ways to choose which these horses will be, and then two ways for the race to end (the tied horses win or they lose), so there are $4 \cdot 2 = 8$ possibilities. (5) There is only one way for all the horses to tie. Putting this all together, the answer is $24 + 36 + 6 + 8 + 1 = 75$.

46. **a)** The complicating factor here is the rule that the penalty kick round (or "group") is over once one team has clinched a victory. For example, if the first team to shoot has missed all of its first four shots and the other team has made two of its first three shots, then the round is over after only seven kicks. There are $2^{10} = 1024$ possible scenarios without this rule (and without worrying yet about whether the score is tied at the end of this round), but it seems rather tedious and dangerous (in the sense of your being likely to make a mistake and leave something out) to try to analyze the more complicated situation by writing out all the possibilities by hand. (This is not impossible, though, and the author has obtained the correct answer in this way.) Rather than do this, one can write a computer program to simulate the situation and do the counting. The result is that there are 672 possible scoring scenarios for a round of penalty kicks, including the possibility that the score is still tied at the end of that round.

Next we need to count the number of ways for the score to end up tied at the end of the round. For this to happen, both teams must score p points, where p is some integer between 0 and 5, inclusive. The scoring scenario is determined by the positions of the kickers who did the scoring. There are $C(5,p)$ ways to choose these positions for each team, or $C(5,p)^2$ ways in all. We need to sum this over the values of p from 0 to 5. The sum is 252. So there are 252 ways for the score to end up tied. We already noted in the paragraph above that there are 672 different scoring scenarios, so there are $672 - 252 = 420$ scenarios in which the score is not tied. This answers the question for this part of the exercise.

b) This is easy after what we've found above. There are 252 ways for the score to be tied at the end of the first group of penalty kicks, and there are 420 ways for the game to be settled in the second group. So there are $252 \cdot 420 = 105{,}840$ ways for the game to end during the second round.

c) We have already seen that there are 420 ways for the game to end in the first round, and 105,840 more ways for it to end in the second round. In order for it to go into a sudden death period, the first two rounds must have ended tied, which can happen in $420 \cdot 420 = 176{,}400$ ways. Thereafter, the game can end after two more kicks in 2 ways (either team can make their kick and have the other team miss theirs), after four more kicks in $2 \cdot 2 = 4$ ways (the first pair of kicks must have the same result, either both made or both missed, and then either team can win), after six more kicks in $2^2 \cdot 2 = 8$ ways (the first two pairs of kicks must have the same results, and then either team can win), after eight more kicks in 16 ways, and after ten more kicks in 32 ways. Thus there are $2+4+8+16+32 = 62$ ways for the sudden death round to end within ten kicks. This needs to be multiplied by the 176,400 ways we can reach sudden death, for a total of 10,936,800 scoring scenarios. So the answer to this last question is $420 + 105840 + 10936800 = 11{,}043{,}060$.

SECTION 6.4 Binomial Coefficients

2. **a)** When $(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, a term in the third sum, a term in the fourth sum, and a term in the fifth sum are added. Terms of the form x^5, x^4y, x^3y^2, x^2y^3, xy^4 and y^5 arise. To obtain a term of the form x^5, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^5 term in the product has a coefficient of 1. (We can think of this coefficient as $\binom{5}{5}$.) To obtain a term of the form x^4y, an x must be chosen in four of the five sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 4-combinations of five objects, namely $\binom{5}{4} = 5$. Similarly, the number of terms of the form x^3y^2 is the number of ways to pick three of the five sums to obtain x's (and consequently take a y from each of the other two factors). This can be done in $\binom{5}{3} = 10$ ways. By the same reasoning there are $\binom{5}{2} = 10$ ways to obtain the x^2y^3 terms, $\binom{5}{1}=5$ ways to obtain the xy^4 terms, and only one way (which we can think of as $\binom{5}{0}$) to obtain a y^5 term. Consequently, the product is $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

b) This is explained in Example 2. The expansion is $\binom{5}{0}x^5+\binom{5}{1}x^4y+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}xy^4+\binom{5}{5}y^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$. Note that it does not matter whether we think of the bottom of the binomial coefficient expression as corresponding to the exponent on x, as we did in part **(a)**, or the exponent on y, as we do here.

4. $\binom{13}{8}=1287$

6. $\binom{11}{7}1^4=330$

8. $\binom{17}{9}3^8 2^9 = 24310\cdot 6561\cdot 512 = 81{,}662{,}929{,}920$

10. By the binomial theorem, the typical term in this expansion is $\binom{100}{j}x^{100-j}(1/x)^j$, which can be rewritten as $\binom{100}{j}x^{100-2j}$. As j runs from 0 to 100, the exponent runs from 100 down to -100 in decrements of 2. If we let k denote the exponent, then solving $k=100-2j$ for j we obtain $j=(100-k)/2$. Thus the values of k for which x^k appears in this expansion are $-100,\ -98,\ \ldots,\ -2,\ 0,\ 2,\ 4,\ \ldots,\ 100$, and for such values of k the coefficient is $\binom{100}{(100-k)/2}$.

12. We just add adjacent numbers in this row to obtain the next row (starting and ending with 1, of course):

$$1 \quad 11 \quad 55 \quad 165 \quad 330 \quad 462 \quad 462 \quad 330 \quad 165 \quad 55 \quad 11 \quad 1$$

14. Using the factorial formulae for computing binomial coefficients, we see that $\binom{n}{k-1}=\frac{k}{n-k+1}\binom{n}{k}$. If $k\le n/2$, then $\frac{k}{n-k+1}<1$, so the "less than" signs are correct. Similarly, if $k>n/2$, then $\frac{k}{n-k+1}>1$, so the "greater than" signs are correct. The middle equality is Corollary 2 in Section 6.3, since $\lfloor n/2\rfloor+\lceil n/2\rceil=n$. The equalities at the ends are clear.

16. **a)** By Exercise 14, we know that $\binom{n}{\lfloor n/2\rfloor}$ is the largest of the $n-1$ binomial coefficients $\binom{n}{1}$ through $\binom{n}{n-1}$. Therefore it is at least as large as their average, which is $(2^n-2)/(n-1)$. But since $2n\le 2^n$ for $n\ge 2$, it follows that $(2^n-2)/(n-1)\ge 2^n/n$, and the proof is complete.

b) This follows from part **(a)** by replacing n with $2n$ when $n\ge 2$, and it is immediate when $n=1$.

18. The numeral 11 in base b represents the number $b+1$. Therefore the fourth power of this number is $b^4+4b^3+6b^2+4b+1$, where the binomial coefficients can be read from Pascal's triangle. As long as $b\ge 7$, these coefficients are single digit numbers in base b, so this is the meaning of the numeral $(14641)_b$. In short, the numeral formed by concatenating the symbols in the fourth row of Pascal's triangle is the answer.

20. It is easy to see that both sides equal

$$\frac{(n-1)!n!(n+1)!}{(k-1)!k!(k+1)!(n-k-1)!(n-k)!(n-k+1)!}.$$

22. **a)** Suppose that we have a set with n elements, and we wish to choose a subset A with k elements and another, disjoint, subset with $r-k$ elements. The left-hand side gives us the number of ways to do this, namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of *these* elements are to go into the first of the subsets. The right-hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

b) On the one hand,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}\,,$$

and on the other hand

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}\,.$$

24. We know that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}\,.$$

Clearly p divides the numerator. On the other hand, p cannot divide the denominator, since the prime factorizations of these factorials contains only numbers less than p. Therefore the factor p does not cancel when this fraction is reduced to lowest terms (i.e., to a whole number), so p divides $\binom{p}{k}$.

26. First, use Exercise 25 to rewrite the right-hand side of this identity as $\binom{2n}{n+1}$. We give a combinatorial proof, showing that both sides count the number of ways to choose from collection of n men and n women, a subset that has one more man than woman. For the left-hand side, we note that this subset must have k men and $k-1$ women for some k between 1 and n, inclusive. For the (modified) right-hand side, choose any set of $n+1$ people from this collection of n men and n women; the desired subset is the set of men chosen and the women left behind.

28. **a)** To choose 2 people from a set of n men and n women, we can either choose 2 men ($\binom{n}{2}$ ways to do so) or 2 women ($\binom{n}{2}$ ways to do so) or one of each sex ($n\cdot n$ ways to do so). Therefore the right-hand side counts the number of ways to do this (by the sum rule). The left-hand side counts the same thing, since we are simply choosing 2 people from $2n$ people.

b) $2\binom{n}{2} + n^2 = n(n-1) + n^2 = 2n^2 - n = n(2n-1) = 2n(2n-1)/2 = \binom{2n}{2}$

30. We follow the hint. The number of ways to choose this committee is the number of ways to choose the chairman from among the n mathematicians (n ways) times the number of ways to choose the other $n-1$ members of the committee from among the other $2n-1$ professors. This gives us $n\binom{2n-1}{n-1}$, the expression on the right-hand side. On the other hand, for each k from 1 to n, we can have our committee consist of k mathematicians and $n-k$ computer scientists. There are $\binom{n}{k}$ ways to choose the mathematicians, k ways to choose the chairman from among these, and $\binom{n}{n-k}$ ways to choose the computer scientists. Since this last quantity equals $\binom{n}{k}$, we obtain the expression on the left-hand side of the identity.

32. For $n=0$ we want

$$(x+y)^0 = \sum_{j=0}^{0}\binom{0}{j}x^{0-j}y^j = \binom{0}{0}x^0y^0\,,$$

which is true, since $1=1$. Assume the inductive hypothesis. Then we have

$$\begin{aligned}(x+y)^{n+1} &= (x+y)\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j\\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n+1-j}y^j + \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^{j+1}\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n+1-k} y^k$$

$$= \binom{n}{0} x^{n+1} + \left(\sum_{k=1}^{n} [\binom{n}{k} + \binom{n}{k-1}] x^{n+1-k} y^k \right) + \binom{n}{n} y^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k ,$$

as desired. The key point was the use of Pascal's identity to simplify the expression in brackets in the fourth line of this calculation.

34. By Exercise 33 there are $\binom{n-k+k}{k} = \binom{n}{k}$ paths from $(0,0)$ to $(n-k,k)$ and $\binom{k+n-k}{n-k} = \binom{n}{n-k}$ paths from $(0,0)$ to $(k, n-k)$. By symmetry, these two quantities must be the same (flip the picture around the 45° line).

36. A path ending up at $(n+1-k,k)$ must have made its last step either upward or to the right. If the last step was made upward, then it came from $(n+1-k,k-1)$; if it was made to the right, then it came from $(n-k,k)$. The path cannot have passed through both of these points. Therefore the number of paths to $(n+1-k,k)$ is the sum of the number of paths to $(n+1-k,k-1)$ and the number of paths to $(n-k,k)$. By Exercise 33 this tells us that $\binom{n+1-k+k}{k} = \binom{n+1-k+k-1}{k-1} + \binom{n-k+k}{k}$, which simplifies to $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, Pascal's identity.

38. We follow the hint, first noting that we can start the summation with $k=1$, since the term with $k=0$ is 0. The left-hand side counts the number of ways to choose a subset as described in the hint by breaking it down by the number of elements in the subset; note that there are k ways to choose each of the distinguished elements if the subset has size k. For the right-hand side, first note that $n(n+1)2^{n-2} = n(n-1+2)2^{n-2} = n(n-1)2^{n-2} + n2^{n-1}$. The first term counts the number of ways to make this choice if the two distinguished elements are different (choose them, then choose any subset of the remaining elements to be the rest of the subset). The second term counts the number of ways to make this choice if the two distinguished elements are the same (choose it, then choose any subset of the remaining elements to be the rest of the subset). Note that this works even if $n=1$.

SECTION 6.5 Generalized Permutations and Combinations

2. There are 5 choices each of 5 times, so the answer is $5^5 = 3125$.

4. There are 6 choices each of 7 times, so the answer is $6^7 = 279{,}936$.

6. By Theorem 2 the answer is $C(3+5-1,5) = C(7,5) = C(7,2) = 21$.

8. By Theorem 2 the answer is $C(21+12-1,12) = C(32,12) = 225{,}792{,}840$.