25. We can do this problem either by working directly with probabilities or by counting ways to satisfy the condition. We choose to do the former. First we need to determine the probability that all the numbers are odd. There are $C(100,4)$ ways to choose the numbers, and there are $C(50,4)$ ways to choose them all to be odd (since there are 50 odd numbers in the given interval). Therefore the probability that they are all odd is $C(50,4)/C(100,4)$. Similarly, since there are 33 multiples of 3 in the given interval, the probability of having all four numbers divisible by 3 is $C(33,4)/C(100,4)$. Finally, the probability that all four are divisible by 5 is $C(20,4)/C(100,4)$.

Next we need to know the probabilities that two of these events occur simultaneously. A number is both odd and divisible by 3 if and only if it is divisible by 3 but not by 6; therefore, since there are $\lfloor 100/6 \rfloor = 16$ multiples of 6 in the given interval, there are $33 - 16 = 17$ numbers that are both odd and divisible by 3. Thus the probability is $C(17,4)/C(100,4)$. Similarly there are 10 odd numbers divisible by 5, so the probability that all four numbers meet those conditions is $C(10,4)/C(100,4)$. Finally, the probability that all four numbers are divisible by both 3 and 5 is $C(6,4)/C(100,4)$, since there are only $\lfloor 100/15 \rfloor = 6$ such numbers.

Finally, the only numbers satisfying all three conditions are the odd multiplies of 15, namely 15, 45, and 75. Since there are only 3 such numbers, it is impossible that all chosen four numbers are divisible by 2, 3, and 5; in other words, the probability of that event is 0. We are now ready to apply the result of Exercise 23 (i.e., inclusion–exclusion viewed in terms of probabilities). We get

$$\begin{aligned}
&\frac{C(50,4)}{C(100,4)}+\frac{C(33,4)}{C(100,4)}+\frac{C(20,4)}{C(100,4)}-\frac{C(17,4)}{C(100,4)}-\frac{C(10,4)}{C(100,4)}-\frac{C(6,4)}{C(100,4)}+0\\
&\qquad=\frac{230300+40920+4845-2380-210-15}{3921225}\\
&\qquad=\frac{273460}{3921225}=\frac{4972}{71295}\approx 0.0697.
\end{aligned}$$

27. We are asked to write down inclusion–exclusion for five sets, just as in Exercise 19, except that intersections of more than three sets can be omitted. Furthermore, we are to use event notation, rather than set notation. Thus we have $p(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = p(E_1) + p(E_2) + p(E_3) + p(E_4) + p(E_5) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_1 \cap E_5) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_2 \cap E_5) - p(E_3 \cap E_4) - p(E_3 \cap E_5) - p(E_4 \cap E_5) + p(E_1 \cap E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_4) + p(E_1 \cap E_2 \cap E_5) + p(E_1 \cap E_3 \cap E_4) + p(E_1 \cap E_3 \cap E_5) + p(E_1 \cap E_4 \cap E_5) + p(E_2 \cap E_3 \cap E_4) + p(E_2 \cap E_3 \cap E_5) + p(E_2 \cap E_4 \cap E_5) + p(E_3 \cap E_4 \cap E_5)$.

29. We are simply asked to rephrase Theorem 1 in terms of probabilities of events. Thus we have

$$\begin{aligned}
p(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{1 \le i \le n} p(E_i) - \sum_{1 \le i < j \le n} p(E_i \cap E_j) + \sum_{1 \le i < j < k \le n} p(E_i \cap E_j \cap E_k)\\
\cdots + (-1)^{n+1} p(E_1 \cap E_2 \cap \cdots \cap E_n).
\end{aligned}$$

SECTION 8.6 Applications of Inclusion–Exclusion

Some of these applications are quite subtle and not easy to understand on first encounter. They do point out the power of the inclusion–exclusion principle. Many of the exercises are closely tied to the examples, so additional study of the examples should be helpful in doing the exercises. It is often helpful, in organizing your work, to write down (in complete English sentences) exactly what the properties of interest are, calling them the P_i's. To find the number of elements lacking all the properties (as you need to do in Exercise 2, for example), use the formula above Example 1.

1. We want to find the number of apples that have neither of the properties of having worms or of having bruises. By inclusion–exclusion, we know that this is equal to the number of apples, minus the numbers with each of the properties, plus the number with both properties. In this case, this is $100 - 20 - 15 + 10 = 75$.

3. We need first to find the number of solutions with no restrictions. By the results of Section 6.5, there are $C(3+13-1, 13) = C(15, 13) = C(15, 2) = 105$. Next we need to find the number of solutions in which each restriction is violated. There are three variables that can fail to be less than 6, and the situation is symmetric, so the total number of solutions in which each restriction is violated is 3 times the number of solutions in which $x_1 \geq 6$. By the trick we used in Section 6.5, this is the same as the number of nonnegative integer solutions to $x_1' + x_2 + x_3 = 7$, where $x_1 = x_1' + 6$. This of course is $C(3+7-1, 7) = C(9, 7) = C(9, 2) = 36$. Therefore there are $3 \cdot 36 = 108$ solutions in which at least one of the restrictions is violated (with some of these counted more than once).

Next we need to find the number of solutions with at least two of the restrictions violated. There are $C(3, 2) = 3$ ways to choose the pair to be violated, so the number we are seeking is 3 times the number of solutions in which $x_1 \geq 6$ and $x_2 \geq 6$. Again by the trick we used in Section 6.5, this is the same as the number of nonnegative integer solutions to $x_1' + x_2' + x_3 = 1$, where $x_1 = x_1' + 6$ and $x_2 = x_2' + 6$. This of course is $C(3+1-1, 1) = C(3, 1) = 3$. Therefore there are $3 \cdot 3 = 9$ solutions in which two of the restrictions are violated. Finally, we note that there are no solutions in which all three of the solutions are violated, since if each of the variables is at least 6, then their sum is at least 18, and hence cannot equal 13.

Thus by inclusion–exclusion, we see that there are $105 - 108 + 9 = 6$ solutions to the original problem. (We can check this on an ad hoc basis. The only way the sum of three numbers, not as big as 6, can be 13, is to have either two 5's and one 3, or else one 5 and two 4's. There are three variables that can be the "odd man out" in each case, for a total of 6 solutions.)

5. We follow the procedure described in the text. There are 198 positive integers less than 200 and greater than 1. The ones that are not prime are divisible by at least one of the primes in the set $\{2, 3, 5, 7, 11, 13\}$. The number of integers in the given range divisible by the prime p is given by $\lfloor 199/p \rfloor$. Therefore we apply inclusion–exclusion and obtain the following number of integers from 2 to 199 that are not divisible by at least one of the primes in our set. (We have only listed those terms that contribute to the result, deleting all those that equal 0.)

$$\begin{aligned}
198 &- \left\lfloor \frac{199}{2} \right\rfloor - \left\lfloor \frac{199}{3} \right\rfloor - \left\lfloor \frac{199}{5} \right\rfloor - \left\lfloor \frac{199}{7} \right\rfloor - \left\lfloor \frac{199}{11} \right\rfloor - \left\lfloor \frac{199}{13} \right\rfloor + \left\lfloor \frac{199}{2\cdot 3} \right\rfloor \\
&+ \left\lfloor \frac{199}{2\cdot 5} \right\rfloor + \left\lfloor \frac{199}{2\cdot 7} \right\rfloor + \left\lfloor \frac{199}{2\cdot 11} \right\rfloor + \left\lfloor \frac{199}{2\cdot 13} \right\rfloor + \left\lfloor \frac{199}{3\cdot 5} \right\rfloor + \left\lfloor \frac{199}{3\cdot 7} \right\rfloor + \left\lfloor \frac{199}{3\cdot 11} \right\rfloor \\
&+ \left\lfloor \frac{199}{3\cdot 13} \right\rfloor + \left\lfloor \frac{199}{5\cdot 7} \right\rfloor + \left\lfloor \frac{199}{5\cdot 11} \right\rfloor + \left\lfloor \frac{199}{5\cdot 13} \right\rfloor + \left\lfloor \frac{199}{7\cdot 11} \right\rfloor + \left\lfloor \frac{199}{7\cdot 13} \right\rfloor + \left\lfloor \frac{199}{11\cdot 13} \right\rfloor \\
&- \left\lfloor \frac{199}{2\cdot 3\cdot 5} \right\rfloor - \left\lfloor \frac{199}{2\cdot 3\cdot 7} \right\rfloor - \left\lfloor \frac{199}{2\cdot 3\cdot 11} \right\rfloor - \left\lfloor \frac{199}{2\cdot 3\cdot 13} \right\rfloor - \left\lfloor \frac{199}{2\cdot 5\cdot 7} \right\rfloor - \left\lfloor \frac{199}{2\cdot 5\cdot 11} \right\rfloor \\
&- \left\lfloor \frac{199}{2\cdot 5\cdot 13} \right\rfloor - \left\lfloor \frac{199}{2\cdot 7\cdot 11} \right\rfloor - \left\lfloor \frac{199}{2\cdot 7\cdot 13} \right\rfloor - \left\lfloor \frac{199}{3\cdot 5\cdot 7} \right\rfloor - \left\lfloor \frac{199}{3\cdot 5\cdot 11} \right\rfloor - \left\lfloor \frac{199}{3\cdot 5\cdot 13} \right\rfloor \\
&= 198 - 99 - 66 - 39 - 28 - 18 - 15 + 33 + 19 + 14 + 9 + 7 + 13 + 9 + 6 + 5 \\
&\quad + 5 + 3 + 3 + 2 + 2 + 1 - 6 - 4 - 3 - 2 - 2 - 1 - 1 - 1 - 1 - 1 - 1 - 1 = 40
\end{aligned}$$

These 40 numbers are therefore all prime, as are the 6 numbers in our set. Therefore there are exactly 46 prime numbers less than 200.

7. We can apply inclusion–exclusion if we reason as follows. First, we restrict ourselves to numbers greater than 1. If the number N is the power of an integer, then it is certainly the prime power of an integer, since if

$N = x^k$, where $k = mp$, with p prime, then $N = (x^m)^p$. Thus we need to count the number of perfect second powers, the number of perfect third powers, the number of perfect fifth powers, etc., less than 10,000. Let us first determine how many positive integers greater than 1 and less than 10,000 are the square of an integer. Since $\lfloor\sqrt{9999}\rfloor = 99$, there must be $99 - 1 = 98$ such numbers (namely 2^2 through 99^2). Similarly, since $\lfloor\sqrt[3]{9999}\rfloor - 1 = 20$, there are 20 cubes of integers less than 10,000. Similarly, there are $\lfloor\sqrt[5]{9999}\rfloor - 1 = 5$ fifth powers, $\lfloor\sqrt[7]{9999}\rfloor - 1 = 2$ seventh powers, $\lfloor\sqrt[11]{9999}\rfloor - 1 = 1$ eleventh power, and $\lfloor\sqrt[13]{9999}\rfloor - 1 = 1$ thirteenth power. There are no higher prime powers, since $\lfloor\sqrt[17]{9999}\rfloor - 1 = 0$ (and indeed, $2^{17} = 131072 > 9999$).

Now we need to account for the double counting. There are $\lfloor\sqrt[6]{9999}\rfloor - 1 = 3$ sixth powers, and these were counted as both second powers and third powers. Similarly, there is $\lfloor\sqrt[10]{9999}\rfloor - 1 = 1$ tenth power ($10 = 2 \cdot 5$). These are the only two cases of double counting, since all other combinations give a count of 0. Therefore among the 9998 numbers from 2 to 9999, inclusive, we found that there were $98+20+5+2+1+1-3-1 = 123$ powers. Therefore there are $9998 - 123 = 9875$ numbers that are not powers.

9. This exercise is just asking for the number of onto functions from a set with 6 elements (the toys) to a set with 3 elements (the children), since each toy is assigned a unique child. By Theorem 1 there are $3^6 - C(3,1)2^6 + C(3,2)1^6 = 540$ such functions.

11. Here is one approach. Let us ignore temporarily the stipulation about the most difficult job being assigned to the best employee (we assume that this language uniquely specifies a job and an employee). Then we are looking for the number of onto functions from the set of 7 jobs to the set of 4 employees. By Theorem 1 there are $4^7 - C(4,1)3^7 + C(4,2)2^7 - C(4,1)1^7 = 8400$ such functions. Now by symmetry, in exactly one fourth of those assignments should the most difficult job be given to the best employee, as opposed to one of the other three employees. Therefore the answer is $8400/4 = 2100$.

13. We simply apply Theorem 2:

$$D_7 = 7!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$$
$$= 5040 - 5040 + 2520 - 840 + 210 - 42 + 7 - 1 = 1854$$

15. **a)** An arrangement in which no letter is put into the correct envelope is a derangement. There are by definition D_{100} derangements. Since there are $P(100,100) = 100!$ equally likely permutations altogether, the probability of a derangement is $D_{100}/100!$. Numerically, this is almost exactly equal to $1/e$, which is about 0.368.

b) We need to count the number of ways to put exactly one letter into the correct envelope. First, there are $C(100,1) = 100$ ways to choose the letter that is to be correctly stuffed. Then there are D_{99} ways to insert the remaining 99 letters so that none of them go into their correct envelopes. By the product rule, there are $100D_{99}$ such arrangements. As in part **(a)** the denominator is $P(100,100) = 100!$. Therefore the answer is $100D_{99}/100! = D_{99}/99!$. Again this is almost exactly $1/e \approx 0.368$.

c) This time, to count the number of ways that exactly 98 letters can be put into their correct envelopes, we need simply to choose the two letters that are to be misplaced, since there is only one way to misplace them. There are of course $C(100,2) = 4950$ ways to do this. As in part **(a)** the denominator is $P(100,100) = 100!$. Therefore the answer is $4950/100!$. This is substantially less than 10^{-100}, so for all practical purposes, the answer is 0.

d) There is no way that exactly 99 letters can be inserted into their correct envelopes, since as soon as 99 letters have been correctly inserted, there is only one envelope left for the remaining letter, and it is the correct one. Therefore the answer is exactly 0. (The probability of an event that cannot happen is 0.)

e) Only one of the 100! permutations is the correct stuffing, so the answer is $1/100!$. As in part **(c)** this is 0 for all practical purposes.

17. We can derive this answer by mimicking the derivation of the formula for the number of derangements, but worrying only about the even digits. (There are 10! permutations altogether. Let e be one of the 5 even digits. The number of permutations in which e is in its original position is 9! (the other 9 digits need to be permuted). Therefore we need to subtract from 10! the $\boxed{5 \cdot 9!}$ ways in which the even digits can end up in their original positions. (However, we have overcounted, since there are $C(5,2)8!$ ways in which 2 of the even digits can end up in their original positions, $C(5,3)7!$ ways in which 3 of them can, $C(5,4)6!$ ways in which 4 of them can, and $C(5,5)5!$ ways in which they can all retain their original positions. Applying inclusion–exclusion, we therefore have the answer

$^5C_4 \times L6$

$$10! - 5 \cdot 9! + 10 \cdot 8! - 10 \cdot 7! + 5 \cdot 6! - 5! = 2{,}170{,}680\,.$$

$^5C_1 * L9)$ $^5C_2 \times L8$ $^5C_3 \times L7$

19. We want to show that $D_n - nD_{n-1} = (-1)^n$. We will use an iterative approach, taking advantage of the result of Exercise 18, which can be rewritten algebraically as $D_k - kD_{k-1} = -(D_{k-1} - (k-1)D_{k-2})$ for all $k \geq 2$. We have

$$\begin{aligned} D_n - nD_{n-1} &= -(D_{n-1} - (n-1)D_{n-2}) \\ &= -(-(D_{n-2} - (n-2)D_{n-3})) \\ &= (-1)^2(D_{n-2} - (n-2)D_{n-3}) \\ &\ \ \vdots \\ &= (-1)^{n-2}(D_2 - 2D_1) \\ &= (-1)^n \end{aligned}$$

since $D_2 = 1$ and $D_1 = 0$, and since $(-1)^{n-2} = (-1)^n$.

21. We can solve this problem by looking at the explicit formula we have for D_n from Theorem 2 (multiplying through by $n!$):

$$D_n = n! - n! + \frac{n!}{2} - \frac{n!}{3!} + \cdots + (-1)^{n-1}\frac{n!}{(n-1)!} + (-1)^n\frac{n!}{n!}$$

Now all of these terms are even except possibly for the last two, since (after being reduced to natural numbers) they all contain the factors n and $n-1$, at least one of which must be even. Therefore to determine whether D_n is even or odd, we need only look at these last two terms, which are $\pm n \mp 1$. If n is even, then this difference is odd; but if n is odd, then this difference is even. Therefore D_n is even precisely when n is odd.

23. Recall that $\phi(n)$, for a positive integer $n > 1$, denotes the number of positive integers less than (or, vacuously, equal to) n and relatively prime to n (in other words, that have no common prime factors with n). We will derive a formula for $\phi(n)$ using inclusion–exclusion. We are given that the prime factorization of n is $n = p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$. Let P_i be the property that a positive integer less than or equal to n has p_i as a factor. Then $\phi(n)$ is precisely the number of positive integers less than or equal to n that have none of the properties P_i. By the alternative form of the principle of inclusion–exclusion, we have the following formula for this quantity:

$$\begin{aligned} N(P_1'P_2'\cdots P_m') = n - \sum_{1\leq i\leq m} N(P_i) + \sum_{1\leq i<j\leq m} N(P_iP_j) - \sum_{1\leq i<j<k\leq m} N(P_iP_jP_k) \\ + \cdots + (-1)^m N(P_1P_2\cdots P_m) \end{aligned}$$

Our only remaining task is to find a formula for each of these sums. This is not hard. First $N(P_i)$, the number of positive integers less than or equal to n divisible by p_i, is equal to n/p_i, just as in the discussion of the sieve of Eratosthenes (we need no floor function symbols since n/p_i is necessarily an integer). Similarly, $N(P_iP_j)$, the number of positive integers less than or equal to n divisible by both p_i and p_j, i.e., by the product p_ip_j, is equal to $n/(p_ip_j)$, and so on. Making these substitutions, we can rewrite the formula displayed above as

$$N(P_1'P_2'\cdots P_m') = n - \sum_{1\le i\le m}\frac{n}{p_i} + \sum_{1\le i<j\le m}\frac{n}{p_ip_j} - \sum_{1\le i<j<k\le m}\frac{n}{p_ip_jp_k} + \cdots + (-1)^m\frac{n}{p_1p_2\cdots p_m}.$$

This formula can be written in a more useful form. If we factor out the n from every term, then it is not hard to see that what remains is the product $(1-1/p_1)(1-1/p_2)\cdots(1-1/p_m)$. Therefore our answer is

$$n\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right).$$

25. A permutation meeting these conditions must be a derangement of 123 followed by a derangement of 456 in positions 4, 5, and 6. Since there are $D_3 = 2$ derangements of the first 3 elements to choose from for the first half of our permutation and $D_3 = 2$ derangements of the last 3 elements to choose from for the second half, there are, by the product rule, $2\cdot 2 = 4$ derangements satisfying the given conditions. Indeed, these 4 derangements are 231564, 231645, 312564, and 312645.

27. Let P_i be the property that a function from a set with m elements to a set with n elements does not have the i^{th} element of the codomain included in its range. We want to compute $N(P_1'P_2'\cdots P_n')$. In order to use the principle of inclusion–exclusion we need to determine $\sum N(P_i)$, $\sum N(P_iP_j)$, etc. By the product rule, there are n^m functions from the set with m elements to the set with n elements. If we want the function not to have the i^{th} element of the codomain in its range, then there are only $n-1$ choices at each stage, rather than n, to assign to each element of the domain; therefore $N(P_i) = (n-1)^m$, for each i. Furthermore, there are $C(n,1)$ different i's. Therefore $\sum N(P_i) = C(n,1)(n-1)^m$. Similarly, to compute $\sum N(P_iP_j)$, we note that there are $C(n,2)$ ways to specify i and j, and that once we have determined which 2 elements are to be omitted from the codomain, there are $(n-2)^m$ different functions with this smaller codomain. Therefore $\sum N(P_iP_j) = C(n,2)(n-2)^m$. We continue in this way, until finally we need to find $N(P_1P_2\cdots P_n)$, which is clearly equal to 0, since the function must have at least one element in its range. The formula given in the statement of Theorem 1 therefore follows from the inclusion–exclusion principle.

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 8

1. **a)** See pp. 158 and 501. **b)** $\$1{,}000{,}000\cdot 1.09^n$

2. See Example 1 in Section 8.1.

3. See Example 2 in Section 8.1.

4. **a)** See Example 3 in Section 8.1 (interchange the roles of 0 and 1). **b)** See Exercise 11 in Section 8.1.

5. **a)** See pp. 407–408. **b)** This application is discussed in detail at the end of Section 8.1.

6. an equation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$

7. **a)** See Theorem 1 and Example 3 in Section 8.2 if the roots of the characteristic equation are distinct; otherwise see Theorem 2 and Example 5.
b) The characteristic equation is $r^2 - 13r + 22 = 0$, leading to roots 2 and 11. This gives the general solution $a_n = \alpha_1 2^n + \alpha_2 11^n$. Substituting in the initial conditions gives $\alpha_1 = 2$ and $\alpha_2 = 1$. Therefore the solution is $a_n = 2^{n+1} + 11^n$.
c) The characteristic equation is $r^2 - 14r + 49 = 0$, leading to the repeated root 7. This gives the general solution $a_n = \alpha_1 7^n + \alpha_2 n7^n$. Substituting in the initial conditions gives $\alpha_1 = 3$ and $\alpha_2 = 2$. Therefore the solution is $a_n = (3+2n)7^n$.