onto function from $\mathbf{Z}^+$ to B. Define a function $h: B \to \mathbf{Z}^+$ as follows: Suppose x is any element of B. Since $g \circ f$ is onto, $\{m \in \mathbf{Z}^+ \mid (g \circ f)(m) = x\} \neq \emptyset$. Thus, by the well-ordering principle for the integers, this set has a least element. In other words, there is a least positive integer n with $(g \circ f)(n) = x$. Let $h(x)$ be this integer.

We claim that h is a one-to-one. For suppose $h(x_1) = h(x_2) = n$. By definition of h, n is the least positive integer with $(g \circ f)(n) = x_1$. But also by definition of h, n is the least positive integer with $(g \circ f)(n) = x_2$. Hence $x_1 = (g \circ f)(n) = x_2$.

Thus h is a one-to-one correspondence between B and a subset S of positive integers (the range of h). Since any subset of a countable set is countable (Theorem 7.5.3), S is countable, and so there is a one-to-one correspondence between B and a countable set. Hence, by the transitive property of cardinality, B is countable.

28. *Hint:* Suppose that A and B are countably infinite. Then there are one-to-one correspondences $f_A: \mathbf{Z}^+ \to A$ and $f_B: \mathbf{Z}^+ \to B$. Define a function $g: \mathbf{Z}^+ \to A \cup B$ as follows:

$$g(n) = \begin{cases} f_A\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ f_B\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \end{cases}.$$

Show that g is onto, and finish by using the result of exercise 27.

29. *Hint:* Use proof by contradiction and the fact that the set of all real numbers is uncountable.

32. *Hint:* Use the one-to-one correspondence $F: \mathbf{Z}^+ \to \mathbf{Z}$ of Example 7.5.2 to define a function $G: \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z} \times \mathbf{Z}$ by the formula $G(m, n) = (F(m), F(n))$. Show that G is a one-to-one correspondence, and use the result of exercise 22 and the transitive property of cardinality.

34. *Hint for Solution 1:* Define a function $f: \mathscr{P}(S) \to T$ as follows: For each subset A of S, let $f(A) = \chi_A$, the *characteristic function* of A, where $\chi_A: S \to \{0, 1\}$ is defined by the rule

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \text{ for all } x \in S \end{cases}.$$

Show that f is one-to-one (for all $A_1, A_2 \subseteq S$, if $\chi_{A_1} = \chi_{A_2}$ then $A_1 = A_2$) and that f is onto (given any function $g: S \to \{0, 1\}$, there is a subset A of S such that $g = \chi_A$).

Hint for Solution 2: Define $H: T \to \mathscr{P}(S)$ by letting $H(f) = \{x \in S \mid f(x) = 1\}$. Show that H is a one-to-one correspondence?

35. *Partial proof (by contradiction):* Suppose not. Suppose there is a one-to-one, onto function $f: S \to \mathscr{P}(S)$. Let

$$A = \{x \in S \mid x \notin f(x)\}.$$

Then $A \in \mathscr{P}(S)$ and since f is onto, there is a $z \in S$ such that $A = f(z)$. [*Now derive a contradiction!*]

37. *Hint:* Since A and B are countable, their elements can be listed as

$$A: a_1, a_2, a_3, \ldots \quad \text{and} \quad B: b_1, b_2, b_3, \ldots$$

Represent the elements of $A \times B$ in a grid:

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \ldots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \ldots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Now use a counting method similar to that of Example 7.5.4.

Section 8.1

1. $a_1 = 1, a_2 = 2a_1 + 2 = 2 \cdot 1 + 2 = 4,$
$a_3 = 2a_2 + 3 = 2 \cdot 4 + 3 = 11,$
$a_4 = 2a_3 + 4 = 2 \cdot 11 + 4 = 26$

3. $c_0 = 1, c_1 = 1 \cdot (c_0)^2 = 1 \cdot (1)^2 = 1,$
$c_2 = 2(c_1)^2 = 2 \cdot (1)^2 = 2,$
$c_3 = 3(c_2)^2 = 3 \cdot (2)^2 = 12$

5. $s_0 = 1, s_1 = 1, s_2 = s_1 + 2s_0 = 1 + 2 \cdot 1 = 3,$
$s_3 = s_2 + 2s_1 = 3 + 2 \cdot 1 = 5$

7. $u_1 = 1, u_2 = 1, u_3 = 3u_2 - u_1 = 3 \cdot 1 - 1 = 2,$
$u_4 = 4u_3 - u_2 = 4 \cdot 2 - 1 = 7$

9. By definition of $a_0, a_1, a_2, \ldots$, for each integer $k \geq 1$,

(*) $\quad a_k = 3k + 1$ and

(**) $\quad a_{k-1} = 3(k-1) + 1.$

Then $a_{k-1} + 3$

$$\begin{aligned} &= 3(k-1) + 1 + 3 \\ &= 3k - 3 + 1 + 3 \\ &= 3k + 1 \\ &= a_k \end{aligned}$$

11. Call the nth term of the sequence c_n. Then, by definition, $c_n = 2^n - 1$, for each integer $n \geq 0$. Substitute k and $k-1$ in place of n to get

(*) $\quad c_k = 2^k - 1$ and

(**) $\quad c_{k-1} = 2^{k-1} - 1$

for all integers $k \geq 1$. Then

$$\begin{aligned} 2c_{k-1} + 1 &= 2(2^{k-1} - 1) + 1 && \text{by substitution from (**)} \\ &= 2^k - 2 + 1 \\ &= 2^k - 1 && \text{by basic algebra} \\ &= c_k && \text{by substitution from (*)} \end{aligned}$$

13. Call the nth term of the sequence t_n. Then, by definition, $t_n = 2 + n$, for each integer $n \geq 0$. Substitute k, $k-1$, and $k-2$ in place of n to get

(*) $\quad t_k = 2 + k,$

(**) $\quad t_{k-1} = 2 + (k-1),\quad$ and

(***) $\quad t_{k-2} = 2 + (k-2)$

for each integer $k \geq 2$. Then

$$\begin{aligned}
&2t_{k-1} - t_{k-2} \\
&\quad = 2(2 + (k-1) - (2 + (k-2)) && \text{by substitution from (**) and (***)} \\
&\quad = 2(k+1) - k \\
&\quad = 2 + k && \text{by basic algebra} \\
&\quad = t_k && \text{by substitution from (*).}
\end{aligned}$$

15. Let k be an integer and $k \geq 2$.

Case 1 (k is even): Then

$$a_k = (-2)^{k/2} \quad \text{and} \quad a_{k-2} = (-2)^{(k-2)/2}.$$

So

$$\begin{aligned}
-2a_{k-2} &= -2 \cdot (-2)^{(k-2)/2} \\
&= (-2)^{1+(k-2)/2} \\
&= (-2)^{k/2} \quad \text{since} \quad 1 + \frac{k-2}{2} = \frac{2}{2} + \frac{k-2}{2} = \frac{2+k-2}{2} = \frac{k}{2} \\
&= a_k.
\end{aligned}$$

Case 2 (k is odd): Then

$$a_k = (-2)^{(k-1)/2}$$

and

$$a_{k-2} = (-2)^{(k-3)/2} \quad \text{since} \quad \frac{(k-2)-1}{2} = \frac{k-3}{2}$$

So

$$\begin{aligned}
-2a_{k-2} &= -2 \cdot (-2)^{(k-3)/2} \\
&= (-2)^{1+(k-3)/2} \\
&= (-2)^{(k-1)/2} \quad \text{since} \quad 1 + \frac{k-3}{2} = \frac{2}{2} + \frac{k-3}{2} = \frac{2+k-3}{2} = \frac{k-1}{2} \\
&= a_k.
\end{aligned}$$

Hence in either case, $a_k = -2a_{k-2}$, as was to be shown.

18. a. $a_1 = 2$

$a_2 = 2$ (moves to move the top disk from pole A to pole C)
$\quad + 1$ (move to move the bottom disk from pole A to pole B)
$\quad + 2$ (moves to move the top disk from pole C to pole A)
$\quad + 1$ (move to move the bottom disk from pole B to pole C)
$\quad + 2$ (moves to move top disk from pole A to pole C)
$\quad = 8$

$a_3 = 8 + 1 + 8 + 1 + 8 = 26$

c. For all integers $k \geq 2$.

$a_k = a_{k-1}$ (moves to move the top $k-1$ disks from pole A to pole C)
$\quad + 1$ (move to move the bottom disk from pole A to pole B)
$\quad + a_{k-1}$ (moves to move the top disk from pole C to pole A)
$\quad + 1$ (move to move the bottom disks from pole B to pole C)
$\quad + a_{k-1}$ (moves to move the top disks from pole A to pole C)
$\quad = 3a_{k-1} + 2.$

19. b. $b_4 = 40$

e. *Hint:* One solution is to use mathematical induction and apply the formula from part (c). Another solution is to prove by mathematical induction that when a most efficient transfer of n disks from one end pole to the other end pole is performed, at some point all the disks are on the middle pole.

20. a. $s_1 = 1,\ s_2 = 1 + 1 + 1 = 3,$
$s_3 = s_1 + (1 + 1 + 1) + s_1 = 5$

b. $s_4 = s_2 + (1 + 1 + 1) + s_2 = 9$

21. b. $t_3 = 14$

22. b. $r_0 = 1,\ r_1 = 1,\ r_2 = 1 + 4 \cdot 1 = 5,\ r_3 = 5 + 4 \cdot 1 = 9,$
$r_4 = 9 + 4 \cdot 5 = 29,\ r_5 = 29 + 4 \cdot 9 = 65,$
$r_6 = 65 + 4 \cdot 29 = 181$

23. c. There are 904 rabbit pairs, or 1,808 rabbits, after 12 months.

25. a. Each term of the Fibonacci sequence beyond the second equals the sum of the previous two. For any integer $k \geq 1$, the two terms previous to F_{k+1} are F_k and F_{k-1}. Hence, for all integers $k \geq 1$, $F_{k+1} = F_k + F_{k-1}$.

26. By repeated use of definition of the Fibonacci sequence, for all integers $k \geq 4$,

$$\begin{aligned} F_k &= F_{k-1} + F_{k-2} = (F_{k-2} + F_{k-3}) + (F_{k-3} + F_{k-4}) \\ &= ((F_{k-3} + F_{k-4}) + F_{k-3}) + (F_{k-3} + F_{k-4}) \\ &= 3F_{k-3} + 2F_{k-4}. \end{aligned}$$

27. For all integers $k \geq 1$,

$F_k^2 - F_{k-1}^2$

$= (F_k - F_{k-1})(F_k + F_{k-1})$ by basic algebra (difference of two squares)

$= (F_k - F_{k-1})F_{k+1}$ by definition of the Fibonacci sequence

$= F_k F_{k+1} - F_{k-1} F_{k+1}$

31. Let $L = \lim_{n\to\infty} \dfrac{F_{n+1}}{F_n}$. Since each $F_{n+1} > F_n > 0$, $L > 0$. Then, by definition of the Fibonacci sequence,

$$\begin{aligned} L &= \lim_{n\to\infty}\left(\frac{F_{n-1}+F_n}{F_n}\right) = \lim_{n\to\infty}\left(\frac{F_{n-1}}{F_n} + \frac{F_n}{F_n}\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{\frac{F_n}{F_{n-1}}} + 1\right) = \frac{1}{\lim_{n\to\infty}\left(\frac{F_n}{F_{n-1}}\right)} + 1 \\ &= \frac{1}{L} + 1. \end{aligned}$$

Hence $L = \frac{1}{L} + 1$. Multiply both sides by L to obtain $L^2 = 1 + L$, or, equivalently, $L^2 - L - 1 = 0$. By the quadratic formula, then, $L = \frac{1 \pm \sqrt{5}}{2}$. But one of these numbers, $\frac{1-\sqrt{5}}{2}$, is less than zero, and $L > 0$. Hence $L = \frac{1+\sqrt{5}}{2}$.

32. *Hint:* Use the result of exercise 30 to prove that the infinite sequence $\frac{F_0}{F_1}, \frac{F_2}{F_3}, \frac{F_4}{F_5}, \ldots$ is strictly decreasing and that the infinite sequence $\frac{F_1}{F_2}, \frac{F_3}{F_4}, \frac{F_5}{F_6}, \ldots$ is strictly increasing. The first sequence is bounded below by 0, and the second sequence is bounded above by 1. Deduce that the limits of both sequences exist, and show that they are equal.

34. a. Because the 4% annual interest is compounded quarterly, the quarterly interest rate is $(4\%)/4 = 1\%$. Then $R_k = R_{k-1} + 0.01R_{k-1} = 1.01R_{k-1}$.

b. Because one year equals four quarters, the amount on deposit at the end of one year is $R_4 = \$5203.02$ (rounded to the nearest cent).

c. The annual percentage rate (APR) for the account is $\frac{\$5203.02 - \$5000.00}{\$5000.00} = 4.0604\%$.

36. a. Length 0: ϵ
Length 1: 0, 1
Length 2: 00, 01, 10, 11
Length 3: 000, 001, 010, 011, 100, 101, 110
Length 4: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101

b. By part (a), $d_0 = 1$, $d_1 = 2$, $d_2 = 4$, $d_3 = 7$, and $d_4 = 13$.

c. Let k be an integer with $k \geq 3$. Any string of length k that does not contain the bit pattern 111 starts either with a 0 or with a 1. If it starts with a 0, this can be followed by any string of $k-1$ bits that does not contain the pattern 111. There are d_{k-1} of these. If the string starts with a 1, then the first two bits are 10 or 11. If the first two bits are 10, then these can be followed by any string of $k-2$ bits that does not contain the pattern 111. There are d_{k-2} of these. If the string starts with a 11, then the third bit must be 0 (because the string does not contain 111), and these three bits can be followed by any string of $k-3$ bits that does not contain the pattern 111. There are d_{k-3} of these. Therefore, for all integers $k \geq 3$, $d_k = d_{k-1} + d_{k-2} + d_{k-3}$.

d. By parts (b) and (c), $d_5 = d_4 + d_3 + d_2 = 13 + 7 + 4 = 24$. This is the number of bit strings of length five that do not contain the pattern 111.

37. c. *Hint:* $s_k = 2s_{k-1} + 2s_{k-2}$

~~39.~~ When one is climbing a staircase consisting of n stairs, the last step taken is either a single stair or two stairs together. The number of ways to climb the staircase and have the final step be a single stair is c_{n-1}; the number of ways to climb the staircase and have the final step be two stairs is c_{n-2}. Therefore, by the addition rule, $c_n = c_{n-1} + c_{n-2}$. Note also that $c_1 = 1$ and $c_2 = 2$ [*because either the two stairs can be climbed one by one or they can be climbed as a unit*].

41. a. $a_3 = 3$ (The three permutations that do not move more than one place from their "natural" positions are 213, 132, and 123.)

43. Call the set X, and suppose that $X = \{x_1, x_2, \ldots, x_n\}$. For each integer $i = 0, 1, 2, \ldots, n-1$, we can consider the set of all partitions of X (let's call them *partitions of type i*) where one of the subsets of the partition is an $(i+1)$-element set that contains x_n and i elements chosen from $\{x_1, \ldots, x_{n-1}\}$. The remaining subsets of the partition will be a partition of the remaining $(n-1) - i$ elements of $\{x_1, \ldots, x_{n-1}\}$. For instance, if $X = \{x_1, x_2, x_3\}$, there are five partitions of the various types, namely,

Type 0: two partitions where one set is a 1-element set containing x_3: $[\{x_3\}, \{x_1\}, \{x_2\}]$, $[\{x_3\}, \{x_1, x_2\}]$

Type 1: two partitions where one set is a 2-element set containing x_3: $[\{x_1, x_3\}, \{x_2\}]$, $[\{x_2, x_3\}, \{x_1\}]$

Type 2: one partition where one set is a 3-element set containing x_3: $\{x_1, x_2, x_3\}$

In general, we can imagine constructing a partition of type i as a two-step process:

Step 1: Select out the i elements of $\{x_1, \ldots, x_{n-1}\}$ to put together with x_n,

Step 2: Choose any partition of the remaining $(n-1) - i$ elements of $\{x_1, \ldots, x_{n-1}\}$ to put with the set formed in step 1.

There are $\binom{n-1}{i}$ ways to perform step 1 and $P_{(n-1)-i}$ ways to perform step 2. Therefore, by the multiplication rule, there are $\binom{n-1}{i} \cdot P_{(n-1)-i}$ partitions of type i. Because any partition of X is of type i for some $i = 0, 1, 2, \ldots, n-1$, it follows from the addition rule that the total number of partitions is

$$\binom{n-1}{0}P_{n-1}+\binom{n-1}{1}P_{n-2}+\binom{n-1}{2}P_{n-3}+\cdots+\binom{n-1}{n-1}P_0.$$

45. $S_{5,2}=S_{4,1}+2S_{4,2}=1+2\cdot 7=15$

48. *Proof* (*by mathematical induction*): Let the property $P(n)$ be the formula $S_{n,2}=2^{n-1}-1$.

Show that the property is true for n = 2:

We must show that $S_{2,2}=2^{2-1}-1$. By Example 8.1.11, $S_{2,2}=1$, and $2^{2-1}-1=2-1=1$ also. So the property is true for $n=2$.

Show that for all integers k ≥ 2, if the property is true for n = k, then it is true for n = k + 1:

Suppose that for some integer $k\geq 2$, $S_{k,2}=2^{k-1}-1$. [*Inductive hypothesis.*] We must show that $S_{k+1,2}=2^{(k+1)-1}-1=2^k-1$. But according to Example 8.1.11, $S_{k+1,2}=S_{k,1}+2S_{k,2}$ and $S_{k,1}=1$. So by substitution and the inductive hypothesis,

$$\begin{aligned}S_{k+1,2}&=1+2S_{k,2}=1+2(2^{k-1}-1)\\&=1+2^k-2=2^k-1\end{aligned}$$

[*as was to be shown*].

50. *Hint:* Observe that the number of onto functions from $X=\{x_1,x_2,x_3,x_4\}$ to $Y=\{y_1,y_2,y_3\}$ is $S_{4,3}\cdot 3!$ because the construction of an onto function can be thought of as a two-step process where step 1 is to choose a partition of X into three subsets and step 2 is to choose, for each subset of the partition, an element of Y for the elements of the subset to be sent to.

52. *Hint:* Use mathematical induction. In the inductive step, use Lemma 3.8.2 and the fact that $F_{k+2}=F_{k+1}+F_k$ to deduce that

$$\gcd(F_{k+2},F_{k+1})=\gcd(F_{k+1},F_k).$$

53. c. *Hint:* If $k\geq 6$, any sequence of k games must begin with W, LW, or LLW, where L stands for "lose" and W stands for "win."

54. c. *Hint:* Divide the set of all derangements into two subsets: one subset consists of all derangements in which the number 1 changes places with another number, and the other subset consists of all derangements in which the number 1 goes to position $i\neq 1$ but i does not go to position 1. The answer is $d_k=(k-1)d_{k-1}+(k-1)d_{k-2}$. Can you justify it?

Section 8.2

1. a. $1+2+3+\cdots+(k-1)$

$$=\frac{(k-1)((k-1)+1)}{2}=\frac{(k-1)k}{2}$$

b. $3+2+4+6+8+\cdots+2n$

$$\begin{aligned}&=3+2(1+2+3+\cdots+n)\\&=3+2\frac{n(n+1)}{2}=3+n(n+1)\\&=n^2+n+3\end{aligned}$$

2. a. $1+2+2^2+\cdots+2^{i-1}=\dfrac{2^{(i-1)+1}-1}{2-1}=2^i-1$

c. $2^n+2^{n-2}\cdot 3+2^{n-3}\cdot 3+\cdots+2^2\cdot 3+2\cdot 3+3$

$$\begin{aligned}&=2^n+3(2^{n-2}+2^{n-3}+\cdots+2^2+2+1)\\&=2^n+3(1+2+2^2+\cdots+2^{n-3}+2^{n-2})\\&=2^n+3\left(\frac{2^{(n-2)+1}-1}{2-1}\right)\\&=2^n+3(2^{n-1}-1)\\&=2\cdot 2^{n-1}+3\cdot 2^{n-1}-3\\&=5\cdot 2^{n-1}-3\end{aligned}$$

3. $a_0=1$

$a_1=1\cdot a_0=1\cdot 1=1$

$a_2=2a_1=2\cdot 1$

$a_3=3a_2=3\cdot 2\cdot 1$

$a_4=4a_3=4\cdot 3\cdot 2\cdot 1$

$\vdots$

Guess:

$a_n=n(n-1)\cdots 3\cdot 2\cdot 1=n!$

5. $c_1=1$

$c_2=3c_1+1=3\cdot 1+1=3+1$

$c_3=3c_2+1=3\cdot(3+1)+1=3^2+3+1$

$c_4=3c_3+1=3\cdot(3^2+3+1)+1$

$=3^3+3^2+3+1$

$\vdots$

Guess:

$$\begin{aligned}c_n&=3^{n-1}+3^{n-2}+\cdots+3^3+3^2+3+1\\&=\frac{3^n-1}{3-1}\quad\text{by Theorem 4.2.3 with } r=3\\&=\frac{3^n-1}{2}\end{aligned}$$

6. *Hint:*

$d_n=2^n+2^{n-2}\cdot 3+2^{n-3}\cdot 3+\cdots+2^2\cdot 3+2\cdot 3+3$

$=5\cdot 2^{n-1}-3$ for all integers $n\geq 1$

9. *Hint:* For any positive real numbers a and b,

$$\frac{\frac{a}{b}}{\frac{a}{b}+2}=\frac{\frac{a}{b}}{\frac{a}{b}+2}\cdot\frac{b}{b}=\frac{a}{a+2b}.$$