61. Let m be a positive integer. Let X_m be the random variable whose value is n if the mth success occurs on the $(n+m)$th trial when independent Bernoulli trials are performed, each with probability of success p.

a) Using Exercise 32 in the Supplementary Exercises of Chapter 7, show that the probability generating function G_{X_m} is given by $G_{X_m}(x) = p^m/(1-qx)^m$, where $q = 1-p$.

b) Find the expected value and the variance of X_m using Exercise 59 and the closed form for the probability generating function in part (a).

62. Show that if X and Y are independent random variables on a sample space S such that $X(s)$ and $Y(s)$ are nonnegative integers for all $s \in S$, then $G_{X+Y}(x) = G_X(x)G_Y(x)$.

8.5 Inclusion–Exclusion

8.5.1 Introduction

A discrete mathematics class contains 30 women and 50 sophomores. How many students in the class are either women or sophomores? This question cannot be answered unless more information is provided. Adding the number of women in the class and the number of sophomores probably does not give the correct answer, because women sophomores are counted twice. This observation shows that the number of students in the class that are either sophomores or women is the sum of the number of women and the number of sophomores in the class minus the number of women sophomores. A technique for solving such counting problems was introduced in Section 6.1. In this section we will generalize the ideas introduced in that section to solve problems that require us to count the number of elements in the union of more than two sets.

8.5.2 The Principle of Inclusion–Exclusion

How many elements are in the union of two finite sets? In Section 2.2 we showed that the number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

As we showed in Section 6.1, the formula for the number of elements in the union of two sets is useful in counting problems. Examples 1–3 provide additional illustrations of the usefulness of this formula.

EXAMPLE 1 In a discrete mathematics class every student is a major in computer science or mathematics, or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

Solution: Let A be the set of students in the class majoring in computer science and B be the set of students in the class majoring in mathematics. Then $A \cap B$ is the set of students in the class who are joint mathematics and computer science majors. Because every student in the class is majoring in either computer science or mathematics (or both), it follows that the number of students in the class is $|A \cup B|$. Therefore,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 25 + 13 - 8 = 30. \end{aligned}$$

Therefore, there are 30 students in the class. This computation is illustrated in Figure 1. ◄

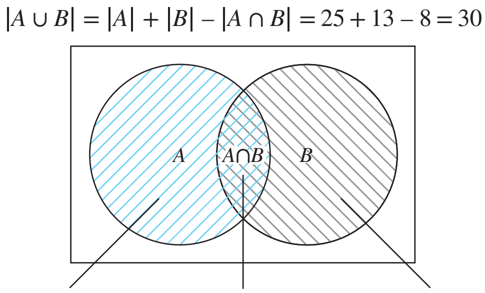


FIGURE 1 **The set of students in a discrete mathematics class.**

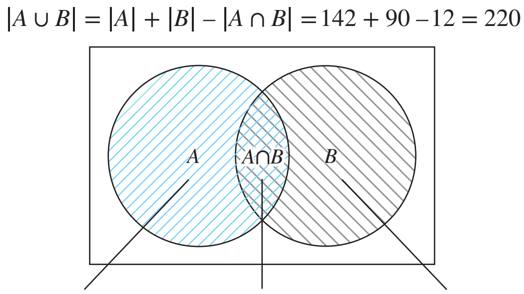


FIGURE 2 **The set of positive integers not exceeding 1000 divisible by either 7 or 11.**

EXAMPLE 2 How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution: Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of integers not exceeding 1000 that are divisible by either 7 or 11, and $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11. From Example 2 of Section 4.1, we know that among the positive integers not exceeding 1000 there are $\lfloor 1000/7 \rfloor$ integers divisible by 7 and $\lfloor 1000/11 \rfloor$ divisible by 11. Because 7 and 11 are relatively prime, the integers divisible by both 7 and 11 are those divisible by $7 \cdot 11$. Consequently, there are $\lfloor 1000/(11 \cdot 7) \rfloor$ positive integers not exceeding 1000 that are divisible by both 7 and 11. It follows that there are

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor \\
&= 142 + 90 - 12 = 220
\end{aligned}
$$

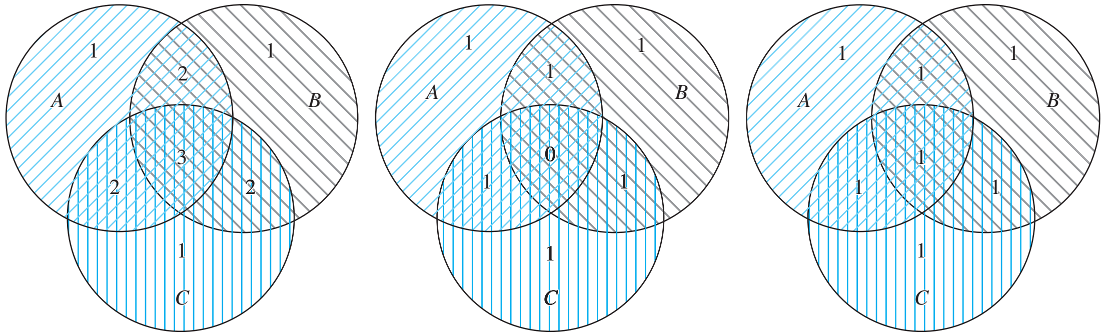
positive integers not exceeding 1000 that are divisible by either 7 or 11. This computation is illustrated in Figure 2. ◀

Example 3 shows how to find the number of elements in a finite universal set that are outside the union of two sets.

EXAMPLE 3 Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

Solution: To find the number of freshmen who are not taking a course in either mathematics or computer science, subtract the number that are taking a course in either of these subjects from the total number of freshmen. Let A be the set of all freshmen taking a course in computer science, and let B be the set of all freshmen taking a course in mathematics. It follows that $|A| = 453$, $|B| = 567$, and $|A \cap B| = 299$. The number of freshmen taking a course in either computer science or mathematics is

$$|A \cup B| = |A| + |B| - |A \cap B| = 453 + 567 - 299 = 721.$$



(a) Count of elements by $|A| + |B| + |C|$

(b) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

FIGURE 3 **Finding a formula for the number of elements in the union of three sets.**

Consequently, there are $1807 - 721 = 1086$ freshmen who are not taking a course in computer science or mathematics. ◀

We will now begin our development of a formula for the number of elements in the union of a finite number of sets. The formula we will develop is called the **principle of inclusion–exclusion**. For concreteness, before we consider unions of n sets, where n is any positive integer, we will derive a formula for the number of elements in the union of three sets A, B, and C. To construct this formula, we note that $|A| + |B| + |C|$ counts each element that is in exactly one of the three sets once, elements that are in exactly two of the sets twice, and elements in all three sets three times. This is illustrated in the first panel in Figure 3.

To remove the overcount of elements in more than one of the sets, we subtract the number of elements in the intersections of all pairs of the three sets. We obtain

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|.$$

This expression still counts elements that occur in exactly one of the sets once. An element that occurs in exactly two of the sets is also counted exactly once, because this element will occur in one of the three intersections of sets taken two at a time. However, those elements that occur in all three sets will be counted zero times by this expression, because they occur in all three intersections of sets taken two at a time. This is illustrated in the second panel in Figure 3.

To remedy this undercount, we add the number of elements in the intersection of all three sets. This final expression counts each element once, whether it is in one, two, or three of the sets. Thus,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

This formula is illustrated in the third panel of Figure 3.

Example 4 illustrates how this formula can be used.

EXAMPLE 4 A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both

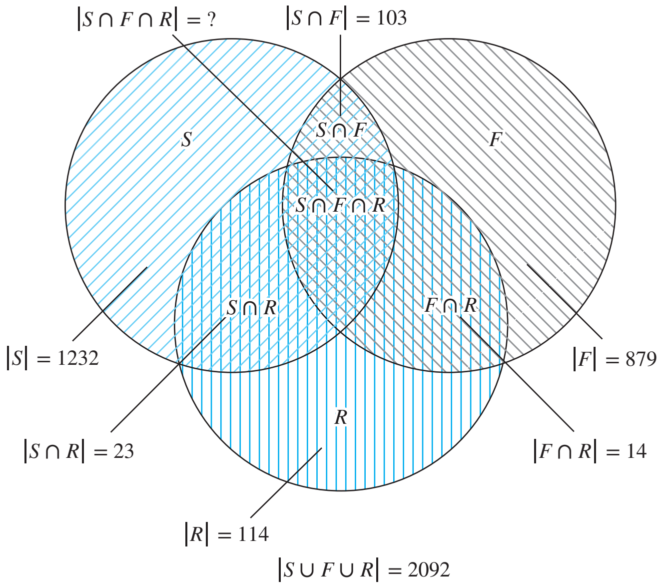


FIGURE 4 The set of students who have taken courses in Spanish, French, and Russian.

French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then

$$|S| = 1232, \quad |F| = 879, \quad |R| = 114,$$
$$|S \cap F| = 103, \ |S \cap R| = 23, \ |F \cap R| = 14,$$

and

$$|S \cup F \cup R| = 2092.$$

When we insert these quantities into the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

we obtain

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|.$$

We now solve for $|S \cap F \cap R|$. We find that $|S \cap F \cap R| = 7$. Therefore, there are seven students who have taken courses in Spanish, French, and Russian. This is illustrated in Figure 4. ◀

We will now state and prove the **inclusion–exclusion principle** for n sets, where n is a positive integer. This priniciple tells us that we can count the elements in a union of n sets by adding the number of elements in the sets, then subtracting the sum of the number of elements in all intersections of two of these sets, then adding the number of elements in all intersections of three of these sets, and so on, until we reach the number of elements in the intersection of all the sets. It is added when there is an odd number of sets and added when there is an even number of sets.

THEOREM 1

THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof: We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation. Suppose that a is a member of exactly r of the sets $A_1, A_2, \ldots, A_n$ where $1 \le r \le n$. This element is counted $C(r, 1)$ times by $\Sigma|A_i|$. It is counted $C(r, 2)$ times by $\Sigma|A_i \cap A_j|$. In general, it is counted $C(r, m)$ times by the summation involving m of the sets A_i. Thus, this element is counted exactly

$$C(r, 1) - C(r, 2) + C(r, 3) - \cdots + (-1)^{r+1}C(r, r)$$

times by the expression on the right-hand side of this equation. Our goal is to evaluate this quantity. By Corollary 2 of Section 6.4, we have

$$C(r, 0) - C(r, 1) + C(r, 2) - \cdots + (-1)^r C(r, r) = 0.$$

Hence,

$$1 = C(r, 0) = C(r, 1) - C(r, 2) + \cdots + (-1)^{r+1}C(r, r).$$

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the principle of inclusion–exclusion. ◁

The inclusion–exclusion principle gives a formula for the number of elements in the union of n sets for every positive integer n. There are terms in this formula for the number of elements in the intersection of every nonempty subset of the collection of the n sets. Hence, there are $2^n - 1$ terms in this formula.

EXAMPLE 5 Give a formula for the number of elements in the union of four sets.

Extra Examples

Solution: The inclusion–exclusion principle shows that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| \\ & - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ & + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

Note that this formula contains 15 different terms, one for each nonempty subset of $\{A_1, A_2, A_3, A_4\}$. ◀

Exercises

1. How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and
   a) $A_1 \cap A_2 = \emptyset$?
   b) $|A_1 \cap A_2| = 1$?
   c) $|A_1 \cap A_2| = 6$?
   d) $A_1 \subseteq A_2$?
2. There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?
3. A survey of households in the United States reveals that 96% have at least one television set, 98% have telephone service, and 95% have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?
4. A marketing report concerning personal computers states that 650,000 owners will buy a printer for their machines next year and 1,250,000 will buy at least one software package. If the report states that 1,450,000 owners will buy either a printer or at least one software package, how many will buy both a printer and at least one software package?
5. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if
   a) the sets are pairwise disjoint.
   b) there are 50 common elements in each pair of sets and no elements in all three sets.
   c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
   d) the sets are equal.
6. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1, 1000 in A_2, and 10,000 in A_3 if
   a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$.
   b) the sets are pairwise disjoint.
   c) there are two elements common to each pair of sets and one element in all three sets.
7. There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages?
8. In a survey of 270 college students, it is found that 64 like Brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both Brussels sprouts and broccoli, 28 like both Brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?
9. How many students are enrolled in a course either in calculus, discrete mathematics, data structures, or programming languages at a school if there are 507, 292, 312, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and programming languages; 211 in both discrete mathematics and data structures; 43 in both discrete mathematics and programming languages; and no student may take calculus and discrete mathematics, or data structures and programming languages, concurrently?
10. Find the number of positive integers not exceeding 100 that are not divisible by 5 or by 7.
11. Find the number of positive integers not exceeding 1000 that are not divisible by 3, 17, or 35.
12. Find the number of positive integers not exceeding 10,000 that are not divisible by 3, 4, 7, or 11.
13. Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.
14. Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.
15. How many bit strings of length eight do not contain six consecutive 0s?

*16. How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat* or *bird*?

17. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?
18. How many elements are in the union of four sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets?
19. How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?
20. How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion–exclusion?
21. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of five sets.
22. How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1000 common elements, each triple of sets has 100 common elements, every four of the sets have 10 common elements, and there is 1 element in all five sets?
23. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of six sets when it is known that no three of these sets have a common intersection.

***24.** Prove the principle of inclusion–exclusion using mathematical induction.

25. Let E_1, E_2, and E_3 be three events from a sample space S. Find a formula for the probability of $E_1 \cup E_2 \cup E_3$.

26. Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

27. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, either all are odd, all are divisible by 3, or all are divisible by 5.

28. Find a formula for the probability of the union of four events in a sample space if no three of them can occur at the same time.

29. Find a formula for the probability of the union of five events in a sample space if no four of them can occur at the same time.

30. Find a formula for the probability of the union of n events in a sample space when no two of these events can occur at the same time.

31. Find a formula for the probability of the union of n events in a sample space.

8.6 Applications of Inclusion–Exclusion

8.6.1 Introduction

Many counting problems can be solved using the principle of inclusion–exclusion. For instance, we can use this principle to find the number of primes less than a positive integer. Many problems can be solved by counting the number of onto functions from one finite set to another. The inclusion–exclusion principle can be used to find the number of such functions. The well-known hatcheck problem can be solved using the principle of inclusion–exclusion. This problem asks for the probability that no person is given the correct hat back by a hatcheck person who gives the hats back randomly.

8.6.2 An Alternative Form of Inclusion–Exclusion

There is an alternative form of the principle of inclusion–exclusion that is useful in counting problems. In particular, this form can be used to solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, \ldots, P_n$.

Let A_i be the subset containing the elements that have property P_i. The number of elements with all the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ will be denoted by $N(P_{i_1} P_{i_2} \ldots P_{i_k})$. Writing these quantities in terms of sets, we have

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = N(P_{i_1} P_{i_2} \ldots P_{i_k}).$$

If the number of elements with none of the properties $P_1, P_2, \ldots, P_n$ is denoted by $N(P'_1 P'_2 \ldots P'_n)$ and the number of elements in the set is denoted by N, it follows that

$$N(P'_1 P'_2 \ldots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

From the inclusion–exclusion principle, we see that

$$\begin{aligned} N(P'_1 P'_2 \ldots P'_n) = N &- \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) \\ &- \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n). \end{aligned}$$

Example 1 shows how the principle of inclusion–exclusion can be used to determine the number of solutions in integers of an equation with constraints.

EXAMPLE 1 How many solutions does

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$?

Solution: To apply the principle of inclusion–exclusion, let a solution have property P_1 if $x_1 > 3$, property P_2 if $x_2 > 4$, and property P_3 if $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3).$$

Using the same techniques as in Example 5 of Section 6.5, it follows that

- N = total number of solutions $= C(3 + 11 - 1, 11) = 78$,
- $N(P_1)$ = (number of solutions with $x_1 \geq 4$) $= C(3 + 7 - 1, 7) = C(9, 7) = 36$,
- $N(P_2)$ = (number of solutions with $x_2 \geq 5$) $= C(3 + 6 - 1, 6) = C(8, 6) = 28$,
- $N(P_3)$ = (number of solutions with $x_3 \geq 7$) $= C(3 + 4 - 1, 4) = C(6, 4) = 15$,
- $N(P_1P_2)$ = (number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$) $= C(3 + 2 - 1, 2) = C(4, 2) = 6$,
- $N(P_1P_3)$ = (number of solutions with $x_1 \geq 4$ and $x_3 \geq 7$) $= C(3 + 0 - 1, 0) = 1$,
- $N(P_2P_3)$ = (number of solutions with $x_2 \geq 5$ and $x_3 \geq 7$) $= 0$,
- $N(P_1P_2P_3)$ = (number of solutions with $x_1 \geq 4$, $x_2 \geq 5$, and $x_3 \geq 7$) $= 0$.

Inserting these quantities into the formula for $N(P_1'P_2'P_3')$ shows that the number of solutions with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ equals

$$N(P_1'P_2'P_3') = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6.$$

◀

8.6.3 The Sieve of Eratosthenes

In Section 4.3 we showed how to use the sieve of Eratosthenes to find all primes less than a specified positive integer n. Using the principle of inclusion–exclusion, we can find the number of primes not exceeding a specified positive integer with the same reasoning as is used in the sieve of Eratosthenes. Recall that a composite integer is divisible by a prime not exceeding its square root. So, to find the number of primes not exceeding 100, first note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes not exceeding 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, or 7. To apply the principle of inclusion–exclusion, let P_1 be the property that an integer is divisible by 2, let P_2 be the property that an integer is divisible by 3, let P_3 be the property that an integer is divisible by 5, and let P_4 be the property that an integer is divisible by 7. Thus, the number of primes not exceeding 100 is

$$4 + N(P_1'P_2'P_3'P_4').$$

$N(P_1'P_2'P_3'P_4') : 99$

Because there are 99 positive integers greater than 1 and not exceeding 100, the principle of inclusion–exclusion shows that

$$
\begin{aligned}
N(P_1'P_2'P_3'P_4') = 99 &- N(P_1) - N(P_2) - N(P_3) - N(P_4) \\
&+ N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_3P_4) \\
&- N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4) \\
&+ N(P_1P_2P_3P_4).
\end{aligned}
$$

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of $\{2, 3, 5, 7\}$ is $\lfloor 100/N \rfloor$, where N is the product of the primes in this subset. (This follows because any two of these primes have no common factor.) Consequently,

$$
\begin{aligned}
N(P_1'P_2'P_3'P_4') = 99 &- \left\lfloor \frac{100}{2} \right\rfloor - \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor \\
&+ \left\lfloor \frac{100}{2\cdot 3} \right\rfloor + \left\lfloor \frac{100}{2\cdot 5} \right\rfloor + \left\lfloor \frac{100}{2\cdot 7} \right\rfloor + \left\lfloor \frac{100}{3\cdot 5} \right\rfloor + \left\lfloor \frac{100}{3\cdot 7} \right\rfloor + \left\lfloor \frac{100}{5\cdot 7} \right\rfloor \\
&- \left\lfloor \frac{100}{2\cdot 3\cdot 5} \right\rfloor - \left\lfloor \frac{100}{2\cdot 3\cdot 7} \right\rfloor - \left\lfloor \frac{100}{2\cdot 5\cdot 7} \right\rfloor - \left\lfloor \frac{100}{3\cdot 5\cdot 7} \right\rfloor + \left\lfloor \frac{100}{2\cdot 3\cdot 5\cdot 7} \right\rfloor \\
&= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 \\
&= 21.
\end{aligned}
$$

Hence, there are $4 + 21 = 25$ primes not exceeding 100.

8.6.4 The Number of Onto Functions

The principle of inclusion–exclusion can also be used to determine the number of onto functions from a set with m elements to a set with n elements. First consider Example 2.

EXAMPLE 2 How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively. Note that a function is onto if and only if it has none of the properties P_1, P_2, or P_3. By the inclusion–exclusion principle it follows that the number of onto functions from a set with six elements to a set with three elements is

$$
\begin{aligned}
N(P_1'P_2'P_3') = N &- [N(P_1) + N(P_2) + N(P_3)] \\
&+ [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3),
\end{aligned}
$$

where N is the total number of functions from a set with six elements to one with three elements. We will evaluate each of the terms on the right-hand side of this equation.

From Example 6 of Section 6.1, it follows that $N = 3^6$. Note that $N(P_i)$ is the number of functions that do not have b_i in their range. Hence, there are two choices for the value of the function at each element of the domain. Therefore, $N(P_i) = 2^6$. Furthermore, there are $C(3, 1)$ terms of this kind. Note that $N(P_iP_j)$ is the number of functions that do not have b_i and b_j in their range. Hence, there is only one choice for the value of the function at each element of the domain. Therefore, $N(P_iP_j) = 1^6 = 1$. Furthermore, there are $C(3, 2)$ terms of this kind. Also, note that $N(P_1P_2P_3) = 0$, because this term is the number of functions that have none of b_1, b_2, and b_3 in their range. Clearly, there are no such functions, so the number of onto functions from a set with six elements to one with three elements is

$$3^6 - C(3, 1)2^6 + C(3, 2)1^6 = 729 - 192 + 3 = 540.$$ ◀

The general result that tells us how many onto functions there are from a set with m elements to one with n elements will now be stated. The proof of this result is left as an exercise for the reader.

THEOREM 1 Let m and n be positive integers with $m \geq n$. Then, there are

$$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - \cdots + (-1)^{n-1}C(n, n-1) \cdot 1^m$$

onto functions from a set with m elements to a set with n elements.

An onto function from a set with m elements to a set with n elements corresponds to a way to distribute the m elements in the domain to n indistinguishable boxes so that no box is empty, and then to associate each of the n elements of the codomain to a box. This means that the number of onto functions from a set with m elements to a set with n elements is the number of ways to distribute m distinguishable objects to n indistinguishable boxes so that no box is empty multiplied by the number of permutations of a set with n elements. Consequently, the number of onto functions from a set with m elements to a set with n elements equals $n!S(m, n)$, where $S(m, n)$ is a *Stirling number of the second kind* defined in Section 6.5. This means that we can use Theorem 1 to deduce the formula given in Section 6.5 for $S(m, n)$. (See Chapter 6 of [MiRo91] for more details about Stirling numbers of the second kind.)

Counting onto functions is much harder than counting one-to-one functions!

One of the many different applications of Theorem 1 will now be described.

EXAMPLE 3 How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job?

Solution: Consider the assignment of jobs as a function from the set of five jobs to the set of four employees. An assignment where every employee gets at least one job is the same as an onto function from the set of jobs to the set of employees. Hence, by Theorem 1 it follows that there are

$$4^5 - C(4, 1)3^5 + C(4, 2)2^5 - C(4, 3)1^5 = 1024 - 972 + 192 - 4 = 240$$

ways to assign the jobs so that each employee is assigned at least one job. ◀

8.6.5 Derangements

The principle of inclusion–exclusion will be used to count the permutations of n objects that leave no objects in their original positions. Consider Example 4.

EXAMPLE 4 **The Hatcheck Problem** A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

Remark: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats. We will return to this example after we find the number of permutations of n objects that leave no objects in their original position.

Links

A **derangement** is a permutation of objects that leaves no object in its original position. To solve the problem posed in Example 4 we will need to determine the number of derangements of a set of n objects.

EXAMPLE 5 The permutation 21453 is a derangement of 12345 because no number is left in its original position. However, 21543 is not a derangement of 12345, because this permutation leaves 4 fixed.

Let D_n denote the number of derangements of n objects. For instance, $D_3 = 2$, because the derangements of 123 are 231 and 312. We will evaluate D_n, for all positive integers n, using the principle of inclusion–exclusion.

THEOREM 2 The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

$!n : \lfloor n - {}^nC_1 \times (n-1)!$
$+ {}^nC_2 \times (n-2)!$

Proof: Let a permutation have property P_i if it fixes element i. The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$. This means that

$$D_n = N(P_1' P_2' \ldots P_n').$$

$-{}^nC_3$

Using the principle of inclusion–exclusion, it follows that

$$D_n = N - \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) - \sum_{i<j<k} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n),$$

where N is the number of permutations of n elements. This equation states that the number of permutations that fix no elements equals the total number of permutations, less the number that fix at least one element, plus the number that fix at least two elements, less the number that fix at least three elements, and so on. All the quantities that occur on the right-hand side of this equation will now be found.

First, note that $N = n!$, because N is simply the total number of permutations of n elements. Also, $N(P_i) = (n-1)!$. This follows from the product rule, because $N(P_i)$ is the number of permutations that fix element i, so the ith position of the permutation is determined, but each of the remaining positions can be filled arbitrarily. Similarly,

$$N(P_i P_j) = (n-2)!,$$

Links

HISTORICAL NOTE In *rencontres* (matches), an old French card game, the 52 cards in a deck are laid out in a row. The cards of a second deck are laid out with one card of the second deck on top of each card of the first deck. The score is determined by counting the number of matching cards in the two decks. In 1708 Pierre Raymond de Montmort (1678–1719) posed *le problème de rencontres:* What is the probability that no matches take place in the game of rencontres? The solution to Montmort's problem is the probability that a randomly selected permutation of 52 objects is a derangement, namely, $D_{52}/52!$, which, as we will see, is approximately $1/e$.

because this is the number of permutations that fix elements i and j, but where the other $n-2$ elements can be arranged arbitrarily. In general, note that

$$N(P_{i_1}P_{i_2}\dots P_{i_m}) = (n-m)!,$$

because this is the number of permutations that fix elements $i_1, i_2, \dots, i_m$, but where the other $n-m$ elements can be arranged arbitrarily. Because there are $C(n, m)$ ways to choose m elements from n, it follows that

$$\sum_{1\le i\le n} N(P_i) = C(n,1)(n-1)!,$$

$$\sum_{1\le i<j\le n} N(P_iP_j) = C(n,2)(n-2)!,$$

and in general,

$$\sum_{1\le i_1<i_2<\cdots<i_m\le n} N(P_{i_1}P_{i_2}\dots P_{i_m}) = C(n,m)(n-m)!.$$

Consequently, inserting these quantities into our formula for D_n gives

$$\begin{aligned} D_n &= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - \cdots + (-1)^n C(n,n)(n-n)! \\ &= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!\,0!}0!. \end{aligned}$$

Simplifying this expression gives

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right].$$

◁

It is now straightforward to find D_n for a given positive integer n. For instance, using Theorem 2, it follows that

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2,$$

as we have previously remarked.

The solution of the problem in Example 4 can now be given.

Solution: The probability that no one receives the correct hat is $D_n/n!$. By Theorem 2, this probability is

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}.$$

The values of this probability for $2 \le n \le 7$ are displayed in Table 1.

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

By the identity $e^x = \sum_{j=0}^{\infty} x^j/j!$ for all real numbers x (from calculus), we know that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} + \cdots \approx 0.368.$$

Because this is an alternating series with terms tending to zero, it follows that as n grows without bound, the probability that no one receives the correct hat converges to $e^{-1} \approx 0.368$. In fact, this probability can be shown to be within $1/(n+1)!$ of e^{-1}. ◀

Exercises

1. Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?

2. Of 1000 applicants for a mountain-climbing trip in the Himalayas, 450 get altitude sickness, 622 are not in good enough shape, and 30 have allergies. An applicant qualifies if and only if this applicant does not get altitude sickness, is in good shape, and does not have allergies. If there are 111 applicants who get altitude sickness and are not in good enough shape, 14 who get altitude sickness and have allergies, 18 who are not in good enough shape and have allergies, and 9 who get altitude sickness, are not in good enough shape, and have allergies, how many applicants qualify?

3. How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where x_1, x_2, and x_3 are nonnegative integers less than 6?

4. Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_i, $i = 1, 2, 3, 4$, are nonnegative integers such that $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 5$, and $x_4 \le 8$.

5. Find the number of primes less than 200 using the principle of inclusion–exclusion.

6. An integer is called **squarefree** if it is not divisible by the square of a positive integer greater than 1. Find the number of squarefree positive integers less than 100.

7. How many positive integers less than 10,000 are not the second or higher power of an integer?

8. How many onto functions are there from a set with seven elements to one with five elements?

9. How many ways are there to distribute six different toys to three different children such that each child gets at least one toy?

$3^8 - 3C_1 \times 2^7 + 3C_2 \times 1^7$

10. In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?

11. In how many ways can seven different jobs be assigned to four different employees so that each employee is assigned at least one job and the most difficult job is assigned to the best employee?

12. List all the derangements of $\{1, 2, 3, 4\}$.

$\lfloor n \times S_2(7,4)$ | (n)

13. How many derangements are there of a set with seven elements?

14. What is the probability that none of 10 people receives the correct hat if a hatcheck person hands their hats back randomly?

15. A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters

a) no letter is put into the correct envelope?

b) exactly one letter is put into the correct envelope?

c) exactly 98 letters are put into the correct envelopes?

d) exactly 99 letters are put into the correct envelopes?

e) all letters are put into the correct envelopes?

$100C_{98} \times D_2$

16. A group of n students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?

***17.** How many ways can the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 be arranged so that no even digit is in its original position?

$10! - (5C_1 \times \lfloor 9 - 5C_2 \times \lfloor 8 + 5C_3 \times \lfloor 7 \ldots)$

***18.** Use a combinatorial argument to show that the sequence $\{D_n\}$, where D_n denotes the number of derangements of n objects, satisfies the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

for $n \ge 2$. [*Hint:* Note that there are $n - 1$ choices for the first element k of a derangement. Consider separately the derangements that start with k that do and do not have 1 in the kth position.]

***19.** Use Exercise 18 to show that

$$D_n = nD_{n-1} + (-1)^n$$

for $n \ge 1$.

20. Use Exercise 19 to find an explicit formula for D_n.

21. For which positive integers n is D_n, the number of derangements of n objects, even?

22. Suppose that p and q are distinct primes. Use the principle of inclusion–exclusion to find $\phi(pq)$, the number of positive integers not exceeding pq that are relatively prime to pq.

***23.** Use the principle of inclusion–exclusion to derive a formula for $\phi(n)$ when the prime factorization of n is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

*24. Show that if n is a positive integer, then

$$n! = C(n, 0)D_n + C(n, 1)D_{n-1} + \cdots + C(n, n-1)D_1 + C(n, n)D_0,$$

where D_k is the number of derangements of k objects.

25. How many derangements of $\{1, 2, 3, 4, 5, 6\}$ begin with the integers 1, 2, and 3, in some order?

26. How many derangements of $\{1, 2, 3, 4, 5, 6\}$ end with the integers 1, 2, and 3, in some order?

27. Prove Theorem 1.

Key Terms and Results

TERMS

recurrence relation: a formula expressing terms of a sequence, except for some initial terms, as a function of one or more previous terms of the sequence

initial conditions for a recurrence relation: the values of the terms of a sequence satisfying the recurrence relation before this relation takes effect

dynamic programming: an algorithmic paradigm that finds the solution to an optimization problem by recursively breaking down the problem into overlapping subproblems and combining their solutions with the help of a recurrence relation

linear homogeneous recurrence relation with constant coefficients: a recurrence relation that expresses the terms of a sequence, except initial terms, as a linear combination of previous terms

characteristic roots of a linear homogeneous recurrence relation with constant coefficients: the roots of the polynomial associated with a linear homogeneous recurrence relation with constant coefficients

linear nonhomogeneous recurrence relation with constant coefficients: a recurrence relation that expresses the terms of a sequence, except for initial terms, as a linear combination of previous terms plus a function that is not identically zero that depends only on the index

divide-and-conquer algorithm: an algorithm that solves a problem recursively by splitting it into a fixed number of smaller nonoverlapping subproblems of the same type

generating function of a sequence: the formal series that has the nth term of the sequence as the coefficient of x^n

sieve of Eratosthenes: a procedure for finding the primes less than a specified positive integer

derangement: a permutation of objects such that no object is in its original place

RESULTS

the formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

the formula for the number of elements in the union of three finite sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

the principle of inclusion–exclusion:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

the number of onto functions from a set with m elements to a set with n elements:

$$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - \cdots + (-1)^{n-1}C(n, n-1) \cdot 1^m$$

the number of derangements of n objects:

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right]$$

Review Questions

1. a) What is a recurrence relation?
   b) Find a recurrence relation for the amount of money that will be in an account after n years if \$1,000,000 is deposited in an account yielding 9% annually.

2. Explain how the Fibonacci numbers are used to solve Fibonacci's problem about rabbits.

3. a) Find a recurrence relation for the number of steps needed to solve the Tower of Hanoi puzzle.
   b) Show how this recurrence relation can be solved using iteration.

4. a) Explain how to find a recurrence relation for the number of bit strings of length n not containing two consecutive 1s.
   b) Describe another counting problem that has a solution satisfying the same recurrence relation.

$0\ D_{n-1} +$