \rfloor$, $\lceil -1 \rceil$, $\lfloor \frac{7}{8} \rfloor$, $\lceil -\frac{7}{8} \rceil$, $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$, $\lceil \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rceil$.

(b) Prove or disprove the following statement $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real number x.

(c) Show that if x is a real number and m is an integer number, then

$$\lceil x + m \rceil = \lceil x \rceil + m.$$

Solution:

(a) $$\begin{array}{llll} \lfloor 2.25 \rfloor = 2, & \lceil 2.25 \rceil = 3, & \lfloor 3 \rfloor = 3, & \lceil -1 \rceil = -1, \\ \lfloor \frac{7}{8} \rfloor = 0, & \lceil -\frac{7}{8} \rceil = 0, & \lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = 2, & \lceil \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rceil = 1. \end{array}$$

(b) Let $x \in \mathbb{R}$ and $m = \lfloor x \rfloor$ with $m \in \mathbb{Z}$. Then, $\lceil \lfloor x \rfloor \rceil = \lceil m \rceil = m = \lfloor x \rfloor$.

(c) Let x is a real number, then $\exists n \in \mathbb{Z}$ such that $\lceil x \rceil = n$. It implies that $x = n - \epsilon$ with $0 \leq \epsilon < 1$. Therefore,

$$\lceil x + m \rceil = \lceil (n - \epsilon) + m \rceil = \lceil n + m - \epsilon \rceil = n + m = \lceil x \rceil + m.$$

$\square$

25. Let $f : D \to C$ be a function and S be a subset of the codomain C. Determine the preimage $f^{-1}(S)$ for the following functions and sets S.

   (a) $D = \mathbb{Z}$, $C = \mathbb{R}$, $f(x) = x^2$, $S = (0, \infty)$.

   (b) $D = \{\text{car}, \text{motorbike}, \text{bicycle}, \text{boat}, \text{tricycle}\}$, $C = \{0, 1, 2, 3, 4\}$, $f(x) =$ amount of wheels of x, $S = \{1, 2, 3\}$.

   (c) $D = \mathbb{N}$, $C = \{0, 1, \ldots, 9\}$, $f(n) = n^{\text{th}}$ decimal place of $\frac{1}{7}$, $S = \{1, 7\}$.

   (d) $D = \mathbb{R}$, $C = [-1, 1]$, $f(x) = \sin(x)$, $S = \{0\}$.

   Solution:

   (a) $f^{-1}(S) = \mathbb{Z}$.

   (b) $f^{-1}(S) = \{\text{motorbike}, \text{bicycle}, \text{tricycle}\}$.

   (c) $f^{-1}(S) = \{6n - 5, 6n : n \in \mathbb{N}\}$.

   (d) $f^{-1}(S) = \{n\pi : n \in \mathbb{Z}\}$.

$\square$

26. (a) Suppose there is a box full of 10 different coloured balls. How many ways are there of choosing two balls if we (i) choose a ball, put it back into the box and then choose another, or (ii) choose a ball, remove it from the box, then choose another from the box?

(b) Prove without the binomial theorem that $2^n = \sum_{k=0}^{n} \binom{n}{k}$ for all positive integers n.
Hint: Try induction with the statement $P(n)$: "$2^n = \sum_{k=0}^{n} \binom{n}{k}$". Remember that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ and $\binom{n}{n} = \binom{n}{0} = 1$ for all positive integers $n \geq k$.

Solution:

(a) (i) $10.10 = 100$, and (ii) $\binom{10}{2} = \frac{10.9}{2} = 45$.

(b) $P(1)$ can easily be checked to be true. Suppose $P(n)$ is true for some positive integer n. We now see that

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \binom{n+1}{n+1} + \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1}{k} \\
&= 2 + \sum_{k=0}^{n-1} \binom{n+1}{k+1} \\
&= 2 + \sum_{k=0}^{n-1} \binom{n}{k} + \sum_{k=0}^{n-1} \binom{n}{k+1} \\
&= \binom{n}{n} + \sum_{k=0}^{n-1} \binom{n}{k} + \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} \\
&= 2 \sum_{k=0}^{n} \binom{n}{k} \\
&= 2^{n+1}.
\end{aligned}$$

As $P(1)$ is true and $(P(n) \Rightarrow P(n+1))$ is true for every positive integer n, $P(n)$ is true for every positive integer n.

□

27. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

(a) Show via induction that the statement $P(n)$: "$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$" holds for every positive integer n.

(b) Show via induction that the statement $P(n)$: "$2^n + 1$ is divisible by 3" holds for every odd number n.
Hint: Every odd number can be written as $n = 2k-1$ for some positive integer k, so we will want to show that $P(1)$ holds and $P(2k-1)$ implies $P(2k+1)$.

Solution:

(a) The statement $P(1)$ holds as $\sum_{k=1}^{n} k^2 = 1$ and $\frac{n(n+1)(2n+1)}{6} = 1.2.3/6 = 1$ for $n = 1$. Suppose $P(n-1)$ is true. Then

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= n^2 + \sum_{k=1}^{n-1} k^2 \\
&= n^2 + \frac{(n-1)(n)(2n-1)}{6} \\
&= \frac{n\left((n-1)(2n-1) - 6n\right)}{6} \\
&= \frac{n\left(2n^2 + 3n + 1\right)}{6} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

As $P(1)$ is true and $(P(n-1) \Rightarrow P(n))$ is true for every positive integer n, $P(n)$ is true for every positive integer n.

(b) The statement $P(1)$ holds as $2^1 + 1 = 3$. Suppose $P(2k-1)$ is true for some positive integer k. We note that

$$2^{2k+1} + 1 = 4.2^{2k-1} + 1 = 4(2^{2k-1} + 1) - 3,$$

hence $2^{2k+1} + 1$ is divisible by 3. As $P(1)$ is true and $(P(2k-1) \Rightarrow P(2k+1))$ is true for every positive integer k, $P(n)$ is true for every odd number n.

$\square$

28. Let A and B be non-empty and bounded subsets of $\mathbb{R}$.

    (a) Suppose the set A has a minimum. Show that $\inf A = \min A$.
    Hint: Both $\inf A$ *and* $\min A$ *are lower bounds, and* $\min A$ *is an element of* A.

    (b) Define $A+B := \{a+b : a \in A, b \in B\}$. Show that $\sup(A+B) = \sup A + \sup B$.
    Hint: First show that $\sup(A+B) \leq \sup A + \sup B$, *then show that* $\sup(A+B) < \sup A + \sup B$ *causes a contradiction. You may also wish to use the alternative definition of supremum that uses epsilons.*

Solution:

(a) The minimum of a set is a lower bound, hence $\inf A \geq \min A$. If $\inf A > \min A$ then $\inf A$ is not a lower bound, since $\min A$ is an element of A. Hence $\inf A = \min A$ as required.

(b) Let u_A and u_B be upper bounds of A and B. For any $a + b \in A + B$ we have $a + b \leq u_A + u_B$, hence $\sup(A + B) \leq \sup A + \sup B$.
Now suppose $\sup(A + B) < \sup A + \sup B$. Fix

$$\varepsilon = \frac{\sup A + \sup B - \sup(A + B)}{2}.$$

By the alternate definition of a supremum, there exists $a \in A$ and $b \in B$ so that $a > \sup A - \varepsilon$ and $b > \sup B - \varepsilon$. However we now note that

$$a + b > \sup A + \sup B - 2\varepsilon = \sup(A + B),$$

contradicting that $\sup(A + B)$ is the supremum of $A + B$.

$\square$

29. Define the two bijective functions $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$ and

$$g : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} -x^2 & \text{if } x < -2, \\ 2x & \text{if } -2 \le x \le 2, \\ x^2 & \text{if } x > 2. \end{cases}$$

(a) Explain why both $f \circ g$ and $g \circ f$ are invertible.

(b) Describe the inverses of $f \circ g$ and $g \circ f$.

Solution:

(a) As f and g are bijective, then so too is $f \circ g$ and $g \circ f$. The result now follows as every bijective function is invertible.

(b) The inverse of f is the function $f^{-1} : \mathbb{R} \to \mathbb{R},\ x \mapsto \sqrt[3]{x}$ and the inverse of g is the function

$$g^{-1} : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} -\sqrt{(-x)} & \text{if } x < -4, \\ x/2 & \text{if } -4 \le x \le 4, \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

(here we are assuming $\sqrt{\cdot}$ is the non-negative square root of any non-negative real number). As $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, we have

$$(f \circ g)^{-1} : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} -\sqrt[6]{(-x)} & \text{if } x < -64, \\ (\sqrt[3]{x})/2 & \text{if } -64 \le x \le 64, \\ \sqrt[6]{x} & \text{if } x > 64. \end{cases}$$

As $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, we have

$$(g \circ f)^{-1} : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} -\sqrt[6]{(-x)} & \text{if } x < -4, \\ \sqrt[3]{(x/2)} & \text{if } -4 \le x \le 4, \\ \sqrt[6]{x} & \text{if } x > 4. \end{cases}$$

$\square$

30. A classroom has a chalkboard with a finite amount of positive integers written on it. One at a time, a student erases a positive integer off the board and replaces it with any (finite and possibly zero) amount of positive integers that are strictly less than the original number. For example, a 10 could be replaced by $1, 1, 2, 4, 8$ and 9, but if a 1 is erased then no new numbers are added to the board since no positive integer is strictly less than 1.

    Prove that if the students play this game, it will eventually terminate (i.e., the game cannot go on forever).
    Hint: Use induction on the statement "$P(n)$: if every number on the board is at most n then the game will terminate".

    Solution:
    The statement $P(1)$ is true as each 1 will be erased and not replaced by anything. Suppose the statement $P(n-1)$ is true for some $n \geq 2$. If we erase all instances of n and replace them with whatever numbers we please (which we can do in a finite number of turns, since no new n's are being added to the board), we will have a board where every number is $n - 1$ or less. By our assumption that $P(n - 1)$ is true, we will now only have a finite amount of moves remaining before the game will terminate. $\square$

(Optional). This exercise does not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solution.

Exercise A

Define the supremum and infimum of the following sets, and state whether or not they have maximums and minimums.

1. $A_1 = (-1, 3] \cap ((0, 3)^c \cup (1, 2))$.
2. $A_2 = \{x \in [\pi, 2\pi] : 0 \leq \cos(x) < 1\}$.
3. $A_3 = \left\{\sum_{k=0}^{n} \frac{1}{k!} : n \in \mathbb{N}_0\right\}$.

Solution:

Refer to the following chart.

	Supremum	Infimum	Maximum?	Minimum?
A_1	3	-1	Yes	No
A_2	2π	$3\pi/2$	No	Yes
A_3	e (i.e., Euler's constant)	1	No	Yes

The infimum/minimum of A_3 is 1 as $\sum_{k=0}^{n} \frac{1}{k!}$ is increasing, and the supremum of A_3 is e, as given in the lecture notes. As $\sum_{k=0}^{n} \frac{1}{k!}$ is strictly increasing, there is no maximum of A_3. □

(Optional). This exercise does not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solution.

Exercise B

For any p be a positive integer, we define the equivalence relation $\equiv_p \subset \mathbb{Z}^2$ where $a \equiv_p b$ if and only if $a - b$ is divisible by p. Given $[a]_p := \{x \in \mathbb{Z} : a \equiv_p x\}$ is an equivalence class of $\mathbb{Z}/\equiv_p$, we define the operation

$$[a]_p + [b]_p := \{a' + b' : a' \in [a]_p\},$$

for all integers $a, b \in \mathbb{Z}$.

1. Write down the set of equivalence classes of $\mathbb{Z}/\equiv_5$ in the form $[a]_p$ so that each a is the smallest non-negative integer possible.
2. For each pair of equivalence classes $[a]_5, [b]_5$ of $\mathbb{Z}/\equiv_5$, find the equivalence class described in part (a) that is equal to $[a]_5 + [b]_5$.
3. Show that $[a]_p + [b]_p = [a + b]_p$ for any choice of positive integer p.
Hint: If $m - n$ is divisible by p, then $m = qp + n$ for some integer q.

Solution:

1. $[0]_5$, $[1]_5$, $[2]_5$, $[3]_5$, $[4]_5$.
2.

$+$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[0]_5$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[1]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$	$[0]_5$
$[2]_5$	$[2]_5$	$[3]_5$	$[4]_5$	$[0]_5$	$[1]_5$
$[3]_5$	$[3]_5$	$[4]_5$	$[0]_5$	$[1]_5$	$[2]_5$
$[4]_5$	$[4]_5$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$

3. ($[a]_p + [b]_p = [a + b]_p$): First choose $c \in [a]_p + [b]_p$, i.e., $c = a' + b'$ for some $a' = q_a p + a$ and $b' = q_b p + b$. As $c = a' + b' = (q_a + q_b)p + (a + b)$, then $c \in [a + b]_p$. Now choose an element $c \in [a + b]_p$. Then $c = qp + (a + b)$ for some for some integer q. Fix $a' := qp + a$ and $b' = b$. Then $c = a' + b'$, $a' \in [a]_p$ and $b' \in [b]_p$, hence $c \in [a]_p + [b]_p$.

$\square$

(Optional). This exercise does not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solution.

Exercise C

1. Describe a bijective function $f : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$.
   Hint: Try treating the odds and evens differently.

2. Given the set $S := \{(n, 1) : n \in \mathbb{N}\} \cup \{(n, 2) : n \in \mathbb{N}\}$, describe a bijective function $g : S \to \mathbb{N}$.
   Hint: We can interlace two tuples $(1, 2, 3)$ and (a, b, c) to form the set $(1, a, 2, b, 3, c)$. How can we apply a similar method for interlacing the elements of S into an infinite list? How can we then apply this method to form g?

3. Describe a bijective function $h : (0, 1) \times (0, 1) \to (0, 1)$.
   Hint: Try using something similar to the previous function with the decimal places of the coordinates.

Solution:

1. $$f(n) := \begin{cases} n/2 & \text{if } n \text{ is even,} \\ -(n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

2. Method 1:
   $$g(n, i) := \begin{cases} 2n & \text{if } i = 1, \\ 2n - 1 & \text{if } i = 2. \end{cases}$$
   Method 2: Define the bijection $b : S \to \mathbb{Z} \setminus \{0\}$ with $b(n, j) = (-1)^j n$. We now define $g := f^{-1} \circ b$, where f is whatever function that was chosen from part (a).

3. Choose any $(x, y) \in (0, 1) \times (0, 1)$. We can write x and y as the decimals $x = 0.x_1 x_2 \ldots$ and $y = 0.y_1 y_2 \ldots$ respectively. With this we now define h as follows:
   $$h(x, y) := 0.x_1 y_1 x_2 y_2 \ldots$$

$\square$

31. For $A, B \subseteq \mathbb{R}$ prove or disprove the following statements:

(a)
$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}.$$

(b)
$$\sup(A + B) \geq \max\{\sup(A), \sup(B)\}.$$

Solution:

(a) Assume $\sup(A) \geq \sup(B)$ ($\sup(A) \leq \sup(B)$ is done analogously). Then for all $a \in A$ and $b \in B$ we have $a \leq \sup(A)$ and $b \leq \sup(A)$ and therefore $c \leq \sup(A)$ for all $c \in A \cup B$.
Thus, $\sup(A \cup B) \leq \sup(A)$. By $A \cup B \supset A$ we have $\sup(A \cup B) \geq \sup(A)$.
Finally arriving at $\sup(A \cup B) = \sup(A)$.

(b) This is a wrong statement. Let $A = \{-3\}$ and $B = \{3\}$ then it is

$$0 = \sup(\{0\}) = \sup(\{-3\} + \{3\}) = \sup(A + B)$$

but
$$3 = \max\{-3, 3\} = \max\{\sup(A), \sup(B)\}.$$

$\square$

32. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

Solve the following in $\mathbb{R}$

(a) $|4x + 8| + 8 \leq 20$

(b) $|16 - 10x| + |2x + 4| > 24$

(c) $|5 - |8 + x|| < 3$

(d) $|x^2 - 4x| \geq 5$

Solution:

(a) The absolute value is negative for $x < -2$ and non-negative for $x \geq -2$. Thus, we get two case:

i. $x < -2$: $-(4x + 8) + 8 \leq 20$ leading to $x \geq -5$

ii. $x \geq -2$: $4x + 8 + 8 \leq 20$ leading to $x \leq 1$

From this follows that the inequality is true for $[-5, -2) \cup [-2, 1] = [-5, 1]$.

(b) The first absolute value has a zero at $x = \frac{16}{10}$ and the second one a zero at $x = -2$. Thus, we have to consider the cases $x < -2$, $-2 \leq x < \frac{16}{10}$ and $\frac{16}{10} \leq x$.

i. $x < -2$: $16 - 10x - (2x + 4) > 24$ leading to $x < -1$ which is always true in that range.

ii. $-2 \leq x < \frac{16}{10}$: $16 - 10x + 2x + 4 > 24$ leading to $x < -\frac{1}{2}$

iii. $\frac{16}{10} \leq x$: $-(16 - 10x) + 2x + 4 > 24$ leading to $x > 3$

From this follows that the inequality is true for $(-\infty, -2) \cup [-2, -\frac{1}{2}) \cup (3, \infty) = (-\infty, -\frac{1}{2}) \cup (3, \infty)$.

(c) In this exercise we consider two cases with possibly two sub-cases (it can happen that the requirements on x cancel each other out, already. each as we have nested absolute values. First the case $x < -8$, with sub-cases and then $x - 8$ afterwards.

i. $x < -8$: $|5 + 8 + x| = |13 + x| < 3$ leading to possible sub-cases $x < -13$ and $-13 \leq x < -8$.

A. $x < -13$: $-(13 + x) < 3$ leading to $x > -16$ or in conjunction $(-16, -13)$

B. $-13 \leq x < -8$: $13 + x < 3$ leading to $x < -10$ or in conjunction $[-13, -10)$

ii. $x \geq -8$: $|5 - (8 + x)| = |3 + x| < 3$ leading to possible sub-cases $-8 \leq x < -3$ and $-3 \leq x$.

A. $-8 \leq x < -3$: $-(3 + x) < 3$ leading to $x > -6$ or in conjunction $(-6, -3)$

B. $-3 \leq x$: $3 + x < 3$ leading to $x < 0$ or in conjunction $[-3, 0)$

From this follows that the inequality is true for $(-16, -13) \cup [-13, -10) \cup (-6, -3) \cup [-3, 0) = (-16, -10) \cup (-6, 0)$.

(d) We first simplify $|x^2 - 4x| = |x||x - 4|$ and get this time the cases $x < 0$, $0 \leq x < 4$ and $4 \leq x$

   i. $x < 0$: $-(x)(-(x-4)) = x(x-4) \geq 5$ leading to $x \leq -1$.

   ii. $0 \leq x < 4$: $-x(x-4) = 4x - x^2 \geq 5$, which has no solution.

   iii. $4 \leq x$: $x(x-4) \geq 5$ leading to $x \geq 5$.

   From this follows that the inequality is true for $(-\infty, -1] \cup [5, \infty)$.

$\square$

33. Let $x, y \in \mathbb{R}$. Show that

$$\max\{x, y\} = \frac{x + y + |x - y|}{2} \qquad \text{and} \qquad \min\{x, y\} = \frac{x + y - |x - y|}{2}$$

Solution:

By symmetry it suffices to only consider $x \geq y$. We calculate

$$\frac{x + y + |x - y|}{2} = \frac{x + y + (x - y)}{2} = \frac{2x}{2} = x = \max\{x, y\}$$

and

$$\frac{x + y - |x - y|}{2} = \frac{x + y - (x - y)}{2} = \frac{2y}{2} = y = \min\{x, y\}$$

$\square$

34. Determine the set of all $n \in \mathbb{N}_0$ such that

$$2^n < n^3$$

Solution:

Calculate and see that the inequality does not hold for $n = 0, 1$; is true for $n = 2, 3, \ldots, 9$ and again not for $n = 10, 11$. Guess: wrong for all $n \geq 10$. Look at the opposite $2^n \geq n^3$. Proof this by induction: Let it be wrong for some $n \geq 10$

$$2^{n+1} = 2 \cdot 2^n \overset{\text{(IA)}}{\geq} 2n^3 \geq (n+1)^3,$$

since

$$\begin{aligned} &(n+1)^3 = n^3 + 3n^2 + 3n + 1 \leq 2n^3 \\ \iff &3n^2 + 3n + 1 \leq n^3 \\ \iff &\frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3} \leq 1, \end{aligned}$$

which is true for $n \geq 10$.

$\square$

35. Prove the following basic properties concerning the complex conjugate.
Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ be complex numbers. Then we have

(a) $\overline{z} = z$ if and only if $z \in \mathbb{R}$,

(b) $\overline{\overline{z}} = z$,

(c) $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ and $\overline{\left(\frac{1}{z}\right)} = \frac{1}{(\overline{z})}$,

(d) $|z| = |\overline{z}|$,

(e) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \frac{1}{2\mathrm{i}}(z - \overline{z})$.

(f) Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n \in \mathbb{N}_0$ with real coefficients $a_k \in \mathbb{R}$ for all $k \in \{0, \dots, n\}$ and complex arguments z. Then we have $\overline{p(z)} = p(\overline{z})$ for all $z \in \mathbb{C}$.

Solution:

(a) $\overline{z} = z \Leftrightarrow x + \mathrm{i}y = x - \mathrm{i}y \Leftrightarrow 2\mathrm{i}y = 0 \Leftrightarrow y = 0 \Leftrightarrow z \in \mathbb{R}$.

(b) $\overline{\overline{z}} = \overline{x - \mathrm{i}y} = \overline{x + \mathrm{i}(-y)} = x - \mathrm{i}(-y) = x + \mathrm{i}y = z$.

(c) $\overline{z + w} = \overline{(x + u) + (y + v)\mathrm{i}} = (x + u) - (y + v)\mathrm{i} = (x - \mathrm{i}y) + (u - \mathrm{i}v) = \overline{z} + \overline{w}$,
$\overline{z \cdot w} = \overline{(xu - yv) + (xv + yu)\mathrm{i}} = (xu - yv) - (xv + yu)\mathrm{i} = (x - \mathrm{i}y)(u - \mathrm{i}v) = \overline{z} \cdot \overline{w}$,
$$\overline{\left(\frac{1}{z}\right)} = \overline{\left(\frac{\overline{z}}{|z|^2}\right)} = \overline{\left(\frac{1}{|z|^2} \cdot \overline{z}\right)} = \underbrace{\overline{\left(\frac{1}{|z|^2}\right)}}_{\in\mathbb{R}} \cdot \overline{\overline{z}} = \frac{1}{|z|^2} \cdot z = \frac{1}{z\overline{z}} \cdot z = \frac{1}{(\overline{z})}.$$

(d) $|\overline{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$,

(e) $\frac{1}{2}(z + \overline{z}) = \frac{1}{2}(x + \mathrm{i}y + x - \mathrm{i}y) = x = \mathrm{Re}(z)$,
$\frac{1}{2\mathrm{i}}(z - \overline{z}) = \frac{1}{2\mathrm{i}}(x + \mathrm{i}y - (x - \mathrm{i}y)) = y = \mathrm{Im}(z)$.

(f) It is clear that $\overline{z^n} = \overline{z}^n$, which follows from the multiplication rule in (c) by induction on n. Similarly, for complex numbers $z_1, \dots, z_n$ the rule $\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z_k}$ follows from the addition rule in (c). Finally, the rule $\overline{r \cdot z} = r \cdot \overline{z}$ for $r \in \mathbb{R}$ and $z \in \mathbb{C}$ is a direct consequence of (c) and (a). Using these properties, we get
$$\overline{p(z)} = \overline{\sum_{k=0}^{n} a_k z^k} = \sum_{k=0}^{n} \overline{a_k z^k} = \sum_{k=0}^{n} a_k \overline{z^k} = \sum_{k=0}^{n} a_k \overline{z}^k = p(\overline{z}).$$

Application of this rule: If $z \in \mathbb{C}$ is a root of a polynomial p with real coefficients (as above), then $\overline{z}$ is also a root, because $p(\overline{z}) = \overline{p(z)} = \overline{0} = 0$. As a consequence, we can guarantee that every such polynomial with odd degree has at least one real root.

$\square$

36. The parallelogram law

    (a) Check the validity of the identity $|z+w|^2+|z-w|^2 = 2(|z|^2+|w|^2)$ for $z = \mathrm{i}$ and $w = 1-\mathrm{i}$.

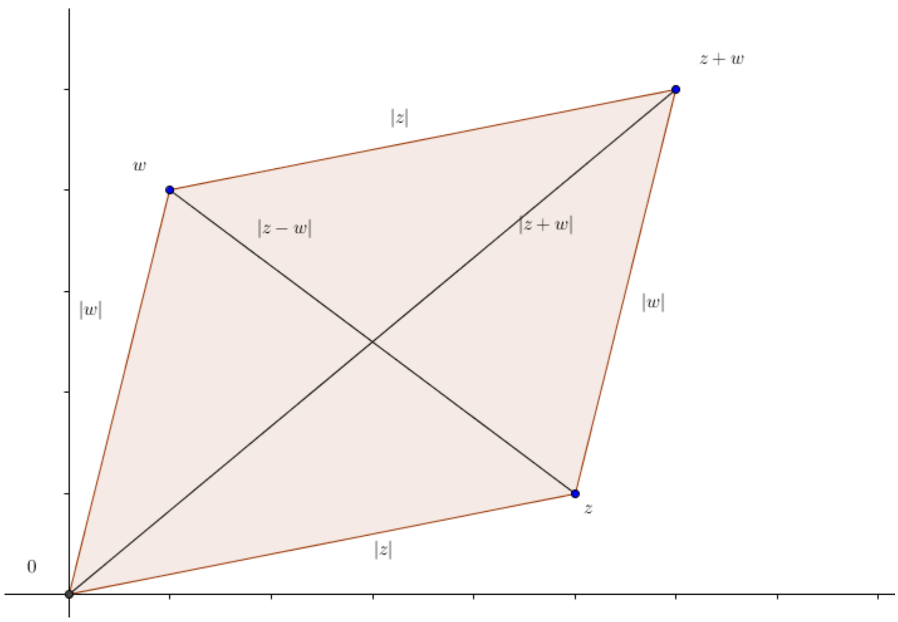
    (b) Prove that this identity actually holds for all complex numbers z and w.

    (c) This identity is known as *parallelogram law*. Justify this name by giving a geometric interpretation of the formula.

Solution:
For $z = \mathrm{i}$ and $w = 1-\mathrm{i}$ the left hand side yields $|1|^2+|2-\mathrm{i}|^2 = 1+2^2+1 = 6$ and the right hand side equals $2(|\mathrm{i}|^2+|1-\mathrm{i}|^2) = 2(1+1^2+(-1)^2) = 6$. General case: Using $|z|^2 = z\overline{z}$ for all $z \in \mathbb{C}$, we find

$$\begin{aligned}
|z+w|^2+|z-w|^2 =&(z+w)\overline{(z+w)}+(z-w)\overline{(z-w)}\\
=&(z+w)(\overline{z}+\overline{w})+(z-w)(\overline{z}-\overline{w})\\
=&z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w}+z\overline{z}-z\overline{w}-w\overline{z}+w\overline{w}\\
=&2z\overline{z}+2w\overline{w} = 2(|z|^2+|w|^2).
\end{aligned}$$

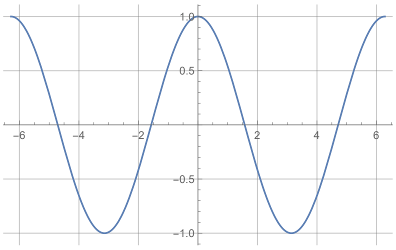
Note that the points 0, z, w and $z+w$ in the complex plane are the vertices of a parallelogram, where the sides have lengths $|z|$, $|w|$, $|z|$ and $|w|$ and the diagonals have lengths $|z+w|$ and $|z-w|$ (Figure below). Therefore, the equation yields the following general rule for parallelograms: *The sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.*
Further comment: If the parallelogram is a rectangle, then the diagonals have same length, i.e. $|z+w| = |z-w|$. Hence in this case the parallelogram law yields $2|z+w|^2 = 2(|z|^2+|w|^2)$ or $|z+w|^2 = |z|^2+|w|^2$, i.e. the Pythagorean theorem.



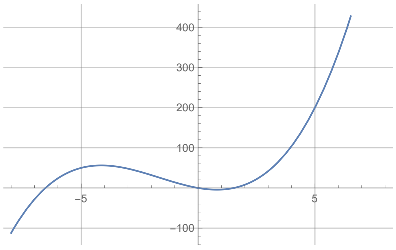
□

(Optional). The following exercises do not count for crosses and will not necessarily be discussed in the exercise class, but you still get the solutions.

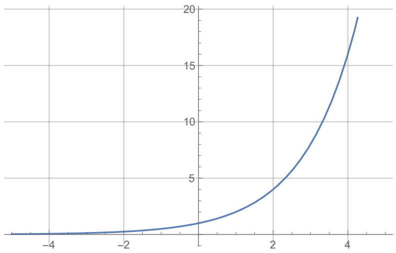
A. Consider the following graphs and determine the types of functions. You may want to use e.g. GeoGebra, Mathematica or similar software to try some examples.



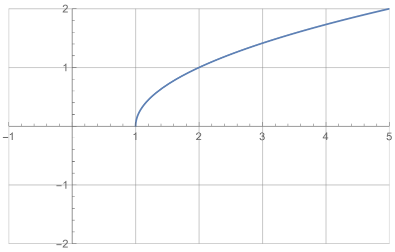
(a) $\cos(x)$



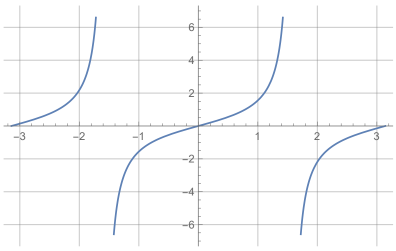
(b) $x^3 - 10x + 5x^2$



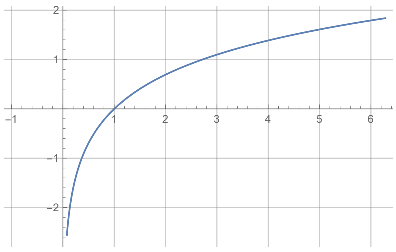
(c) 2^x



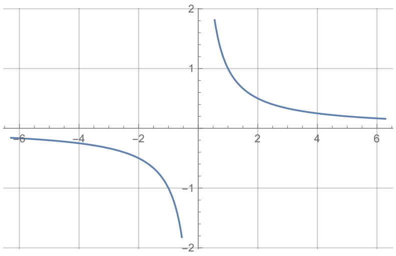
(d) $\sqrt{x-1}$



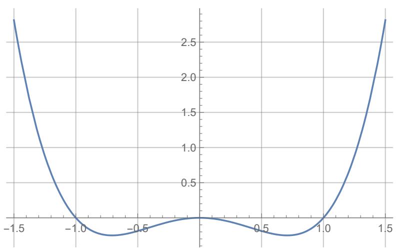
(e) $\tan(x)$



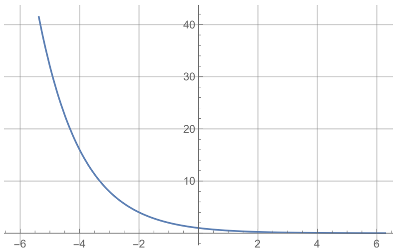
(f) $\log(x)$



(g) x^{-1}



(h) $x^4 - x^2$



(i) 0.5^x

B. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Describe in words or with some examples, how the following transformations change the function f, i.e. how the graph of the function $\bar{f}(x)$ looks like compared to the graph of $f(x)$. (For example, do they shift the function, or does the graph get steeper, etc.) You again may use software of your choice.

(a) $\bar{f}(x) = f(x) + a$, $a \in \mathbb{R}$

(b) $\bar{f}(x) = f(x + a)$, $a \in \mathbb{R}$

(c) $\bar{f}(x) = k \cdot f(x)$, $k \in \mathbb{R}$

(d) $\bar{f}(x) = f(k \cdot x)$, $k \in \mathbb{R}$

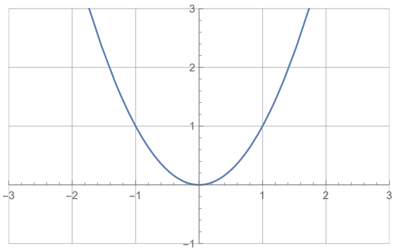
(e) $\bar{f}(x) = |f(x)|$

(f) $\bar{f}(x) = f(|x|)$

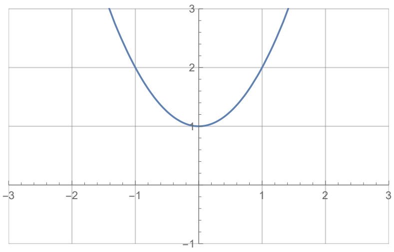
Solution:

(a) The graph of $f(x)$ is shifted in direction of the y-axis.

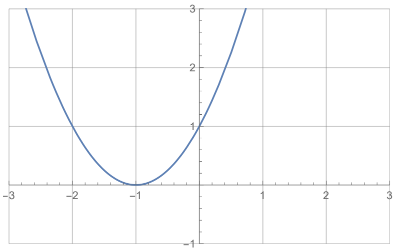
(b) The graph of $f(x)$ is shifted in direction of the x-axis.



$f(x) = x^2$

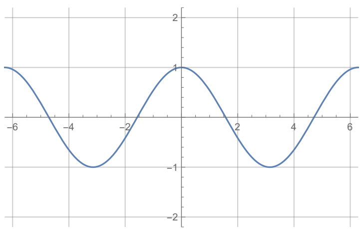


$\bar{f}(x) = x^2 + 1$

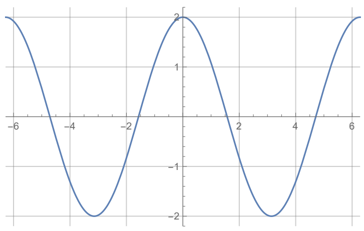


$\bar{f}(x) = (x+1)^2$

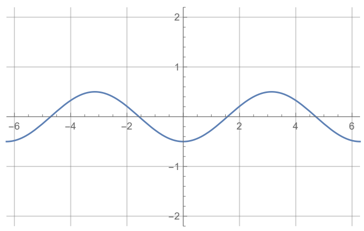
(c) The graph of $f(x)$ is stretched ($|k| > 1$) or compressed ($|k| < 1$) in direction of the y-axis; if $k < 0$ the graph is additionally mirrored at the x-axis.



$f(x) = \cos x$

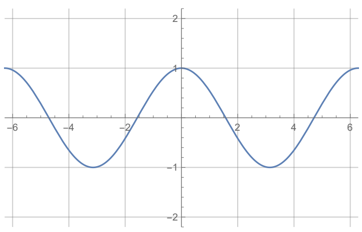


$\bar{f}(x) = 2\cos x$

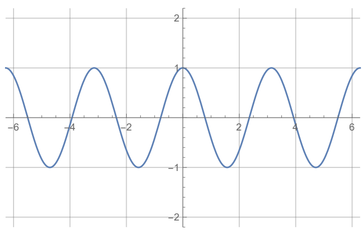


$\bar{f}(x) = -0.5\cos x$

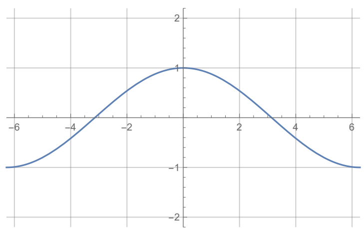
(d) The graph of $f(x)$ is stretched ($|k| < 1$) or compressed ($|k| > 1$) in direction of the x-axis; if $k < 0$ the graph is additionally mirrored at the y-axis.



$f(x) = \cos x$



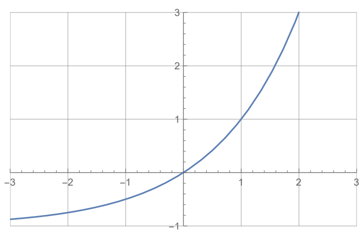
$\bar{f}(x) = \cos 2x$



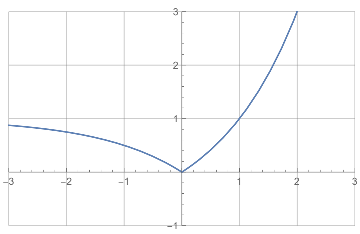
$\bar{f}(x) = \cos -0.5x$

(e) The parts of the graph of $f(x)$ with $f(x) > 0$ are mirrored at the x-axis

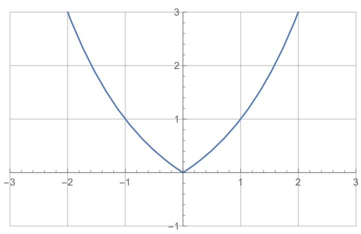
(f) The graph of $f(x)$ with $x < 0$ is mirrored at the y-axis



$f(x) = 2^x - 1$



$\bar{f}(x) = 2^{|x|} - 1$



$\bar{f}(x) = |2^x - 1|$

$\square$

C. Let $a > 0$. Write the following numbers in ascending order and justify your answer.

$$a^{-2}; \quad \sqrt[3]{a^2}; \quad a^{-\frac{1}{2}}; \quad a^{\frac{3}{2}}; \quad a^{-1}$$

Solution:
Let $0 < a < 1$. Then $f(x) = a^x$ is a decreasing function, i.e. $f(x) > f(y)$ for $x < y$. Hence,

$$a^{\frac{3}{2}} < \sqrt[3]{a^2} = a^{\frac{2}{3}} < a^{-\frac{1}{2}} < a^{-1} < a^{-2}.$$

If $a = 1$, all the numbers are the same since $f(x) = 1^x = 1$ is a constant function.

If $a > 1$, $f(x) = a^x$ is an increasing function, i.e. $f(x) > f(y)$ for $x > y$. Hence,

$$a^{-2} < a^{-1} < a^{-\frac{1}{2}} < \sqrt[3]{a^2} = a^{\frac{2}{3}} < a^{\frac{3}{2}}.$$

□

37. Find the 5^{th} roots of unity, i.e. solve the equation

$$z^5 = 1, \tag{1}$$

where $z \in \mathbb{C}$.
Hint: Use trigonometric or polar form and note that (**??**) has 5 different roots.

Solution:
Consider $z^n = 1$ and let $z = r\mathrm{e}^{\mathrm{i}\varphi}$. Then

$$z^n = r^n \mathrm{e}^{\mathrm{i}n\varphi} = |r^n| \mathrm{e}^{\mathrm{i}\arg(z^n)} = \mathrm{e}^{\mathrm{i}\arg(1)}.$$

Hence $r^n = 1$ and $n\varphi = \arg(1) + 2k\pi$ for $k = 0, \ldots, n$.
Further $\arg(1) = \arg(1 + 0\mathrm{i}) = \arctan(0) = 0$. Thus

$$z_k = \mathrm{e}^{\frac{2k\pi}{n}\mathrm{i}} = \cos\left(\frac{2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{n}\right) \qquad \text{for } k = 0, \ldots, n.$$

So for $n = 5$ we get

$$\begin{aligned}
z_0 &= \mathrm{e}^0 = 1 \\
z_1 &= \mathrm{e}^{\frac{2\pi}{5}\mathrm{i}} = \cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right) \\
z_2 &= \mathrm{e}^{\frac{4\pi}{5}\mathrm{i}} = \cos\left(\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(\frac{4\pi}{5}\right) \\
z_3 &= \mathrm{e}^{\frac{6\pi}{5}\mathrm{i}} = \cos\left(\frac{6\pi}{5}\right) + \mathrm{i}\sin\left(\frac{6\pi}{5}\right) \\
z_4 &= \mathrm{e}^{\frac{8\pi}{5}\mathrm{i}} = \cos\left(\frac{8\pi}{5}\right) + \mathrm{i}\sin\left(\frac{8\pi}{5}\right)
\end{aligned}$$

$\square$

38. Sketch the set A in the complex plane:

$$A = \{z \in \mathbb{C}\colon\ |z + 4 + 3\mathrm{i}| \leq |5z + 10 - 9\mathrm{i}|\}$$

Solution:
Let $z = x + \mathrm{i}y$. Then the condition in A can be rewritten as follows:

$$\begin{aligned}
|z + 4 + 3\mathrm{i}| \leq |5z + 10 - 9\mathrm{i}| &\Leftrightarrow \\
(x + 4)^2 + (y + 3)^2 \leq (5x + 10)^2 + (5y - 9)^2 &\Leftrightarrow \\
24x^2 + 92x + 24y^2 - 96y \geq -156 &\Leftrightarrow \\
x^2 + \frac{23}{6}x + y^2 - 4y \geq -\frac{39}{6} &\Leftrightarrow \\
\left(x + \frac{23}{12}\right)^2 + (y - 2)^2 \geq \frac{169}{144} = \left(\frac{13}{12}\right)^2 &.
\end{aligned}$$

Hence set A in the complex plane is the exterior and boundary of a circle with mid point $\left(-\frac{23}{12}, 2\right)$ and radius $\frac{13}{12}$. $\square$

39. **Please hand in this exercise as an extra pdf-file and name the file something like [Name]-exercise-[number].pdf. Please write especially neatly, as you will get written feedback on this exercise, if you hand it in.**
Calculate and give the result in canonical form:

(a) $\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{100}$

(b) $\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)^{35}$

Solution (with some magic):

(a) As $\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{2}=-4\mathrm{i}$ we have

$$\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{100}=\left(\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{2}\right)^{50}=(-4\mathrm{i})^{50}=2^{100}\left(\mathrm{i}^{4}\right)^{12}\mathrm{i}^{2}=-2^{100}.$$

(b) As $\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)^{3}=-1$ we have

$$\begin{aligned}\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)^{35}&=\left(\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)^{3}\right)^{11}\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)^{2}\\&=-\left(\frac{1}{4}-\frac{\sqrt{3}\mathrm{i}}{2}+\frac{3\mathrm{i}^2}{4}\right)=\frac{1}{2}+\frac{\sqrt{3}\mathrm{i}}{2}.\end{aligned}$$

□ **Solution:**

We use the following solution strategy:

- We are given the complex number in canonical form, we transform it into polar form, i.e. we use the notation, which uses Euler's formula. Thus we need to compute the radius and argument.
- Then we do the exponentiation in polar form.
- Next we find an equivalent angle between 0 and 2π by subtracting (or adding) 2π from the resulting angle as often as needed.
- Finally we transform the result back into canonical form.

(a) We transform $\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)$ into polar form. For this we first compute the radius

$$r=\sqrt{\left(-\sqrt{2}\right)^{2}+\left(\sqrt{2}\right)^{2}}=2.$$

Thus we get

$$\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)=2\cdot\left(\underbrace{-\frac{\sqrt{2}}{2}}_{\cos(\varphi)}+\mathrm{i}\underbrace{\frac{\sqrt{2}}{2}}_{\sin(\varphi)}\right),$$

for some angle $\varphi \in [0, 2\pi)$. We find this angle using the table on page 44, so $\varphi = \frac{3\pi}{4}$. Thus we have

$$\begin{aligned}\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{100} &= \left(2\cdot\left(-\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}\right)\right)^{100}\\ &= \left(2\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right)^{100}\\ &= 2^{100}\mathrm{e}^{\mathrm{i}\frac{300\pi}{4}} = 2^{100}\mathrm{e}^{\mathrm{i}(2\cdot 37+1)\pi} = 2^{100}\mathrm{e}^{\mathrm{i}\pi}\\ &= 2^{100}(-1) = -2^{100}.\end{aligned}$$

(b) We transform $\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)$ into polar form. For this we first compute the radius

$$r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1.$$

Thus we get

$$\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right) = 1\cdot\left(\underbrace{\frac{1}{2}}_{\cos(\varphi)}-\mathrm{i}\underbrace{\frac{\sqrt{3}}{2}}_{\sin(\varphi)}\right),$$

for some angle $\varphi \in [0, 2\pi)$. We find this angle using the table on page 44, so $\varphi = \frac{5\pi}{3}$ (Verify this!). Thus we have

$$\begin{aligned}\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)^{35} &= \left(1\cdot\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)\right)^{35}\\ &= \left(1\cdot\mathrm{e}^{\mathrm{i}\frac{5\pi}{3}}\right)^{35}\\ &= 1^{35}\mathrm{e}^{\mathrm{i}\frac{175\pi}{3}} = \mathrm{e}^{\mathrm{i}(2\cdot 29+\frac{1}{3})\pi} = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\\ &= \frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}.\end{aligned}$$

$\square$

40. Verify that for the inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ the following properties hold:

    - $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$
    - $\langle x, \lambda y + \mu z \rangle = \overline{\lambda} \langle x, y \rangle + \overline{\mu} \langle x, z \rangle$,

    where $x, y, z \in \mathbb{C}^d$ and $\lambda, \mu \in \mathbb{C}$.

    Solution:

    - $$\begin{aligned} \langle \lambda x + \mu y, z \rangle &= \langle \lambda x, z \rangle + \langle \mu y, z \rangle \\ &= \lambda \langle x, z \rangle + \mu \langle y, z \rangle . \end{aligned}$$

    - $$\begin{aligned} \langle x, \lambda y + \mu z \rangle &= \overline{\langle \lambda y + \mu z, x \rangle} \\ &= \overline{\langle \lambda y, x \rangle + \langle \mu z, x \rangle} \\ &= \overline{\lambda \langle y, x \rangle + \mu \langle z, x \rangle} \\ &= \overline{\lambda}\, \overline{\langle y, x \rangle} + \overline{\mu}\, \overline{\langle z, x \rangle} \\ &= \overline{\lambda} \langle x, y \rangle + \overline{\mu} \langle x, z \rangle . \end{aligned}$$

$\square$

41. Consider $\mathbb{R}^d$ equipped with the Euclidean norm and the corresponding inner product $\langle \cdot, \cdot \rangle$. Let $x, y \in \mathbb{R}^d$ with $x, y \neq 0$. Show that for the angle $\alpha \in [0, \pi]$ in-between the vectors x and y the following equation holds:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$$

Hint:

- Use the Cauchy-Schwarz inequality to show that the RHS is in $[-1, 1]$.
- Use the law of cosines and Exercise 36 c) to show the equality.

Solution:
From the Cauchy-Schwarz inequality we have

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

and thus

$$\frac{|\langle x, y \rangle|}{\|x\| \cdot \|y\|} \leq 1$$

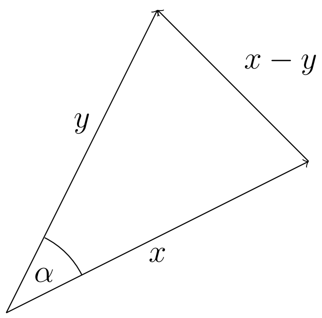
or put differently

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} \leq 1.$$

That means there do exist unique angles $\alpha \in [0, \pi]$ such that

$$\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}.$$

We are left to verify that those α correspond with the angle in-between x and y.



By the law of cosines we get

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos(\alpha)$$

or if we write it with inner products

$$\begin{aligned} \langle x - y, x - y \rangle &= \langle x, x \rangle + \langle y, y \rangle - 2\|x\|\,\|y\|\cos(\alpha) \Leftrightarrow \\ \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle &= \langle x, x \rangle + \langle y, y \rangle - 2\|x\|\,\|y\|\cos(\alpha) \Leftrightarrow \\ 2\langle x, y \rangle &= 2\|x\|\,\|y\|\cos(\alpha) \end{aligned}$$

and hence

$$\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}.$$

$\square$

42. (a) Use the Cauchy-Schwarz inequality to find an upper bound for $|\sin(x) + \cos(x)|$. When does equality hold in the resulting inequality?
    (b) Show that the triangle inequality is equivalent to

$$|\|x\| - \|y\|| \leq \|x - y\| \, .$$

Solution:

(a) Consider $\begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then by the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\sin(x) + \cos(x)| &= \left| \left\langle \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \right| \leq \left\| \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \\ &= \sqrt{\sin^2(x) + \cos^2(x)}\sqrt{2} = \sqrt{2}, \end{aligned}$$

with equality, if and only if $x \in \{\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi : k \in \mathbb{Z}\}$.

(b) We need to show that

$$|\|x\| - \|y\|| \leq \|x - y\| \Leftrightarrow \|x + y\| \leq \|x\| + \|y\| \, .$$

We start by proving the implication from left to right. So, suppose that

$$|\|a\| - \|b\|| \leq \|a - b\| \, ,$$

which is equivalent to

$$- \|a - b\| \leq \|a\| - \|b\| \leq \|a - b\| \, .$$

Case 1: $- \|a - b\| \leq \|a\| - \|b\|$.
Then

$$\|b\| \leq \|a\| + \|a - b\| \, .$$

Choose $a = x$ and $b = x + y$. Then

$$\|x + y\| \leq \|x\| + \|x - (x + y)\| = \|x\| + \|-y\| = \|x\| + \|y\| \, .$$

Case 2: $\|a\| - \|b\| \leq \|a - b\|$.
Then

$$\|a\| \leq \|a - b\| + \|b\| \, .$$

Choose $a = x + y$ and $b = y$. Then

$$\|x + y\| \leq \|x + y - y\| + \|y\| = \|x\| + \|y\| \, .$$

Next we need to prove the implication from right to left. Let thus $\|a + b\| \leq \|a\| + \|b\|$, or, put differently,

$$\|a + b\| - \|b\| \leq \|a\| \, .$$

If we choose $a = x + y$ and $b = -y$ we get

$$\|x + y - y\| - \|-y\| \leq \|x + y\|,$$

which is the same as

$$\|x\| - \|y\| \leq \|x + y\|.$$

If we choose $a = x + y$ and $b = -x$ we get

$$\|x + y - x\| - \|-x\| \leq \|x + y\|,$$

which is the same as

$$\|y\| - \|x\| \leq \|x + y\|,$$

or

$$-\|x + y\| \leq \|x\| - \|y\|.$$

Putting the two together we obtain

$$|\|x\| - \|y\|| \leq \|x - y\|$$

$\square$

(Optional). The following exercises do not count for ticks and will not necessarily be discussed in the exercise class, but you still get the solutions.

A. Solve the following inequalities:

- $5|3x-7| - 3|2x-10| \leq 12x - 31$
- $|15 - |2x+5|| \geq 11$

Solution:

- $5|3x-7| - 3|2x-10| \leq 12x - 31$.

$$3x - 7 \geq 0 \Leftrightarrow x \geq \frac{7}{3}$$
$$2x - 10 \geq 0 \Leftrightarrow x \geq 5.$$

Case 1: $x < \frac{7}{3}$.

$$5(7-3x) - 3(10-2x) \leq 12x - 31 \Leftrightarrow x \geq \frac{12}{7}.$$

Hence $S_1 = \left[\frac{12}{7}, \frac{7}{3}\right)$.
Case 2: $\frac{7}{3} \leq x < 5$.

$$5(3x-7) - 3(10-2x) \leq 12x - 31 \Leftrightarrow x \leq \frac{34}{9}.$$

Hence $S_2 = \left[\frac{7}{3}, \frac{34}{9}\right)$.
Case 3: $x \geq 5$.

$$5(3x-7) - 3(2x-10) \leq 12x - 31 \Leftrightarrow x \geq \frac{26}{3}.$$

Hence $S_3 = \left[\frac{26}{3}, \infty\right)$ and thus the overall solution set is given as

$$S = \left[\frac{12}{7}, \frac{34}{9}\right] \cup \left[\frac{26}{3}, \infty\right).$$

- $|15 - |2x+5|| \geq 11 \Leftrightarrow 15 - |2x-5| \leq -11 \vee 15 - |2x+5| \geq 11$.
Case 1: $15 - |2x-5| \leq -11$
Subcase 1.1: $2x + 5 \geq 0 \Leftrightarrow x \geq -\frac{5}{2}$.

$$15 - 2x - 5 \leq -11 \Leftrightarrow x \geq \frac{21}{2}.$$

Hence $S_1 = \left[\frac{21}{2}, \infty\right)$.
Subcase 1.2: $2x + 5 < 0 \Leftrightarrow x < -\frac{5}{2}$.

$$15 + 2x + 5 \leq -11 \Leftrightarrow x \leq -\frac{31}{2}.$$

Hence $S_2 = \left(-\infty, -\frac{31}{2}\right]$. **Case 2:** $15 - |2x-5| \geq 11$
Subcase 2.1: $2x + 5 \geq 0 \Leftrightarrow x \geq -\frac{5}{2}$.

$$15 - 2x - 5 \geq 11 \Leftrightarrow x \leq -\frac{1}{2}.$$

Hence $S_3 = \left[-\frac{5}{2}, -\frac{1}{2}\right]$.
Subcase 2.2: $2x + 5 < 0 \Leftrightarrow x < -\frac{5}{2}$.

$$15 + 2x + 5 \geq 11 \Leftrightarrow x \geq -\frac{9}{2}.$$

Hence $S_4 = \left[-\frac{9}{2}, -\frac{5}{2}\right)$ and the overall solution set is

$$S = \left(-\infty, -\frac{31}{2}\right] \cup \left[-\frac{9}{2}, -\frac{1}{2}\right] \cup \left[\frac{21}{2}, \infty\right).$$

$\square$

B. Let $z_1 = 7 + 3\mathrm{i}$, $z_2 = -3 + 4\mathrm{i}$ and $z_3 = 1 - 4\mathrm{i}$. Determine

- $z_1 \cdot z_2$
- $\mathrm{Re}(\overline{z_1} + z_3^2)$
- $\frac{z_1 + \overline{z_3}}{z_2} - |z_2|$

Solution:

- $(7 + 3\mathrm{i})(-3 + 4\mathrm{i}) = -21 - 12 + \mathrm{i}(28 - 9) = -33 + 19\mathrm{i}$
- $\mathrm{Re}(7 - 3\mathrm{i} + 1 - 8\mathrm{i} - 16) = -8$
- $\frac{8+7\mathrm{i}}{-3+4\mathrm{i}} \cdot \frac{-3-4\mathrm{i}}{-3-4\mathrm{i}} - 5 = \frac{-24+28-\mathrm{i}(32+21)}{9+16} - 5 = \frac{-121-53\mathrm{i}}{25} = -\frac{121}{25} - \frac{53}{25}\mathrm{i}$

$\square$

C. Show that

$$|z + w| = |z| + |w|$$

for $z, w \in \mathbb{C}$ if, and only if, z and w have the same argument.
Hint: Consider Lemma 1.67 together with the polar form of z and w.

Solution:
We know from Lemma 1.67 that

$$|z + w| = |z| + |w|$$

for $z, w \in \mathbb{C}$ if, and only if $z\overline{w} > 0$. Note that this means that $z\overline{w} \in \mathbb{R}$. Let $z = r_1 e^{i\varphi_1}$ and $w = r_2 e^{i\varphi_2}$. Hence $\overline{w} = r_2 e^{-i\varphi_2}$. Note further, that $r_1, r_2 \geq 0$.
Case 1: $r_1 = 0$. Then the equation reduces to $|w| = |w|$.
Case 2: $r_2 = 0$. Then the equation reduces to $|z| = |z|$.
Case 3: $r_1, r_2 > 0$. Then

$$\begin{aligned} z\overline{w} = r_1 r_2 e^{i(\varphi_1 - \varphi_2)} > 0 &\Leftrightarrow \\ e^{i(\varphi_1 - \varphi_2)} = 1 &\Leftrightarrow \\ \varphi_1 = \varphi_2&, \end{aligned}$$

which means that z and w have the same argument. □

43. Let $A := \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$, $B := \begin{pmatrix} -3 & 1 & 2 \\ 0 & 4 & 2 \\ -4 & -1 & 1 \end{pmatrix}$, $C := \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -4 & 1 \end{pmatrix}$

Which of the following expressions are well-defined? Compute the result if possible.

(a) AB

(b) ABC

(c) CBA

(d) $(A+B)C$

(e) $2(A^T + C)$

(f) AA^T

(g) $A^T A$

Solution:

(a) $AB = \begin{pmatrix} 5 & 3 & 0 \\ -4 & 3 & 3 \end{pmatrix}$

(b) $ABC = \begin{pmatrix} 11 & 5 \\ -10 & -1 \end{pmatrix}$

(c) inner matrix dimensions do not agree

(d) inner matrix dimensions do not agree

(e) $2(A^T + C) = \begin{pmatrix} 4 & 2 \\ 4 & 2 \\ -12 & 4 \end{pmatrix}$

(f) $AA^T = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$

(g) $A^T A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -2 & 1 & 5 \end{pmatrix}$

$\square$

44. (a) Provide an example that shows that matrix multiplication in $\mathbb{R}^{3\times 3}$ is not commutative.

    (b) Show that matrix multiplication and addition are left-distributive, i.e. $A(B+C) = AB + AC$ for any $A \in \mathbb{R}^{m\times p}, \quad B, C \in \mathbb{R}^{p\times n}$.
    Hint: Represent A, B and C using row and column vectors.

Solution:

(a) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, but $BA = 0 \quad \big(= 0_{33}\big)$.

(b) Let $A = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix}, \quad B = (b_1, b_2, \cdots, b_n), \quad C = (c_1, c_2, \cdots, c_n)$,

where $a_i^T, b_j, c_j \in \mathbb{R}^p$ for $i = 1, \cdots, m, j = 1, \cdots, n$. Then $A(B+C)$

$$= \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix} \Big[(b_1, b_2, \cdots, b_n) + (c_1, c_2, \cdots, c_n)\Big]$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix} (b_1 + c_1, b_2 + c_2, \cdots, b_n + c_n)$$

$$= \begin{pmatrix} a_1(b_1+c_1) & a_1(b_2+c_2) & \cdots & a_1(b_n+c_n) \\ a_2(b_1+c_1) & a_2(b_2+c_2) & \cdots & a_2(b_n+c_n) \\ \cdots & \cdots & \cdots & \cdots \\ a_m(b_1+c_1) & a_m(b_2+c_2) & \cdots & a_m(b_n+c_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1 + a_1c_1 & a_1b_2 + a_1c_2 & \cdots & a_1b_n + a_1c_n \\ a_2b_1 + a_2c_1 & a_2b_2 + a_2c_2 & \cdots & a_2b_n + a_2c_n \\ \cdots & \cdots & \cdots & \cdots \\ a_mb_1 + a_mc_1 & a_mb_2 + a_mc_2 & \cdots & a_mb_n + a_mc_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \cdots & \cdots & \cdots & \cdots \\ a_mb_1 & a_mb_2 & \cdots & a_mb_n \end{pmatrix} + \begin{pmatrix} a_1c_1 & a_1c_2 & \cdots & a_1c_n \\ a_2c_1 & a_2c_2 & \cdots & a_2c_n \\ \cdots & \cdots & \cdots & \cdots \\ a_mc_1 & a_mc_2 & \cdots & a_mc_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix} (b_1, b_2, \cdots, b_n) + \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix} (c_1, c_2, \cdots, c_n)$$

$$= AB + AC$$

$\square$

45. (a) Let A and C be the matrices from Exercise 43. Verify that Lemma 2.9 holds for this example, i.e. show that $(AC)^T = C^T A^T$ and $(CA)^T = A^T C^T$.

(b) Let $A, B \in \mathbb{R}^{n \times n}$ be quadratic matrices. Prove or disprove the following statements:

i. $A + A^T$ and $A^T A$ are symmetric

ii. If A and B are symmetric, then AB is also symmetric

Hint: You may use $(A + B)^T = A^T + B^T$.

Solution:

(a) $A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \end{pmatrix}$, $\quad C^T = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 0 & 1 \end{pmatrix}$

$AC = \begin{pmatrix} 9 & -1 \\ -2 & 1 \end{pmatrix}$, $\quad C^T A^T = \begin{pmatrix} 9 & -2 \\ -1 & 1 \end{pmatrix}$, $\quad$ hence $(AC)^T = C^T A^T$.

$CA = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -4 \\ -4 & 1 & 9 \end{pmatrix}$, $\quad A^T C^T = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 0 & 1 \\ -1 & -4 & 9 \end{pmatrix}$, $\quad$ hence $(CA)^T = A^T C^T$.

(b) i. This statement is correct.
$(A + A^T)^T = A^T + A^{TT} = A^T + A = A + A^T$.
Further,
$(A^T A)^T = A^T A^{TT} = A^T A$, where Lemma 2.9 was used for the first identity.

ii. This statement is wrong. Let e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which is not symmetric.

$\square$

46. A quadratic matrix $A \in \mathbb{R}^{n\times n}$ is called *skew-symmetric* if $A^T = -A$.

   (a) Provide two example of skew-symmetric matrices.

   (b) How many matrices are both symmetric and skew-symmetric?

   (c) Prove that the diagonal elements of a skew-symmetric matrix are always zero.

Solution:

(a) e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$.

(b) Let $A \in \mathbb{R}^{n\times n}$ be both symmetric and skew-symmetric, i.e. $A = A^T$ and $A = -A^T$. It follows that $2A = 0$, hence only the zero matrix is both symmetric and skew-symmetric.

(c) Let $A \in \mathbb{R}^{n\times n}$ be skew-symmetric. Then,

$$0 = A + A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 2a_{11} & a_{12}+a_{21} & \cdots & a_{1n}+a_{n1} \\ a_{21}+a_{12} & 2a_{22} & \cdots & a_{2n}+a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}+a_{1n} & a_{n2}+a_{2n} & \cdots & 2a_{nn} \end{pmatrix}.$$

By doing element-wise comparison, we can see that $2a_{ii} = 0$ needs to hold for $i = 1, \cdots, n$, i.e. the diagonal elements must be zero.

□

47. Determine the set of solutions for each of the following linear systems:

(a)
$$\begin{aligned} x + 3y &= 2 \\ y - 2z &= -2 \\ -y + z &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x - y + z &= 1 \\ x - z &= 0 \\ 3x - 2y + z &= 2 \end{aligned}$$

(c)
$$\begin{aligned} 2x + y &= 1 \\ x - 2z &= 1 \\ y + 4z &= 2 \end{aligned}$$

Solution:

(a) The system in matrix-vector form is $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

By using e.g. substitution we can easily determine that the system has a single solution, $L = \{(2, 0, 1)^T\}$

(b) The system in matrix-vector form is $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

From the second equation we can immediately see that $x = z$. Plugging that into the first equation (or third, since they are linearly dependent) yields $2x - y = 1$, or $y = 2x - 1$. Hence the solution is $L = \{(\lambda, 2\lambda - 1, \lambda) : \lambda \in \mathbb{R}\}$

(c) The system in matrix-vector form is $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

From the second equation we have that $x = 1 + 2z$. From the third equation we have that $y = 2 - 4z$. Plugging that into equation one yields

$$2(1 + 2z) + (2 - 4z) = 1$$

$$\Leftrightarrow\ 4 = 1$$

Hence this system has no solutions and $L = \emptyset$.

$\square$

48. Let $A = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 1 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ 0 & 0 & \sqrt{2} & 1 \end{pmatrix}$.

    Show that for any $b \in \mathbb{R}^4$, it holds that $|L(A, b)| \leq 1$

    Solution:
    We consider the homogenous system $Ax = 0$ with $x = (x_1, x_2, x_3, x_4)^T$. Going through the rows of the system from top to bottom, we observe that

    $$x_1 = -\sqrt{2}x_4$$
    $$x_2 = -\sqrt{2}x_1 = \sqrt{2}^2 x_4$$
    $$x_3 = -\sqrt{2}x_2 = -\sqrt{2}^3 x_4$$
    $$x_4 = -\sqrt{2}x_3 = \sqrt{2}^4 x_4,$$

    which only holds for $x = 0$. Hence the homogenous system only has the trivial solution. Using Lemma 2.25, we can conclude that $Ax = b$, $b \in \mathbb{R}^4$ can have at most one solution. Therefore, $|L(A, b)| \leq 1$. $\square$

(Optional). The following exercise does not count for ticks and will not necessarily be discussed in the exercise class, but you still get the solution.

A. Let $A = \begin{pmatrix} 1+i & -i \\ -1+i & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2+i & 2i & 0 \\ i & 0 & 2 \end{pmatrix}$.

Calculate AB, A^*A and BB^*.

Solution:

$$AB = \begin{pmatrix} 1+i & -i \\ -1+i & 0 \end{pmatrix} \begin{pmatrix} 2+i & 2i & 0 \\ i & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2+3i & -2+2i & -2i \\ -3+i & -2-2i & 0 \end{pmatrix}$$

$$A^*A = \begin{pmatrix} 1-i & -1-i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1+i & -i \\ -1+i & 0 \end{pmatrix} = \begin{pmatrix} 4 & -1-i \\ -1+i & 1 \end{pmatrix}$$

$$BB^* = \begin{pmatrix} 2+i & 2i & 0 \\ i & 0 & 2 \end{pmatrix} \begin{pmatrix} 2-i & -i \\ -2i & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 1-2i \\ 1+2i & 5 \end{pmatrix} \qquad \square$$

49. **Please hand in this exercise as an extra pdf-file and name the file something like [Name]-exercise-[number].pdf. Please write especially neatly, as you will get written feedback on this exercise, if you hand it in.**

Solve each of the following systems by Gaussian elimination:

(a)
$$\begin{aligned} -w + x - y + 2z &= -1 \\ -2w + 2x + y - 2z &= -2 \\ w - x + 2y - 4z &= 1 \\ -3w + 3x &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ -3x_1 + 7x_2 - 4x_3 &= 10 \end{aligned}$$

Solution:

(a) Writing the first linear system of equation in matrix notation, we have

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 2 & -1 \\ -2 & 2 & 1 & -2 & -2 \\ 1 & -1 & 2 & -4 & 1 \\ -3 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} -1 & 1 & -1 & 2 & -1 \\ 0 & 0 & 3 & -6 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & -6 & 4 \end{array}\right)$$
$$\to \left(\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right).$$

Here we can already see that the solution set is the empty set.

(b) Writing the second system of linear equations in matrix notation, we have

$$\begin{pmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ -3 & 7 & -4 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 10 & 2 & 34 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 5 & 1 & 17 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 26 & 62 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 13 & 31 \end{pmatrix} \to \begin{pmatrix} 13 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{38}{13} \\ 0 & 0 & 1 & \frac{31}{13} \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & \frac{4}{13} \\ 0 & 1 & 0 & \frac{38}{13} \\ 0 & 0 & 1 & \frac{31}{13} \end{pmatrix},$$

where we can immediately see that $S = \left\{ \frac{1}{13} \begin{pmatrix} 4 \\ 38 \\ 31 \end{pmatrix} \right\}$.

$\square$

50. For which $a \in \mathbb{R}$ is the following system of linear equations solvable?

$$\begin{aligned} x + y \quad\ \ &= 2 \\ x + y + z &= a \\ y + z &= 2 \end{aligned}$$

Solution:
Written in matrix form we get

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & a \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & a-2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 4-a \\ 0 & 0 & 1 & a-2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & a-2 \\ 0 & 1 & 0 & 4-a \\ 0 & 0 & 1 & a-2 \end{pmatrix}.$$

Hence the linear equation system is solvable for any real a. $\square$

51. For each of the following matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ -1 & -3 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 1 & 4 & 2 & 1 & 5 \\ 3 & 4 & 8 & 1 & 2 \end{pmatrix},$$

compute their reduced row echelon forms and ranks.

Solution:

(a)

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \xrightarrow{II-2\,I} \underbrace{\begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix}}_{\text{echelon form}} \xrightarrow{\frac{II}{-5}} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \xrightarrow{I-3\,II} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $\operatorname{rank}(A) = 2$.

(b)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ -1 & -3 & -3 \end{pmatrix} \xrightarrow[III\cdot -1]{II*\frac{1}{2}} \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 3 \end{pmatrix} \xrightarrow[III-I]{II-I} \begin{pmatrix} 1 & 3 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[\frac{III}{2}]{I+II} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[II-III]{I-2\,III} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{II}{-3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, $\operatorname{rank}(B) = 3$.

(c)

$$\begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 1 & 4 & 2 & 1 & 5 \\ 3 & 4 & 8 & 1 & 2 \end{pmatrix} \xrightarrow[III-3\,I]{II-I} \begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 4 & -1 & -2 & -4 \end{pmatrix} \xrightarrow{III-II}$$

$$\begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & -2 & -7 \end{pmatrix} \xrightarrow[\frac{III}{-2}]{\frac{II}{4}} \begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{7}{2} \end{pmatrix} \xrightarrow{I-III} \begin{pmatrix} 1 & 0 & 3 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{-1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{7}{2} \end{pmatrix}$$

Thus, $\operatorname{rank}(C) = 3$.

□

52. Let $A_n \in \mathbb{R}^{n\times n}$ be the matrix of the first n^2 positive integers ordered from smallest to largest, e.g.

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Show that the rank of A_n equals 2 for all $n \geq 2$.

Solution:
We start by writing the matrix A_n in the following form:

$$A_n = \begin{pmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & n+n \\ 2n+1 & 2n+2 & \dots & 2n+n \\ 3n+1 & 3n+2 & \dots & 3n+n \\ \vdots & \vdots & & \vdots \\ (n-2)n+1 & (n-2)n+2 & \dots & (n-2)n+n \\ (n-1)n+1 & (n-1)n+2 & \dots & (n-1)n+n \end{pmatrix}$$

Now we perform the following elementary matrix transformations:

- new n-th row = old n-th row − old $(n-1)$-st row
- new $(n-1)$-st row = old $(n-1)$-st row − old $(n-2)$-nd row
- $\vdots$
- new 3-rd row = old 3-rd row − old 2-nd row
- new 2-nd row = old 2-nd row − old 1-st row
- new 1-st row = old 1-st row

We obtain the following matrix:

$$\begin{pmatrix} 1 & 2 & \dots & n \\ n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & & \vdots \\ n & n & \dots & n \end{pmatrix} \to \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & \dots & n \\ 0 & 1 & \dots & n-1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

from which we can easily see that $\operatorname{rank}(A_n) = 2$. □

53. Evaluate the determinant of the following matrices

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} x-2 & x-3 & x-4 \\ x+1 & x-1 & x-3 \\ x-4 & x-7 & x-10 \end{pmatrix}.$$

Solution:

(a) $\det(A) = 4$

(b) $\det(B) = 0 + 5\,\det\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} + 2\,\det\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} = 5 \cdot 7 + 2 \cdot (-7) = 21.$

(c)

$$\begin{aligned}\det(C) = -(x+1)\,\det\begin{pmatrix} x-3 & x-4 \\ x-7 & x-10 \end{pmatrix} + (x-1)\,\det\begin{pmatrix} x-2 & x-4 \\ x-4 & x-10 \end{pmatrix} \\ -(x-3)\,\det\begin{pmatrix} x-2 & x-3 \\ x-4 & x-7 \end{pmatrix} \\ = (2\,x^2-2) + (-4\,x^2+8\,x-4) + (2\,x^2-8\,x+6) = 0\end{aligned}$$

□

54. Show that

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & \dots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \dots & x_n^n \end{pmatrix} = \prod_{0 \le i < k \le n} (x_k - x_i),$$

for real numbers $x_0, x_1, \dots, x_n$.
Hint: This determinant is called Vandermonde determinant.

Solution:
We use induction to prove this statement.
Induction base: $n = 1$

$$\det \begin{pmatrix} 1 & 1 \\ x_0 & x_1 \end{pmatrix} = x_1 - x_0 = \prod_{0 \le i < k \le 1} (x_k - x_i)$$

Induction hypothesis:

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & \dots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \dots & x_n^n \end{pmatrix} = \prod_{0 \le i < k \le n} (x_k - x_i)$$

Induction step: $n \to n + 1$

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_0 & x_1 & x_2 & \dots & x_n & x_{n+1} \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 & x_{n+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^n & x_1^n & x_2^n & \dots & x_n^n & x_{n+1}^n \\ x_0^{n+1} & x_1^{n+1} & x_2^{n+1} & \dots & x_n^{n+1} & x_{n+1}^{n+1} \end{pmatrix}$$

$$\overset{(\star)}{=} \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_0 - x_{n+1} & x_1 - x_{n+1} & x_2 - x_{n+1} & \dots & x_n - x_{n+1} & 0 \\ x_0^2 - x_0 x_{n+1} & x_1^2 - x_1 x_{n+1} & x_2^2 - x_2 x_{n+1} & \dots & x_n^2 - x_n x_{n+1} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_0^{n+1} - x_0^n x_{n+1} & x_1^{n+1} - x_1^n x_{n+1} & x_2^{n+1} - x_2^n x_{n+1} & \dots & x_n^{n+1} - x_n^n x_{n+1} & 0 \end{pmatrix},$$

where in $(\star)$ we have performed the following matrix transformations which have no impact on the determinant: From bottom to top, each row (except the first) is replaced by the difference from the original row with the multiple of the row above and x_{n+1}.

Next we use Laplace expansion along the last column and obtain

$$
\begin{aligned}
&(-1)^{n+2+1}\det\begin{pmatrix} x_0-x_{n+1} & x_1-x_{n+1} & \dots & x_n-x_{n+1}\\ (x_0-x_{n+1})x_0 & (x_1-x_{n+1})x_1 & \dots & (x_n-x_{n+1})x_n\\ \vdots & \vdots & \ddots & \vdots\\ (x_0-x_{n+1})x_0^n & (x_1-x_{n+1})x_1^n & \dots & (x_n-x_{n+1})x_n^n \end{pmatrix}\\
&\overset{\text{Lemma 2.55}}{=} (-1)^{n+3}\det\begin{pmatrix} x_0-x_{n+1} & (x_0-x_{n+1})x_0 & \dots & (x_0-x_{n+1})x_0^n\\ x_1-x_{n+1} & (x_1-x_{n+1})x_1 & \dots & (x_1-x_{n+1})x_1^n\\ \vdots & \vdots & \ddots & \vdots\\ x_n-x_{n+1} & (x_n-x_{n+1})x_n & \dots & (x_n-x_{n+1})x_n^n \end{pmatrix}\\
&\overset{\text{Lemma 2.47}}{=} (-1)^{n+3}(x_0-x_{n+1})(x_1-x_{n+1})\dots(x_n-x_{n+1})\det\begin{pmatrix} 1 & x_0 & \dots & x_0^n\\ 1 & x_1 & \dots & x_1^n\\ \vdots & \vdots & & \vdots\\ 1 & x_n & \dots & x_n^n \end{pmatrix}\\
&\overset{\text{Lemma 2.55}}{=} (-1)^{n+3}(x_0-x_{n+1})(x_1-x_{n+1})\dots(x_n-x_{n+1})\det\begin{pmatrix} 1 & 1 & \dots & 1\\ x_0 & x_1 & \dots & x_n\\ \vdots & \vdots & & \vdots\\ x_0^n & x_1^n & \dots & x_n^n \end{pmatrix}\\
&\overset{IH}{=} (-1)^{n+3}(-1)^{n+1}(x_{n+1}-x_0)(x_{n+1}-x_1)\dots(x_{n+1}-x_n)\prod_{0\le i<k\le n}(x_k-x_i)\\
&= (-1)^{2n+4}\prod_{0\le i<k\le n+1}(x_k-x_i)\\
&= \prod_{0\le i<k\le n+1}(x_k-x_i)
\end{aligned}
$$

$\square$

(Optional). The following exercises do not count for ticks and will not necessarily be discussed in the exercise class, but you still get the solutions.

A. For a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ the following formula for calculating the determinante holds:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

It is called the rule of Sarrus. Prove that the rule of Sarrus holds.

Solution:
To prove the rule of Sarrus we use Laplace expansion to calculate the determinant:

$$\begin{aligned} \det(A) &= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{31} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

$\square$

B. (a) Solve the linear equation system given by

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & -1 \\ -1 & 1 & 2 & -3 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ -1 & -2 & -1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 0 \\ 2 \\ -3 \end{pmatrix}.$$

(b) Solve the following linear equation system:

$$\begin{pmatrix} 2 & 8 & 1 \\ 4 & 4 & -1 \\ -1 & 2 & 12 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 32 \\ 16 \\ 52 \end{pmatrix}.$$

Solution:

(a)

$$\begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 3 \\ -1 & 1 & 2 & -3 & 1 & 0 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -1 & -2 & -1 & 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 3 \\ 0 & 3 & 3 & -3 & 0 & 3 \\ 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we can easily deduce the solution set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}.$$

(b)

$$\begin{pmatrix} 2 & 8 & 1 & 32 \\ 4 & 4 & -1 & 16 \\ -1 & 2 & 12 & 52 \end{pmatrix} \to \begin{pmatrix} 2 & 8 & 1 & 32 \\ 0 & -12 & -3 & -48 \\ 0 & 12 & 25 & 136 \end{pmatrix} \to \begin{pmatrix} 2 & 8 & 1 & 32 \\ 0 & -4 & -1 & -16 \\ 0 & 0 & 22 & 88 \end{pmatrix}$$
$$\to \begin{pmatrix} 2 & 4 & 0 & 16 \\ 0 & -4 & 0 & -12 \\ 0 & 0 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix},$$

from which we can see that the solution set is given as $S = \left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

□

C. Calculate the determinante of

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 1 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

Solution:

$$\begin{aligned} \det \begin{pmatrix} -1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 1 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} &= -\det \begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} + \det \begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\ &\quad - \det \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix} \\ &= -(4+2-1) + (-1-2+4-2) - (4+1+2+1) = -14 \end{aligned}$$

□

55. Compute the determinant of the following matrices using the Laplace expansion.

$$A = \begin{pmatrix} 0 & 0 & -2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Solution:

(a)

$$\begin{aligned} \det(A) = -2 \begin{vmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & -3 \end{vmatrix} &= -2 \left(\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} \right) \\ &\quad - 3 \left(\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} \right) \\ &= 12 + 27 \\ &= 39. \end{aligned}$$

(b)

$$\begin{aligned} \det(B) &= \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \\ &= 0 - \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 2 \left(\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \right) \\ &= 0 + 2 - 6 + 0 - 3 - 4 + 12 \\ &= 1 \end{aligned}$$

□

56. Use Cramer's rule to solve

(a)

$$\begin{aligned} x + y + z &= -1 \\ x + y \quad\;\; &= \gamma \\ y + z &= \alpha \end{aligned}$$

(b)

$$\begin{pmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/t \\ 1/t^2 \end{pmatrix}$$

Solution:

(a) The first system translates to

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ \gamma \\ \alpha \end{pmatrix}.$$

By Cramer's rule,

$$x = \frac{\begin{vmatrix} -1 & 1 & 1 \\ \gamma & 1 & 0 \\ \alpha & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}} = -(1+\alpha), \quad y = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & \gamma & 0 \\ 0 & \alpha & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}} = \alpha+\gamma+1, \quad z = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & \gamma \\ 0 & 1 & \alpha \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}} = -(1+\gamma).$$

(b) Solving for $x_1(t)$, $x_2(t)$, and $x_3(t)$ using Cramer's rule, we have that

$$x_1(t) = \frac{\begin{vmatrix} 1 & 0 & 1/t \\ 1/t & t & t^2 \\ 1/t^2 & t^2 & t^3 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = \frac{1}{t^2} - 1$$

$$x_2(t) = \frac{\begin{vmatrix} t & 1 & 1/t \\ 0 & 1/t & t^2 \\ 1 & 1/t^2 & t^3 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = -\left(t^2 - 1\right)t - t^2 + \frac{1}{t^2}$$

$$x_3(t) = \frac{\begin{vmatrix} t & 0 & 1 \\ 0 & t & 1/t \\ 1 & t^2 & 1/t^2 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = t^2 + t - 1.$$

□

57. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

(a) Use Cramer's rule together with the Laplace expansion to compute the inverse of the matrix given below

$$\mathbf{A} = \begin{pmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{pmatrix}$$

(b) Compute the inverse of the matrix below using the Gauss-Jordan algorithm

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Solution:
Recall
Let $\mathbf{P}$ by an $n \times n$ matrix, then using Cramer's rule together with the Laplace expansion to compute $\mathbf{P}^{-1}$, yields the formula:

$$\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\mathbf{C}^T \tag{1}$$

where $(\mathbf{C}) = (c_{i,j})_{i,j=1}^n$ is the cofactor matrix of $\mathbf{P}$ with $c_{i,j} = (-1)^{i+j}m_{i,j}$, where $m_{i,j}$ are the minors of $\mathbf{P}$.

(a) By (1) one must compute the determinant of $\mathbf{A}$ and the matrix of cofactors of $(\mathbf{A})$.

$$\det(\mathbf{A}) = 4\begin{vmatrix} -7 & 4 \\ -4 & 2 \end{vmatrix} - 4\begin{vmatrix} -8 & 5 \\ -4 & 2 \end{vmatrix} + 3\begin{vmatrix} -8 & 5 \\ -7 & 4 \end{vmatrix} = 8 - 16 + 9 = 1.$$

To compute $\text{adj}(\mathbf{A})$, we need to do the following:

i. Compute the minor matrix $\mathbf{M} = (m_{i,j})_{i,j=1}^3$ of $\mathbf{A}$.

In this case we have that

$$\mathbf{M} = \begin{pmatrix} 2 & 4 & 5 \\ 4 & -7 & 8 \\ 3 & -4 & 4 \end{pmatrix}.$$

ii. Compute the cofactor matrix $\mathbf{C} = (c_{i,j})_{i,j=1}^3$ of $\mathbf{A}$, with $c_{i,j} = (-1)^{i+j}\, m_{i,j}$ for $1 \leq i, j \leq 3$.

In this case,

$$\mathbf{C} = \begin{pmatrix} 2 & -4 & 5 \\ -4 & -7 & -8 \\ 3 & 4 & 4 \end{pmatrix}.$$

Now we have that

$$\mathbf{C}^T = \begin{pmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{pmatrix}.$$

By (1)

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{pmatrix}.$$

(b) Apply the reduced row-echelon procedure to the augmented matrix $(\mathbf{B} \mid \mathbf{I})$ we have the following:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right).$$

Thus

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

$\square$

58. Given the following matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

(a) Compute $(\mathbf{A}\,\mathbf{B})^{-1}$ and $\mathbf{B}^{-1}\,\mathbf{A}^{-1}$.

(b) Given that matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are non-singular matrices, derive a formula for $(\mathbf{A}\,\mathbf{B}\,\mathbf{C})^{-1}$ in terms of $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, and $\mathbf{C}^{-1}$.

Solution:

(a)

$$\begin{aligned}
(\mathbf{A}\,\mathbf{B})^{-1} &= \left(\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\right)^{-1} &&= \begin{pmatrix} 4 & 7 \\ 5 & 11 \end{pmatrix}^{-1} &&= \frac{1}{9}\begin{pmatrix} 11 & -7 \\ -5 & 4 \end{pmatrix} \\
\mathbf{B}^{-1}\,\mathbf{A}^{-1} &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} &&= \frac{1}{9}\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} &&= \frac{1}{9}\begin{pmatrix} 11 & -7 \\ -5 & 4 \end{pmatrix}.
\end{aligned}$$

(b) Given that the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are non-singular matrices, our gaol is to derive a formula for $(\mathbf{A}\,\mathbf{B}\,\mathbf{C})^{-1}$ in terms of $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, and $\mathbf{C}^{-1}$. Let $\mathbf{D} = \mathbf{B}\,\mathbf{C}$, then $\mathbf{D}$ is non-singular.

$$\mathbf{A}\,\mathbf{D} = \mathbf{D}^{-1}\,\mathbf{A}^{-1} = (\mathbf{B}\,\mathbf{C})^{-1}\,\mathbf{A}^{-1} = \mathbf{C}^{-1}\,\mathbf{B}^{-1}\,\mathbf{A}^{-1}.$$

$\square$

59. Prove the following statements:

    (a) If the matrices $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{M}+\mathbf{N}$ are non-singular, i.e., the determinant of those matrices are non-zero, then the matrix identity

    $$\mathbf{M}\left(\mathbf{M}+\mathbf{N}\right)^{-1}\mathbf{N} = \mathbf{N}\left(\mathbf{M}+\mathbf{N}\right)^{-1}\mathbf{M} = \left(\mathbf{M}^{-1}+\mathbf{N}^{-1}\right)^{-1}$$

    holds.

    (b) If $\mathbf{A}$ is a square and non-singular upper triangular matrix, then $\mathbf{A}^{-1}$ is also an upper triangular matrix.

Solution:

(a) We show that

$$\mathbf{M}\left(\mathbf{M}+\mathbf{N}\right)^{-1}\mathbf{N} = \left(\mathbf{M}^{-1}+\mathbf{N}^{-1}\right)^{-1} \tag{2}$$

Note that proving the matrix identity (2) is equivalent to showing that

$$\left(\mathbf{M}\left(\mathbf{M}+\mathbf{N}\right)^{-1}\mathbf{N}\right)^{-1} = \left(\left(\mathbf{M}^{-1}+\mathbf{N}^{-1}\right)^{-1}\right)^{-1} = \mathbf{M}^{-1}+\mathbf{N}^{-1} \tag{3}$$

holds. The right hand side of (3) gives

$$\left(\mathbf{M}\left(\mathbf{M}+\mathbf{N}\right)^{-1}\mathbf{N}\right)^{-1} = \mathbf{N}^{-1}\left(\mathbf{M}+\mathbf{N}\right)\mathbf{M}^{-1} = \mathbf{N}^{-1}+\mathbf{M}^{-1} = \mathbf{M}^{-1}+\mathbf{N}^{-1}.$$

Similarly,

$$\left(\mathbf{N}\left(\mathbf{M}+\mathbf{N}\right)^{-1}\mathbf{M}\right)^{-1} = \mathbf{M}^{-1}\left(\mathbf{M}+\mathbf{N}\right)\mathbf{N}^{-1} = \mathbf{N}^{-1}+\mathbf{M}^{-1} = \mathbf{M}^{-1}+\mathbf{N}^{-1}.$$

(b) We prove the statement by induction on the size, n, of the matrix $\mathbf{A}$. If $n=0$, there is nothing to prove. Suppose the statement holds for n. We prove that it holds for $n+1$. Let

$$\begin{gathered}
\mathbf{A} = \begin{array}{cc} \overbrace{\hphantom{a_{1,1}\; a_{1,2}\; a_{1,3}\; \dots\; a_{1,n}}}^{A} & \overbrace{\hphantom{a_{n+1,n+1}}}^{\boldsymbol{w}} \\ \multicolumn{2}{c}{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} & a_{1,n+1} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} & a_{2,n+1} \\ 0 & 0 & a_{3,3} & \dots & a_{3,n} & a_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} & a_{n,n+1} \\ 0 & 0 & 0 & \dots & 0 & a_{n+1,n+1} \end{pmatrix}} \\ \underbrace{\hphantom{a_{1,1}\; a_{1,2}\; a_{1,3}\; \dots\; a_{1,n}}}_{\mathbf{0}} & \underbrace{\hphantom{a_{n+1,n+1}}}_{\alpha} \end{array} = \begin{pmatrix} A & \boldsymbol{w} \\ \mathbf{0} & \alpha \end{pmatrix} \quad \text{and} \\
\mathbf{B} = \begin{array}{cc} \overbrace{\hphantom{b_{1,1}\; b_{1,2}\; b_{1,3}\; \dots\; b_{1,n}}}^{B} & \overbrace{\hphantom{b_{n+1,n+1}}}^{\boldsymbol{s}} \\ \multicolumn{2}{c}{\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n} & b_{1,n+1} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n} & b_{2,n+1} \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & b_{3,n} & b_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n} & b_{n,n+1} \\ b_{n+1,1} & b_{n+1,2} & b_{n+1,3} & \dots & b_{n+1,n} & b_{n+1,n+1} \end{pmatrix}} \\ \underbrace{\hphantom{b_{1,1}\; b_{1,2}\; b_{1,3}\; \dots\; b_{1,n}}}_{\boldsymbol{t}^T} & \underbrace{\hphantom{b_{n+1,n+1}}}_{\beta} \end{array} = \begin{pmatrix} B & \boldsymbol{s} \\ \boldsymbol{t}^T & \beta \end{pmatrix}
\end{gathered} \tag{4}$$

be an $(n+1)\times(n+1)$ non singular upper triangular matrix and its inverse respectively, where α, β are scalars, $\boldsymbol{w}$, $\boldsymbol{s}$ and $\boldsymbol{t}$ are $(n\times 1)$ vectors, and A and B are $n\times n$ matrices. In particular, A is an upper triangular matrix. We prove that $\mathbf{B}$ is upper triangular matrix, i.e., the entries $b_{i,j}=0$ for all $i>j$. Since $\mathbf{B}$ is the inverse of $\mathbf{A}$, it follows that

$$\mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A} = \mathbf{I}_{n+1}, \tag{5}$$

where $\mathbf{I}_{n+1}$ is the $(n+1)\times(n+1)$ identity matrix. Substituting (4) into (5) we have that

$$\begin{aligned}\begin{pmatrix} A & \boldsymbol{w} \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} B & \boldsymbol{s} \\ \boldsymbol{t}^T & \beta \end{pmatrix} = \begin{pmatrix} B & \boldsymbol{s} \\ \boldsymbol{t}^T & \beta \end{pmatrix}\begin{pmatrix} A & \boldsymbol{w} \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \mathbf{I}_{n+1} \\ = \begin{pmatrix} B\,A & B\,\boldsymbol{w}+\boldsymbol{s}\,\alpha \\ \boldsymbol{t}^T A & \boldsymbol{t}^T\boldsymbol{w}+\beta\,\alpha \end{pmatrix} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}\end{aligned} \tag{6}$$

Comparing the terms we have the following:

i. $B\,A = \mathbf{I}_n$. Thus by the induction hypothesis, B is upper triangular of dimension $n\times n$.
ii. $\boldsymbol{t}^T A = \mathbf{0} \Rightarrow \boldsymbol{t} = 0$ since A is invertible by the induction hypothesis.
iii. $\boldsymbol{t}^T\boldsymbol{w}+\beta\,\alpha = 1 \Rightarrow \beta = \frac{1}{\alpha}$. Since $\mathbf{A}$ is invertible, $\alpha\neq 0$.

Therefore,

$$\mathbf{B} = \begin{pmatrix} B & \boldsymbol{s} \\ 0 & \beta \end{pmatrix}$$

is upper triangular.

□

60. Use the definition of convergence to check that the sequence

    (a) $a_n = \frac{n^2-3n+1}{2n^2+1}$ has limit $\frac{1}{2}$, and

    (b) $b_n = \frac{n^3-1}{n^2-n}$ diverges.

Solution:

(a) Let $\epsilon > 0$ be arbitrary but fixed. Our goal is to find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the inequality

$$\left|\frac{n^2-3n+1}{2n^2+1} - \frac{1}{2}\right| = \left|\frac{1-6\,n}{2\,(2\,n^2+1)}\right| < \epsilon$$

holds. Note that

$$\left|\frac{1-6\,n}{2\,(2\,n^2+1)}\right| \leq \frac{6\,n}{4\,n^2} = \frac{3}{2\,n} < \epsilon.$$

By the Archimedean Principle, such a natural number n_0 exist, with $n_0 > \left\lceil \frac{3}{2\,\epsilon} \right\rceil$.

(b) Suppose that the sequence b_n converges to L. Then for any given $\epsilon > 0$, there is a natural number n_0 such that for all $n \geq n_0$, the inequality

$$|b_n - L| = \left|\frac{n^3-1}{n^2-n} - L\right| = \left|\frac{n^2+n+1}{n} - L\right| < \epsilon$$

holds. From the inequality, we have that

$$L - \epsilon < \frac{n^2+n+1}{n} < L + \epsilon.$$

Note that $\frac{n^2+n+1}{n} > n$. Since there is no way for the inequality

$$L - \epsilon < n < L + \epsilon$$

to be satisfied for sufficiently large n, as n will eventually exceed $L + \epsilon$, and hence divergent, it follows that b_n is also divergent since $\frac{n^2+n+1}{n} > n$.

$\square$

A. Prove that

$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \colon |a_n - a| \leq \varepsilon$$

is equivalent to

$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \colon |a_n - a| < \varepsilon$$

for any sequence $(a_n)_{n\in\mathbb{N}}$, see Definition **3.6** in the manuscript.

Solution:

Suppose that

$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \colon |a_n - a| < \varepsilon$$

then it is obvious that

$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \colon |a_n - a| \leq \varepsilon$$

holds since $|a_n - a| < \varepsilon$ implies $|a_n - a| \leq \varepsilon$. Conversely, suppose that

$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \colon |a_n - a| \leq \varepsilon$$

holds. Let $\varepsilon > 0$ be arbitrary but fixed. Then $\frac{\varepsilon}{2} > 0$ also holds. So there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|a_n - a| \leq \frac{\varepsilon}{2}$ holds. But $\frac{\varepsilon}{2} < \varepsilon$, so it follows that $|a_n - a| < \varepsilon$ for all $n \geq n_0$. $\square$

61. Determine the limits of the sequences (a_n), (b_n), (c_n) and (d_n), $n \in \mathbb{N}$, defined by

$$\text{(a)}\quad a_n := \frac{10n+3}{n^2-2}, \qquad \text{(c)}\quad c_n := \sqrt[n]{2n(a^2+2)}, \quad a \in \mathbb{R}$$

$$\text{(b)}\quad b_n = \frac{n^3-3n+1}{2n^3+1} \qquad \text{(d)}\quad d_n := \frac{2\sqrt{n}+ni(n+1)}{n^2}$$

Solution:

(a) By using Theorem 3.24 (i) and (iv), we can conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2(\frac{10}{n}+\frac{3}{n^2})}{n^2(1-\frac{2}{n^2})} = \lim_{n\to\infty} \frac{\frac{10}{n}+\frac{3}{n^2}}{1-\frac{2}{n^2}} = \frac{0+0}{1-0} = 0$$

(b) We again use Theorem 3.24 (i) and (iv),

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^3(1-\frac{3}{n^2}+\frac{1}{n^3})}{n^3(2+\frac{1}{n^3})} = \lim_{n\to\infty} \frac{1-\frac{3}{n^2}+\frac{1}{n^3}}{2+\frac{1}{n^3}} = \frac{1-0+0}{2+0} = \frac{1}{2}$$

(c) We observe that

$$c_n = \sqrt[n]{n}\,\sqrt[n]{2(a^2+2)}$$

From Example 3.22 we know that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

From Example 3.21, we know that

$$\lim_{n\to\infty} \sqrt[n]{2(a^2+2)} = 1$$

Using Theorem 3.24(iii) we can conclude that the limit of the entire sequence is given by

$$\lim_{n\to\infty} c_n = 1 \cdot 1 = 1$$

(d) We again use Theorem 3.24,

$$\lim_{n\to\infty} d_n = 2\lim_{n\to\infty} \frac{\sqrt{n}}{n^2} + i\lim_{n\to\infty} \frac{n+1}{n} = 0 + i\cdot 1 = i$$

$\square$

62. Use the Sandwich rule to determine the limit of the following sequences,

$$(a) \quad a_n = \sqrt[n]{2 - \frac{n-1}{n+1}} \qquad \text{(b)} \quad b_n = \frac{(1+i)^n}{2^n}\sin(n\pi)$$

Solution:

(a) Since $0 \le \frac{n-1}{n+1} < 1$ we have

$$\sqrt[n]{1} < \sqrt[n]{2 - \frac{n-1}{n+1}} \le \sqrt[n]{2}.$$

From the lecture we know that for any $a > 0$ it holds that $\lim_{n\to\infty} \sqrt[n]{a} = 1$. Thus,

$$\lim_{n\to\infty} \sqrt[n]{2} = 1 = \lim_{n\to\infty} \sqrt[n]{1}.$$

By the Sandwich rule it follows that also

$$\lim_{n\to\infty} \sqrt[n]{2 - \frac{n-1}{n+1}} = 1.$$

(b) We observe that

$$|b_n| \le \left|\frac{(1+i)^n}{2^n}\right| = \left|\frac{1+i}{2}\right|^n = \left(\frac{\sqrt{2}}{2}\right)^n.$$

For the right hand side we can see that

$$\lim_{n\to\infty} \left(\frac{\sqrt{2}}{2}\right)^n = 0.$$

Therefore, by the sandwich rule we can deduce that

$$\lim_{n\to\infty} b_n = 0.$$

$\square$

63. Check if the following sequences are bounded and compute the limit if possible.

$$(a) \quad a_n := 1 - \frac{(-2)^n}{2^{n+1}} \qquad (b) \quad b_n := (-1)^n \frac{(n-1)}{\sqrt{n^3+1}} \qquad \text{(c)} \quad a_n := \frac{n^2(n+2)}{(1+i)^n}$$

Solution:

(a) We observe that

$$a_n = 1 - \frac{(-1)^n}{2} = \begin{cases} \frac{1}{2} & \text{for } n \text{ even} \\ \frac{3}{2} & \text{for } n \text{ odd} \end{cases}$$

The sequence is obviously bounded. However, the subsequence with odd index and the subsequence with even index do not converge to the same limit. Therefore, (a_n) is divergent.

(b) We use the sandwich rule. We first make the following estimate,

$$|b_n| = \frac{n-1}{\sqrt{n^3+1}} \leq \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}} \to 0$$

Using the sandwich rule we can now conclude that $b_n \to 0$. Since all convergent sequences are bounded, we know that b_n is also bounded.

(a) Using Theorem 3.24 (i) and (ii), we know that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{(1+i)^n} + 2 \lim_{n\to\infty} \frac{n^2}{(1+i)^n}$$

The absolute value of the complex number in the denominator is $|1+i| = \sqrt{2}$. Therefore we can apply Example 3.23 and immediately deduce that

$$\lim_{n\to\infty} a_n = 0 + 2 \cdot 0 = 0.$$

Boundedness again follows from convergence.

□

64. Find an example for a sequence with the following properties, if possible:

    (a) a bounded sequence that is divergent

    (b) a nondecreasing bounded sequence that is divergent

    (c) a strictly decreasing sequence converging to 1

    (d) an unbounded null sequence

Solution:

(a) e.g. $a_n = (-1)^n$

(b) this is not possible, as the monotonicity principle (Theorem 3.35) asserts that any monotonous bounded sequence is convergent.

(c) e.g. $a_n = 1 + \frac{1}{n}$

(d) this is not possible, as any convergent sequence is bounded (Theorem 3.15)

$\square$

65. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**
Let the sequence (a_n) be recursively defined by

$$a_1 = 0, \quad a_{n+1} = \frac{a_n + 2}{2}.$$

Show that (a_n) is bounded. Use the monotonicity principle to show that it is convergent.

Solution:
We calculate the first couple of values and see that $a_n = (0, 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \cdots)$. Our educated guess is that a_n is bounded by 2. We show this by induction:

- Induction basis: $a_0 = 0 < 2$
- Induction hypothesis: Assume that $a_n < 2$ holds for some $n \in \mathbb{N}$.
- Induction step: We show that also $a_{n+1} < 2$.

$$a_{n+1} = \frac{a_n + 2}{2} = \frac{a_n}{2} + 1 \overset{IH}{<} \frac{2}{2} + 1 = 2.$$

Therefore, we know that (a_n) is bounded by 2. Next, we show that (a_n) is monotonously increasing:

$$a_{n+1} - a_n = \frac{a_n + 2}{2} - a_n = -\frac{a_n}{2} + 1$$

From $a_n < 2$, it follows that $-\frac{a_n}{2} > -1$. We plug that into the above equation and deduce that

$$a_{n+1} - a_n > -1 + 1 = 0.$$

This means that (a_n) is indeed increasing. We can now apply the monotonicity principle (Theorem 3.35) and immediately get that (a_n) is convergent. $\square$

66. Determine the limit of the following sequence

$$a_n = \prod_{k=1}^{n} \left(1 - \frac{1}{(k+1)^2}\right)$$

Solution:
We observe that

$$1 - \frac{1}{(k+1)^2} = \frac{(k+1)^2}{(k+1)^2} - \frac{1}{(k+1)^2} = \frac{k^2+2k+1-1}{(k+1)^2} = \frac{k(k+2)}{(k+1)^2} = \frac{k}{k+1} \cdot \frac{k+2}{k+1}$$

This reminds us of a telescope sum, as most factors will cancel each other out. We show this by splitting up the product and performing an index shift:

$$\begin{aligned} a_n &= \prod_{k=1}^{n} \frac{k}{k+1} \cdot \frac{k+2}{k+1} \\ &= \prod_{k=1}^{n} \frac{k}{k+1} \prod_{k=1}^{n} \frac{k+2}{k+1} \\ &= \prod_{k=1}^{n} \frac{k}{k+1} \prod_{k=2}^{n+1} \frac{k+1}{k} \end{aligned}$$

Now we see that all the factors except at $k = 1$ and $k = n+1$ cancel each other out. Therefore,

$$a_n = \frac{1}{1+1} \cdot \frac{(n+1)+1}{n+1} = \frac{1}{2} \cdot \frac{n+2}{n+1}$$

Obviously, the factor $\frac{n+2}{n+1}$ converges to 1. We show this using the standard trick and immediately arrive at the limit of the entire sequence:

$$\lim_{n\to\infty} a_n = \frac{1}{2} \cdot \lim_{n\to\infty} \frac{n+2}{n+1} = \frac{1}{2} \cdot \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

□

(Optional). The following exercises do not count for ticks and will not necessarily be discussed in the exercise class, but you still get the solutions.

A. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be defined by

$$a_n := \frac{i}{n} \quad \text{and} \quad b_n := \frac{1}{n} - \frac{i}{n^2}.$$

If existent, calculate the limits of (a_n), (b_n),$\left(\frac{a_n}{b_n}\right)$ and $\left(\frac{b_n}{a_n}\right)$.

Solution:
Using Theorem 3.24, we calculate the following limits:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(i\frac{1}{n}\right) = i \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{-i}{n^2} = \lim_{n\to\infty} \frac{1}{n} - i \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{i}{n}}{\frac{n-i}{n^2}} = \lim_{n\to\infty} \frac{ni}{n-i} = i \lim_{n\to\infty} \frac{n}{n-i} = i \cdot 1 = i$$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{n-i}{ni} = \frac{1}{i} \lim_{n\to\infty} \frac{n-i}{n} = \frac{1}{i} \cdot 1 = -i$$

Alternatively, using part (iv) of Theorem 3.24, we can immediately see that the limit of $\left(\frac{b_n}{a_n}\right)$ is the inverse of the limit of $\left(\frac{a_n}{b_n}\right)$,

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{i} = -i$$

□

B. Show that (a_n) is decreasing and determine its limit

$$a_n := \frac{1}{n^2}\sum_{k=1}^{n} k$$

Solution:
We use the well-known identity (e.g. Example 1.47)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Therefore we can rewrite a_n as

$$a_n = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2n}$$

We first show that the sequence is decreasing:

$$a_{n+1} - a_n = \frac{n+2}{2(n+1)} - \frac{n+1}{2n}$$

Bringing this to a common denominator we see that

$$= \frac{n(n+2)-(n+1)^2}{2n(n+1)} = -\frac{1}{2n(n+1)} < 0,$$

proving that (a_n) is indeed decreasing. Next, we compute the limit of (a_n):

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2} = \frac{1}{2}.$$

$\square$

C. We define a sequence $(a_n)_{n>1}$ with a_n being the unique solution of $A_n x = b$, where

$$A_n := \begin{pmatrix} -1 & 1 & 0 \\ 1 & n & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad b := \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

Determine the limit of this sequence.

Solution:
Let n be a.b.f. We bring $(A_n|b)$ into row echelon form:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 1 & n & 0 & 0 \\ 3 & 0 & 1 & 4 \end{array}\right)$$

Adding the first row to the second row, and three times the first row to the third row,

$$\longrightarrow \left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & n+1 & 0 & 1 \\ 0 & 3 & 1 & 7 \end{array}\right)$$

Subtracting $1/(n+1)$ times the second row from the first row, and $3/(n+1)$ times the second row from the third row,

$$\longrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 0 & 1 - \frac{1}{n+1} \\ 0 & n+1 & 0 & 1 \\ 0 & 0 & 1 & 7 - \frac{3}{n+1} \end{array}\right)$$

Multiplying the first row by -1 and the second row by $\frac{1}{n+1}$, we arrive at the reduced row echelon form:

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 + \frac{1}{n+1} \\ 0 & 1 & 0 & \frac{1}{n+1} \\ 0 & 0 & 1 & 7 - \frac{3}{n+1} \end{array}\right)$$

Our solution is therefore given by

$$a_n = \left(-1 + \frac{1}{n+1}, \frac{1}{n+1}, 7 - \frac{3}{n+1}\right)^T$$

To determine the limit of the sequence, we take the component-wise limit,

$$\lim_{n\to\infty} a_n = (-1, 0, 7)^T$$

$\square$

D. As secretly you are a special agent, hiding your true identity by pretending to be a regular Math for AI student, you take another secret mission to save the planet during the holidays. On this mission, while raiding a secret lab during the night, you find a "Post-It" on a desk. Holding it under the light of a candle you see that slowly the following numbers appear

$$\begin{matrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{matrix} .$$

Furthermore, there is a stack of paper on the desk. Touching the first sheet, you suddenly feel some grooves. Gently hatching this sheet of paper makes the following string visible. Could it be a phone number? Maybe it is related to a bank account?
382, 54, -269, -395, -674, 973, 643, 5, -150, -177, -313, -32, 241, 349, 567, -796, -519, 12, 144, 153, 107, 37, -33, -63, -113, 217, 154, 25, -8, -8
In order to investigate this later, you put the Post-It and the sheet, which contains the string of numbers, in your pocket. Just before leaving the lab, you realize that one of the computers on the desk is still running. Unfortunately, there is nothing important on the screen, it just looks one employee got bored and browsed the internet for random topics. The only website that is opened contains an ASCII table. The noise of approaching night guards interrupts your search for more data on the computer; you decide that you have to leave the place quickly!

Solution:
We start by inverting the matrix

$$A := \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} -24 & 18 & 5 & 1 & 0 & 0 \\ 20 & -15 & -4 & 0 & 1 & 0 \\ -5 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & -1 \\ 20 & -15 & -4 & 0 & 1 & 0 \\ -5 & 4 & 1 & 0 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 5 & -4 & -20 & -19 & 20 \\ 0 & -1 & 1 & 5 & 5 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & -1 & 1 & 5 & 5 & -4 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 5 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 5 & 6 & 0 \end{array}\right).$$

Thus

$$A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}.$$

$\square$

If we want to multiply the numbers by this matrix, we need to organize them in 3 rows:

$$B := \begin{pmatrix} 382 & 54 & -269 & -395 & -674 & 973 & 643 & 5 & -150 & -177 \\ -313 & -32 & 241 & 349 & 567 & -796 & -519 & 12 & 144 & 153 \\ 107 & 37 & -33 & -63 & -113 & 217 & 154 & 25 & -8 & -8 \end{pmatrix}.$$

When we compute $A^{-1}B$, we get

$$A^{-1}B = \begin{pmatrix} 77 & 101 & 114 & 114 & 121 & 32 & 67 & 104 & 114 & 105 \\ 115 & 116 & 109 & 97 & 115 & 72 & 97 & 112 & 112 & 121 \\ 32 & 78 & 101 & 119 & 32 & 89 & 101 & 97 & 114 & 33 \end{pmatrix}.$$

Using an ASCII-table (e.g. `https://www.asciitable.com/`) we can convert every entry of the matrix to get

$$\begin{pmatrix} \text{M} & \text{e} & \text{r} & \text{r} & \text{y} & & \text{C} & \text{h} & \text{r} & \text{i} \\ \text{s} & \text{t} & \text{m} & \text{a} & \text{s} & \text{H} & \text{a} & \text{p} & \text{p} & \text{y} \\ & \text{N} & \text{e} & \text{w} & & \text{Y} & \text{e} & \text{a} & \text{r} & \text{!} \end{pmatrix}.$$

67. Let (a_n) be a sequence in $\mathbb{R}_0^+$ with $a_n \to a > 0$ for some $a \in \mathbb{R}^+$. Show that then

$$\sqrt[3]{a_n} \to \sqrt[3]{a}.$$

Solution:
We use the formula

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

to obtain

$$0 \leq \left| a_n^{\frac{1}{3}} - a^{\frac{1}{3}} \right| = \left| \frac{a_n - a}{a_n^{\frac{2}{3}} + a_n^{\frac{1}{3}} a^{\frac{1}{3}} + a^{\frac{2}{3}}} \right| \leq \left| \frac{a_n - a}{a^{\frac{2}{3}}} \right| \to 0.$$

Thus by the sandwich theorem

$$\left| a_n^{\frac{1}{3}} - a^{\frac{1}{3}} \right| \to 0$$

or equivalently

$$\sqrt[3]{a_n} \to \sqrt[3]{a}.$$

$\square$

68. Consider the recursively defined sequence given as

$$a_1 = 1 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}.$$

Proof that (a_n) is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

Solution:
We need to show that

(a) $(a_n) \subseteq \mathbb{Q}$
(b) (a_n) is a Cauchy sequence
(c) (a_n) does not converge in $\mathbb{Q}$.

We prove (a) by induction.
IB: $a_1 = 1 \in \mathbb{Q}$.
IH: $a_1, \ldots, a_n \in \mathbb{Q}$.
IS: The IH implies that $\frac{a_n}{2}$ and $\frac{1}{a_n}$ are elements of $\mathbb{Q}$. Thus, $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \in \mathbb{Q}$ as well.

To prove that (a_n) is a Cauchy sequence we start by showing that $a_n \geq \sqrt{2}$ for any $n \geq 2$. Therefore

$$a_n^2 - 2 = \left(\frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}\right)^2 - 2 = \frac{1}{4}\left(a_{n-1}^2 - 4 + \frac{4}{a_{n-1}^2}\right) = \frac{1}{4}\left(a_{n-1} - \frac{2}{a_{n-1}}\right)^2 \geq 0.$$

Next we show that (a_n) is monotonically decreasing for $n \geq 2$, as

$$a_{n+1} - a_n = \frac{a_n}{2} + \frac{1}{a_n} - a_n = \frac{2 - a_n^2}{2a_n} \leq \frac{2-2}{2a_n} = 0,$$

since $a_n \geq \sqrt{2}$. Hence (a_n) is bounded from below and monotonically decreasing. Thus it is convergent and thereby a Cauchy sequence (cf. Thm. 3.57).

Finally we need to show (c), that (a_n) does not converge in $\mathbb{Q}$. We know already that it is convergent. That means that there exists some $a \in \mathbb{R}$ such that $a_n \to a$. Thus

$$a = \frac{a}{2} + \frac{1}{a} \Leftrightarrow a = \frac{a^2+2}{2a} \Leftrightarrow a^2 = 2 \Leftrightarrow a = \sqrt{2} \notin \mathbb{Q}.$$

$\square$

69. **Please hand in this exercise as an extra pdf-file and name the file something like [Name]-exercise-[number].pdf. Please write especially neatly, as you will get written feedback on this exercise, if you hand it in.**
For the following sequences, determine all accumulation points as well as lim inf and lim sup:

(a) $a_n = \sin\left(n\frac{\pi}{4}\right)$

(b) $b_n = \frac{3n^2-1}{10n+5n^2}$

(c) $c_n = (-1)^n n$.

Solution:

(a) We know that sine is 2π-periodic. Hence we distinguish the 8 subsequences

$$(a_{8k+r})_{k\geq 1},$$

where $r \in \{0, 1, \ldots, 7\}$.
$\underline{r = 0}$: $a_{8k} = \sin\left(8k\frac{\pi}{4}\right) = \sin(2k\pi) = 0$.
$\underline{r = 1}$: $a_{8k+1} = \sin\left((8k+1)\frac{\pi}{4}\right) = \sin\left(2k\pi + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
$\underline{r = 2}$: $a_{8k+2} = \sin\left(\frac{\pi}{2}\right) = 1$.
$\underline{r = 3}$: $a_{8k+3} = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
$\underline{r = 4}$: $a_{8k+4} = \sin(\pi) = 0$.
$\underline{r = 5}$: $a_{8k+5} = \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.
$\underline{r = 6}$: $a_{8k+6} = \sin\left(\frac{3\pi}{2}\right) = -1$.
$\underline{r = 7}$: $a_{8k+7} = \sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.
Thus we have the accumulation points $-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1$ and $\liminf(a_n) = -1$ and $\limsup(a_n) = 1$.

(b) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{3-\frac{1}{n^2}}{\frac{10}{n}+5} = \frac{3}{5}$, which simulaneously is the only accumulation point and the liminf and the limsup of the sequence.

(c) We distinguish the two subsequences

$$(c_{2k})_{k\geq 1} \text{ and } (c_{2k+1})_{k\geq 1}.$$

$c_{2k} = 2k \to \infty$ and $c_{2k+1} = -(2k+1) \to -\infty$, hence we have no accumulation points but by the remark underneath Def. 3.46 we have $\liminf c_n = -\infty$ and $\limsup c_n = \infty$.

$\square$

70. Consider the sequence

$$a_n = \left((-1)^n + \frac{2n^2+3}{n^2+7n}\right)^{n \bmod 3},$$

where n mod 3 gives the remainder of n of the division by 3 in $\mathbb{N}$. Find all accumulation points of (a_n) as well as $\liminf(a_n)$ and $\limsup(a_n)$.

Solution:
We distinguish three cases.
Case 1: $n \equiv 0 \bmod 3$.
In this case we have the exponent equal to zero and thus

$$(a_{3k})_{k\geq 1} = (1, 1, 1, ...),$$

which leads to the accumulation point 1.
Case 2: $n \equiv 1 \bmod 3$.
Here we distinguish two subcases, where n is once congruent to zero and once congruent to 1 modulo 2. (Being congruent to r modulo m, means again having remainder r when divided by m in $\mathbb{N}$.)
Subcase 2.1: $n \equiv 0 \bmod 2$.
Then we have

$$a_{3k+1} = 1 + \frac{2(3k+1)^2+3}{(3k+1)^2+7(3k+1)} = \frac{27k^2+39k+13}{9k^2+27k+8} = \frac{27+\frac{39}{k}+\frac{13}{k^2}}{9+\frac{27}{k}+\frac{8}{k^2}} \to \frac{27}{9} = 3,$$

which leads to the accumulation point 3.
Subcase 2.2: $n \equiv 1 \bmod 2$.
Then we have

$$a_{3k+1} = -1 + \frac{2(3k+1)^2+3}{(3k+1)^2+7(3k+1)} = \frac{9k^2-15k-3}{9k^2+27k+8} = \frac{9-\frac{15}{k}-\frac{3}{k^2}}{9+\frac{27}{k}+\frac{8}{k^2}} \to \frac{9}{9} = 1,$$

which leads again to the accumulation point 1.
Case 3: $n \equiv 2 \bmod 3$.
Here we distinguish again the two subcases, where n is once congruent to zero and once congruent to 1 modulo 2.
Subcase 3.1: $n \equiv 0 \bmod 2$.
Then we have

$$\begin{aligned} a_{3k+2} &= \left(1 + \frac{2(3k+2)^2+3}{(3k+2)^2+7(3k+2)}\right)^2 = \left(\frac{27k^2+57k+29}{9k^2+33k+18}\right)^2 \\ &= \left(\frac{27+\frac{57}{k}+\frac{29}{k^2}}{9+\frac{33}{k}+\frac{18}{k^2}}\right)^2 \to \left(\frac{27}{9}\right)^2 = 3^2 = 9, \end{aligned}$$

which leads to the accumulation point 9.
Subcase 3.2: $n \equiv 1 \bmod 2$.
Then we have

$$\begin{aligned} a_{3k+2} &= \left(-1 + \frac{2(3k+2)^2+3}{(3k+2)^2+7(3k+2)}\right)^2 = \left(\frac{9k^2-9k-7}{9k^2+33k+18}\right)^2 \\ &= \left(\frac{9-\frac{9}{k}-\frac{7}{k^2}}{9+\frac{33}{k}+\frac{18}{k^2}}\right)^2 \to \left(\frac{9}{9}\right)^2 = 1, \end{aligned}$$

which again leads to the accumulation point 1.

Summing up all the cases we have found the accumulation points 1, 3 and 9. Thus we have $\liminf(a_n) = 1$ and $\limsup(a_n) = 9$. $\square$

71. Let (z_n) be a sequence in $\mathbb{C}$ satisyfying $|z_n| = 3$ and $\mathrm{Im}(z_n)$ is monotonous. Show that (z_n) has at least one and at most two accumulation points.
Hint: Use Bolzano-Weierstrass.

Solution:
Let $z_n = x_n + \mathrm{i}y_n$. Then we know that $9 = |z_n|^2 = x_n^2 + y_n^2$ and thus $|x_n|$ and $|y_n|$ both need to be smaller than or equal to 3. From this we know from Thm. 3.43 that (z_n) has got at least one accumulation point.
Now we know that (y_n) is bounded and monotonous. Thus it is convergent, say $y_n \to y$.
Furthermore we have that

$$x_n^2 = 9 - y_n^2 \to 9 - y^2$$

and thus any convergent subsequence of (x_n) can only converge either to $\sqrt{9 - y^2}$ or to $-\sqrt{9 - y^2}$. So, in total (z_n) can have at most two accumulation points, namely $\sqrt{9 - y^2} + \mathrm{i}y$ or $-\sqrt{9 - y^2} + \mathrm{i}y$. $\square$

72. Check the validity of the following assertions:

    (a) Every sequence $(a_n) \subseteq \mathbb{R}$ with $-10 < a_n < 3$ for all $n \in \mathbb{N}$ has an accumulation point.

    (b) Every sequence $(a_n) \subseteq \mathbb{R}$ with $a_n < 17$ for all $n \in \mathbb{N}$ has an accumulation point.

    Solution:

    (a) Since $a_n \in (-10, 3)$ for any $n \in \mathbb{N}$, the sequence is bounded and has hence an accumulation point, due to Thm. 3.43.

    (b) The sequence $(a_n) \subseteq \mathbb{R}$ is bounded from above but not necessarily from below. Consider the example $a_n = -n$, then $a_n < 17$ for all $n \in \mathbb{N}$, but (a_n) has no accumulation point, as it is divergent to $-\infty$.

$\square$

(Optional). The following exercises do not count for ticks and will not necessarily be discussed in the exercise class, but you still get the solutions.

A. Use the monotonicity principle to show that the sequence given by

$$a_0 = 1 \text{ and } a_{n+1} = \frac{6(1 + a_n)}{7 + a_n}$$

is convergent and determine the limit.

Solution:
We start be showing that $0 < a_n < 2$, where the first inequalitiy $a_n > 0$ is trivially true. We show the second inequality $a_n < 2$ by induction.
IB: $a_0 = 1 < 2$.
IH: $a_0, \dots, a_n < 2$.
IS: We give an indirect proof. Suppose $a_{n+1} = k \geq 2$. Then we have

$$k = \frac{6(1 + a_n)}{7 + a_n} \Leftrightarrow a_n = \frac{7k - 6}{6 - k} \geq \frac{8}{4} = 2,$$

which is a contradiction to $a_n < 2$.
Next we show, that (a_n) is monotonically increasing:

$$a_{n+1} - a_n = \frac{6(1 + a_n)}{7 + a_n} - a_n = \frac{6 - a_n - a_n^2}{7 + a_n} > \frac{6 - 2 - 4}{9} = 0.$$

Thus (a_n) is convergent, say $a_n \to a$ and hence we have

$$a = \frac{6(1 + a)}{7 + a} \Leftrightarrow a = -3 \text{ or } a = 2.$$

As $a_n > 0$, we have $a_n \to 2$. □

B. Use the monotonicity principle to show that

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

converges for any real x.
Remark: In fact, $a_n \to \mathrm{e}^x$.

Solution:
Let $x \in \mathbb{R}$. As n tends to infinity, we know that from some n_0 onwards we have $n > |x|$. We consider only such n and thus know that all our a_n are positive.
Next we prove that a_n is monotonically increasing for those n, using the inequality of arithmetic and geometric means:

$$\sqrt[n+1]{a_n} = \sqrt[n+1]{\left(1 + \frac{x}{n}\right)^n \cdot 1} \leq \frac{n\left(1 + \frac{x}{n}\right) + 1}{n+1} = 1 + \frac{x}{n+1},$$

from which we can conlude that

$$a_n \leq \left(1 + \frac{x}{n+1}\right)^{n+1} = a_{n+1}.$$

Now we distinguish two cases:
Case 1: $x \leq 0$.
From our assumption, $n > |x|$, we get $n > -x$ and thus $\frac{x}{n} > -1$. This we can sum up to $0 < 1 + \frac{x}{n} < 1$ and consequently $0 \leq \left(1 + \frac{x}{n}\right)^n \leq 1$. So $\left(1 + \frac{x}{n}\right)^n$ is bounded and monotonically increasing and thus converging to a limit different from zero.
Case 2: $x > 0$.
We know from Case 1 that $b_n = \left(1 - \frac{x}{n}\right)^n$ converges. We consider

$$a_n b_n = \left(1 + \frac{x}{n}\right)^n \left(1 - \frac{x}{n}\right)^n = \left(1 - \frac{x^2}{n^2}\right)^n.$$

From the Bernoulli inequality we get

$$1 \geq a_n b_n \geq 1 - n\frac{x^2}{n^2} = 1 - \frac{x^2}{n} \to 1$$

and thus by the sandwich theorem $a_n b_n \to 1$. Now we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n b_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n b_n}{\lim\limits_{n\to\infty} b_n} = \frac{1}{\lim\limits_{n\to\infty} b_n},$$

which exists, as $\lim\limits_{n\to\infty} b_n \neq 0$. $\square$

C. Compute the limit of

$$a_n = \sqrt{\frac{6n^4 + 3n^2 + 2}{7n^4 + 12n^3 + 6}}.$$

Solution:
As all the a_n are positive we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{6 + \frac{3}{n^2} + \frac{2}{n^4}}{7 + \frac{12}{n} + \frac{6}{n^4}}} = \sqrt{\frac{6}{7}}.$$

□

73. Determine whether the following series are convergent, and quote any theorems you use to do so:

    (a)
    $$\sum_{n=1}^{\infty}(-1)^n \tan(1/n).$$

    (b)
    $$\sum_{n=1}^{\infty}\frac{n+5}{2n^2-1}.$$

    (c)
    $$\sum_{n=1}^{\infty}\frac{\sum_{k=0}^{m} a_k n^k}{e^n}$$
    for some $m \in \mathbb{N}_0$ and some coefficients $a_1, \dots, a_m \in \mathbb{C}$.

    (d)
    $$\sum_{n=2}^{\infty}\log\left(1+\frac{n^2-4n-5}{n^4+2}\right).$$
    Hint: You may use that $e^x \geq 1+x$ *for all* $x \geq 0$. *Remember that* $\log(a) \leq \log(b)$ *if and only if* $a \leq b$.

Solution:

(a) As $(\tan(1/n))_{n\in\mathbb{N}}$ is a monotonic null sequence, the sum is convergent by Leibniz Criterion (Theorem 3.93).

(b) Since
$$\frac{n+5}{2n^2-1} \geq \frac{n}{2n^2} = \frac{1}{2n}$$
for each $n \in \mathbb{N}$, the series is divergent by the Comparison Test (Theorem 3.80) since the harmonic series is divergent.

(c) Fix $p(n) = \sum_{k=1}^{m} a_k n^k$. Both $p(n)$ and $p(n+1)$ are polynomials with the same degree and leading coefficient, hence $p(n+1)/p(n) \to 1$ as $n \to \infty$. As
$$\frac{p(n+1)}{e^{n+1}}\frac{e^n}{p(n)} = \frac{1}{e}\frac{p(n+1)}{p(n)} \to \frac{1}{e} < 1$$
as $n \to \infty$, the series is absolutely convergent by the Ratio Test (Theorem 3.87).

(d) As $e^x \geq 1+x$ for all $x \geq 0$, we have that $\log(1+x) \leq x$ for all $x \geq 0$. Hence
$$\log\left(1+\frac{n^2-4n-5}{n^4+2}\right) \leq \frac{n^2-4n-5}{n^4+2} \leq \frac{n^2}{n^4} = \frac{1}{n^2}$$
for each $n \geq 5$. Since $\sum_{n=1}^{\infty} 1/n^2$ is absolutely convergent (Lemma 3.92), the series in question is also absolutely convergent by the Comparison Test (Theorem 3.80).

$\square$

74. Find an example (if possible) of each of the following statements:

    (a) A convergent series that is not absolutely convergent.

    (b) A convergent series with non-negative terms that is not absolutely convergent.

    (c) A convergent series $\sum_{n=1}^{\infty} x_n + iy_n$ with $x_n, y_n \in \mathbb{R}$ where $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ diverge.

    (d) A convergent series $\sum_{n=1}^{\infty} x_n + iy_n$ with $x_n, y_n \in \mathbb{C}$ where $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ diverge.

Solution:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

(b) Not possible; any series with non-negative terms is convergent if and only if it is absolutely convergent.

(c) Not possible; a complex series $\sum_{n\in\mathbb{N}} a_n + ib_n$ with all $a_n, b_n \in \mathbb{R}$ is defined to be convergent if and only if the series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ are both convergent.

(d) Choose $x_n = 1$ and $y_n = i$ for all $n \in \mathbb{N}$.

□

75. The *alternating harmonic series* is the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \log(2).$$

(Here log is the natural logarithm.) We shall, for this question, call the following Series A:

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) + \ldots.$$

(a) Write Series A in its closed form (you do not need to sum the insides of the brackets).

(b) How is Series A related to the alternating harmonic series?

(c) Both the alternating harmonic series and Series A are *conditionally convergent*; they are convergent but not absolutely convergent. What does Series A converge to?

First hint: Try rearranging each term in the closed form so that it looks similar to the alternating harmonic series.

Second hint: The series does not converge to $\log(2)$.

Solution:

(a) The closed form of Series A is

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2(2n-1)} - \frac{1}{4n}.$$

(b) Series A can be formed from the alternating harmonic series by rearranging its terms.

(c) We note that

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2(2n-1)} - \frac{1}{4n} = \sum_{n=1}^{\infty} \frac{1}{2(2n-1)} - \frac{1}{4n} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n}\right),$$

hence Series A converges to $\log(2)/2$. (For more information regarding this property, see the *Riemann series theorem.*)

$\square$

76. **Please hand in this exercise as an extra pdf and name the file something like [Name]-exercise-[number].pdf. Please write especially nice, you will get written feedback on this exercise, if you hand it in.**

Determine the limits of the following convergent series:

(a)
$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{2^n}.$$

(b)
$$\sum_{n=2}^{\infty} \frac{2}{n^2-1}.$$

Hint: You will want to use telescoping at some point.

Solution:

(a) We first note that
$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} - \frac{1}{2^{2n+1}}.$$
As
$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}, \qquad \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{2}{3},$$
our series converges to $2/3$ by Theorem 3.68.

(b) Since $n^2-1 = (n-1)(n+1)$, first observe the equation
$$\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{A(n+1)+B(n-1)}{n^2-1}.$$
The only solution is $A = 1$ and $B = -1$, hence
$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}.$$
By telescoping we see that for all $N \geq 3$ we have
$$\sum_{n=2}^{N} \frac{1}{n-1} - \frac{1}{n+1} = \frac{1}{1} + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1},$$
hence
$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{N\to\infty} \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1} = \frac{3}{2}.$$

□

77. Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. For every $z \in \mathbb{R}$, define the series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n.$$

It is immediate that $f(0)$ converges to 0 for any choice of sequence. Define for which values of $z \in \mathbb{R}\backslash\{0\}$ the series $f(z)$ is convergent for the following sequences.

Hint: Use the Ratio Test (Theorem 3.87) and then check the boundary cases.

(a) $a_n = 2^n$ for all $n \geq 0$.
(b) $a_n = n^2$ for all $n \geq 0$.
(c) $a_n = n!$ for all $n \geq 0$ (remember: $0! = 1$).
(d) $a_n = 1/(n!)^2$ for all $n \geq 0$.

Solution:

(a) For any $z \in \mathbb{R} \setminus \{0\}$,

$$\frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \frac{2^{n+1}}{2^n}|z| = 2|z|,$$

hence the Ratio Test says that $f(z)$ is absolutely convergent when $0 < |z| < 1/2$ and divergent when $|z| > 1/2$. We now have two case to check. When $z = 1/2$ we have $\sum_{n=1}^{\infty} 1$ which is divergent. When $z = -1/2$ we have $\sum_{n=1}^{\infty}(-1)^n$ which is also divergent. Hence $f(z)$ is convergent for all $z \in (-1/2, 1/2)$ and divergent elsewhere.

(b) For any $z \in \mathbb{R} \setminus \{0\}$,

$$\frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \frac{n^2+2n+1}{n^2}|z| = \frac{1+2/n+1/n^2}{1}|z| \to |z|,$$

hence the Ratio Test says that $f(z)$ is absolutely convergent when $0 < |z| < 1$ and divergent when $|z| > 1$. Both $f(1) = \sum_{n=0}^{\infty} n^2$ and $f(-1) = \sum_{n=0}^{\infty}(-1)^n n^2$ are divergent as their terms do not form null sequences. Hence $f(z)$ is convergent for all $z \in (-1, 1)$ and divergent elsewhere.

(c) For any $z \in \mathbb{R} \setminus \{0\}$,

$$\frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \frac{(n+1)!}{n!}|z| = (n+1)|z| \to \infty,$$

and so the Ratio Test says that $f(z)$ is divergent for all $z \in \mathbb{R} \setminus \{0\}$. Hence $f(z)$ is only convergent when $z = 0$.

(d) For any $z \in \mathbb{R} \setminus \{0\}$,

$$\frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \frac{n!n!}{(n+1)!(n+1)!}|z| = \frac{|z|}{(n+1)^2} \to 0,$$

and so the Ratio Test says that $f(z)$ is absolutely convergent for all $z \in \mathbb{R} \setminus \{0\}$. Hence $f(z)$ is convergent for all $z \in \mathbb{R}$.

$\square$

78. Given a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, we say that the infinite product $\prod_{i=1}^{\infty} a_n$ *converges* if the limit of the sequence $(\prod_{n=1}^{N} a_n)_{N\in\mathbb{N}}$ converges to a positive real number; otherwise we say that it *diverges*. If the sequence $(\prod_{n=1}^{N} a_n)_{N\in\mathbb{N}}$ converges to 0, we say that $\prod_{i=1}^{\infty} a_n$ *diverges to 0*. If an infinite product converges, we say that

$$\prod_{n=1}^{\infty} a_n = \lim_{N\to\infty} \prod_{n=1}^{N} a_n.$$

(a) A sequence of positive real numbers $(x_n)_{n\in\mathbb{N}}$ converges to a positive real number x if and only if $(\log(x_n))_{n\in\mathbb{N}}$ converges to $\log(x)$. Using this fact, show that $\prod_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \log(a_n)$ converges.

(b) If $\prod_{n=1}^{\infty} a_n$ converges, what can we say about the sequence $(a_n)_{n\in\mathbb{N}}$?

(c) Give an example of an infinite product that diverges to 0 where the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a positive number.

(d) Give an example of an infinite product that diverges to 0 where the sequence $(a_n)_{n\in\mathbb{N}}$ does not converge.

Solution:

(a)

$$\begin{aligned}
\prod_{n=1}^{\infty} a_n \text{ converges} \quad &\Leftrightarrow \quad \left(\prod_{n=1}^{N} a_n\right)_{N\in\mathbb{N}} \quad \text{converges} \\
&\Leftrightarrow \quad \left(\log\left(\prod_{n=1}^{N} a_n\right)\right)_{N\in\mathbb{N}} \quad \text{converges} \\
&\Leftrightarrow \quad \left(\sum_{n=1}^{N} \log(a_n)\right)_{N\in\mathbb{N}} \quad \text{converges} \\
&\Leftrightarrow \quad \sum_{n=1}^{\infty} \log(a_n) \text{ converges.}
\end{aligned}$$

(b) By part (a), we have that $\log(a_n) \to 0$ as $n \to \infty$. Hence by using the fact stated in part (a), we have that $a_n \to 1$ as $n \to \infty$.

(c) Example: Set $a_n = 1/2$ for each $n \in \mathbb{N}$.

(d) Example: Set $a_{2n-1} = 1$ and $a_{2n} = 1/2$ for each $n \in \mathbb{N}$.

□