8. $v_1 = 1, \quad v_2 = 3, \quad v_3 = v_2 + v_1 + 1 = 3 + 1 + 1 = 5, \quad v_4 = v_3 + v_2 + 1 = 5 + 3 + 1 = 9$

10. By definition of $b_0, b_1, b_2, \ldots$, for all integers $k \geq 1$, $b_k = 4^k$ and $b_{k-1} = 4^{k-1}$. So for all integers $k \geq 1$, $4 \cdot b_{k-1} = 4 \cdot 4^{k-1} = 4^k = b_k$.

12. Call the nth term of the sequence s_n. Then $s_n = \dfrac{(-1)^n}{n!}$ for all integers $n \geq 0$. So for all integers $k \geq 1$, $s_k = \dfrac{(-1)^k}{k!}$ and $s_{k-1} = \dfrac{(-1)^{k-1}}{(k-1)!}$. It follows that for all integers $k \geq 1$, $\dfrac{-s_{k-1}}{k} = \dfrac{-\frac{(-1)^{k-1}}{(k-1)!}}{k} = \dfrac{-(-1)^{k-1}}{k \cdot (k-1)!} = \dfrac{(-1)^k}{k!} = s_k$.

14. Call the nth term of the sequence d_n. Then $d_n = 3^n - 2^n$ for all integers $n \geq 0$. So for all integers $k \geq 2$, $d_k = 3^k - 2^k$, $d_{k-1} = 3^{k-1} - 2^{k-1}$, and $d_{k-2} = 3^{k-2} - 2^{k-2}$. It follows that for all integers $k \geq 2$, $5d_{k-1} - 6d_{k-2} = 5(3^{k-1} - 2^{k-1}) - 6(3^{k-2} - 2^{k-2}) = 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 2 \cdot 3 \cdot 3^{k-2} + 2 \cdot 3 \cdot 2^{k-2}$ $= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 2 \cdot 3^{k-1} + 3 \cdot 2^{k-1} = (5-2) \cdot 3^{k-1} + (-5+3) \cdot 2^{k-1} = 3 \cdot 3^{k-1} - 2 \cdot 2^{k-1} = 3^k - 2^k = d_k$.

16. According to exercise 17 of Section 6.6, for each integer $n \geq 1$, $C_n = \dfrac{1}{4n+2}\dbinom{2n+2}{n+1}$. Substituting $k-1$ in place of n gives

$$C_{k-1} = \frac{1}{4(k-1)+2}\binom{2(k-1)+2}{(k-1)+1} = \frac{1}{4k-2}\binom{2k}{k}.$$

Then for each integer $k \geq 2$,

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \frac{1}{k+1} \cdot \frac{4k-2}{4k-2}\binom{2k}{k} = \frac{4k-2}{k+1}C_{k-1}.$$

17. $m_7 = 2m_6 + 1 = 2 \cdot 63 + 1 = 127, \quad m_8 = 2m_7 + 1 = 2 \cdot 127 + 1 = 255$

18. *b.* $a_4 = 26 + 1 + 26 + 1 + 26 = 80$ 3

$a_k = 3a_{k-1} + 2$

19. *a.* $b_1 = 1, \quad b_2 = 1 + 1 + 1 + 1 = 4, \quad b_3 = 4 + 4 + 1 + 4 = 13$

c. Note that it takes just as many moves to move a stack of disks from the middle pole to an outer pole as from an outer pole to the middle pole: the moves are the same except that their order and direction are reversed. For all integers $k \geq 2$,

$$\begin{aligned} b_k &= a_{k-1} && \text{(moves to transfer the top } k-1 \text{ disks from pole } A \text{ to pole } C\text{)} \\ &\quad +1 && \text{(move to transfer the bottom disk from pole } A \text{ to pole } B\text{)} \\ &\quad +b_{k-1} && \text{(moves to transfer the top } k-1 \text{ disks from pole } C \text{ to pole } B\text{).} \\ &= \boxed{a_{k-1} + 1 + b_{k-1}}. \end{aligned}$$

d. One way to transfer a tower of k disks from pole A to pole B is first to transfer the top $k-1$ disks from pole A to pole B *[this requires b_{k-1} moves]*, then transfer the top $k-1$ disks from pole B to pole C *[this also requires b_{k-1} moves]*, then transfer the bottom disk from pole A to pole B *[this requires one move]*, and finally transfer the top $k-1$ disks from pole C to pole B *[this again requires b_{k-1} moves]*. This sequence of steps need not necessarily, however, result in a minimum number of moves. Therefore, at this point, all we can say for sure is that for all integers $k \geq 2$,

$$b_k \leq b_{k-1} + b_{k-1} + 1 + b_{k-1} = 3b_{k-1} + 1.$$

e. *Proof 1*: In part 1 of the proof, we show that for any integer $k \geq 1$, in a tower of Hanoi with adjacency requirement, when a transfer of k disks from one end pole to the other end pole is performed, at some point all the disks are piled on the middle pole. In part 2 of the proof, we use the result of part 1 together with the result of part (c) of the problem to deduce the equation $b_k = 3b_{k-1} + 1$ for all integers $k \geq 2$.

Part 1 *(by mathematical induction)*: Let the property $P(k)$ be the sentence "In a tower of Hanoi with adjacency requirement, when a transfer of k disks from one end pole to the other end pole is performed, at some point all the disks are piled on the middle pole."

Show that the property is true for $k = 1$: The property is true for $k = 1$ because when one disk is transferred from one end pole to the other end pole with an adjacency requirement, it must first be placed on the middle pole before it can be moved to the pole at the other end.

Show that for all integers $k \geq 1$, if the property is true for $k = i$, then it is true for $k = i + 1$: Let i be an integer with $i \geq 1$, and suppose that in a tower of Hanoi with adjacency requirement, when a tower of i disks is transferred from one end pole to the other end pole, at some point all the disks are piled on the middle pole. *[This is the inductive hypothesis.]* We must show that in a tower of Hanoi with adjacency requirement, when a tower of $i + 1$ disks is transferred from one end pole to the other end pole, at some point all the disks are piled on the middle pole. So suppose $i+1$ disks are piled on one end pole, say pole A. Call the middle pole B and the third pole C. In order to move the bottom disk from pole A, the top i disks must previously have been moved to pole C. Because of the adjacency requirement, the bottom disk must then be moved to the middle pole. Furthermore, to transfer the entire tower of $i + 1$ disks to pole C, the bottom disk must be moved to pole C. To achieve this, the top i disks must be transferred back to pole A. By inductive hypothesis, at some point while making this transfer, the top i disks will all be piled on the middle pole. But at that time, the bottom disk will be at the bottom of the middle pole *[because if it were back on pole A, transferring the top $i - 1$ disks to pole A would simply recreate the initial position of the disks]*, and so the entire tower of $i + 1$ disks will be on the middle pole. *[This is what was to be shown.]*

Part 2: By part (c) of this exercise, we know that $b_k = a_{k-1} + 1 + b_{k-1}$. Now a_{k-1} is the minimum number of moves needed to transfer a tower of $k - 1$ disks from end pole A to end pole C. By part 1 of this proof, we know that at some point during the transfer all $k - 1$ disks will be on the middle pole. But the minimum number of moves needed to put them there is, by definition, b_{k-1}. Moreover, from their position on the middle pole, the top $k - 1$ disks must be moved to pole C in order to be able to place the bottom disk on the middle pole. By symmetry, the minimum number of moves needed to transfer the top $k - 1$ disks from pole B to pole C is also b_{k-1}. Thus $a_{k-1} = b_{k-1} + b_{k-1}$, and so $b_k = b_{k-1} + b_{k-1} + 1 + b_{k-1} = 3b_{k-1} + 1$.

Proof 2 (by mathematical induction): Let the property $P(k)$ be the equation $b_k = 3b_{k-1} + 1$.

Show that the property is true for $k = 2$: The property is true for $k = 2$ because for $k = 2$ the left-hand side is 4 (by part (a)) and the right-hand side is $3 \cdot 1 + 1 = 4$ also.

Show that for all integers $i \geq 2$, if the property is true for $k = i$ then it is true for $k = i + 1$: Let i be an integer with $i \geq 2$, and suppose that $b_i = 3b_{i-1} + 1$. *[This is the inductive hypothesis.]* We must show that $b_{i+1} = 3b_i + 1$. But $b_{i+1} = a_i + 1 + b_i$ *[by part (c)]* $= a_i + 1 + 3b_{i-1} + 1$ *[by inductive hypothesis]* $= (3a_{i-1} + 2) + 1 + 3b_{i-1} + 1$ *[by exercise 18(c)]* $= 3a_{i-1} + 3 + 3b_{i-1} + 1 = 3(a_{i-1} + 1 + b_{i-1}) + 1 = 3b_i + 1$ *[by part (c) of this exercise]*. *[This is what was to be shown.]*

20. c. Name the poles A, B, C, and D going from left to right. Because disks can be moved from one pole to any other pole, the number of moves needed to transfer a tower of disks from any one pole to any other pole is the same for any two poles. One way to transfer a tower of k disks from pole A to pole D is to first transfer the top $k - 2$ disks from pole A to pole B, then transfer the second largest disk from pole A to pole C, then transfer the largest disk from pole A to pole D, then transfer the second largest disk from pole C to pole D, and finally transfer

the top $k-2$ disks from pole B to pole D. This might not result in a minimal number of moves, however. So for all integers $k \geq 3$,

$$\begin{aligned} s_k \leq\ & s_{k-2} && \text{(moves to transfer the top } k-2 \text{ disks from pole } A \text{ to pole } B) \\ & +1 && \text{(move to transfer the second largest disk from pole } A \text{ to pole } C) \\ & +1 && \text{(move to transfer the largest disk from pole } A \text{ to pole } D) \\ & +1 && \text{(move to transfer the second largest disk from pole } C \text{ to pole } D) \\ & +s_{k-2} && \text{(moves to transfer the top } k-2 \text{ disks from pole } B \text{ to pole } D) \\ \leq\ & 2s_{k-2}+3. \end{aligned}$$

21. *a.* $t_1 = 2$, $\quad t_2 = 2+2+2 = 6$

c. For all integers $k \geq 2$,

$$\begin{aligned} t_k =\ & t_{k-1} && \text{(moves to transfer the top } 2k-2 \text{ disks from pole } A \text{ to pole } B) \\ & +2 && \text{(moves to transfer the bottom two disks from pole } A \text{ to pole } C) \\ & +t_{k-1} && \text{(moves to transfer the top } 2k-2 \text{ disks from pole } B \text{ to pole } C) \\ =\ & 2t_{k-1}+2. \end{aligned}$$

Note that transferring the stack of $2k$ disks from pole A to pole C requires at least two transfers of the top $2(k-1)$ disks: one to transfer them off the bottom two disks to free the bottom disks so that they can be moved to pole C and another to transfer the top $2(k-1)$ disks back on top of the bottom two disks. Thus at least $2t_{k-1}$ moves are needed to effect these two transfers. Two more moves are needed to transfer the bottom two disks from pole A to pole C, and this transfer cannot be effected in fewer than two moves. It follows that the sequence of moves indicated in the description of the equation above is, in fact, minimal.

22. *a.* $r_k = r_{k-1} + 4r_{k-2}$ for all integers $k \geq 3$

c. $r_7 = r_6 + 4r_5 = 181 + 4 \cdot 65 = 441$; $r_8 = r_7 + 4r_6 = 441 + 4 \cdot 181 = 1,165$; $r_9 = r_8 + 4r_7 = 1165+4\cdot 441 = 2,929$; $r_{10} = r_9+4r_8 = 2929+4\cdot 1165 = 7,589$; $r_{11} = r_{10}+4r_9 = 7589+4\cdot 2929 = 19,305$; $r_{12} = r_{11} + 4r_{10} = 19305 + 4 \cdot 7589 = 49,661$

At the end of the year there will be $r_{12} = 49,661$ rabbit pairs or 99,322 rabbits.

23. *a.* $\boxed{s_k = s_{k-1} + 3s_{k-3} \text{ for all integers } k \geq 3}$

b. $s_0 = 1$, $s_1 = 1$, $s_2 = 1$, $s_3 = 1 + 3 \cdot 1 = 4$, $s_4 = 4 + 3 \cdot 1 = 7$, $s_5 = 7 + 3 \cdot 1 = 10$

c. $s_6 = s_5 + 3s_3 = 22$, $s_7 = s_6 + 3s_4 = 43$, $s_8 = s_7 + 3s_5 = 73$, $s_9 = s_8 + 3s_6 = 139$, $s_{10} = s_9 + 3s_7 = 268$, $s_{11} = s_{10} + 3s_9 = 487$, $s_{12} = s_{11} + 3s_{10} = 904$.

24. $F_{13} = F_{12} + F_{11} = 233 + 144 = 377$, $\quad F_{14} = F_{13} + F_{12} = 377 + 233 = 610$

25. *b.* $F_{k+2} = F_{k+1} + F_k \quad$ *c.* $F_{k+3} = F_{k+2} + F_{k+1}$

28. By definition of the Fibonacci sequence, for any integer $k \geq 1$, $F_{k+1}^2 - {F_k}^2 - F_{k-1}^2 = (F_k + F_{k-1})^2 - {F_k}^2 - F_{k-1}^2 = {F_k}^2 + 2F_kF_{k-1} + F_{k-1}^2 - {F_k}^2 - F_{k-1}^2 = 2F_kF_{k-1}$.

29. By definition of the Fibonacci sequence, for any integer $k \geq 1$, $F_{k+1}^2 - {F_k}^2 = (F_{k+1}-F_k)(F_{k+1}+F_k) = (F_{k+1}-F_k)F_{k+2}$. But since $F_{k+1} = F_k + F_{k-1}$, then $F_{k+1} - F_k = F_{k-1}$. By substitution, $F_{k+1}^2 - {F_k}^2 = F_{k-1}F_{k+2}$.

30. *d.* *Proof (by mathematical induction)*: Let the property $P(n)$ be the equation $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$.

Show that the property is true for $n = 0$: The property is true for $n = 0$ because for $n = 0$ the left-hand side is $F_{0+2}F_0 - {F_1}^2 = 2 \cdot 1 - 1^2 = 1$, and the right-hand side is $(-1)^0 = 1$ also.

Show that for all integers $k \geq 0$, if the property is true for $n = k$ then it is true for $n = k + 1$: Let k be an integer with $k \geq 0$, and suppose that $F_{k+2}F_k - F_{k+1}^2 = (-1)^k$ for some integer $k \geq 0$. *[This is the inductive hypothesis.]* We must show that $F_{k+3}F_{k+1} - F_{k+2}^2 = (-1)^k$. But by inductive hypothesis,

$$F_{k+1}^2 = F_{k+2}F_k - (-1)^k = F_{k+2}F_k + (-1)^{k+1}. \quad \text{(We call this equation (*).)}$$

Hence,

$$\begin{aligned}
&F_{k+3}F_{k+1} - F_{k+2}^2 \\
&= (F_{k+1} + F_{k+2})F_{k+1} - F_{k+2}^2 && \text{by definition of the Fibonacci sequence} \\
&= F_{k+1}^2 + F_{k+2}F_{k+1} - F_{k+2}^2 \\
&= F_{k+2}F_k + (-1)^{k+1} + F_{k+2}F_{k+1} - F_{k+2}^2 && \text{by substitution from equation (*)} \\
&= F_{k+2}(F_k + F_{k+1} - F_{k+2}) + (-1)^{k+1} && \text{by factoring out } F_{k+2} \\
&= F_{k+2}(F_{k+2} - F_{k+2}) + (-1)^{k+1} && \text{by definition of the Fibonacci sequence} \\
&= F_{k+2} \cdot 0 + (-1)^{k+1} \\
&= (-1)^{k+1}.
\end{aligned}$$

32. *Proof*: In part 1 of the proof, we will show that $\lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}}$ exists and $\lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}} \geq 0$. In part 2 of the proof, we will show that $\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+2}}$ exists and $\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+2}} \leq 1$. In part 3, we will show that because both $\lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}}$ and $\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+2}}$ exist and are finite, $\lim_{n\to\infty} \frac{F_n}{F_{n+1}}$ exists and equals $\frac{\sqrt{5}-1}{2}$. In parts 1 and 2, we use the result of exercise 30 that $F_{m+2}F_m - F_{m+1}^2 = (-1)^m$ for all integers $m \geq 0$. Adding F_{m+1}^2 to both sides of the equation gives that for all integers $m \geq 0$,

$$F_{m+2}F_m = F_{m+1}^2 + (-1)^m \qquad \text{(We call this equation (1).)}$$

Part 1: Because all values of the Fibonacci sequence are positive, we may apply properties of inequalities, the definition of the Fibonacci sequence, and equation (1) to obtain the following sequence of if-and-only-if statements: For any integer $n \geq 0$,

$$\frac{F_{2n}}{F_{2n+1}} > \frac{F_{2n+2}}{F_{2n+3}} \Leftrightarrow F_{2n}F_{2n+3} > F_{2n+1}F_{2n+2} \quad \textit{[by cross-multiplying]}$$

$$\Leftrightarrow F_{2n}(F_{2n+2} + F_{2n+1}) > F_{2n+1}(F_{2n+1} + F_{2n})$$

$$\Leftrightarrow F_{2n}F_{2n+2} > F_{2n+1}^2 \Leftrightarrow F_{2n+1}^2 + (-1)^{2n} > F_{2n+1}^2 \quad \textit{[by equation (1) with } m = 2n\textit{]}$$

$$\Leftrightarrow F_{2n+1}^2 + 1 > F_{2n+1}^2 \quad \textit{[because } 2n \textit{ is even]} \Leftrightarrow 1 > 0, \text{ which is true.}$$

Thus, since the original inequality is equivalent to an inequality that is true, the original inequality is also true. Therefore, $\frac{F_{2n}}{F_{2n+1}} > \frac{F_{2n+2}}{F_{2n+3}}$ for all integers $n \geq 0$, and hence the infinite sequence $\left\{\frac{F_{2n}}{F_{2n+1}}\right\}_{n=0}^{\infty}$ is strictly decreasing. Because the terms of the sequence are bounded below by 0, this implies (by a theorem from calculus) that $\lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}}$ exists and is greater than or equal to 0.

Part 2: As in part 1, because all values of the Fibonacci sequence are positive, we may apply properties of inequalities, the definition of the Fibonacci sequence, and equation (1) to obtain the following sequence of if-and-only-if statements: For any integer $n \geq 0$,

$$\frac{F_{2n+1}}{F_{2n+2}} < \frac{F_{2n+3}}{F_{2n+4}} \Leftrightarrow F_{2n+1}F_{2n+4} < F_{2n+2}F_{2n+3} \quad \textit{[by cross-multiplying]}$$

$$\Leftrightarrow F_{2n+1}(F_{2n+3} + F_{2n+2}) < F_{2n+2}(F_{2n+2} + F_{2n+1}) \Leftrightarrow F_{2n+1}F_{2n+3} < F_{2n+2}^2$$

$$\Leftrightarrow F_{2n+2}^2 + (-1)^{2n+1} < F_{2n+2}^2 \textit{ [by equation (1) with } m = 2n+1\textit{]}$$

$$\Leftrightarrow F_{2n+2}^2 - 1 < F_{2n+1}^2 \textit{ [because } 2n+1 \textit{ is odd]} \Leftrightarrow -1 < 0, \text{ which is true.}$$

Thus, since the original inequality is equivalent to an inequality that is true, the original inequality is also true. Therefore, $\frac{F_{2n+1}}{F_{2n+2}} < \frac{F_{2n+3}}{F_{2n+4}}$ for all integers $n \geq 0$, and hence the infinite sequence $\left\{\frac{F_{2n+1}}{F_{2n+2}}\right\}_{n=0}^{\infty}$ is strictly increasing. Because the terms of the sequence are bounded above by 1, this implies (by a theorem from calculus) that $\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+2}}$ exists and is less than or equal to 1.

Part 3: Let $L_1 = \lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}}$ and $L_2 = \lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+2}}$. Then

$$L_1 = \lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}} = \lim_{n\to\infty} \frac{1}{\frac{F_{2n+1}}{F_{2n}}} = \lim_{n\to\infty} \frac{1}{\frac{F_{2n}+F_{2n-1}}{F_{2n}}} = \lim_{n\to\infty} \frac{1}{1+\frac{F_{2n-1}}{F_{2n}}}$$

$$= \frac{1}{1+\lim_{n\to\infty}\frac{F_{2n-1}}{F_{2n}}} = \frac{1}{1+L_2}.$$

Multiplying both sides by $1 + L_2$ gives that

$$L_1 + L_1L_2 = 1. \qquad \text{(Call this equation (2).)}$$

Now $L_2 = \lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+2}} = \lim_{n\to\infty} \frac{1}{\frac{F_{2n+2}}{F_{2n+1}}} = \lim_{n\to\infty} \frac{1}{\frac{F_{2n+1}+F_{2n}}{F_{2n+1}}}$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{F_{2n}}{F_{2n+1}}} = \frac{1}{1+\lim_{n\to\infty}\frac{F_{2n}}{F_{2n+1}}} = \frac{1}{1+L_1}$$

Multiplying both sides by $1 + L_1$ gives that

$$L_2 + L_1L_2 = 1. \qquad \text{(Call this equation (3).)}$$

By substituting from equation (3) into equation (2), we have

$$L_1 + L_1L_2 = L_2 + L_1L_2$$

and subtracting L_1L_2 from both sides gives that $L_1 = L_2$. Thus both subsequences $\left\{\frac{F_{2n}}{F_{2n+1}}\right\}_{n=0}^{\infty}$ and $\left\{\frac{F_{2n+1}}{F_{2n+2}}\right\}_{n=0}^{\infty}$ have the same limit and this is the limit for the entire sequence.

To discover the value of the limit, substitute L_1 in place of L_2 in equation (2) to obtain

$$L_1 + L_1^2 = 1, \text{ or equivalently, } L_1^2 + L_1 - 1 = 0.$$

Solving this equation with the quadratic formula and using the fact that $L_1 \geq 0$ gives that $L_1 = \frac{\sqrt{5}-1}{2}$. So the limit of the sequence is $\frac{\sqrt{5}-1}{2}$.

33. Let $L = \lim_{n\to\infty} x_n$. By definition of $x_0, x_1, x_2, \ldots$ and by the continuity of the square root function,

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{2 + x_{n-1}} = \sqrt{2 + \lim_{n\to\infty} x_{n-1}} = \sqrt{2+L}.$$

Hence $L^2 = 2 + L$, and so $L^2 - L - 2 = 0$. Factoring gives $(L-2)(L+1) = 0$, and so $L = 2$ or $L = -1$. But $L \geq 0$ because each $x_i \geq 0$. Thus $L = 2$.

35. *a.* When 3% interest is compounded monthly, the interest rate per month is $0.03/12 = 0.0025$. If S_k is the amount on deposit at the end of month k, then $S_k = S_{k-1} + 0.0025S_{k-1} = (1 + 0.0025)S_{k-1} = (1.0025)S_{k-1}$ for each integer $k \geq 1$.

b. $S_{12} = (1.0025)S_{11}$

$= (1.0025)[(1.0025)S_{10}] = (1.0025S_{10} = (1.0025)^2[(1.0025)S_9] = (1.0025)^3 S_9$

$= (1.0025)^3[(1.0025)S_8] = (1.0025)^4 S_8 = (1.0025)^4[(1.0025)S_7] = (1.0025)^5 S_7$

$= (1.0025)^5[(1.0025)S_6] = (1.0025)^6 S_6 = (1.0025)^6[(1.0025)S_5] = (1.0025)^7 S_5$

$= (1.0025)^7[(1.0025)S_4] = (1.0025)^8 S_4 = (1.0025)^8[(1.0025)S_3] = (1.0025)^9 S_3$

$= (1.0025)^9[(1.0025)S_2] = (1.0025)^{10} S_2 = (1.0025)^{10}[(1.0025)S_1] = (1.0025)^{11} S_1$

$= (1.0025)^{11}[(1.0025)S_0] = (1.0025)^{12} S_0 = (1.0025)^{12} \cdot 10000 \cong 10{,}304.16$ dollars.

c. The APR $= \dfrac{10304.16 - 10000}{10000} = 0.030416 = 3.0416\%$.

37. *a.* length 0: ϵ
length 1: *a, b, c*
length 2: *ab, ac, ba, bb, bc, ca, cb, cc*
length 3: *aba, abb, abc, aca, acb, acc, bab, bac, bba, bbb, bbc, bca, bcb, bcc, cab, cac cba, cbb, cbc, cca, ccb, ccc*

b. By part (a), $s_0 = 1$, $s_1 = 3$, $s_2 = 8$, and $s_3 = 22$

c. Let k be an integer with $k \geq 2$. Any string of length k that does not contain the pattern *aa* starts with an *a*, with a *b*, or with a *c*. If it starts with an *b* or a *c*, this can be followed by any string of length $k-1$ that does not contain the pattern *aa*. There are s_{k-1} such strings, and so there are $2s_{k-1}$ strings that start either with a *b* or with a *c*. If the string starts with an *a*, then the first two characters must be *ab* or *ac*. In either case, the remaining $k-2$ characters can be any string of length $k-2$ that does not contain the pattern *aa*. There are s_{k-2} such strings, and so there are $2s_{k-2}$ strings that start either *ab* or *ac*. It follows that for all integers $k \geq 2$, $s_k = 2s_{k-1} + 2s_{k-2}$.

c. By part (b) $s_2 = 8$ and $s_3 = 22$, and so $s_4 = 2s_3 + 2s_2 = 44 + 16 = 60$.

38. *a.* Let k be an integer with $k \geq 3$. The set of bit strings of length k that do not contain the pattern 101 can be partitioned into $k+1$ subsets: the subset of strings that start with 0 and continue with any bit string of length $k-1$ not containing 101 *[there are a_{k-1} of these]*, the subset of strings that start with 100 and continue with any bit string of length $k-3$ not containing 101 *[there are a_{k-3} of these]*, the subset of strings that start with 1100 and continue with any bit string of length $k-4$ not containing 101 *[there are a_{k-4} of these]*, the subset of strings that start with 11100 and continue with any bit string of length $k-5$ not containing 101 *[there are a_{k-5} of these]*, until the following subset of strings is obtained: $\{\underbrace{11\ldots1}_{k-3\ 1's}001, \underbrace{11\ldots1}_{k-3\ 1's}000\}$ *[there are 2 of these and a_1 equals 2]*. In addition, the three single-element sets $\{\underbrace{11\ldots1}_{k-2\ 1's}00\}$, $\{\underbrace{11\ldots1}_{k-1\ 1's}0\}$, and $\{\underbrace{11\ldots1}_{k-1\ 1's}1\}$ are in the partition, and since $a_0 = 1$

P
100
1100
11100

(because the only bit string of length zero that satisfies the condition is ϵ), $3 = a_0 + 2$. Thus by the addition rule,

$$a_k = a_{k-1} + a_{k-3} + a_{k-4} + \cdots + a_1 + a_0 + 2.$$

b. By part (a), if $k \geq 4$,

$$\begin{aligned} a_k &= a_{k-1} + a_{k-3} + a_{k-4} + \cdots + a_1 + a_0 + 2 \\ a_{k-1} &= a_{k-2} + a_{k-4} + a_{k-5} + \cdots + a_1 + a_0 + 2. \end{aligned}$$

Subtracting the second equation from the first gives

$$\begin{aligned} a_k - a_{k-1} &= a_{k-1} + a_{k-3} - a_{k-2} \\ \Rightarrow \quad a_k &= 2a_{k-1} + a_{k-3} - a_{k-2}. \text{ (Call this equation (*).)} \end{aligned}$$

Note that $a_2 = 4$ (because all four bit strings of length 2 satisfy the condition) and $a_3 = 7$ (because all eight bit strings of length 3 satisfy the condition except 101). Thus equation (*) is also satisfied when $k = 3$ because in that case the right-hand side of the equation becomes $2a_2 + a_0 - a_1 = 2 \cdot 4 + 1 - 2 = 7$, which equals the left-hand side of the equation.

40. Imagine a tower of height k inches. If the bottom block has height one inch, then the remaining blocks make up a tower of height $k-1$ inches. There are t_{k-1} such towers. If the bottom block has height two inches, then the remaining blocks make up a tower of height $k-2$ inches. There are t_{k-2} such towers. If the bottom block has height four inches, then the remaining blocks make up a tower of height $k-4$ inches. There are t_{k-4} such towers. Therefore, $t_k = t_{k-1} + t_{k-2} + t_{k-4}$ for all integers $k \geq 5$.

41. *b.* Let $k \geq 3$ and consider a permutation of $\{1, 2, \ldots, k\}$ that does not move any number more than one place from its "natural" position. Such a permutation either leaves 1 fixed or it interchanges 1 and 2. If it leaves 1 fixed, then the remaining $k-1$ numbers can be permuted in any way except that they must not be moved more than one place from their natural positions. There are a_{k-1} ways to do this. If it interchanges 1 and 2, then the remaining $k-2$ numbers can be permuted in any way except that they must not be moved more than one place from their natural positions. There are a_{k-2} ways to do this. Therefore, $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 2$.

42. To get a sense of the problem, we compute s_4 directly. If there are four seats in the row, there can be a single student in any one of the four seats or there can be a pair of students in seats 1&3, 1&4, or 2&4. No other arrangements are possible because with more than two students, two would have to sit next to each other. Thus $s_4 = 4 + 3 = 7$. In general, if there are k chairs in a row, then

$s_k = s_{k-1}$ (the number of ways a nonempty set of students can sit in the row with no two students adjacent and chair k empty)

$+ s_{k-2}$ (the number of ways students can sit in the row with chair k occupied, chair $k-1$ empty, and chairs 1 through $k-2$ occupied by a nonempty set of students in such a way that no two students are adjacent)

$+1$ (for the seating in which chair k is occupied and all the other chairs are empty

$= s_{k-1} + s_{k-2} + 1$ for all integers $k \geq 3$.

44. The partitions are

$\{x_1\}\{x_2\}\{x_3\}\{x_4, x_5\}$ $\{x_1\}\{x_2\}\{x_4\}\{x_3, x_5\}$ $\{x_1\}\{x_3\}\{x_4\}\{x_2, x_5\}$ $\{x_2\}\{x_3\}\{x_4\}\{x_1, x_5\}$
$\{x_1\}\{x_2\}\{x_5\}\{x_3, x_4\}$ $\{x_1\}\{x_3\}\{x_5\}\{x_2, x_4\}$ $\{x_2\}\{x_3\}\{x_5\}\{x_1, x_4\}$ $\{x_1\}\{x_4\}\{x_5\}\{x_2, x_3\}$
$\{x_2\}\{x_4\}\{x_5\}\{x_1, x_3\}$ $\{x_3\}\{x_4\}\{x_5\}\{x_1, x_2\}$

So $S_{5,4} = 10$.

46. By the recurrence relation from Example 8.1.11 and the values computed in Example 8.1.10, $S_{5,3} = S_{4,2} + 3 \cdot S_{4,3} = 7 + 3 \cdot 6 = 25$.

47. By the definition and initial conditions for Stirling numbers of the second kind and the results of exercises 44–46, the total number of partitions of a set with five elements is $S_{5,1} + S_{5,2} + S_{5,3} + S_{5,4} + S_{5,5} = 1 + 15 + 25 + 10 + 1 = 52$.

49. *Proof (by mathematical induction)*: Let the property $P(n)$ be the equation $\sum_{k=2}^{n} 3^{n-k} S_{k,2} = S_{n+1,3}$.

Show that the property is true for $n = 2$: The property is true for $n = 2$ because for $n = 2$ the left-hand side of the equation is $\sum_{k=2}^{2} 3^{2-k} S_{k,2} = 3^{2-2} S_{2,2} = 1$, and the right-hand side is $S_{2+1,3} = S_{3,3} = 1$ also.

Show that for all integers $m \geq 2$, if the property is true for $n = m$ then it is true for $n = m + 1$: Let m be an integer with $m \geq 2$, and suppose that $\sum_{k=2}^{m} 3^{m-k} S_{k,2} = S_{m+1,3}$. *[This is the inductive hypothesis.]* We must show that

$$\sum_{k=2}^{m+1} 3^{(m+1)-k} S_{k,2} = S_{m+2,3}.$$

But

$$\begin{aligned}
\sum_{k=2}^{m+1} 3^{(m+1)-k} S_{k,2} &= \sum_{k=2}^{m} 3 \cdot 3^{m-k} S_{k,2} + 3^0 S_{m+1,2} \\
&= 3 \sum_{k=2}^{m} 3^{m-k} S_{k,2} + S_{m+1,2} \\
&= S_{m+1,2} + 3 S_{m+1,3} && \text{by inductive hypothesis} \\
&= S_{m+2,3} && \text{by the recurrence relation for Stirling numbers of the second kind.}
\end{aligned}$$

50. If X is a set with n elements and Y is a set with m elements, then the number of onto functions from X to Y is $m!S_{n,m}$, where $S_{n,m}$ is a Stirling number of the second kind. The reason is that we can construct all possible onto functions from X to Y as follows: For each partition of X into m subsets, order the subsets of the partition; call them, say, $S_1, S_2, \ldots, S_m$. Define an onto function from X to Y by first choosing an element of Y to be the image of all the elements in S_1 (there are m ways to do this), then choosing another element of Y to be the image of all the elements in S_2 (there are $m - 1$ ways to do this), then choosing another element of Y to be the image of all the elements in S_3 (there are $m - 2$ ways to do this), and so forth. Each of the $m!$ functions constructed in this way is onto because since Y has m elements and there are m subsets in the partition, eventually every element in Y will be the image of at least one element in X. Thus for each partition of X into m subsets, there are $m!$ onto functions, and so the total number of onto functions is the number of partitions, $S_{n,m}$, times $m!$, or $m!S_{n,m}$.

51. *Proof (by strong mathematical induction)*: Let the property $P(n)$ be the inequality $F_n < 2^n$ where F_n is the nth Fibonacci number.

Show that the property is true for $n = 1$ and $n = 2$: $F_1 = 1 < 2 = 2^1$ and $F_2 = 3 < 4 = 2^2$.

Show that for all integers $k > 2$, if the property is true for all integers i with $1 \le i < k$ then it is true for k: Let k be an integer with $k > 2$, and suppose that $F_i < 2^i$ for all integers i with $0 \le i < k$. *[This is the inductive hypothesis.]* We must show that $F_k < 2^k$. Now by definition of the Fibonacci numbers, $F_k = F_{k-1} + F_{k-2}$. But by inductive hypothesis *[since $k > 2$]*, $F_{k-1} < 2^{k-1}$ and $F_{k-2} < 2^{k-2}$. Hence $F_k = F_{k-1} + F_{k-2} < 2^{k-1} + 2^{k-2} = 2^{k-2} \cdot (2+1) = 3 \cdot 2^{k-2} < 4 \cdot 2^{k-2} = 2^k$. Thus $F_k < 2^k$ *[as was to be shown]*.

[Since both the basis and inductive steps have been proved, we conclude that $F_n < 2^n$ for all integers $n \ge 1$.]

52. *Proof (by mathematical induction)*: Let the property $P(n)$ be the equation $\gcd(F_{n+1}, F_n) = 1$.

Show that the property is true for $n = 0$: To prove the property for $n = 0$, we must show that $\gcd(F_1, F_0) = 1$. But $F_1 = 1$ and $F_0 = 1$ and $\gcd(1, 1) = 1$.

Show that for all integers $k \ge 0$, if the property is true for $n = k$ then it is true for $n = k + 1$: Let k be an integer with $k \ge 0$, and suppose that $\gcd(F_{k+1}, F_k) = 1$. *[This is the inductive hypothesis.]* We must show that $\gcd(F_{k+2}, F_{k+1}) = 1$. But by definition of the Fibonacci sequence $F_{k+2} = F_{k+1} + F_k$. It follows from Lemma 3.8.2 that $\gcd(F_{k+2}, F_{k+1}) = \gcd(F_{k+1}, F_k)$. But by inductive hypothesis, $\gcd(F_{k+1}, F_k) = 1$. Hence $\gcd(F_{k+2}, F_{k+1}) = 1$ *[as was to be shown]*.

[Since both the basis and the inductive steps have been proved, we conclude that $\gcd(F_{n+1}, F_n) = 1$ for all integers $n \ge 0$.]

53. *a.* $g_3 = 1$, $g_4 = 1$, $g_5 = 2$ ($LWLLL$ and $WWLLL$)

b. $g_6 = 4$ ($WWWLLL$, $WLWLLL$, $LWWLLL$, $LLWLLL$)

c. If $k \ge 6$, then any sequence of k games must begin with exactly one of the possibilities: W, LW, or LLW. The number of sequences of k games that begin with W is g_{k-1} because the succeeding $k-1$ games can consist of any sequence of wins and losses except that the first sequence of three consecutive losses occurs at the end. Similarly, the number of sequences of k games that begin with LW is g_{k-2} and the number of sequences of k games that begin with LLW is g_{k-3}. Therefore, $g_k = g_{k-1} + g_{k-2} + g_{k-3}$ for all integers $k \ge 6$.

54. *a.* $d_1 = 0$, $d_2 = 1$, $d_3 = 2$ (231 and 312)

b. $d_4 = 9$ (2143, 3412, 4321, 3142, 4123, 2413, 4312, 2341, 3421)

c. Divide the set of all derangements into two subsets: one subset, S, consists of all derangements in which the number 1 changes places with another number, and the other subset, T, consists of all derangements in which the number 1 goes to position $i \ne 1$ but i does not go to position 1. Forming a derangement in S can be regarded as a two-step process: step 1 is to choose a position i and to interchange 1 and i and step 2 is to derange the remaining $k-2$ numbers. Now there are $k-1$ numbers with which 1 can trade places in step 1, and so by the product rule there are $(k-1)d_{k-2}$ derangements in S. Forming a derangement in T can also be regarded as a two-step process: step 1 is to derange the $k-1$ numbers $2, 3, \ldots, k$ in positions $2, 3, \ldots, k$, and step 2 is to interchange the number 1 in position 1 with any of the numbers in the derangement. Now there are $k-1$ choices of numbers to interchange 1 with in step 2, and so by the multiplication rule there are $d_{k-1}(k-1)$ derangements in T. It follows that the total number of derangements of the given k numbers is $d_k = (k-1)d_{k-1} + (k-1)d_{k-2}$ for all integers $k \ge 3$.

55. For each integer $k = 1, 2, \ldots, n-1$, consider the product $(x_1x_2 \ldots x_k)(x_{k+1}x_{k+2} \ldots x_n)$. The factor $x_1x_2 \ldots x_k$ can be parenthesized in P_k ways, and the factor $x_{k+1}x_{k+2} \ldots x_n$ can be parenthesized in P_{n-k} ways. Therefore, the product $x_1x_2x_3 \ldots x_{n-1}x_n$ can be parenthesized in P_kP_{n-k} ways if the final multiplication is $(x_1x_2 \ldots x_k) \cdot (x_{k+1}x_{k+2} \ldots x_n)$. Now when $x_1x_2x_3 \ldots x_{n-1}x_n$ is fully parenthesized, the final multiplication can be any one of the following: $(x_1) \cdot (x_2x_3 \ldots x_n)$, $(x_1x_2) \cdot (x_3x_4 \ldots x_n)$, $(x_1x_2x_3) \cdot (x_4x_5 \ldots x_n)$, $\ldots$, $(x_1x_2 \ldots x_{n-2}) \cdot (x_{n-1}x_n)$,

$(x_1x_2\ldots x_{n-1})\cdot(x_n)$. So the total number of ways to parenthesize the product is the sum of all the numbers P_kP_{n-k} for $k=1,2,\ldots,n-1$. In symbols: $P_n=\sum_{k=1}^{n-1}P_kP_{n-k}$.

Section 8.2

1. *c.* $3+3\cdot2+3\cdot3+\cdots+3\cdot n+n = 3(1+2+3+\ldots+n)+n = 3\left(\frac{n(n+1)}{2}\right)+n = \frac{3n(n+1)}{2}+\frac{2n}{2}=\frac{3n^2+5n}{2}$.

2. *b.* $3^{n-1}+3^{n-2}+\cdots+3^2+3+1=1+3+3^2+\cdots+3^{n-2}+3^{n-1}=\frac{3^{(n-1)+1}-1}{3-1}=\frac{3^n-1}{2}$.

d. Note that $\frac{1}{(-1)^n}=(-1)^n$ and $\frac{1}{(-1)^n}\cdot(-1)^k=\frac{1}{(-1)^{n-k}}=(-1)^{n-k}$. Thus

$$\begin{aligned}
&2^n-2^{n-1}+2^{n-2}-2^{n-3}+\cdots+(-1)^{n-2}\cdot2^2+(-1)^{n-1}\cdot2+(-1)^n\\
&= \frac{1}{(-1)^n}((-1)^n2^n+(-1)^{n-1}2^{n-1}+(-1)^{n-2}2^{n-2}+\cdots+(-1)^2\cdot2^2+(-1)^1\cdot2^1+1)\\
&= \frac{1}{(-1)^n}(1-2+2^2+\cdots+(-1)^{n-1}2^{n-1}+(-1)^n2^n)\\
&= (-1)^n(1+(-2)+(-2)^2+\cdots+(-2)^{n-1}+(-2)^n)\\
&= (-1)^n\left(\frac{(-2)^{n+1}-1}{(-2)-1}\right)\\
&= (-1)^n\left(\frac{1-(-2)^{n+1}}{3}\right)\\
&= \frac{(-1)^n+2^{n+1}}{3}.
\end{aligned}$$

4.
$$\begin{aligned}
b_0 &= 1\\
b_1 &= \frac{b_0}{1+b_0}=\frac{1}{1+1}=\frac{1}{2}\\
b_2 &= \frac{b_1}{1+b_1}=\frac{\frac12}{1+\frac12}=\frac{1}{2+1}=\frac{1}{3}\\
b_3 &= \frac{b_2}{1+b_2}=\frac{\frac13}{1+\frac13}=\frac{1}{3+1}=\frac{1}{4}\\
b_4 &= \frac{b_3}{1+b_3}=\frac{\frac14}{1+\frac14}=\frac{1}{4+1}=\frac{1}{5}\\
&\ \ \vdots
\end{aligned}$$

Guess: $b_n=\frac{1}{n+1}$ for all integers $n\geq0$

6.
$$\begin{aligned}
d_1 &= 2\\
d_2 &= 2d_1+3=2\cdot2+3=2^2+3\\
d_3 &= 2d_2+3=2(2^2+3)+3=2^3+2\cdot3+3\\
d_4 &= 2d_3+3=2(2^3+2\cdot3+3)+3=2^4+2^2\cdot3+2\cdot3+3\\
d_5 &= 2d_4+3=2(2^4+2^2\cdot3+2\cdot3+3)+3=2^5+2^3\cdot3+2^2\cdot3+2\cdot3+3\\
&\ \ .
\end{aligned}$$