3. We can find the eigenvectors associated with the eigenvalues we already found. $\lambda = 10+\sqrt{97},\ 10-\sqrt{97},\ 0.$

For $\lambda = 10+\sqrt{97}$:

$$A^T A - (10+\sqrt{97})\, I = \begin{bmatrix} -5-\sqrt{97} & 8 & 3 \\ 8 & 3-\sqrt{97} & 5 \\ 3 & 5 & -8-\sqrt{97} \end{bmatrix}$$

Reduce to RREF:

$$\begin{bmatrix} -5-\sqrt{97} & 8 & 3 \\ 8 & 3-\sqrt{97} & 5 \\ 3 & 5 & -8-\sqrt{97} \end{bmatrix} \quad R1/(-5-\sqrt{97}) \quad = \begin{bmatrix} 1 & \frac{5-\sqrt{97}}{9} & \frac{5-\sqrt{97}}{24} \\ 8 & 3-\sqrt{97} & 5 \\ 3 & 5 & -8-\sqrt{97} \end{bmatrix} \begin{array}{l} \\ R2-8R1 \\ R3-3R1 \end{array}$$

$$= \begin{bmatrix} 1 & \frac{5-\sqrt{97}}{9} & \frac{5-\sqrt{97}}{24} \\ 0 & \frac{-(13+\sqrt{97})}{9} & \frac{10+\sqrt{97}}{3} \\ 0 & \frac{10+\sqrt{97}}{3} & \frac{-69-7\sqrt{97}}{8} \end{bmatrix} \quad R2 \times -\frac{9}{13+\sqrt{97}} \ \text{then}\ R1 - \frac{5-\sqrt{97}}{9} R2$$

$$= \begin{bmatrix} 1 & 0 & \frac{-3-\sqrt{97}}{8} \\ 0 & 1 & \frac{-11-\sqrt{97}}{8} \\ 0 & \frac{10+\sqrt{97}}{3} & \frac{-7\sqrt{97}-69}{8} \end{bmatrix} \quad R3 - \frac{10+\sqrt{97}}{3} R2$$

$$= \begin{bmatrix} 1 & 0 & \frac{-3-\sqrt{97}}{8} \\ 0 & 1 & \frac{-11-\sqrt{97}}{8} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[A^T A - (10+\sqrt{97})\, I\right] \vec{v}_1 = 0$$

$$\begin{bmatrix} 1 & 0 & \frac{-3-\sqrt{97}}{8} \\ 0 & 1 & \frac{-11-\sqrt{97}}{8} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_3 = t$

$v_2 = \frac{11+\sqrt{97}}{8} t$

$v_1 = \frac{3+\sqrt{97}}{8} t$

$$\vec{v}_1 = \begin{bmatrix} 3+\sqrt{97} \\ 11+\sqrt{97} \\ 8 \end{bmatrix}$$

For $\lambda = 10 - \sqrt{97}$:

$$A^TA - (10-\sqrt{97})I = \begin{bmatrix} -5+\sqrt{97} & 8 & 3 \\ 8 & 3+\sqrt{97} & 5 \\ 3 & 5 & -8+\sqrt{97} \end{bmatrix} R1/(-5+\sqrt{97})$$

$$= \begin{bmatrix} 1 & \frac{5+\sqrt{97}}{9} & \frac{5+\sqrt{97}}{24} \\ 8 & 3+\sqrt{97} & 5 \\ 3 & 5 & -8+\sqrt{97} \end{bmatrix} \begin{matrix} \\ R2-8R1 \\ R3-3R1 \end{matrix} = \begin{bmatrix} 1 & \frac{5+\sqrt{97}}{9} & \frac{5+\sqrt{97}}{24} \\ 0 & \frac{-13+\sqrt{97}}{9} & \frac{10-\sqrt{97}}{3} \\ 0 & \frac{10-\sqrt{97}}{3} & \frac{-69+7\sqrt{97}}{8} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{5+\sqrt{97}}{9} & \frac{5+\sqrt{97}}{24} \\ 0 & 1 & \frac{-11+\sqrt{97}}{8} \\ 0 & \frac{10-\sqrt{97}}{3} & \frac{-69+7\sqrt{97}}{8} \end{bmatrix} \begin{matrix} R1 - \frac{5+\sqrt{97}}{9}R2 \\ \\ R3 - \frac{10-\sqrt{97}}{3}R2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{-3+\sqrt{97}}{8} \\ 0 & 1 & \frac{-11+\sqrt{97}}{8} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A^TA - (10-\sqrt{97})I]\,\vec{v}_2 = 0$$

$$\begin{bmatrix} 1 & 0 & \frac{-3+\sqrt{97}}{8} \\ 0 & 1 & \frac{-11+\sqrt{97}}{8} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$v_3 = t$

$v_2 = \frac{11-\sqrt{97}}{8}t$

$v_1 = \frac{3-\sqrt{97}}{8}t$

$$\therefore \vec{v}_2 = \begin{bmatrix} 3-\sqrt{97} \\ 11-\sqrt{97} \\ 8 \end{bmatrix}$$

For $\lambda = 0$:

$A^T A - 0(I) = A^T A$

$(A^T A)\vec{v_3} = 0$

$$\begin{bmatrix} 5 & 8 & 3 \\ 8 & 13 & 5 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5v_1 + 8v_2 + 3v_3 = 0$

$v_1 = \dfrac{-8v_2 - 3v_3}{5}$

$8v_1 + 13v_2 + 5v_3 = 0$

$8\left(\dfrac{-8v_2 - 3v_3}{5}\right) + 13v_2 + 5v_3 = 0$

$\dfrac{-64v_2 - 24v_3}{5} + 13v_2 + 5v_3 = 0$

$\frac{1}{5}v_2 + \frac{1}{5}v_3 = 0$

$v_2 = -v_3$

$3v_1 + 5v_2 + 2v_3 = 0$

$3v_1 - 5v_3 + 2v_3 = 0$

$3v_1 - 3v_3 = 0$

$v_1 = v_3$

By induction, $v_1 = v_3 = 1$

$v_2 = -1$

$$\vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

To normalize $\vec{v}$, find $|\vec{v_1}|$, $|\vec{v_2}|$ and $|\vec{v_3}|$

$|\vec{v_1}| = \sqrt{(3+\sqrt{97})^2 + (11+\sqrt{97})^2 + 8^2} = 25.76369$

$|\vec{v_2}| = 10.59396$

$|\vec{v_3}| = \sqrt{3}$

$$\boxed{V = \begin{bmatrix} 0.499 & -0.646 & 0.577 \\ 0.809 & 0.109 & -0.577 \\ 0.310 & 0.755 & 0.577 \end{bmatrix}}$$

To find U, we know $\sigma_i U_i = AV_i$

$$U_i = \frac{1}{\sigma_i} AV_i$$

Two σ are $\sigma_1 = \sqrt{10+\sqrt{97}}$ and $\sigma_2 = \sqrt{10-\sqrt{97}}$

For $\sigma_1 = \sqrt{10+\sqrt{97}}$:

$$U_1 = \frac{1}{\sqrt{10+\sqrt{97}}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.499 \\ 0.809 \\ 0.310 \end{bmatrix}$$

$$= \frac{1}{\sqrt{10+\sqrt{97}}} \begin{bmatrix} 2.43 \\ 3.74 \end{bmatrix}$$

$$= \begin{bmatrix} 0.545 \\ 0.838 \end{bmatrix}$$

For $\sigma_2 = \sqrt{10-\sqrt{97}}$:

$$U_2 = \frac{1}{\sqrt{10-\sqrt{97}}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -0.646 \\ 0.109 \\ 0.755 \end{bmatrix}$$

$$= \frac{1}{\sqrt{10-\sqrt{97}}} \begin{bmatrix} 0.327 \\ -0.21 \end{bmatrix}$$

$$= \begin{bmatrix} 0.841 \\ -0.540 \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} 0.545 & 0.841 \\ 0.838 & -0.540 \end{bmatrix}}$$