7. For the given functions (expressions)

   (a) $f(x) = (2 - x)^4$,
   (b) $g(x) = \frac{1}{1+\sqrt{x}}$,
   (c) $h(x) = 3 - |x|$,
   (d) $u(x) = \frac{1}{1-\sqrt{x}}$,

   find the largest possible domain in $\mathbb{R}$ and the corresponding range.

   Solution:

   (a) $f : \mathbb{R} \to \mathbb{R}_0^+$,
   (b) $g : \mathbb{R}_0^+ \to (0, 1]$,
   (c) $h : \mathbb{R} \to (-\infty, 3]$,
   (d) $u : \mathbb{R}_0^+ \backslash \{1\} \to (-\infty, 0) \cup [1, \infty)$.

   Remark. It is fine, if you found it difficult to come up with the range of u. This is one of the reasons that we may only specify the codomain and not the range for a real function in the definition. We will learn theorems and techniques (continuity, extrema, etc.) to answer this type of questions (range) for arbitrary univariate elementary functions soon.

   Finding the largest possible domain is easier even for a composite function. For u, we have to consider two conditions: $x \geq 0$, because of the square root and the denominator should not vanish: $1 - \sqrt{x} \neq 0$, so we have to exclude 1 from the domain, i.e., $D_u = \mathbb{R}_0^+ \backslash \{1\}$. $\square$

8. Let $f\colon D \to C$, defined as $x \mapsto x^2$, with $D, C \subset \mathbb{R}$.

   (a) Give an example for $D, C \subset \mathbb{R}$, such that f is injective. Find another example for $D, C \subset \mathbb{R}$, such that f is not injective.

   (b) Give an examples for $D, C \subset \mathbb{R}$, such that f is surjective. Find another example for $D, C \subset \mathbb{R}$, such that f is not surjective.

   (c) Consider $D = \mathbb{R}$ and $C = \mathbb{R}$.

      i. For $S_1 := \{-1, 0, 1, 2, \pi\}$, $S_2 := \{x \in \mathbb{R} : \; -3 \leq x \leq 4\}$ and $S_3 := \{x \in \mathbb{R} : \; x < 0 \vee x > 16\}$, determine the image of S_i under f, i.e. compute $f(S_i)$, for $i = 1, 2, 3$.

      ii. For $T_1 := \{0, 1, 2, \pi\}$, $T_2 := \{x \in \mathbb{R} : \; x \geq 0\}$ and $T_3 := \{x \in \mathbb{R} : \; x < 0\}$, determine the pre-image of T_i under f, i.e. compute $f^{-1}(T_i)$, for $i = 1, 2, 3$.

Solution:

(a) Let $D = \mathbb{R}_0^+ := \{x \in \mathbb{R} : \; x \geq 0\}$ and $C = \mathbb{R}$. Then f is injective: Assume for $x_1, x_2 \in D$ we have that $f(x_1) = f(x_2)$. This means $x_1^2 = x_2^2$. But then $x_1 = x_2$ (we only allow nonnegative values).

If $D = \{-1, 1\}$ and $C = \mathbb{R}$, then $-1 \neq 1$, but $f(-1) = f(1) = 1$.

(b) Let $D = \mathbb{R}$ and $C = \mathbb{R}_0^+$. Let $y \in C$ be arbitrary but fixed. Then $x := \sqrt{y}$ is well defined (as $y \geq 0$) and $f(x) = (\sqrt{y})^2 = y$.

If $D = \mathbb{R}$ and $C = \mathbb{R}$, then there is no $x \in D$, s.t. $f(x) = x^2 = -1$.

(c) i. We have

$$
\begin{aligned}
f(S_1) &= \{f(x) \in C : \; x \in S_1\} = \{(-1)^2, 0^2, 1^2, 2^2, \pi^2\} = \{0, 1, 4, \pi^2\}, \\
f(S_2) &= \{f(x) \in C : \; x \in S_2\} = \{y \in C : \; 0 \leq y \leq 16\} = [0, 16], \\
f(S_3) &= \{f(x) \in C : \; x \in S_3\} = \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}.
\end{aligned}
$$

ii. We have

$$
\begin{aligned}
f^{-1}(T_1) &= \{x \in D : \; f(x) \in T_1\} = \{-\sqrt{\pi}, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \sqrt{\pi}\}, \\
f^{-1}(T_2) &= \{x \in D : \; f(x) \in T_2\} = \mathbb{R}, \\
f^{-1}(T_3) &= \{x \in D : \; f(x) \in T_3\} = \emptyset.
\end{aligned}
$$

□

9. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$, with $f(x) = x + 3$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x - 5$. Determine $(g \circ f)(x)$ and $(f \circ g)(x)$.

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$, with $f(x) = 3x$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = -5x$. Determine $(g \circ f)(x)$ and $(f \circ g)(x)$.

   (c) Is function composition commutative, i.e. does $(g \circ f)(x) = (f \circ g)(x)$ hold in general? Justify your answer!

Solution:

(a) We have

$$(g \circ f)(x) = g(f(x)) = g(x+3) = (x+3) - 5 = x - 2,$$

and

$$(f \circ g)(x) = f(g(x)) = f(x-5) = (x-5) + 3 = x - 2.$$

(b) We have

$$(g \circ f)(x) = g(f(x)) = g(3x) = -5(3x) = -15x,$$

and

$$(f \circ g)(x) = f(g(x)) = f(-5x) = 3(-5x) = -15x.$$

(c) No, let for example $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto 3x$ and $g\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x - 5$. Then

$$(g \circ f)(x) = g(f(x)) = g(3x) = (3x) - 5 = 3x - 5,$$

but

$$(f \circ g)(x) = f(g(x)) = f(x-5) = 3(x-5) = 3x - 15.$$

□

10. Investigate if the following functions are injective, surjective or bijective. Sketch the graph of each function and calculate its inverse if possible.

    (a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^4$,

    (b) $g : \mathbb{R}_0^+ \to \mathbb{R}_0^-$ with $g(x) = -x^2$,

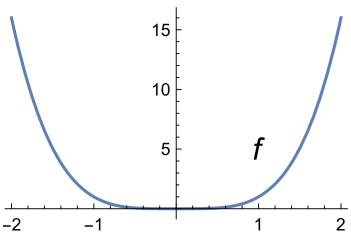
    (c) $h : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ with $h(x) = -x^{-1}$.

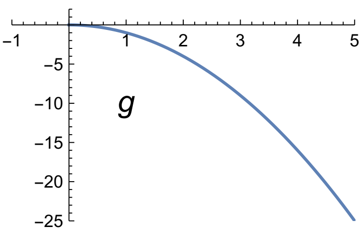
    Solution:

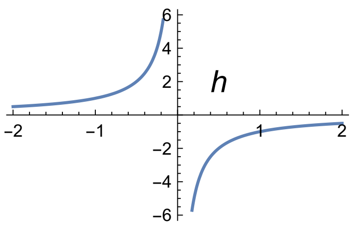
    (a) f is neither injective nor surjective. We see, that for $x_1 = -1 \neq x_2 = 1 (\in \mathbb{R})$ we have $f(-1) = f(1) = 1$, so different elements have the same image, f is not injective. The element $y = -1 \in \mathbb{R}$ has no pre-image, so f is not surjective. Thus f has no inverse.

    (b) g is both injective and surjective. We start again with injectivity. Assume we have $x_1, x_2 \in \mathbb{R}_0^+$ with $g(x_1) = g(x_2)$. Since we consider only nonnegative numbers, from $-x_1^2 = -x_2^2$ follows that $x_1 = x_2$. We claim that an arbitrary $y \in \mathbb{R}_0^-$ has a unique pre-image $\sqrt{-y}$ (note that $-y \geq 0$): $g(\sqrt{-y}) = -(\sqrt{-y})^2 = y$. Thus g is bijective and invertible with the inverse function
    $$g^{-1} : \mathbb{R}_0^- \to \mathbb{R}_0^+ \quad \text{with } g^{-1}(x) = \sqrt{-x}.$$

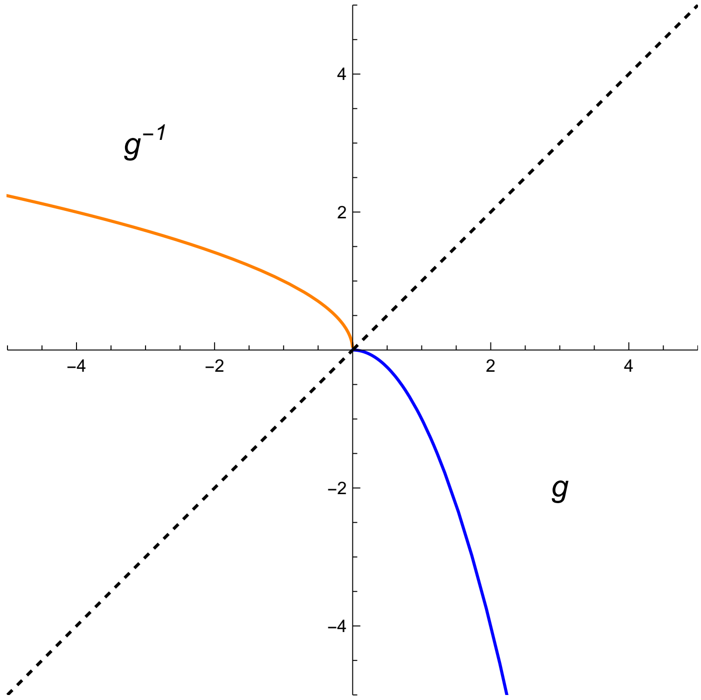
    (c) h is bijective and self-inverse, that is, $h^{-1} = h$. We just check the latter in details:
    $$h(h^{-1}(y)) = -\frac{1}{-\frac{1}{y}} = y.$$







We give the plot for the graph of the inverse function g^{-1} as well.



$\square$

11. (a) Show that the divisibility relation over $\mathbb{N}$, that is, $| \subset \mathbb{N} \times \mathbb{N}$, is a partial order. (Recall that the divisibility relation $|$ is defined in the lecture notes as follows: Let m and n be integers. We say that m divides n, and write $m|n$, if there exists $k \in \mathbb{N}$ such that $mk = n$.)

    (b) Restrict the relation above to the finite set $\{1, 2, 3, 4, 5, 6\}$ and give the pairs of elements which are in relation.

Solution:

(a) A relation ρ (on an arbitrary the set A) is a partial order iff $\rho \subset A^2$ is reflexive, antisymmetric and transitive (see Def. 1.18).

Reflexivity: Let $a \in \mathbb{N}$ arbitrary but fixed. $a|a$, iff there exists $k \in \mathbb{N}$ such that $a \cdot k = a$. This holds, because $a \cdot 1 = a$.

Antisymmetric: Let $a, b \in \mathbb{N}$. $a|b \wedge b|a$ iff there exist $k, l \in N$ such that $a \cdot k = b$ and $b \cdot l = a$. But then $b = a \cdot k = (b \cdot l) \cdot k = b \cdot (l \cdot k)$. This holds iff $l \cdot k = 1$ iff $l = 1 \wedge k = 1$. Which means that $a = b$.

Transitivity: Assume that $a|b \wedge b|c$. Then there exist $k, l \in \mathbb{N}$ such that $b = a \cdot k \wedge c = b \cdot l$. By combining the two equations we get that $c = a \cdot (k \cdot l)$, that is $a|c$ (The product $k \cdot l$ is again a natural number).

(b) The set of pairs which are in relation is: $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\} \subset \{1,2,3,4,5,6\}^2$.

$\square$

12. Let $f : M \to N$ be a function and $C, D \subset N$. By

$$f^{-1}(C) := \{x : f(x) \in C\} \subset M$$

we denote the preimage of C under f (Definition 1.8). Is the following statement true?

$$f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D).$$

If true, provide a proof, and if false, a counterxample!

Solution:
True. Observe that $x \in f^{-1}(C) \cap f^{-1}(D)$ if and only if $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. This is true if and only if $f(x) \in C$ and $f(x) \in D$, i.e. $f(x) \in C \cap D$. This is equivalent to $x \in f^{-1}(C \cap D)$. $\square$