*57. Dynamic programming can be used to develop an algorithm for solving the matrix-chain multiplication problem introduced in Section 3.3. This is the problem of determining how the product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n$ can be computed using the fewest integer multiplications, where $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ are $m_1 \times m_2, m_2 \times m_3, \dots, m_n \times m_{n+1}$ matrices, respectively, and each matrix has integer entries. Recall that by the associative law, the product does not depend on the order in which the matrices are multiplied.

a) Show that the brute-force method of determining the minimum number of integer multiplications needed to solve a matrix-chain multiplication problem has exponential worst-case complexity. [*Hint:* Do this by first showing that the order of multiplication of matrices is specified by parenthesizing the product. Then, use Example 5 and the result of part (c) of Exercise 43 in Section 8.4.]

b) Denote by $\mathbf{A}_{ij}$ the product $\mathbf{A}_i\mathbf{A}_{i+1} \dots, \mathbf{A}_j$, and $M(i, j)$ the minimum number of integer multiplications required to find $\mathbf{A}_{ij}$. Show that if the least number of integer multiplications are used to compute $\mathbf{A}_{ij}$, where $i < j$, by splitting the product into the product of $\mathbf{A}_i$ through $\mathbf{A}_k$ and the product of $\mathbf{A}_{k+1}$ through $\mathbf{A}_j$, then the first k terms must be parenthesized so that $\mathbf{A}_{ik}$ is computed in the optimal way using $M(i, k)$ integer multiplications, and $\mathbf{A}_{k+1,j}$ must be parenthesized so that $\mathbf{A}_{k+1,j}$ is computed in the optimal way using $M(k + 1, j)$ integer multiplications.

c) Explain why part (b) leads to the recurrence relation $M(i, j) = \min_{i \le k < j}(M(i, k) + M(k + 1, j) + m_i m_{k+1} m_{j+1})$ if $1 \le i \le j < j \le n$.

d) Use the recurrence relation in part (c) to construct an efficient algorithm for determining the order the n matrices should be multiplied to use the minimum number of integer multiplications. Store the partial results $M(i, j)$ as you find them so that your algorithm will not have exponential complexity.

e) Show that your algorithm from part (d) has $O(n^3)$ worst-case complexity in terms of multiplications of integers.

8.2 Solving Linear Recurrence Relations

8.2.1 Introduction

Links

A wide variety of recurrence relations occur in models. Some of these recurrence relations can be solved using iteration or some other ad hoc technique. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are recurrence relations that express the terms of a sequence as linear combinations of previous terms.

Definition 1

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation in the definition is **linear** because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation is **homogeneous** because no terms occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on n. The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \dots, a_{k-1} = C_{k-1}.$$

EXAMPLE 1 The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five. ◀

To help clarify the definition of linear homogeneous recurrence relations with constant coefficients, we will now provide examples of recurrence relations each lacking one of the defining properties.

EXAMPLE 2 The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients. ◀

Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved.

8.2.2 Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Recurrence relations may be difficult to solve, but fortunately this is not the case for linear homogenous recurrence relations with constant coefficients. We can use two key ideas to find all their solutions. First, these recurrence relations have solutions of the form $a_n = r^n$, where r is a constant. To see this, observe that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}.$$

When both sides of this equation are divided by r^{n-k} (when $r \neq 0$) and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0.$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ where $r \neq 0$ is a solution if and only if r is a solution of this last equation. We call this the **characteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

The other key observation is that a linear combination of two solutions of a linear homogeneous recurrence relation is also a solution. To see this, suppose that s_n and t_n are both solutions of $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$. Then

$$s_n = c_1s_{n-1} + c_2s_{n-2} + \cdots + c_ks_{n-k}$$

and

$$t_n = c_1t_{n-1} + c_2t_{n-2} + \cdots + c_kt_{n-k}.$$

Now suppose that b_1 and b_2 are real numbers. Then

$$\begin{aligned} b_1s_n + b_2t_n &= b_1(c_1s_{n-1} + c_2s_{n-2} + \cdots + c_ks_{n-k}) + b_2(c_1t_{n-1} + c_2t_{n-2} + \cdots + c_kt_{n-k}) \\ &= c_1(b_1s_{n-1} + b_2t_{n-1}) + c_2(b_1s_{n-2} + b_2t_{n-2}) + \cdots + c_k(b_1s_{n-k} + b_kt_{n-k}). \end{aligned}$$

This means that $b_1s_n + b_2t_n$ is also a solution of the same linear homogeneous recurrence relation.

Using these key observations, we will show how to solve linear homogeneous recurrence relations with constant coefficients.

THE DEGREE TWO CASE We now turn our attention to linear homogeneous recurrence relations of degree two. First, consider the case when there are two distinct characteristic roots.

THEOREM 1 Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: We must do two things to prove the theorem. First, it must be shown that if r_1 and r_2 are the roots of the characteristic equation, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation. Second, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2.

We now show that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Because r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$.

From these equations, we see that

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n. \end{aligned}$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions. This requires that

$$\begin{aligned} a_0 &= C_0 = \alpha_1 + \alpha_2, \\ a_1 &= C_1 = \alpha_1 r_1 + \alpha_2 r_2. \end{aligned}$$

We can solve these two equations for α_1 and α_2. From the first equation it follows that $\alpha_2 = C_0 - \alpha_1$. Inserting this expression into the second equation gives

$$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2.$$

Hence,

$$C_1 = \alpha_1(r_1 - r_2) + C_0 r_2.$$

This shows that

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

and

$$\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

where these expressions for α_1 and α_2 depend on the fact that $r_1 \neq r_2$. (When $r_1 = r_2$, this theorem is not true.) Hence, with these values for α_1 and α_2, the sequence $\{a_n\}$ with $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.

We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$. Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all nonnegative integers n. We have completed the proof by showing that a solution of the linear homogeneous recurrence relation with constant coefficients of degree two must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants. ◁

The characteristic roots of a linear homogeneous recurrence relation with constant coefficients may be complex numbers. Theorem 1 (and also subsequent theorems in this section) still applies in this case. Recurrence relations with complex characteristic roots will not be discussed in the text. Readers familiar with complex numbers may wish to solve Exercises 38 and 39.

Examples 3 and 4 show how to use Theorem 1 to solve recurrence relations.

EXAMPLE 3 What is the solution of the recurrence relation

Extra Examples

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution: Theorem 1 can be used to solve this problem. The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$
$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$ ◀

EXAMPLE 4 Find an explicit formula for the Fibonacci numbers.

Solution: Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Therefore, from Theorem 1 it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2. The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants. We have

$$f_0 = \alpha_1 + \alpha_2 = 0,$$
$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

The solution to these simultaneous equations for α_1 and α_2 is

$$\alpha_1 = 1/\sqrt{5}, \qquad \alpha_2 = -1/\sqrt{5}.$$

Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

◀

Theorem 1 does not apply when there is one characteristic root of multiplicity two. If this happens, then $a_n = nr_0^n$ is another solution of the recurrence relation when r_0 is a root of multiplicity two of the characteristic equation. Theorem 2 shows how to handle this case.

THEOREM 2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

The proof of Theorem 2 is left as Exercise 10. Example 5 illustrates the use of this theorem.

EXAMPLE 5 What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2. Using the initial conditions, it follows that

$$a_0 = 1 = \alpha_1,$$
$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$. Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n.$$

◀

THE GENERAL CASE We will now state the general result about the solution of linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots. The proof of this result will be left as Exercise 16.

THEOREM 3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

We illustrate the use of the theorem with Example 6.

EXAMPLE 6 Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3, \\ a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9. \end{aligned}$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$ ◀

We now state the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots. The key point is that for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m-1$, with m the multiplicity of this root. We leave the proof of this result as Exercise 51.

THEOREM 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = \; & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example 7 illustrates how Theorem 4 is used to find the general form of a solution of a linear homogeneous recurrence relation when the characteristic equation has several repeated roots.

EXAMPLE 7 Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

Solution: By Theorem 4, the general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2) 2^n + (\alpha_{2,0} + \alpha_{2,1} n) 5^n + \alpha_{3,0} 9^n.$$

◀

We now illustrate the use of Theorem 4 to solve a linear homogeneous recurrence relation with constant coefficients when the characteristic equation has a root of multiplicity three.

EXAMPLE 8 Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Because $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1} n(-1)^n + \alpha_{1,2} n^2 (-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$\begin{aligned}a_0 &= 1 = \alpha_{1,0},\\ a_1 &= -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},\\ a_2 &= -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.\end{aligned}$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

◀

8.2.3 Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

We have seen how to solve linear homogeneous recurrence relations with constant coefficients. Is there a relatively simple technique for solving a linear, but not homogeneous, recurrence relation with constant coefficients, such as $a_n = 3a_{n-1} + 2n$? We will see that the answer is yes for certain families of such recurrence relations.

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a **linear nonhomogeneous recurrence relation with constant coefficients**, that is, a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$$

is called the **associated homogeneous recurrence relation**. It plays an important role in the solution of the nonhomogeneous recurrence relation.

EXAMPLE 9 Each of the recurrence relations $a_n = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $a_n = 3a_{n-1} + n3^n$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ is a linear nonhomogeneous recurrence relation with constant coefficients. The associated linear homogeneous recurrence relations are $a_n = a_{n-1}$, $a_n = a_{n-1} + a_{n-2}$, $a_n = 3a_{n-1}$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, respectively. ◀

The key fact about linear nonhomogeneous recurrence relations with constant coefficients is that every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation, as Theorem 5 shows.

THEOREM 5 If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}.$$

Proof: Because $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation, we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^p\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$. Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n. ◁

By Theorem 5, we see that the key to solving nonhomogeneous recurrence relations with constant coefficients is finding a particular solution. Then every solution is a sum of this solution and a solution of the associated homogeneous recurrence relation. Although there is no general method for finding such a solution that works for every function $F(n)$, there are techniques that work for certain types of functions $F(n)$, such as polynomials and powers of constants. This is illustrated in Examples 10 and 11.

EXAMPLE 10 Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: To solve this linear nonhomogeneous recurrence relation with constant coefficients, we need to solve its associated linear homogeneous equation and to find a particular solution for the given nonhomogeneous equation. The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

We now find a particular solution. Because $F(n) = 2n$ is a polynomial in n of degree one, a reasonable trial solution is a linear function in n, say, $p_n = cn + d$, where c and d are constants. To determine whether there are any solutions of this form, suppose that $p_n = cn + d$ is such a solution. Then the equation $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$. Simplifying and combining like terms gives $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. This shows that $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$. Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5 all solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n,$$

where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the formula we obtained for the general solution. We find that $3 = -1 - 3/2 + 3\alpha$, which implies that $\alpha = 11/6$. The solution we seek is $a_n = -n - 3/2 + (11/6)3^n$. ◀

EXAMPLE 11 Find all solutions of the recurrence relation

Extra Examples

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants. Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant. Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$. Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C = 49$, or that $C = 49/20$. Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution. By Theorem 5, all solutions are of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$ ◀

In Examples 10 and 11, we made an educated guess that there are solutions of a particular form. In both cases we were able to find particular solutions. This was not an accident. Whenever $F(n)$ is the product of a polynomial in n and the nth power of a constant, we know exactly what form a particular solution has, as stated in Theorem 6. We leave the proof of Theorem 6 as Exercise 52.

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0)s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0)s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0)s^n.$$

Note that in the case when s is a root of multiplicity m of the characteristic equation of the associated linear homogeneous recurrence relation, the factor n^m ensures that the proposed particular solution will not already be a solution of the associated linear homogeneous recurrence relation. We next provide Example 12 to illustrate the form of a particular solution provided by Theorem 6.

EXAMPLE 12 What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root, 3, of multiplicity two. To apply Theorem 6, with $F(n)$ of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.

Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has the form $p_0 n^2 3^n$ if $F(n) = 3^n$, the form $n^2(p_1 n + p_0)3^n$ if $F(n) =$

$n3^n$, the form $(p_2n^2 + p_1n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2n^2 + p_1n + p_0)3^n$ if $F(n) = (n^2 + 1)3^n$. ◀

Care must be taken when $s = 1$ when solving recurrence relations of the type covered by Theorem 6. In particular, to apply this theorem with $F(n) = b_t n_t + b_{t-1} n_{t-1} + \cdots + b_1 n + b_0$, the parameter s takes the value $s = 1$ (even though the term 1^n does not explicitly appear). By the theorem, the form of the solution then depends on whether 1 is a root of the characteristic equation of the associated linear homogeneous recurrence relation. This is illustrated in Example 13, which shows how Theorem 6 can be used to find a formula for the sum of the first n positive integers.

EXAMPLE 13 Let a_n be the sum of the first n positive integers, so that

$$a_n = \sum_{k=1}^{n} k.$$

Note that a_n satisfies the linear nonhomogeneous recurrence relation

$$a_n = a_{n-1} + n.$$

(To obtain a_n, the sum of the first n positive integers, from a_{n-1}, the sum of the first $n - 1$ positive integers, we add n.) Note that the initial condition is $a_1 = 1$.

The associated linear homogeneous recurrence relation for a_n is

$$a_n = a_{n-1}.$$

The solutions of this homogeneous recurrence relation are given by $a_n^{(h)} = c(1)^n = c$, where c is a constant. To find all solutions of $a_n = a_{n-1} + n$, we need find only a single particular solution. By Theorem 6, because $F(n) = n = n \cdot (1)^n$ and $s = 1$ is a root of degree one of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $n(p_1n + p_0) = p_1n^2 + p_0n$.

Inserting this into the recurrence relation gives $p_1n^2 + p_0n = p_1(n-1)^2 + p_0(n-1) + n$. Simplifying, we see that $n(2p_1 - 1) + (p_0 - p_1) = 0$, which means that $2p_1 - 1 = 0$ and $p_0 - p_1 = 0$, so $p_0 = p_1 = 1/2$. Hence,

$$a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

is a particular solution. Hence, all solutions of the original recurrence relation $a_n = a_{n-1} + n$ are given by $a_n = a_n^{(h)} + a_n^{(p)} = c + n(n+1)/2$. Because $a_1 = 1$, we have $1 = a_1 = c + 1 \cdot 2/2 = c + 1$, so $c = 0$. It follows that $a_n = n(n+1)/2$. (This is the same formula given in Table 2 in Section 2.4 and derived previously.) ◀

Exercises

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
b) $a_n = 2na_{n-1} + a_{n-2}$
c) $a_n = a_{n-1} + a_{n-4}$
d) $a_n = a_{n-1} + 2$
e) $a_n = a_{n-1}^2 + a_{n-2}$
f) $a_n = a_{n-2}$
g) $a_n = a_{n-1} + n$

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$
b) $a_n = 3$
c) $a_n = a_{n-1}^2$
d) $a_n = a_{n-1} + 2a_{n-3}$
e) $a_n = a_{n-1}/n$
f) $a_n = a_{n-1} + a_{n-2} + n + 3$
g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

5. How many different messages can be transmitted in n microseconds using the two signals described in Exercise 19 in Section 8.1?

6. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

7. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

b) Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

9. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

a) Find a recurrence relation for $\{P_n\}$, where P_n is the amount in the account at the end of n years if no money is ever withdrawn.

b) How much is in the account after n years if no money has been withdrawn?

***10.** Prove Theorem 2.

11. The **Lucas numbers** satisfy the recurrence relation

Links
$$L_n = L_{n-1} + L_{n-2},$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

a) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, \ldots$, where f_n is the nth Fibonacci number.

b) Find an explicit formula for the Lucas numbers.

12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

14. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

15. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

***16.** Prove Theorem 3.

17. Prove this identity relating the Fibonacci numbers and the binomial coefficients:

$$f_{n+1} = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k),$$

where n is a positive integer and $k = \lfloor n/2 \rfloor$. [*Hint:* Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$. Show that the sequence $\{a_n\}$ satisfies the same recurrence relation and initial conditions satisfied by the sequence of Fibonacci numbers.]

18. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5$, $a_1 = 4$, and $a_2 = 88$.

19. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, −2, −2, −2, 3, 3, −4?

22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots −1, −1, −1, 2, 2, 5, 5, 7?

23. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 1$.

24. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.

a) Show that $a_n = n2^n$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 2$.

25. a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution of this recurrence relation with $a_0 = 4$.

26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

a) $F(n) = n^2$? **b)** $F(n) = 2^n$?
c) $F(n) = n2^n$? **d)** $F(n) = (-2)^n$?
e) $F(n) = n^2 2^n$? **f)** $F(n) = n^3(-2)^n$?
g) $F(n) = 3$?

27. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

a) $F(n) = n^3$? **b)** $F(n) = (-2)^n$?
c) $F(n) = n2^n$? **d)** $F(n) = n^2 4^n$?
e) $F(n) = (n^2 - 2)(-2)^n$? **f)** $F(n) = n^4 2^n$?
g) $F(n) = 2$?

28. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.
b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

29. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.
b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.
b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.

31. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$. [*Hint:* Look for a particular solution of the form $qn2^n + p_1 n + p_2$, where q, p_1, and p_2 are constants.]

32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.

33. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.

34. Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2$, $a_1 = 0$, and $a_2 = 5$.

35. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.

36. Let a_n be the sum of the first n perfect squares, that is, $a_n = \sum_{k=1}^{n} k^2$. Show that the sequence $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.

37. Let a_n be the sum of the first n triangular numbers, that is, $a_n = \sum_{k=1}^{n} t_k$, where $t_k = k(k+1)/2$. Show that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n(n+1)/2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.

38. a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$. [*Note:* These are complex numbers.]
b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$ and $a_1 = 2$.

***39. a)** Find the characteristic roots of the linear homogeneous recurrence relation $a_n = a_{n-4}$. [*Note:* These include complex numbers.]
b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 1$.

***40.** Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

***41. a)** Use the formula found in Example 4 for f_n, the nth Fibonacci number, to show that f_n is the integer closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

b) Determine for which n f_n is greater than

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and for which n f_n is less than

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

42. Show that if $a_n = a_{n-1} + a_{n-2}$, $a_0 = s$ and $a_1 = t$, where s and t are constants, then $a_n = sf_{n-1} + tf_n$ for all positive integers n.

43. Express the solution of the linear nonhomogenous recurrence relation $a_n = a_{n-1} + a_{n-2} + 1$ for $n \geq 2$ where $a_0 = 0$ and $a_1 = 1$ in terms of the Fibonacci numbers. [*Hint:* Let $b_n = a_n + 1$ and apply Exercise 42 to the sequence b_n.]

***44.** (*Linear algebra required*) Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2s on its main diagonal, 1s in all positions next to a diagonal element, and 0s everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.

45. Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of 1 month and six new pairs of rabbits at the age of 2 months and every month afterward. None of the rabbits ever die or leave the island.
a) Find a recurrence relation for the number of pairs of rabbits on the island n months after one newborn pair is left on the island.
b) By solving the recurrence relation in (a) determine the number of pairs of rabbits on the island n months after one pair is left on the island.

46. Suppose that there are two goats on an island initially. The number of goats on the island doubles every year by natural reproduction, and some goats are either added or removed each year.

a) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that during each year an extra 100 goats are put on the island.

b) Solve the recurrence relation from part (a) to find the number of goats on the island at the start of the nth year.

c) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that n goats are removed during the nth year for each $n \geq 3$.

d) Solve the recurrence relation in part (c) for the number of goats on the island at the start of the nth year.

47. A new employee at an exciting new software company starts with a salary of \$50,000 and is promised that at the end of each year her salary will be double her salary of the previous year, with an extra increment of \$10,000 for each year she has been with the company.

a) Construct a recurrence relation for her salary for her nth year of employment.

b) Solve this recurrence relation to find her salary for her nth year of employment.

Some linear recurrence relations that do not have constant coefficients can be systematically solved. This is the case for recurrence relations of the form $f(n)a_n = g(n)a_{n-1} + h(n)$. Exercises 48–50 illustrate this.

***48. a)** Show that the recurrence relation

$$f(n)a_n = g(n)a_{n-1} + h(n),$$

for $n \geq 1$, and with $a_0 = C$, can be reduced to a recurrence relation of the form

$$b_n = b_{n-1} + Q(n)h(n),$$

where $b_n = g(n+1)Q(n+1)a_n$, with

$$Q(n) = (f(1)f(2)\cdots f(n-1))/(g(1)g(2)\cdots g(n)).$$

b) Use part (a) to solve the original recurrence relation to obtain

$$a_n = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)}.$$

***49.** Use Exercise 48 to solve the recurrence relation $(n+1)a_n = (n+3)a_{n-1} + n$, for $n \geq 1$, with $a_0 = 1$.

50. It can be shown that C_n, the average number of comparisons made by the quick sort algorithm (described in preamble to Exercise 50 in Section 5.4), when sorting n elements in random order, satisfies the recurrence relation

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

for $n = 1, 2, \ldots,$ with initial condition $C_0 = 0$.

a) Show that $\{C_n\}$ also satisfies the recurrence relation $nC_n = (n+1)C_{n-1} + 2n$ for $n = 1, 2, \ldots$.

b) Use Exercise 48 to solve the recurrence relation in part (a) to find an explicit formula for C_n.

****51.** Prove Theorem 4.

****52.** Prove Theorem 6.

53. Solve the recurrence relation $T(n) = nT^2(n/2)$ with initial condition $T(1) = 6$ when $n = 2^k$ for some integer k. [*Hint:* Let $n = 2^k$ and then make the substitution $a_k = \log T(2^k)$ to obtain a linear nonhomogeneous recurrence relation.]

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

8.3.1 Introduction

Links

Many recursive algorithms take a problem with a given input and divide it into one or more smaller problems. This reduction is successively applied until the solutions of the smaller problems can be found quickly. For instance, we perform a binary search by reducing the search for an element in a list to the search for this element in a list half as long. We successively apply this reduction until one element is left. When we sort a list of integers using the merge sort, we split the list into two halves of equal size and sort each half separately. We then merge the two sorted halves. Another example of this type of recursive algorithm is a procedure for multiplying integers that reduces the problem of the multiplication of two integers to three multiplications of pairs of integers with half as many bits. This reduction is successively applied until integers with one bit are obtained. There procedures follow an important algorithmic paradigm known as **divide-and-conquer**, and are called **divide-and-conquer algorithms**, because they *divide* a problem into one or more instances of the same problem of smaller size and they *conquer* the problem by using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work.

"Divide et impera" (translation: *"Divide and conquer"*) —Julius Caesar

In this section we will show how recurrence relations can be used to analyze the computational complexity of divide-and-conquer algorithms. We will use these recurrence relations