CHAPTER 8
Advanced Counting Techniques

SECTION 8.1 Applications of Recurrence Relations

2. a) A permutation of a set with n elements consists of a choice of a first element (which can be done in n ways), followed by a permutation of a set with $n-1$ elements. Therefore $P_n = nP_{n-1}$. Note that $P_0 = 1$, since there is just one permutation of a set with no objects, namely the empty sequence.

b) $P_n = nP_{n-1} = n(n-1)P_{n-2} = \cdots = n(n-1)\cdots 2 \cdot 1 \cdot P_0 = n!$

4. This is similar to Exercise 3 and solved in exactly the same way. The recurrence relation is $a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}$. It would be quite tedious to write down the 100 initial conditions.

6. a) Let s_n be the number of such sequences. A string ending in n must consist of a string ending in something less than n, followed by an n as the last term. Therefore the recurrence relation is $s_n = s_{n-1} + s_{n-2} + \cdots + s_2 + s_1$. Here is another approach, with a more compact form of the answer. A sequence ending in n is either a sequence ending in $n-1$, followed by n (and there are clearly s_{n-1} of these), or else it does not contain $n-1$ as a term at all, in which case it is *identical* to a sequence ending in $n-1$ in which the $n-1$ has been replaced by an n (and there are clearly s_{n-1} of these as well). Therefore $s_n = 2s_{n-1}$. Finally we notice that we can derive the second form from the first (or vice versa) algebraically (for example, $s_4 = 2s_3 = s_3 + s_3 = s_3 + s_2 + s_2 = s_3 + s_2 + s_1$).

b) We need two initial conditions if we use the second formulation above, $s_1 = 1$ and $s_2 = 1$ (otherwise, our argument is invalid, because the first and last terms are the same). There is one sequence ending in 1, namely the sequence with just this 1 in it, and there is only the sequence $1, 2$ ending in 2. If we use the first formulation above, then we can get by with just the initial condition $s_1 = 1$.

c) Clearly the solution to this recurrence relation and initial condition is $s_n = 2^{n-2}$ for all $n \geq 2$.

8. This is very similar to Exercise 7, except that we need to go one level deeper.

a) Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit string of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

b) There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

c) We will compute a_3 through a_7 using the recurrence relation:

$$a_3 = a_2 + a_1 + a_0 + 2^0 = 0 + 0 + +0 + 1 = 1$$
$$a_4 = a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3$$
$$a_5 = a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8$$
$$a_6 = a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20$$
$$a_7 = a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47$$

Thus there are 47 bit strings of length 7 containing three consecutive 0's.

10. First let us solve this problem without using recurrence relations at all. It is clear that the only strings that do not contain the string 01 are those that consist of a string of 1's follows by a string of 0's. The string can consist of anywhere from 0 to n 1's, so the number of such strings is $n+1$. All the rest have at least one occurrence of 01. Therefore the number of bit strings that contain 01 is $2^n - (n+1)$. However, this approach does not meet the instructions of this exercise.

a) Let a_n be the number of bit strings of length n that contain 01. If we want to construct such a string, we could start with a 1 and follow it with a bit string of length $n-1$ that contains 01, and there are a_{n-1} of these. Alternatively, for any k from 1 to $n-1$, we could start with k 0's, follow this by a 1, and then follow this by any $n-k-1$ bits. For each such k there are 2^{n-k-1} such strings, since the final bits are free. Therefore the number of such strings is $2^0 + 2^1 + 2^2 + \cdots + 2^{n-2}$, which equals $2^{n-1} - 1$. Thus our recurrence relation is $a_n = a_{n-1} + 2^{n-1} - 1$. It is valid for all $n \geq 2$.

b) The initial conditions are $a_0 = a_1 = 0$, since no string of length less than 2 can have 01 in it.

c) We will compute a_2 through a_7 using the recurrence relation:

$$a_2 = a_1 + 2^1 - 1 = 0 + 2 - 1 = 1$$
$$a_3 = a_2 + 2^2 - 1 = 1 + 4 - 1 = 4$$
$$a_4 = a_3 + 2^3 - 1 = 4 + 8 - 1 = 11$$
$$a_5 = a_4 + 2^4 - 1 = 11 + 16 - 1 = 26$$
$$a_6 = a_5 + 2^5 - 1 = 26 + 32 - 1 = 57$$
$$a_7 = a_6 + 2^6 - 1 = 57 + 64 - 1 = 120$$

$\frac{1(2^{n-1} - 1)}{2-1} = 2^{n-1} - 1$

$1 \to a_{n-1}$

$01 \to 2^{n-2}$

$001 \to 2^{n-3}$

Thus there are 120 bit strings of length 7 containing 01. Note that this agrees with our nonrecursive analysis, since $2^7 - (7+1) = 120$.

12. This is identical to Exercise 11, one level deeper.

a) Let a_n be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in a_{n-1} ways) or else start with a step of two stairs and then climb $n-2$ stairs (and this can be done in a_{n-2} ways) or else start with a step of three stairs and then climb $n-3$ stairs (and this can be done in a_{n-3} ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

b) The initial conditions are $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb two stairs (one step twice or two steps at once). Note that the recurrence relation is the same as that for Exercise 9.

c) Each term in our sequence $\{a_n\}$ is the sum of the previous three terms, so the sequence begins $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 13$, $a_6 = 24$, $a_7 = 44$, $a_8 = 81$. Thus a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

14. a) Let a_n be the number of ternary strings that contain two consecutive 0's. To construct such a string we could start with either a 1 or a 2 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-1}$ ways), or we could start with 01 or 02 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-2}$ ways), we could start with 00 and follow with any ternary string of length $n-2$ (of which there are clearly 3^{n-2}). Therefore the recurrence relation, valid for all $n \geq 2$, is $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$.

b) Clearly $a_0 = a_1 = 0$.

c) We will compute a_2 through a_6 using the recurrence relation:

$$a_2 = 2a_1 + 2a_0 + 3^0 = 2 \cdot 0 + 2 \cdot 0 + 1 = 1$$
$$a_3 = 2a_2 + 2a_1 + 3^1 = 2 \cdot 1 + 2 \cdot 0 + 3 = 5$$
$$a_4 = 2a_3 + 2a_2 + 3^2 = 2 \cdot 5 + 2 \cdot 1 + 9 = 21$$
$$a_5 = 2a_4 + 2a_3 + 3^3 = 2 \cdot 21 + 2 \cdot 5 + 27 = 79$$
$$a_6 = 2a_5 + 2a_4 + 3^4 = 2 \cdot 79 + 2 \cdot 21 + 81 = 281$$

Thus there are 281 bit strings of length 6 containing two consecutive 0's.

16. a) Let a_n be the number of ternary strings that contain either two consecutive 0's or two consecutive 1's. To construct such a string we could start with a 2 and follow with a string containing either two consecutive 0's or two consecutive 1's, and this can be done in a_{n-1} ways. There are other possibilities, however. For each k from 0 to $n-2$, the string could start with $n-1-k$ alternating 0's and 1's, followed by a 2, and then be followed by a string of length k containing either two consecutive 0's or two consecutive 1's. The number of such strings is $2a_k$, since there are two ways for the initial part to alternate. The other possibility is that the string has no 2's at all. Then it must consist $n-k-2$ alternating 0's and 1's, followed by a pair of 0's or 1's, followed by any string of length k. There are $2 \cdot 3^k$ such strings. Now the sum of these quantities as k runs from 0 to $n-2$ is (since this is a geometric progression) $3^{n-1} - 1$. Putting this all together, we have the following recurrence relation, valid for all $n \geq 2$: $a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0 + 3^{n-1} - 1$. (By subtracting this recurrence relation from the same relation with $n-1$ substituted for n, we can obtain the following closed form recurrence relation for this problem: $a_n = 2a_{n-1} + a_{n-2} + 2 \cdot 3^{n-2}$.)

b) Clearly $a_0 = a_1 = 0$.

c) We will compute a_2 through a_6 using the recurrence relation:

$$a_2 = a_1 + 2a_0 + 3^1 - 1 = 0 + 2 \cdot 0 + 3 - 1 = 2$$
$$a_3 = a_2 + 2a_1 + 2a_0 + 3^2 - 1 = 2 + 2 \cdot 0 + 2 \cdot 0 + 9 - 1 = 10$$
$$a_4 = a_3 + 2a_2 + 2a_1 + 2a_0 + 3^3 - 1 = 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 27 - 1 = 40$$
$$a_5 = a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^4 - 1 = 40 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 81 - 1 = 144$$
$$\begin{aligned} a_6 &= a_5 + 2a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^5 - 1 \\ &= 144 + 2 \cdot 40 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 243 - 1 = 490 \end{aligned}$$

Thus there are 490 ternary strings of length 6 containing two consecutive 0's or two consecutive 1's.

18. a) Let a_n be the number of ternary strings that contain two consecutive symbols that are the same. We will develop a recurrence relation for a_n by exploiting the symmetry among the three symbols. In particular, it must be the case that $a_n/3$ such strings start with each of the three symbols. Now let us see how we might specify a string of length n satisfying the condition. We can choose the first symbol in any of three ways. We can follow this by a string that starts with a different symbol but has in it a pair of consecutive symbols; by what we have just said, there are $2a_{n-1}/3$ such strings. Alternatively, we can follow the initial symbol by another copy of itself and then any string of length $n-2$; there are clearly 3^{n-2} such strings. Thus the recurrence relation is $a_n = 3 \cdot ((2a_{n-1}/3) + 3^{n-2}) = \boxed{2a_{n-1} + 3^{n-1}}$. It is valid for all $n \geq 2$.

b) Clearly $a_0 = a_1 = 0$.

c) We will compute a_2 through a_6 using the recurrence relation:

$$\begin{aligned}
a_2 &= 2a_1 + 3^1 = 2 \cdot 0 + 3 = 3 \\
a_3 &= 2a_2 + 3^2 = 2 \cdot 3 + 9 = 15 \\
a_4 &= 2a_3 + 3^3 = 2 \cdot 15 + 27 = 57 \\
a_5 &= 2a_4 + 3^4 = 2 \cdot 57 + 81 = 195 \\
a_6 &= 2a_5 + 3^5 = 2 \cdot 195 + 243 = 633
\end{aligned}$$

Thus there are 633 bit strings of length 6 containing two consecutive 0's, 1's, or 2's.

20. We let a_n be the number of ways to pay a toll of $5n$ cents. (Obviously there is no way to pay a toll that is not a multiple of 5 cents.)

a) This problem is isomorphic to Exercise 11, so the answer is the same: $a_n = a_{n-1} + a_{n-2}$, with $a_0 = a_1 = 1$.

b) Iterating, we find that $a_9 = 55$.

22. a) We start by computing the first few terms to get an idea of what's happening. Clearly $R_1 = 2$, since the equator, say, splits the sphere into two hemispheres. Also, $R_2 = 4$ and $R_3 = 8$. Let's try to analyze what happens when the n^{th} great circle is added. It must intersect each of the other circles twice (at diametrically opposite points), and each such intersection results in one prior region being split into two regions, as in Exercise 21. There are $n-1$ previous great circles, and therefore $2(n-1)$ new regions. Therefore $R_n = R_{n-1} + 2(n-1)$. If we impose the initial condition $R_1 = 2$, then our values of R_2 and R_3 found above are consistent with this recurrence. Note that $R_4 = 14$, $R_5 = 22$, and so on.

b) We follow the usual technique, as in Exercise 17 in Section 2.4. In the last line we use the familiar formula for the sum of the first $n-1$ positive integers. Note that the formula agrees with the values computed above.

$$\begin{aligned}
R_n &= 2(n-1) + R_{n-1} \\
&= 2(n-1) + 2(n-2) + R_{n-2} \\
&= 2(n-1) + 2(n-2) + 2(n-3) + R_{n-3} \\
&\;\;\vdots \\
&= 2(n-1) + 2(n-2) + 2(n-3) + 2 \cdot 1 + R_1 \\
&= n(n-1) + 2 = n^2 - n + 2
\end{aligned}$$

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24. Let e_n be the number of bit sequences of length n with an even number of 0's. Note that therefore there are $2^n - e_n$ bit sequences with an odd number of 0's. There are two ways to get a bit string of length n with an even number of 0's. It can begin with a 1 and be followed by a bit string of length $n-1$ with an even number of 0's, and there are e_{n-1} of these; or it can begin with a 0 and be followed by a bit string of length $n-1$ with an odd number of 0's, and there are $2^{n-1} - e_{n-1}$ of these. Therefore $e_n = \cancel{e_{n-1}} + 2^{n-1} - \cancel{e_{n-1}}$, or simply $e_n = 2^{n-1}$. See also Exercise 31 in Section 6.4.

26. Let a_n be the number of coverings.

a) We follow the hint. If the right-most domino is positioned vertically, then we have a covering of the left-most $n-1$ columns, and this can be done in a_{n-1} ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

b) Clearly $a_1 = 1$ and $a_2 = 2$.

c) The sequence we obtain is just the Fibonacci sequence, shifted by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ..., so the answer to this part is 2584.