SECTION 8.6 Applications of Inclusion–Exclusion

2. $1000 - 450 - 622 - 30 + 111 + 14 + 18 - 9 = 32$

4. $C(4+17-1,17) - C(4+13-1,13) - C(4+12-1,12) - C(4+11-1,11) - C(4+8-1,8) + C(4+8-1,8) + C(4+7-1,7) + C(4+4-1,4) + C(4+6-1,6) + C(4+3-1,3) + C(4+2-1,2) - C(4+2-1,2) = 20$

6. Square-free numbers are those not divisible by the square of a prime. We count them as follows: $99 - \lfloor 99/2^2 \rfloor - \lfloor 99/3^2 \rfloor - \lfloor 99/5^2 \rfloor - \lfloor 99/7^2 \rfloor + \lfloor 99/(2^2 3^2) \rfloor = 61$.

8. $5^7 - C(5,1)4^7 + C(5,2)3^7 - C(5,3)2^7 + C(5,4)1^7 = 16{,}800$

10. This problem is asking for the number of onto functions from a set with 8 elements (the balls) to a set with 3 elements (the urns). Therefore the answer is $3^8 - C(3,1)2^8 + C(3,2)1^8 = 5796$.

12. 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

14. We use Theorem 2 with $n = 10$, which gives us

$$\frac{D_{10}}{10!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{10!} = \frac{1334961}{3628800} = \frac{16481}{44800} \approx 0.3678794643\,,$$

which is almost exactly $e^{-1} \approx 0.3678794412\ldots$.

16. There are $n!$ ways to make the first assignment. We can think of this first seating as assigning student n to a chair we will label n. Then the next seating must be a derangement with respect to this numbering, so there are D_n second seatings possible. Therefore the answer is $n!D_n$.

18. In a derangement of the numbers from 1 to n, the number 1 cannot go first, so let $k \neq 1$ be the number that goes first. There are $n-1$ choices for k. Now there are two ways to get a derangement with k first. One way is to have 1 in the k^{th} position. If we do this, then there are exactly D_{n-2} ways to derange the rest of the numbers. On the other hand, if 1 does not go into the k^{th} position, then think of the number 1 as being temporarily relabeled k. A derangement is completed in this case by finding a derangement of the numbers 2 through n in positions 2 through n, so there are D_{n-1} of them. Combining all this, by the product rule and the sum rule, we obtain the desired recurrence relation. The initial conditions are $D_0 = 1$ and $D_1 = 0$.

20. We apply iteration to the formula $D_n = nD_{n-1} + (-1)^n$, obtaining

$$\begin{aligned}
D_n &= n\big((n-1)D_{n-2} + (-1)^{n-1}\big) + (-1)^n \\
&= n(n-1)D_{n-2} + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)\big((n-2)D_{n-3} + (-1)^{n-2}\big) + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)(n-2)D_{n-3} + n(n-1)(-1)^{n-2} + n(-1)^{n-1} + (-1)^n \\
&\;\;\vdots \\
&= n(n-1)\cdots 2D_1 + n(n-1)\cdots 3 - n(n-1)\cdots 4 + \cdots + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)\cdots 3 - n(n-1)\cdots 4 + \cdots + n(-1)^{n-1} + (-1)^n\,,
\end{aligned}$$

which yields the formula in Theorem 2 after factoring out $n!$.

22. The numbers not relatively prime to pq are the ones that have p and/or q as a factor. Thus we have

$$\phi(pq) = pq - \frac{pq}{p} - \frac{pq}{q} + \frac{pq}{pq} = pq - q - p + 1 = (p-1)(q-1)\,.$$

24. The left-hand side of course counts the number of permutations of the set of integers from 1 to n. The right-hand side counts it, too, by a two-step process: first decide how many and which elements are to be fixed (this can be done in $C(n,k)$ ways, for each of $k=0,1,\ldots,n$), and in each case derange the remaining elements (which can be done in D_{n-k} ways).

26. This permutation starts with $4,5,6$ in some order ($3!=6$ ways to choose this), followed by $1,2,3$ in some order ($3!=6$ ways to decide this). Therefore the answer is $6\cdot 6=36$.

SUPPLEMENTARY EXERCISES FOR CHAPTER 8

2. **a)** Let a_n be the amount that remains after n hours. Then $a_n=0.99a_{n-1}$.
b) By iteration we find the solution $a_n=(0.99)^n a_0$, where a_0 is the original amount of the isotope.

4. **a)** Let B_n be the number of bacteria after n hours. The initial conditions are $B_0=100$ and $B_1=300$. Thereafter, $B_n=B_{n-1}+2B_{n-1}-B_{n-2}=3B_{n-1}-B_{n-2}$.
b) The characteristic equation is $r^2-3r+1=0$, which has roots $(3\pm\sqrt{5})/2$. Therefore the general solution is $B_n=\alpha_1((3+\sqrt{5})/2)^n+\alpha_2((3-\sqrt{5})/2)^n$. Plugging in the initial conditions we determine that $\alpha_1=50+30\sqrt{5}$ and $\alpha_2=50-30\sqrt{5}$. Therefore the solution is $B_n=(50+30\sqrt{5})((3+\sqrt{5})/2)^n+(50-30\sqrt{5})((3-\sqrt{5})/2)^n$.
c) Plugging in small values of n, we find that $B_9=676{,}500$ and $B_{10}=1{,}771{,}100$. Therefore the colony will contain more than one million bacteria after 10 hours.

6. We can put any of the stamps on first, leaving a problem with a smaller number of cents to solve. Thus the recurrence relation is $a_n=a_{n-4}+a_{n-6}+a_{n-10}$. We need 10 initial conditions, and it is easy to see that $a_0=1$, $a_1=a_2=a_3=a_5=a_7=a_9=0$, and $a_4=a_6=a_8=1$.

8. If we add the equations, we obtain $a_n+b_n=2a_{n-1}$, which means that $b_n=2a_{n-1}-a_n$. If we now substitute this back into the first equation, we have $a_n=a_{n-1}+(2a_{n-2}-a_{n-1})=2a_{n-2}$. The initial conditions are $a_0=1$ (given) and $a_1=3$ (follows from the first recurrence relation and the given initial conditions). We can solve this using the characteristic equation $r^2-2=0$, but a simpler approach, that avoids irrational numbers, is as follows. It is clear that $a_{2n}=2^n a_0=2^n$, and $a_{2n+1}=2^n a_1=3\cdot 2^n$. This is a nice explicit formula, which is all that "solution" really means. We also need a formula for b_n, of course. From $b_n=2a_{n-1}-a_n$ (obtained above), we have $b_{2n}=3\cdot 2^n-2^n=2^{n+1}$, and $b_{2n+1}=2\cdot 2^n-3\cdot 2^n=-2^n$.

10. Following the hint, we let $b_n=\log a_n$. Then the recurrence relation becomes $b_n=3b_{n-1}+2b_{n-2}$, with initial conditions $b_0=b_1=1$. This is solved in the usual manner. The characteristic equation is $r^2-3r-2=0$, which gives roots $(3\pm\sqrt{17})/2$. Plugging the initial conditions into the general solution and doing some messy algebra gives

$$b_n=\frac{17-\sqrt{17}}{34}\left(\frac{3+\sqrt{17}}{2}\right)^n+\frac{17+\sqrt{17}}{34}\left(\frac{3-\sqrt{17}}{2}\right)^n.$$

The solution to the original problem is then $a_n=2^{b_n}$.

12. The characteristic equation is $r^3-3r^2+3r-1=0$. This factors as $(r-1)^3=0$, so there is only one root, 1, and its multiplicity is 3. Therefore the general solution is $a_n=\alpha_1+\alpha_2 n+\alpha_3 n^2$. Plugging in the initial conditions gives us $2=\alpha_1$, $2=\alpha_1+\alpha_2+\alpha_3$, and $4=\alpha_1+2\alpha_2+4\alpha_3$. Solving yields $\alpha_1=2$, $\alpha_2=-1$, and $\alpha_3=1$. Therefore the solution is $a_n=2-n+n^2$.