## SECTION 6.2 The Pigeonhole Principle

**2.** This follows from the pigeonhole principle, with $k = 26$.

**4.** We assume that the woman does not replace the balls after drawing them.

**a)** There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains three pigeons. By the generalized pigeonhole principle, the answer is 5. If five balls are selected, at least $\lceil 5/2 \rceil = 3$ must have the same color. On the other hand four balls is not enough, because two might be red and two might be blue. Note that the number of balls was irrelevant (assuming that it was at least 5).

**b)** She needs to select 13 balls in order to insure at least three blue ones. If she does so, then at most 10 of them are red, so at least three are blue. On the other hand, if she selects 12 or fewer balls, then 10 of them could be red, and she might not get her three blue balls. This time the number of balls did matter.

**6.** There are only $d$ possible remainders when an integer is divided by $d$, namely $0, 1, \ldots, d-1$. By the pigeonhole principle, if we have $d+1$ remainders, then at least two must be the same.

**8.** This is just a restatement of the pigeonhole principle, with $k = |T|$.

**10.** The midpoint of the segment whose endpoints are $(a,b)$ and $(c,d)$ is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if $a$ and $c$ have the same parity (both odd or both even) and $b$ and $d$ have the same parity. Thus what matters in this problem is the parities of the coordinates. There are four possible pairs of parities: (odd, odd), (odd, even), (even, odd), and (even, even). Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

**12.** This is similar in spirit to Exercise 10. Working modulo 5 there are 25 pairs: $(0,0)$, $(0,1)$, $\ldots$, $(4,4)$. Thus we could have 25 ordered pairs of integers $(a,b)$ such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

**14. a)** We can group the first ten positive integers into five subsets of two integers each, each subset adding up to 11: $\{1,10\}$, $\{2,9\}$, $\{3,8\}$, $\{4,7\}$, and $\{5,6\}$. If we select seven integers from this set, then by the pigeonhole principle at least two of them come from the same subset. Furthermore, if we forget about these two in the same group, then there are five more integers and four groups; again the pigeonhole principle guarantees two integers in the same group. This gives us two pairs of integers, each pair from the same group. In each case these two integers have a sum of 11, as desired.

**b)** No. The set $\{1,2,3,4,5,6\}$ has only 5 and 6 from the same group, so the only pair with sum 11 is 5 and 6.

**16.** We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 16, namely $\{1,15\}$, $\{3,13\}$, $\{5,11\}$, and $\{7,9\}$. If we select five numbers from the set $\{1,3,5,7,9,11,13,15\}$, then at least two of them must fall within the same subset, since there are only four subsets. Two numbers in the same subset are the desired pair that add up to 16. We also need to point out that choosing four numbers is not enough, since we could choose $\{1,3,5,7\}$, and no pair of them add up to more than 12.

**18. a)** If not, then there would be 4 or fewer male students and 4 or fewer female students, so there would be $4 + 4 = 8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

**b)** If not, then there would be 2 or fewer male students and 6 or fewer female students, so there would be $2 + 6 = 8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

**20.** One maximal length increasing sequence is $5, 7, 10, 15, 21$. One maximal length decreasing sequence is $22, 7, 3$. See Exercise 25 for an algorithm.

**22.** This follows immediately from Theorem 3, with $n = 10$.

**24.** This problem was on the International Mathematical Olympiad in 2001, a test taken by the six best high school students from each country. Here is a paraphrase of a solution posted on the Web by Steve Olson, author of a book about this competition entitled *Count Down*. Make a table listing the 21 boys at the top of each column and the 21 girls to the left of each row. This table will contain $21 \cdot 21 = 441$ boxes. In each box write the number of a problem solved by both that girl and that boy. From the given information, each box will contain a number. Each contestant solved at most six problems, so only six different numbers can appear in any given row or column of 21 boxes. Because $5 \cdot 2 = 10$, at least $21 - 10 = 11$ of the boxes in any given row or column must contain problem numbers that appear three or more times in that row. (This is an application of the idea of the pigeonhole principle.) In each row color red all the boxes containing problem numbers that appear at least three times in that row. So each row will have at least 11 red boxes, and therefore there will be at least $11 \cdot 21 = 231$ boxes colored red. Repeat the process with the columns, using the color blue. Because at least 231 boxes are red and 231 are blue, and there are only 441 boxes in all, some of the boxes will be both red and blue. (Here is the second place where the pigeonhole principle is used.) The problem number in a doubly-colored box represents a problem solved by at least three girls and at least three boys.

**26.** Let the people be $A$, $B$, $C$, $D$, and $E$. Suppose the following pairs are friends: $A$–$B$, $B$–$C$, $C$–$D$, $D$–$E$, and $E$–$A$. The other five pairs are enemies. In this example, there are no three mutual friends and no three mutual enemies.

**28.** Let $A$ be one of the people. She must have either 10 friends or 10 enemies, since if there were 9 or fewer of each, then that would account for at most 18 of the 19 other people. Without loss of generality assume that $A$ has 10 friends. By Exercise 27 there are either 4 mutual enemies among these 10 people, or 3 mutual friends. In the former case we have our desired set of 4 mutual enemies; in the latter case, these 3 people together with $A$ form the desired set of 4 mutual friends.

**30.** This is clear by symmetry, since we can just interchange the notions of friends and enemies.

**32.** There are 99,999,999 possible positive salaries less than one million dollars, i.e., from \$0.01 to \$999,999.99. By the pigeonhole principle, if there were more than this many people with positive salaries less than one million dollars, then at least two of them must have the same salary.

**34.** This follows immediately from Theorem 2, with $N = 8{,}008{,}278$ and $k = 1{,}000{,}001$ (the number of hairs can be anywhere from 0 to a million).

**36.** Let $K(x)$ be the number of other computers that computer $x$ is connected to. The possible values for $K(x)$ are $1, 2, 3, 4, 5$. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.

**38.** This is similar to Example 9. Label the computers $C_1$ through $C_8$, and label the printers $P_1$ through $P_4$. If we connect $C_k$ to $P_k$ for $k = 1, 2, 3, 4$ and connect each of the computers $C_5$ through $C_8$ to *all* the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables. Clearly this is sufficient, because if computers $C_1$ through $C_4$ need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 19 cables are not enough. Since there are 19 cables and 4 printers, the average number of computers per printer is $19/4$, which is less than 5. Therefore some printer must be connected to fewer than 5 computers (the average of a set of numbers cannot be bigger than each of the numbers in the set). That means it is connected to 4 or fewer computers, so there are at least 4 computers that are not connected to it. If those 4 computers all needed a printer simultaneously, then they would be out of luck, since they are connected to at most the 3 other printers.

**40.** Let $K(x)$ be the number of other people at the party that person $x$ knows. The possible values for $K(x)$ are $0, 1, \ldots, n-1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are $n$ pigeons and $n$ pigeonholes. However, it is impossible for both 0 and $n-1$ to be in the range of $K$, since if one person knows everybody else, then nobody can know no one else (we assume that "knowing" is symmetric). Therefore the range of $K$ has at most $n-1$ elements, whereas the domain has $n$ elements, so $K$ is not one-to-one, precisely what we wanted to prove.

**42.** **a)** The solution of Exercise 41, with 24 replaced by 2 and 149 replaced by 127, tells us that the statement is true.

**b)** The solution of Exercise 41, with 24 replaced by 23 and 149 replaced by 148, tells us that the statement is true.

**c)** We begin in a manner similar to the solution of Exercise 41. Look at $a_1, a_2, \ldots, a_{75}, a_1+25, \ldots, a_{75}+25$, where $a_i$ is the total number of matches played up through and including hour $i$. Then $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$, and $26 \leq a_1 + 25 < a_2 + 25 < \cdots < a_{75} + 25 \leq 150$. Now either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get, as in Exercise 41, $a_i = a_j + 25$ for some $i$ and $j$ and we are done. In the former case, however, since each of the numbers $a_i + 25$ is greater than or equal to 26, the numbers $1, 2, \ldots, 25$ must all appear among the $a_i$'s. But since the $a_i$'s are increasing, the only way this can happen is if $a_1 = 1$, $a_2 = 2$, ..., $a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

**d)** We need a different approach for this part, an approach, incidentally, that works for many numbers besides 30 in this setting. Let $a_1$, $a_2$, ..., $a_{75}$ be as before, and note that $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$. By the pigeonhole principle two of the numbers among $a_1$, $a_2$, ..., $a_{31}$ are congruent modulo 30. If they differ by 30, then we have our solution. Otherwise they differ by 60 or more, so $a_{31} \geq 61$. Similarly, among $a_{31}$ through $a_{61}$, either we find a solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{61} \geq 121$. But this means that $a_{66} \geq 126$, a contradiction.

**44.** Look at the pigeonholes $\{1000, 1001\}$, $\{1002, 1003\}$, $\{1004, 1005\}$, ..., $\{1098, 1099\}$. There are clearly 50 sets in this list. By the pigeonhole principle, if we have 51 numbers in the range from 1000 to 1099 inclusive, then at least two of them must come from the same set. These are the desired two consecutive house numbers.

**46.** Suppose this statement were not true. Then for each $i$, the $i^{\text{th}}$ box contains at most $n_i - 1$ objects. Adding, we have at most $(n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1) = n_1 + n_2 + \cdots + n_t - t$ objects in all, contradicting the fact that there were $n_1 + n_2 + \cdots + n_t - t + 1$ objects in all. Therefore the statement must be true.