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ECE368: Probabilistic Reasoning
Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \ldots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

$$p(z_i \mid (\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1})) = \gamma(z_i) = \frac{\alpha(z_i)\,\beta(z_i)}{\sum_{z_j} \alpha(z_j)\,\beta(z_j)}$$

Forward Message:

$\alpha(z_0) = p(z_0)\, p((\hat{x}_0, \hat{y}_0) \mid z_0)$ when $(\hat{x}_i, \hat{y}_i)$ not observed, $p((\hat{x}_i, \hat{y}_i) \mid z_i) = 1$

$\alpha(z_i) = p((\hat{x}_i, \hat{y}_i) \mid z_i) \sum_{z_{i-1}} p(z_i \mid z_{i-1})\, \alpha(z_{i-1})$ for $i = 1, 2, 3, \ldots, N-1$

Backwards Message:

$\beta(z_{N-1}) = 1$

$\beta(z_i) = \sum_{z_{i+1}} \beta(z_{i+1})\, p((\hat{x}_{i+1}, \hat{y}_{i+1}) \mid z_{i+1})\, p(z_{i+1} \mid z_i)$ for $i = N-2, N-3, \ldots, 0$

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$$p(z_{99} \mid (\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.81026 & z_{99} = (11, 0, \text{stay}) \\ 0.17961 & z_{99} = (11, 0, \text{right}) \\ 0.10128 & z_{99} = (10, 1, \text{down}) \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

$$p(z_{30} \mid (\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.91304 & z_{30} = (6, 7, \text{right}) \\ 0.04348 & z_{30} = (5, 7, \text{right}) \\ 0.04348 & z_{30} = (5, 7, \text{stay}) \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

Forward Path:

$\omega_0(z_0) = \ln(p((\hat{x}_0, \hat{y}_0) \mid z_0)\, p(z_0))$

$\omega_i(z_i) = \ln(p((\hat{x}_i, \hat{y}_i) \mid z_i)) + \max_{z_{i=1}} \{ \ln(p(z_i \mid z_{i-1})) + (\omega_{i-1})(z_{i-1}) \}$

$z_i^* = \operatorname{argmax}_{z_i} (\omega_i)(z_i)$

Backwards Path: Use trackback procedure.

$z_{i-1}^* = \phi_i(z_i) = \operatorname{argmax}(\omega_i)(z_i^*)$

When $(\hat{x}_i, \hat{y}_i)$ not observed: $p((\hat{x}_i, \hat{y}_i) \mid z_i) = 1$.

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

Last 10 Hidden States in MAP Estimate:

z_{90} = (11, 5, down)
z_{91} = (11, 6, down)
z_{92} = (11, 7, down)
z_{93} = (11, 7, stay)
z_{94} = (11, 7, stay)
z_{95} = (10, 7, left)
z_{96} = (9, 7, left)
z_{97} = (8, 7, left)
z_{98} = (7, 7, left)
z_{99} = (6, 7, left)

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

Sequence $\{\check{z}_i\}$ isn't a valid sequence
An example proving this is z_{64} = (3, 7, stay) and z_{65} = (2, 7, stay).
Inherently, the rover stays at (3, 7) at timestamp 64. Timestamp 65 suggests the rover stays at the previous timestamp, but it actually moved 1 unit down, thus violating the transition model.