A Semi-incremental Scheme for Fatigue Damage Computations

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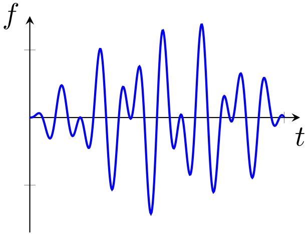
Fatigue damage

- Fluctuating loads



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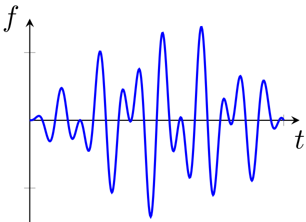

- Material degradation

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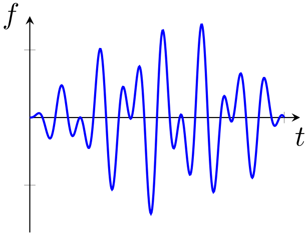

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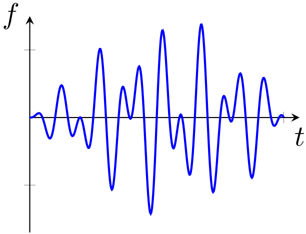

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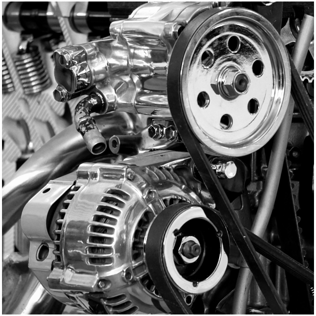

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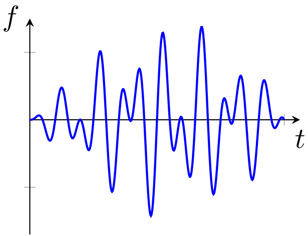

- Material degradation
- Continuum damage model
- Large number of cycles
- Macro crack initiation
- Computationally expensive

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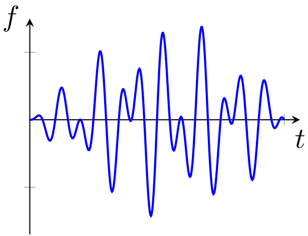

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Model order reduction (MOR) techniques

Outline

1. State of art
2. Reduced order model for cyclic loading
3. Numerical examples
4. Conclusions and future research

Outline

ROM for nonlinear problems

- Short-term plastic/damage computations
  - Proper orthogonal decomposition (POD)
    [Kerfriden et al, 2011-2012; Ryckelynck, 2005-2011].
  - Proper generalised decomposition (PGD)
    [Vitse, 2016 ; Bhattacharyya et al, 2017].

- Long-term fatigue computations
  - (Modified) jump cycle approach
    [Desmorat 2005; Bhattacharyya et al, 2018]
  - Temporal homogenisation
    [Fish and Yu, 2002; Devulder et al, 2010]
  - Space-time finite element method
    [Bhamare, 2014; Fritzen et al, 2018]

Outline

Mechanical problem

Admissibility equations (AE)

- Static admissibility

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \quad \text{in } \Omega \times \mathcal{I}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{\boldsymbol{t}} \quad \text{on } \partial\Omega_{\mathrm{N}} \times \mathcal{I}$$

- Kinematic admissibility

$$\boldsymbol{\varepsilon} = \nabla^{\mathrm{s}}\boldsymbol{u} \quad \text{in } \Omega \times \mathcal{I}$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } \partial\Omega_{\mathrm{D}} \times \mathcal{I}$$

Nonlinear material model (CM)

- State equations

$$\boldsymbol{\sigma} = f(\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \boldsymbol{q}))$$

$$\boldsymbol{Q} = g(\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \boldsymbol{q}))$$

- Evolution equations

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \hat{f}(\phi(\boldsymbol{\sigma}, \boldsymbol{Q}))$$

$$\dot{\boldsymbol{q}} = \hat{g}(\phi(\boldsymbol{\sigma}, \boldsymbol{Q}))$$

LATIN solution algorithm over a time interval

- Start with an elastic initialisation
- Local stage, given initial conditions

  Solve the state and evolution equations (CM) to get $\hat{\square}$
- Global stage

  Solve the admissibility equations (AE) to get $\square_{i+1}$
- Data flow between these stages

$$
\begin{aligned}
(\boldsymbol{\sigma}_{i+1} - \hat{\boldsymbol{\sigma}}) - \mathbb{H}^{-}(\boldsymbol{\varepsilon}_{i+1} - \hat{\boldsymbol{\varepsilon}}) &= \mathbf{0} \\
(\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_{i}) + \mathbb{H}^{+}(\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_{i}) &= \mathbf{0}
\end{aligned}
$$

- Iterate until convergence with an energy error indicator

$$
\xi = \frac{\left\| s_{i+1} - \hat{s} \right\|}{\left\| \left(s_{i+1} + \hat{s} \right) / 2 \right\|}, \quad \|\square\|^2 = \frac{1}{2T} \int_{\Omega \times \mathcal{I}} \left(\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} + \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \right) \, \mathrm{d}\Omega \, \mathrm{d}t.
$$

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$$

Proper Generalised Decomposition

- Low-rank approximation $$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{j=1}^{N} \boldsymbol{v}_j(\boldsymbol{x}) \circ \boldsymbol{\lambda}_j(t)$$
- Enrichment step $$\Delta\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x}) \circ \boldsymbol{\lambda}(t)$$
- Update step $$\Delta\boldsymbol{u}(\boldsymbol{x},t) = \sum_{j=1}^{\mu} \underbrace{\boldsymbol{v}_j(\boldsymbol{x})}_{known} \circ \Delta\boldsymbol{\lambda}_j(t)$$

Choosing the search directions as

$$\mathbb{H}^+ = 0, \qquad \mathbb{H}^- = \alpha\,\mathbb{C} \qquad \alpha \in \,]0,1]$$

Algorithmic point of view

- Enrichment

$$\gamma \, \underline{\underline{K}} \, \underline{v} = \underline{f} \qquad \gamma \in \mathbb{R} \qquad \underline{\underline{K}} \in \mathbb{R}^{n \times n} \qquad \underline{v}, \underline{f} \in \mathbb{R}^{n} \qquad \text{with B.C.}$$

$$a \, \underline{\lambda} = \underline{b} \qquad a \in \mathbb{R} \qquad \underline{\lambda}, \underline{b} \in \mathbb{R}^{n_t} \qquad \text{with I.C.}$$

- Update

$$\underline{\underline{\tilde{A}}} \left[\Delta\underline{\lambda}_1, \cdots, \Delta\underline{\lambda}_\mu\right]^{\mathsf{T}} = \underline{\underline{\tilde{B}}} \qquad \underline{\underline{\tilde{A}}} \in \mathbb{R}^{\mu \times \mu} \qquad \underline{\underline{\tilde{B}}} \in \mathbb{R}^{\mu \times n_t}$$

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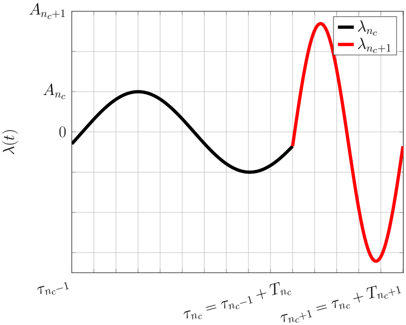
Semi-incremental scheme

- The temporal domain is divided into intervals/cycles
- Temporal modes are defined over $\tilde{t} \in [0, 1]$

$$\tilde{\lambda}_{n_c+1}(\tilde{t}) = m\,\lambda_{n_c}(\tilde{t}) + g\,\tilde{t} + h \quad \text{with}$$
$$\tilde{\lambda}_{n_c+1}(0) = \lambda_{n_c}(1), \quad \tilde{\lambda}_{n_c+1}(1) = \tilde{\lambda}_{n_c+1}(0).$$



SVD Compression of PGD

- Large number of modes
- Slow temporal update step

SVD Compression of PGD

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- Slow temporal update step
- Randomised SVD (RSVD) orthonormalisation step

$$\underline{\underline{\tilde{U}}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}^{\mathsf{T}} \in \mathbb{R}^{n \times n_t}$$

$$\underline{\underline{V}} = \underline{\underline{Q}}_v\,\underline{\underline{R}}_v, \qquad \underline{\underline{\Lambda}} = \underline{\underline{Q}}_\lambda\,\underline{\underline{R}}_\lambda \qquad \sim \mathcal{O}(\mu^2\,(n + n_t))$$

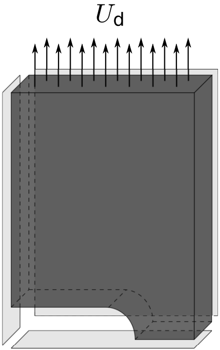
$$\underline{\underline{T}} = \underline{\underline{R}}_v\,\underline{\underline{R}}_\lambda^{\mathsf{T}} \in \mathbb{R}^{\mu \times \mu}, \quad \underline{\underline{T}} \approx \underline{\underline{\tilde{V}}}\,\underline{\underline{\tilde{S}}}\,\underline{\underline{\tilde{\Lambda}}}^{\mathsf{T}}$$

$$\underline{\underline{\tilde{U}}} \approx \underline{\underline{Q}}_v\,\underline{\underline{\tilde{V}}}\,\underline{\underline{\tilde{S}}}\,\underline{\underline{\tilde{\Lambda}}}^{\mathsf{T}}\underline{\underline{Q}}_\lambda^{\mathsf{T}}$$

Outline

Numerical results

- A plate subjected to cyclic loading (Cr-Mo steel at $580^\circ\mathrm{C}$)



Viscoplastic viscodamage material model

- State equations

$$\boldsymbol{\sigma} = (1 - D)\, \mathbb{C} : \boldsymbol{\varepsilon}^{\mathrm{e}}$$

$$\boldsymbol{\beta} = \frac{2}{3}\, c\, \boldsymbol{\alpha}$$

$$R = R_\infty\, (1 - \mathrm{e}^{-b\, r})$$

$$Y = \frac{1}{2}\, \boldsymbol{\varepsilon}^{\mathrm{e}} : \mathbb{C} : \boldsymbol{\varepsilon}^{\mathrm{e}}$$

- Evolution equations

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\, \boldsymbol{N}$$

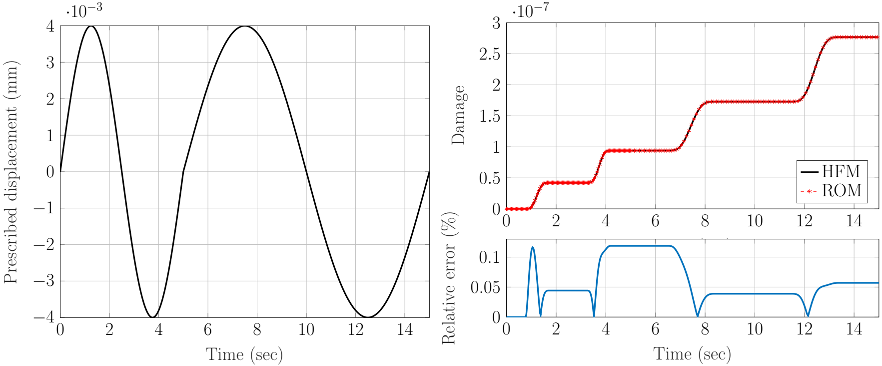
$$\dot{\boldsymbol{\alpha}} = \dot{\lambda}\, \left(\tilde{\boldsymbol{N}} - \frac{3\, a}{2\, c}\, \boldsymbol{\beta} \right)$$

$$\dot{r} = \dot{\lambda}$$

$$\dot{D} = \frac{\dot{\lambda}}{1 - D} \left(\frac{Y}{S} \right)^{s} \text{if } (r > p_{\mathrm{D}})$$

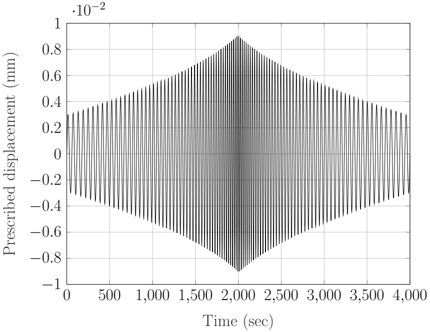
Model verification

- With respect to a classical Newton-Raphson scheme



Variable amplitude and frequency loading

- Amplitudes: $[30, 90] \cdot 10^{-4}\,\mathrm{mm}$ Time periods: $[20, 60]\,\mathrm{sec}$



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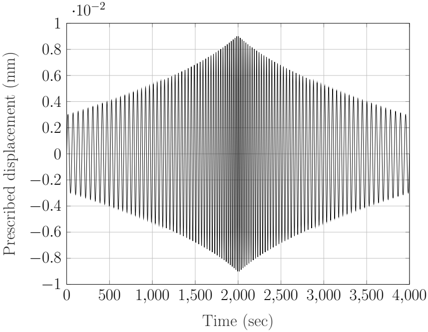
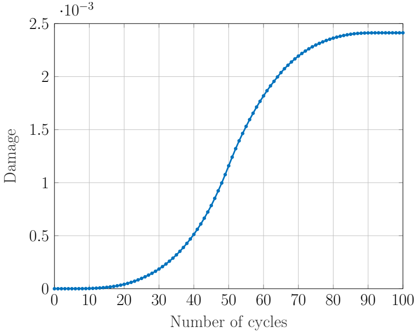
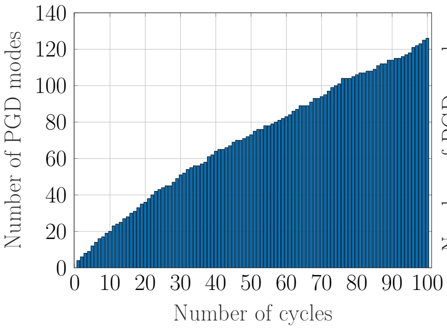




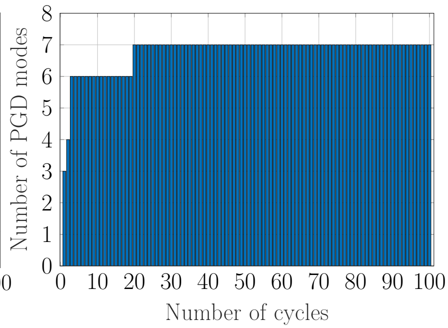
Number of modes is eight

Random amplitude loading

- 100 cycles with different orthonormalisation schemes



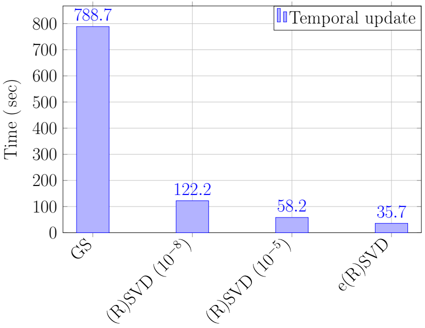
Gram-Schmidt



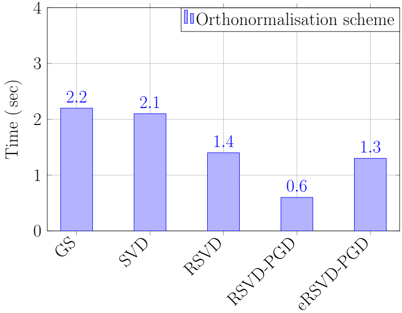
SVD scheme

Random amplitude loading

- Required time



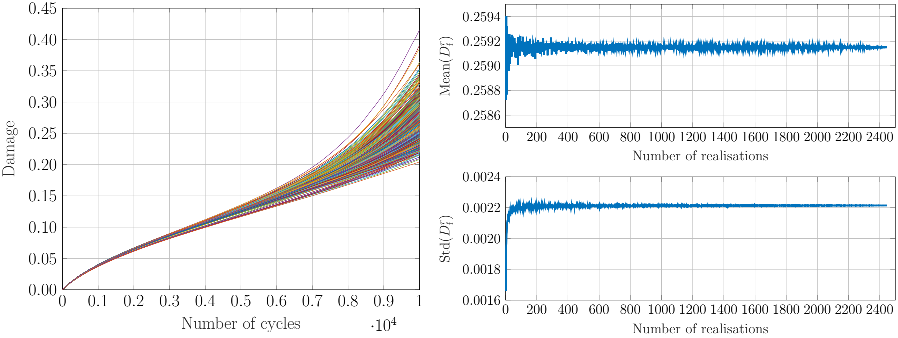
Temporal update



Orthonormalisation step

Random amplitude loading

- 10000 cycles, uniform distribution in $[53, 56] \cdot 10^{-2}\,\mathrm{mm}$



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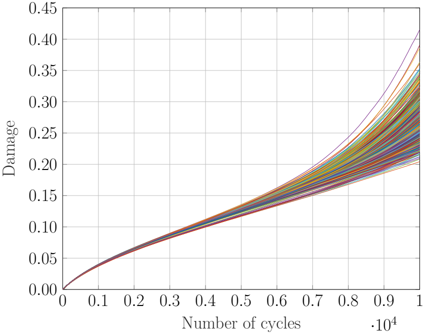

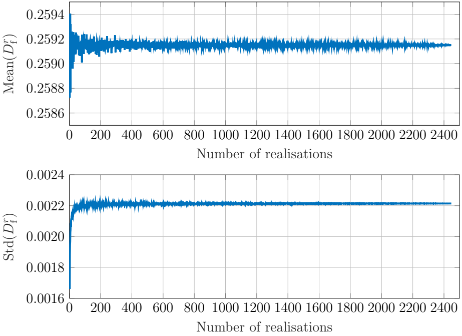

- $D_c = 0.3 \quad \rightarrow \quad P_f = 5.4\%$
- Requirements: $[15 - 35]\,\mathrm{min}$ and $[1 - 1.5]\,\mathrm{GB}$

Random amplitude loading

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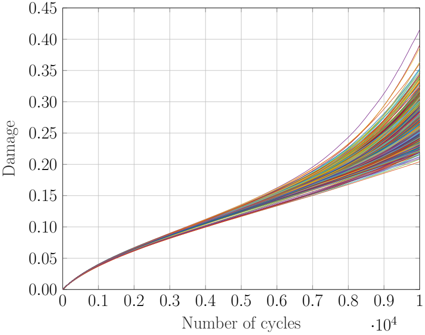

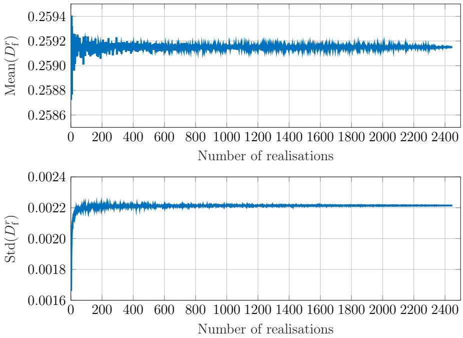

Time-saving factors $50 \sim 100$

Outline

Conclusion and future research

- Efficient cycle by cycle simulation for damage problems
- Open-source: `gitlab.com/shadialameddin/romfem`

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- The computation of the local stage
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Future development

- PGD of all QoI in the local stage
- Machine learning to approximate the constitutive manifold

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Thank you for your attention

The global stage

- Weak form at iteration $i+1$

$$\int\limits_{\Omega\times\mathcal{I}} \boldsymbol{\sigma}_{i+1} : \boldsymbol{\varepsilon}(u^*)\ \mathrm{d}\Omega\,\mathrm{d}t = \int\limits_{\Omega\times\mathcal{I}} \boldsymbol{b}\cdot u^*\,\mathrm{d}\Omega\,\mathrm{d}t + \int\limits_{[0,T]\times\partial\Omega_N} \bar{\boldsymbol{t}}\cdot u^*\,\mathrm{d}S\,\mathrm{d}t, \quad \forall u^* \in \mathcal{U}_0$$

- Correction $\Delta\bullet_{i+1} = \bullet_{i+1} - \bullet_i$

$$\Delta\boldsymbol{\sigma}_{i+1} - \mathbb{H}^{-}\Delta\boldsymbol{\varepsilon}_{i+1} - \hat{f} = \mathbf{0}, \quad \hat{f} = \underbrace{(\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_i) - \mathbb{H}^{-}(\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_i)}_{known}$$

$$\int\limits_{\Omega\times\mathcal{I}} \Delta\boldsymbol{\sigma}_{i+1} : \boldsymbol{\varepsilon}(\Delta u^*)\ \mathrm{d}\Omega\,\mathrm{d}t = 0$$

Proper Generalised Decomposition

- Low-rank approximation of the solution

$$u(\boldsymbol{x}, t) = \sum_{i=1}^{N} \lambda_i(t)\; v_i(\boldsymbol{x})$$

- Enriching with one mode

$$\Delta u = \lambda(t)\; v(\boldsymbol{x}) \qquad \Delta u^* = \lambda^*\; v + \lambda\; v^*$$

- Updating (μ) previously generated time modes

$$\Delta u = \sum_{i=1}^{\mu} \Delta\lambda_i(t)\; \underbrace{v_i(\boldsymbol{x})}_{known}$$

Static admissibility

- Static admissibility and global search direction

$$\int_{\Omega\times\mathcal{I}} \mathbb{H}^{-}\Delta\boldsymbol{\varepsilon}_{i+1} : \boldsymbol{\varepsilon}(\Delta u^{*})\ \mathrm{d}\Omega\,\mathrm{d}t = -\int_{\Omega\times\mathcal{I}} \hat{f} : \boldsymbol{\varepsilon}(\Delta u^{*})\ \mathrm{d}\Omega\,\mathrm{d}t$$

- Space problem: $\langle\bullet\rangle = \int_{[0,T]} \bullet\ \mathrm{d}t$, given λ_j

$$\langle\lambda_j\lambda_j\rangle \int_{\Omega} \nabla v^{*} : \mathbb{H}^{-}\nabla v_{j+1}\ \mathrm{d}\Omega = -\langle\lambda_j\rangle \int_{\Omega} \nabla v^{*} : \hat{f}\,\mathrm{d}\Omega$$

- Time problem, given v_{j+1}:

$$\int_{[0,T]} \lambda^{*} \left[\int_{\Omega} \nabla v_{j+1} : \mathbb{H}^{-}\ \nabla v_{j+1}\,\mathrm{d}\Omega\right] \lambda_{j+1}\ \mathrm{d}t = -\langle\lambda^{*}\rangle \int_{\Omega} \nabla v_{j+1} : \hat{f}\,\mathrm{d}\Omega$$