Thus

$$S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}$$

for all integers $n$ and $r$ with $1 \leq r \leq n$.

The initial conditions for the recurrence relation are

$$S_{n,1} = 1 \quad \text{and} \quad S_{n,n} = 1 \quad \text{for all integers } n \geq 1$$

because there is only one way to partition $\{x_1, x_2, \ldots, x_n\}$ into one subset, namely

$$\{x_1, x_2, \ldots, x_n\},$$

and only one way to partition $\{x_1, x_2, \ldots, x_n\}$ into $n$ subsets, namely

$$\{x_1\}\{x_2\}, \ldots, \{x_n\}.$$

■

## Exercise Set 8.1*

Find the first four terms of each of the recursively defined sequences in 1–8.

**1.** $a_k = 2a_{k-1} + k$, for all integers $k \geq 2$
$a_1 = 1$

2. $b_k = b_{k-1} + 3k$, for all integers $k \geq 2$
$b_1 = 1$

**3.** $c_k = k(c_{k-1})^2$, for all integers $k \geq 1$
$c_0 = 1$

4. $d_k = k(d_{k-1})^2$, for all integers $k \geq 1$
$d_0 = 3$

**5.** $s_k = s_{k-1} + 2s_{k-2}$, for all integers $k \geq 2$
$s_0 = 1, s_1 = 1$

6. $t_k = t_{k-1} + 2t_{k-2}$, for all integers $k \geq 2$
$t_0 = -1, t_1 = 2$

**7.** $u_k = ku_{k-1} - u_{k-2}$, for all integers $k \geq 3$
$u_1 = 1, u_2 = 1$

8. $v_k = v_{k-1} + v_{k-2} + 1$, for all integers $k \geq 3$
$v_1 = 1, v_2 = 3$

**9.** Let $a_0, a_1, a_2, \ldots$ be defined by the formula $a_n = 3n + 1$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = a_{k-1} + 3$, for all integers $k \geq 1$.

10. Let $b_0, b_1, b_2, \ldots$ be defined by the formula $b_n = 4^n$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $b_k = 4b_{k-1}$, for all integers $k \geq 1$.

**11.** Show that the sequence $0, 1, 3, 7, \ldots, 2^n - 1, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$c_k = 2c_{k-1} + 1 \quad \text{for all integers } k \geq 1.$$

12. Show that the sequence $1, -1, \frac{1}{2}, \frac{-1}{3!}, \ldots, \frac{(-1)^n}{n!}, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$s_k = \frac{-s_{k-1}}{k} \quad \text{for all integers } k \geq 1.$$

**13.** Show that the sequence $2, 3, 4, 5, \ldots, 2 + n, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$t_k = 2t_{k-1} - t_{k-2} \quad \text{for all integers } k \geq 2.$$

14. Show that the sequence $0, 1, 5, 19, \ldots, 3^n - 2^n, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$d_k = 5d_{k-1} - 6d_{k-2} \quad \text{for all integers } k \geq 2.$$

**15.** Define a sequence $a_0, a_1, a_2, \ldots$ by the formula

$$a_n = (-2)^{\lfloor n/2 \rfloor} = \begin{cases} (-2)^{n/2} & \text{if } n \text{ is even} \\ (-2)^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = -2a_{k-2}$, for all integers $k \geq 2$.

16. The sequence of Catalan numbers was defined in Exercise Set 6.6 by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$, for each integer $n \geq 1$. Show that this sequence satisfies the recurrence relation $C_k = \frac{4k-2}{k+1}C_{k-1}$, for all integers $k \geq 2$.

17. Use the recurrence relation and values for the Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ discussed in Example 8.1.5 to compute $m_7$ and $m_8$.

18. *Tower of Hanoi with Adjacency Requirement:* Suppose that in addition to the requirement that they never move a larger disk on top of a smaller one, the priests who move the disks

$a_k = 3a_{k-1} + 2$

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

of the Tower of Hanoi are also allowed only to move disks one by one from one pole to an *adjacent* pole. Assume poles $A$ and $C$ are at the two ends of the row and pole $B$ is in the middle. Let

$$a_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } C \end{bmatrix}.$$

**a.** Find $a_1$, $a_2$, and $a_3$. **b.** Find $a_4$.
**c.** Find a recurrence relation for $a_1, a_2, a_3, \ldots$.

19. *Tower of Hanoi with Adjacency Requirement:* Suppose the same situation as in exercise 18. Let

$$b_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } B \end{bmatrix}.$$

a. Find $b_1$, $b_2$, and $b_3$. **b.** Find $b_4$.
c. Show that $b_k = a_{k-1} + 1 + b_{k-1}$ for all integers $k \geq 2$, where $a_1, a_2, a_3, \ldots$ is the sequence defined in exercise 18.
d. Show that $b_k \leq 3b_{k-1} + 1$ for all integers $k \geq 2$.
✶ *H* **e.** Show that $b_k = 3b_{k-1} + 1$ for all integers $k \geq 2$.

$b_k = a_{k-1} + 1 + b_{k-1}$
$= 3b_{k-1} + 1$

20. *Four-Pole Tower of Hanoi:* Suppose that the Tower of Hanoi problem has four poles in a row instead of three. Disks can be transferred one by one from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Let $s_n$ be the minimum number of moves needed to transfer the entire tower of $n$ disks from the left-most to the right-most pole.
**a.** Find $s_1$, $s_2$, and $s_3$. **b.** Find $s_4$.
c. Show that $s_k \leq 2s_{k-2} + 3$ for all integers $k \geq 3$.

21. *Double Tower of Hanoi:* In this variation of the Tower of Hanoi there are three poles in a row and $2n$ disks, two of each of $n$ different sizes, where $n$ is any positive integer. Initially one of the poles contains all the disks placed on top of each other in pairs of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let $t_n$ be the minimum number of moves needed to transfer a tower of $2n$ disks from one pole to another.
a. Find $t_1$ and $t_2$. **b.** Find $t_3$.
c. Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

$s_k: 2s_{k-1} + 2$
$s_k: 2k$ disc

22. *Fibonacci Variation:* A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions (which are more realistic than Fibonacci's):
(1) Rabbit pairs are not fertile during their first month of life but thereafter give birth to four new male/female pairs at the end of every month.
(2) No rabbits die.
a. Let $r_n$ = the number of pairs of rabbits alive at the end of month $n$, for each integer $n \geq 1$, and let $r_0 = 1$. Find a recurrence relation for $r_0, r_1, r_2, \ldots$.
**b.** Compute $r_0$, $r_1$, $r_2$, $r_3$, $r_4$, $r_5$, and $r_6$.
c. How many rabbits will there be at the end of the year?

$S_k = S_{k-1} + 4S_{k-2}$

23. *Fibonacci Variation:* A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:
(1) Rabbit pairs are not fertile during their first *two* months of life, but thereafter give birth to three new male/female pairs at the end of every month.
(2) No rabbits die.
a. Let $s_n$ = the number of pairs of rabbits alive at the end of month $n$, for each integer $n \geq 1$, and let $s_0 = 1$. Find a recurrence relation for $s_0, s_1, s_2, \ldots$.
b. Compute $s_0$, $s_1$, $s_2$, $s_3$, $s_4$, and $s_5$.
**c.** How many rabbits will there be at the end of the year?

In 24–32, $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

24. Use the recurrence relation and values for $F_0, F_1, F_2, \ldots$ given in Example 8.1.6 to compute $F_{13}$ and $F_{14}$.

25. The Fibonacci sequence satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$, for all integers $k \geq 2$.
**a.** Explain why the following is true:
$$F_{k+1} = F_k + F_{k-1} \quad \text{for all integers } k \geq 1.$$
b. Write an equation expressing $F_{k+2}$ in terms of $F_{k+1}$ and $F_k$.
c. Write an equation expressing $F_{k+3}$ in terms of $F_{k+2}$ and $F_{k+1}$

**26.** Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for all integers $k \geq 4$.

**27.** Prove that $F_k^2 - F_{k-1}^2 = F_k F_{k+1} - F_{k+1} F_{k-1}$, for all integers $k \geq 1$.

28. Prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_k F_{k-1}$, for all integers $k \geq 1$.

29. Prove that $F_{k+1}^2 - F_k^2 = F_{k-1} F_{k+2}$, for all integers $k \geq 1$.

30. Use mathematical induction to prove that for all integers $n \geq 0$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$.

**31.** (For students who have studied calculus) Find $\lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$, assuming that the limit exists.

*H* ✶ **32.** (For students who have studied calculus) Prove that $\lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$ exists.

33. (For students who have studied calculus) Define $x_0, x_1, x_2, \ldots$ as follows:
$$x_k = \sqrt{2 + x_{k-1}} \quad \text{for all integers } k \geq 1$$
$$x_0 = 0$$
Find $\lim_{n\to\infty} x_n$. (Assume that the limit exists.)

**34.** *Compound Interest:* Suppose a certain amount of money is deposited in an account paying 4% annual interest compounded quarterly. For each positive integer $n$, let $R_n$ = the amount on deposit at the end of the $n$th quarter, assuming no additional deposits or withdrawals, and let $R_0$ be the initial amount deposited.

a. Find a recurrence relation for $R_0, R_1, R_2, \ldots$.
b. If $R_0 = \$5000$, find the amount of money on deposit at the end of one year.
c. Find the APR for the account.

35. *Compound Interest:* Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer $n$, let $S_n$ = the amount on deposit at the end of the $n$th month, and let $S_0$ be the initial amount deposited.
    a. Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year.
    b. If $S_0 = \$10{,}000$, find the amount of money on deposit at the end of one year.
    c. Find the APR for the account.

36. *Counting Strings:*
    a. Make a list of all bit strings of lengths zero, one, two, three, and four that do not contain the bit pattern 111.
    b. For each integer $n \geq 0$, let $d_n$ = the number of bit strings of length $n$ that do not contain the bit pattern 111. Find $d_0, d_1, d_2, d_3$, and $d_4$.
    c. Find a recurrence relation for $d_0, d_1, d_2, \ldots$.
    d. Use the results of parts (b) and (c) to find the number of bit strings of length five that do not contain the pattern 111.

37. *Counting Strings:* Consider the set of all strings of $a$'s, $b$'s, and $c$'s.
    a. Make a list of all of these strings of lengths zero, one, two, and three that do not contain the pattern $aa$.
    b. For each integer $n \geq 0$, let $s_n$ = the number of strings of $a$'s, $b$'s, and $c$'s of length $n$ that do not contain the pattern $aa$. Find $s_0, s_1, s_2$, and $s_3$.
    
    **H** c. Find a recurrence relation for $s_0, s_1, s_2, \ldots$.
    d. Use the results of parts (b) and (c) to find the number of strings of $a$'s, $b$'s, and $c$'s of length four that do not contain the pattern $aa$.

38. For each integer $n \geq 0$, let $a_n$ be the number of bit strings of length $n$ that do not contain the pattern 101.
    a. Show that $a_k = a_{k-1} + a_{k-3} + \cdots + a_0 + 2$, for all integers $k \geq 3$.
    b. Use the result of part (a) to show that if $k \geq 3$, then $a_k = 2a_{k-1} - a_{k-2} + a_{k-3}$.

39. With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one- and two-stair increments. For each integer $n \geq 1$, if the staircase consists of $n$ stairs, let $c_n$ be the number of different ways to climb the staircase. Find a recurrence relation for $c_1, c_2, c_3, \ldots$.

$t_k$
$= t_{k-1}$
$+ t_{k-2}$

40. A set of blocks contains blocks of heights 1, 2, and 4 inches. Imagine constructing towers by piling blocks of different heights directly on top of one another. (A tower of height 6 inches could be obtained using six 1-inch blocks, three 2-inch blocks, one 2-inch block with one 4-inch block on top, one 4-inch block with one 2-inch block on top, and so forth.) Let $t_n$ be the number of ways to construct a tower of height $n$ inches using blocks from the set. (Assume an infinite supply of block of each size.) Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

(k-3)

★41. For each integer $n \geq 2$ let $a_n$ be the number of permutations of $\{1, 2, 3, \ldots, n\}$ in which no number is more than one place removed from its "natural" position. Thus $a_1 = 1$ since the one permutation of $\{1\}$, namely 1, does not move 1 from its natural position. Also $a_2 = 2$ since neither of the two permutations of $\{1, 2\}$, namely 12 and 21, moves either number more than one place from its natural position.
    **a.** Find $a_3$.
    b. Find a recurrence relation for $a_1, a_2, a_3, \ldots$.

$S_k = S_{k-1}$
$+ S_{k-2}$
$+ 1$

★42. A row in a classroom has $n$ seats. Let $s_n$ be the number of ways nonempty sets of students can sit in the row so that no student is seated directly adjacent to any other student. (For instance, a row of three seats could contain a single student in any of the seats or a pair of students in the two outer seats. Thus $s_3 = 4$.) Find a recurrence relation for $s_1, s_2, s_3, \ldots$.

★**43.** Let $P_n$ be the number of partitions of a set with $n$ elements. Show that

$$P_n = \binom{n-1}{0} P_{n-1} + \binom{n-1}{1} P_{n-2} + \cdots + \binom{n-1}{n-1} P_0$$

for all integers $n \geq 1$.

Exercises 44–50 refer to the sequence of Stirling numbers of the second kind.

44. Find $S_{5,4}$ by exhibiting all the partitions of $\{x_1, x_2, x_3, x_4, x_5\}$ into four subsets.

**45.** Use the values computed in Example 8.1.10 and the recurrence relation and initial conditions found in Example 8.1.11 to compute $S_{5,2}$.

46. Use the values computed in Example 8.1.10 and the recurrence relation and initial conditions found in Example 8.1.11 to compute $S_{5,3}$.

47. Find the total number of different partitions of a set with five elements.

**48.** Use mathematical induction and the recurrence relation found in Example 8.1.11 to prove that for all integers $n \geq 2$, $S_{n,2} = 2^{n-1} - 1$.

49. Use mathematical induction and the recurrence relation found in Example 8.1.11 to prove that for all integers $n \geq 2$, $\sum_{k=1}^{n} (3^{n-k} S_{k,2}) = S_{n+1,3}$.

**H 50.** If $X$ is a set with $n$ elements and $Y$ is a set with $m$ elements, express the number of onto functions from $X$ and $Y$ using Stirling numbers of the second kind. Justify your answer.

In 51 and 52, assume that $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

✶51. Use strong mathematical induction to prove that $F_n < 2^n$ for all integers $n \geq 1$.

*H* ✶**52.** Prove that for all integers $n \geq 0$, $\gcd(F_{n+1}, F_n) = 1$.

53. A gambler decides to play successive games of blackjack until he loses three times in a row. (Thus the gambler could play five games by losing the first, winning the second, and losing the final three or by winning the first two and losing the final three. These possibilities can be symbolized as $LWLLL$ and $WWLLL$.) Let $g_n$ be the number of ways the gambler can play $n$ games.

a. Find $g_3$, $g_4$, and $g_5$.

b. Find $g_6$.

*H* **c.** Find a recurrence relation for $g_3, g_4, g_5, \ldots$.

✶54. A *derangement* of the set $\{1, 2, \ldots, n\}$ is a permutation that moves every element of the set away from its "natural" position. Thus 21 is a derangement of $\{1, 2\}$, and 231 and 312 are derangements of $\{1, 2, 3\}$. For each positive integer $n$, let $d_n$ be the number of derangements of the set $\{1, 2, \ldots, n\}$.

a. Find $d_1$, $d_2$, and $d_3$.

b. Find $d_4$.

*H* **c.** Find a recurrence relation for $d_1, d_2, d_3, \ldots$.

$(n-1)(S(n-2) + S(n-1))$

55. Note that a product $x_1x_2x_3$ may be parenthesized in two different ways: $(x_1x_2)x_3$ and $x_1(x_2x_3)$. Similarly, there are several different ways to parenthesize $x_1x_2x_3x_4$. Two such ways are $(x_1x_2)(x_3x_4)$ and $x_1((x_2x_3)x_4)$. Let $P_n$ be the number of different ways to parenthesize the product $x_1x_2\cdots x_n$. Show that if $P_1 = 1$, then

$$P_n = \sum_{k=1}^{n-1} P_k P_{n-k} \quad \text{for all integers } n \geq 2.$$

(It turns out that the sequence $P_1, P_2, P_3, \ldots$ is the same as the sequence of Catalan numbers.)

# 8.2 Solving Recurrence Relations by Iteration

*The keener one's sense of logical deduction, the less often one makes hard and fast inferences.* — Bertrand Russell, 1872–1970

Suppose you have a sequence that satisfies a certain recurrence relation and initial conditions. It is often helpful to know an explicit formula for the sequence, especially if you need to compute terms with very large subscripts or if you need to examine general properties of the sequence. Such an explicit formula is called a **solution** to the recurrence relation. In this section and the next, we discuss methods for solving recurrence relations. In the text and exercises of this section, we will show that the Tower of Hanoi sequence of Example 8.1.5 satisfies the formula

$$m_n = 2^n - 1,$$

and that the compound interest sequence of Example 8.1.7 satisfies

$$A_n = (1.04)^n \cdot \$100{,}000.$$

In Section 8.3 we will show that the Fibonacci sequence of Example 8.1.6 satisfies the formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

## The Method of Iteration

The most basic method for finding an explicit formula for a recursively defined sequence is **iteration.** Iteration works as follows: Given a sequence $a_0, a_1, a_2, \ldots$ defined by a recurrence relation and initial conditions, you start from the initial conditions and calculate successive terms of the sequence until you see a pattern developing. At that point you guess an explicit formula.