Then for $n = b^{k+1}$ we apply first the recurrence relation, then the inductive hypothesis, and finally some algebra:

$$\begin{aligned}
f(n) &= af\left(\frac{n}{b}\right) + cn^d \\
&= a\left(\frac{b^d c}{b^d - a}\left(\frac{n}{b}\right)^d + \left(f(1) + \frac{b^d c}{a - b^d}\right)\left(\frac{n}{b}\right)^{\log_b a}\right) + cn^d \\
&= \frac{b^d c}{b^d - a} \cdot n^d \cdot \frac{a}{b^d} + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a} + cn^d \\
&= n^d\left(\frac{ac}{b^d - a} + \frac{c(b^d - a)}{b^d - a}\right) + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a} \\
&= \frac{b^d c}{b^d - a} \cdot n^d + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a}
\end{aligned}$$

Thus we have verified that the equation holds for $k+1$, and our induction proof is complete.

33. The equation given in Exercise 31 says that $f(n)$ is the sum of a constant times n^d and a constant times $n^{\log_b a}$. Therefore we need to determine which term dominates, i.e., whether d or $\log_b a$ is larger. But we are given $a > b^d$; hence $\log_b a > \log_b b^d = d$. It therefore follows (we are also using the fact that f is increasing) that $f(n)$ is $O(n^{\log_b a})$.

35. We use the result of Exercise 33, since $a = 5 > 4^1 = b^d$. Therefore $f(n)$ is $O(n^{\log_b a}) = O(n^{\log_4 5}) \approx O(n^{1.16})$.

37. We use the result of Exercise 33, since $a = 8 > 2^2 = b^d$. Therefore $f(n)$ is $O(n^{\log_b a}) = O(n^{\log_2 8}) = O(n^3)$.

SECTION 8.4 Generating Functions

Generating functions are an extremely powerful tool in mathematics (not just in discrete mathematics). This section, as well as some material introduced in these exercises, gives you an introduction to them. The algebra in many of these exercises gets very messy, and you probably want to check your answers, either by computing values when solving recurrence relation problems, or by using a computer algebra package. See the solution to Exercise 11, for example, to learn how to get Maple *to produce the sequence for a given generating function. For more information on generating functions, consult reference [Wi2] given at the end of this* Guide *(in the List of References for the Writing Projects).*

1. By definition we want the function $f(x) = 2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 = 2(1 + x + x^2 + x^3 + x^4 + x^5)$. From Example 2, we see that the expression in parentheses can also be written as $(x^6 - 1)/(x - 1)$. Thus we can write the answer as $f(x) = 2(x^6 - 1)/(x - 1)$.

3. We will use Table 1 in much of this solution.

a) Apparently all the terms are 0 except for the six 2's shown. Thus $f(x) = 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6$. This is already in closed form, but we can also write it more compactly as $f(x) = 2x(1 - x^6)/(1 - x)$ by factoring out $2x$, or as $f(x) = 2(1 - x^7)/(1 - x) - 2$ by subtracting away the missing term. In each case we use the identity from Example 2.

b) Apparently all the terms beyond the first three are 1's. Since $1/(1 - x) = 1 + x + x^2 + x^3 + \cdots$, we can write this generating function as $1/(1-x) - 1 - x - x^2$, or we can write it as $x^3/(1-x)$, depending on whether we want to simplify by adding back the missing terms or by factoring out x^3.

c) This generating function is the sequence $x + x^4 + x^7 + x^{10} + \cdots$. If we factor out an x, we have $x(1 + (x^3) + (x^3)^2 + (x^3)^3 + \cdots) = x/(1 - x^3)$, from Table 1.

d) We factor out a 2 and then include the remaining factors of 2 along with the x terms. Thus our generating function is $2(1 + (2x) + (2x)^2 + (2x)^3 + \cdots) = 2/(1 - 2x)$, again using Table 1.

e) By the binomial theorem (see also Table 1), the generating function is $(1+x)^7$.

f) From Table 1 we know that $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$. That is what we have here, with $a = -1$ (and a factor of 2 in front of it all). Therefore the generating function is $2/(1+x)$.

g) This sequence is all 1's except for a 0 where the x^2 coefficient should be. Therefore the generating function is $(1/(1-x)) - x^2$.

h) If we factor out x^3, then we can use a formula from Table 1: $x^3+2x^4+3x^5+\cdots = x^3(1+2x+3x^2+\cdots) = x^3/(1-x)^2$.

5. As in Exercise 3, we make extensive use of Table 1.

a) Since the sequence with $a_n = 1$ for all n has generating function $1/(1-x)$, this sequence has generating function $5/(1-x)$.

b) By Table 1 the answer is $1/(1-3x)$.

c) We can either subtract the missing terms and write this generating function as $(2/(1-x)) - 2 - 2x - 2x^2$, or we can factor out x^3 and write it as $2x^3/(1-x)$. Note that these two algebraic expressions are equivalent.

d) We need to split this into two parts. Since we know that the generating function for the sequence $\{n+1\}$ is $1/(1-x)^2$, we write $2n+3 = 2(n+1)+1$. Therefore the generating function is $2/(1-x)^2 + 1/(1-x)$. We can combine terms and write this function as $(3-x)/(1-x)^2$, but there is no particular reason to prefer that form in general.

e) By Table 1 the answer is $(1+x)^8$. Note that $C(8,n) = 0$ by definition for all $n > 8$.

f) By Table 1 the generating function is $1/(1-x)^5$.

7. a) We can rewrite this as $(-4(1-\frac{3}{4}x))^3 = -64(1-\frac{3}{4}x)^3$ and then apply the binomial theorem (the second line of Table 1) to get $a_n = -64C(3,n)(-\frac{3}{4})^n$. Explicitly, this says that $a_0 = -64$, $a_1 = 144$, $a_2 = -108$, $a_3 = 27$, and $a_n = 0$ for all $n \geq 4$. Alternatively, we could (by hand or with *Maple*) just multiply out this finite polynomial and note the coefficients.

b) This is like part **(a)**. By the binomial theorem (the third line of Table 1) we get $a_{3n} = C(3,n)$, and the other coefficients are all 0. Alternatively, we could just multiply out this finite polynomial and note the nonzero coefficients: $a_0 = 1$, $a_3 = 3$, $a_6 = 3$, $a_9 = 1$.

c) By Table 1, $a_n = 5^n$.

d) Note that $x^3/(1+3x) = x^3\sum_{n=0}^{\infty}(-3)^n x^n = \sum_{n=0}^{\infty}(-3)^n x^{n+3} = \sum_{n=3}^{\infty}(-3)^{n-3}x^n$. So $a_n = (-3)^{n-3}$ for $n \geq 3$, and $a_0 = a_1 = a_2 = 0$.

e) We know what the coefficients are for the power series of $1/(1-x^2)$, namely 0 for the odd ones and 1 for the even ones. The first three terms of this function force us to adjust the values of a_0, a_1 and a_2. So we have $a_0 = 7+1 = 8$, $a_1 = 3+0 = 3$, $a_2 = 1+1 = 2$, $a_n = 0$ for odd n greater than 2, and $a_n = 1$ for even n greater than 2.

f) Perhaps this is easiest to see if we write it out: $x^4(1+x^4+x^8+x^{12}+\cdots) - x^3 - x^2 - x - 1 = x^4 + x^8 + x^{12} + \cdots - x^3 - x^2 - x - 1$. Therefore we have $a_n = 1$ if n is a positive multiple of 4; $a_n = -1$ if $n < 4$, and $a_n = 0$ otherwise.

g) We know that $x^2/(1-x)^2 = x^2\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}(n+1)x^{n+2} = \sum_{n=2}^{\infty}(n-1)x^n$. Therefore $a_n = n-1$ for $n \geq 2$ and $a_0 = a_1 = 0$.

h) We know that $2e^{2x} = 2\sum_{n=0}^{\infty}(2x)^n/n! = \sum_{n=0}^{\infty}(2^{n+1}/n!)x^n$. Therefore $a_n = 2^{n+1}/n!$.

9. Different approaches are possible for obtaining these answers. One can use brute force algebra and just multiply everything out, either by hand or with computer algebra software such as *Maple*. One can view the problem as asking for the solution to a particular combinatorial problem and solve the problem by other means (e.g., listing all the possibilities). Or one can get a closed form expression for the coefficients, using the generating function theory developed in this section.

a) First we view this combinatorially. To obtain a term x^{10} when multiplying out these three factors, we could either take two x^5's and one x^0, or we could take two x^0's and one x^{10}. In each case there are $C(3,1) = 3$ choices for the factor from which to pick the single value. Therefore the answer is $3 + 3 = 6$. Second, it is clear that we can view this problem as asking for the coefficient of x^2 in $(1 + x + x^2 + x^3 + \cdots)^3$, since each x^5 in the original is playing the role of x here. Since $(1 + x + x^2 + x^3 + \cdots)^3 = 1/(1-x)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, the answer is clearly $C(2+2,2) = C(4,2) = 6$. A third way to get the answer is to ask *Maple* to compute $(1 + x^5 + x^{10})^3$ and look at the coefficient of x^{10}, which will turn out to be 6. Note that we don't have to go beyond x^{10} in each factor, because the higher terms can't contribute to an x^{10} term in the answer.

b) If we factor out x^3 from each factor, we can write this as $x^9(1 + x + x^2 + \cdots)^3$. Thus we are seeking the coefficient of x in $(1 + x + x^2 + \cdots)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, so the answer is $C(1+2,2) = 3$. The other two methods explained in part **(a)** work here as well.

c) If we factor out as high a power of x from each factor as we can, then we can write this as

$$x^7(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + \cdots),$$

and so we seek the coefficient of x^3 in $(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + \cdots)$. By brute force we can list the nine ways to obtain x^3 in this product (where "ijk" means choose an x^i term from the first factor, an x^j term from the second factor, and an x^k term from the third factor): 003, 012, 021, 030, 102, 111, 120, 201, 210. If we want to do this more analytically, let us write our expression in closed form as

$$\frac{1-x^3}{1-x} \cdot \frac{1-x^5}{1-x} \cdot \frac{1}{1-x} = \frac{1 - x^3 - x^5 + x^8}{(1-x)^3} = \frac{1}{(1-x)^3} - x^3 \cdot \frac{1}{(1-x)^3} + \text{irrelevant terms}.$$

Now the coefficient of x^n in $1/(1-x)^3$ is $C(n+2,2)$. Furthermore, the coefficient of x^3 in this power series comes either from the coefficient of x^3 in the first term in the final expression displayed above, or from the coefficient of x^0 in the second factor of the second term of that expression. Therefore our answer is $C(3+2,2) - C(0+2,2) = 10 - 1 = 9$.

d) Note that only even powers appear in the first and third factor, so to get x^{10} when we multiply this out, we can only choose the x^6 term in the second factor. But this would require terms from the first and third factors with a total exponent of 4, and clearly that is not possible. Therefore the desired coefficient is 0.

e) The easiest approach here might be brute force. Using the same notation as explained in part **(c)** above, the ways to get x^{10} are 046, 280, 406, 640, and (10)00. Therefore the answer is 5. We can check this with *Maple*. An analytic approach would be rather messy for this problem.

11. **a)** By Table 1 the coefficient of x^n in this power series is 2^n. Therefore the answer is $2^{10} = 1024$.

b) By Table 1 the coefficient of x^n in this power series is $(-1)^n C(n+1,1)$. Therefore the answer is $(-1)^{10}C(10+1,1) = 11$.

c) By Table 1 the coefficient of x^n in this power series is $C(n+2,2)$. Therefore the answer is $C(10+2,2) = 66$.

d) By Table 1 the coefficient of x^n in this power series is $(-2)^n C(n+3,3)$. Therefore the answer is $(-2)^{10}C(10+3,3) = 292{,}864$. Incidentally, *Maple* can do this kind of problem as well. Typing

```
series(1/(1+2*x)^4, x = 0, 11);
```

will cause *Maple* to give the terms of the power series for this function, including all terms less than x^{11}. The output looks like

$$1 - 8x + 40x^2 - 160x^3 + 560x^4 - 1792x^5 + 5376x^6 - 15360x^7 + 42240x^8 - 112640x^9 + 292864x^{10} + O(x^{11}).$$

(You might wonder why *Maple* says that the terms involving x^{11}, x^{12}, and so on are big-O of x^{11}. That seems backward! The reason is that one thinks of x as approaching 0 here, rather than infinity. Then, indeed, each term with a higher power of x (greater than 11) is smaller than x^{11}, up to a constant multiple.)

e) This is really asking for the coefficient of x^6 in $1/(1-3x)^3$. Following the same idea as in part **(d)**, we see that the answer is $3^6C(6+2,2) = 20{,}412$.

13. Each child will correspond to a factor in our generating function. We can give any number of balloons to the child, as long as it is at least 2; therefore the generating function for each child is $x^2 + x^3 + x^4 + \cdots$. We want to find the coefficient of x^{10} in the expansion of $(x^2 + x^3 + x^4 + \cdots)^4$. This function is the same as $x^8(1 + x + x^2 + x^3 + \cdots)^4 = x^8/(1-x)^4$. Therefore we want the coefficient of x^2 in the generating function for $1/(1-x)^4$, which we know from Table 1 is $C(2+3,3) = 10$. Alternatively, to find the coefficient of x^2 in $(1+x+x^2+x^3+\cdots)^4$, we can multiply out $(1+x+x^2)^4$ (perhaps with a computer algebra package such as *Maple*), and the coefficient of x^2 turns out to be 10. Note that we truncated the series to be multiplied out, since terms higher than x^2 can't contribute to the x^2 term.

15. Each child will correspond to a factor in our generating function. We can give 1, 2, or 3 animals to the child; therefore the generating function for each child is $x + x^2 + x^3$. We want to find the coefficient of x^{15} in the expansion of $(x + x^2 + x^3)^6$. Factoring out an x from each term, we see that this is the same as the coefficient of x^9 in $(1 + x + x^2)^6$. We can multiply this out (preferably with a computer algebra package such as *Maple*), and the coefficient of x^9 turns out to be 50. To solve it analytically, we write our generating function $(1 + x + x^2)^6$ as

$$\left(\frac{1-x^3}{1-x}\right)^6 = \frac{1 - 6x^3 + 15x^6 - 20x^9 + \text{higher order terms}}{(1-x)^6}.$$

There are four contributions to the coefficient of x^9; one for each listed term in the numerator, from the power series for $1/(1-x)^6$. Since the coefficient of x^n in $1/(1-x)^6$ is $C(n+5,5)$, our answer is $C(9+5,5) - 6C(6+5,5) + 15C(3+5,5) - 20C(0+5,5) = 2002 - 2772 + 840 - 20 = 50$.

17. The factor in the generating function for choosing the donuts for each policeman is $x^3 + x^4 + x^5 + x^6 + x^7$. Therefore the generating function for this problem is $(x^3+x^4+x^5+x^6+x^7)^4$. We want to find the coefficient of x^{25}, since we want 25 donuts in all. This is equivalent to finding the coefficient of x^{13} in $(1+x+x^2+x^3+x^4)^4$, since we can factor out $(x^3)^4 = x^{12}$. At this point, we could multiply it out (perhaps with *Maple*), and see that the desired coefficient is 20. Alternatively, we can write our generating function as

$$\left(\frac{1-x^5}{1-x}\right)^4 = \frac{1 - 4x^5 + 6x^{10} + \text{higher order terms}}{(1-x)^4}.$$

There are three contributions to the coefficient of x^{13}, one for each term in the numerator, from the power series for $1/(1-x)^4$. Since the coefficient of x^n in $1/(1-x)^4$ is $C(n+3,3)$, our answer is $C(13+3,3) - 4C(8+3,3) + 6C(3+3,3) = 560 - 660 + 120 = 20$.

19. We want the coefficient of x^k to be the number of ways to make change for k dollars. One-dollar bills contribute 1 each to the exponent of x. Thus we can model the choice of the number of one-dollar bills by the choice of a term from $1+x+x^2+x^3+\cdots$. Two-dollar bills contribute 2 each to the exponent of x. Thus we can model the choice of the number of two-dollar bills by the choice of a term from $1+x^2+x^4+x^6+\cdots$. Similarly, five-dollar bills contribute 5 each to the exponent of x, so we can model the choice of the number of five-dollar bills by the choice of a term from $1+x^5+x^{10}+x^{15}+\cdots$. Similar reasoning applies to ten-dollar bills. Thus the generating function is $f(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+x^{30}+\cdots)$, which can also be written (see Table 1) as

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})}.$$

21. Let e_i, for $i = 1, 2, 3$, be the exponent of x taken from the i^{th} factor in forming a term x^4 in the expansion. Thus $e_1 + e_2 + e_3 = 4$. The coefficient of x^4 is therefore the number of ways to solve this equation with nonnegative integers, which, from Section 6.5, is $C(3+4-1,4) = C(6,4) = 15$.

23. **a)** The restriction on x_1 gives us the factor $x^2 + x^3 + x^4 + \cdots$. The restriction on x_2 gives us the factor $1 + x + x^2 + x^3$. The restriction on x_3 gives us the factor $x^2 + x^3 + x^4 + x^5$. Thus the answer is the product of these: $(x^2 + x^3 + x^4 + \cdots)(1 + x + x^2 + x^3)(x^2 + x^3 + x^4 + x^5)$. We can use algebra and Table 1 to rewrite this in closed form as $x^4(1 + x + x^2 + x^3)^2/(1 - x)$.

b) We want the coefficient of x^6 in this series, which is the same as the coefficient of x^2 in the series for

$$\frac{(1 + x + x^2 + x^3)^2}{1 - x} = \frac{1 + 2x + 3x^2 + \text{higher order terms}}{1 - x}.$$

Since the coefficient of x^n in $1/(1 - x)$ is 1, our answer is $1 + 2 + 3 = 6$.

25. This problem reinforces the point that "and" corresponds to multiplication and "or" corresponds to addition.

a) The only issue is how many stamps of each denomination we choose. The exponent on x will be the number of cents. So the generating function for choosing 3-cent stamps is $1 + x^3 + x^6 + x^9 + \cdots$, the generating function for 4-cent stamps is $1 + x^4 + x^8 + x^{12} + \cdots$, and the generating function for 20-cent stamps is $1 + x^{20} + x^{40} + x^{60} + \cdots$. In closed form this is $1/((1 - x^3)(1 - x^4)(1 - x^{20}))$. The coefficient of x^r gives the answer—the number of ways to choose stamps totaling r cents of postage.

b) Again the exponent on x will be the number of cents, but this time we paste one stamp at a time. For the first pasting, we can choose a 3-cent stamp, a 4-cent stamp, or a 20-cent stamp. Hence the generating function for the number of ways to paste one stamp is $x^3 + x^4 + x^{20}$. For the second pasting, we can make these same choices, so the generating function for the number of ways to paste two stamps is $(x^3 + x^4 + x^{20})^2$. In general, if we use n stamps, the generating function is $(x^3 + x^4 + x^{20})^n$. Since a pasting consists of a pasting of zero or more stamps, the entire generating function will be

$$\sum_{n=0}^{\infty}(x^3 + x^4 + x^{20})^n = \frac{1}{1 - x^3 - x^4 - x^{20}}.$$

c) We seek the coefficient of x^{46} in the power series for our answer to part **(a)**, $1/((1 - x^3)(1 - x^4)(1 - x^{20}))$. Other than working this out by brute force (enumerating the combinations), the best way to get the answer is probably asking *Maple* or another computer algebra package to multiply out these series. If we do so, the answer turns out to be 7. (The choices are $2 \cdot 20 + 2 \cdot 3$, $20 + 5 \cdot 4 + 2 \cdot 3$, $20 + 2 \cdot 4 + 6 \cdot 3$, $10 \cdot 4 + 2 \cdot 3$, $7 \cdot 4 + 6 \cdot 3$, $4 \cdot 4 + 10 \cdot 3$, and $1 \cdot 4 + 14 \cdot 3$.)

d) We seek the coefficient of x^{46} in the power series for our answer to part **(b)**, $1/(1 - x^3 - x^4 - x^{20})$. The best way to get the answer is probably asking *Maple* or another computer algebra package to find this power series using calculus. If we do so, the answer turns out to be 3224. Alternatively, for each of the seven combinations in our answer to part **(c)**, we can find the number of ordered arrangements, as in Section 6.5. Thus the answer is

$$\frac{4!}{2!2!} + \frac{8!}{1!5!2!} + \frac{9!}{1!2!6!} + \frac{12!}{10!2!} + \frac{13!}{7!6!} + \frac{14!}{4!10!} + \frac{15!}{1!14!} = 6 + 168 + 252 + 66 + 1716 + 1001 + 15 = 3224.$$

27. We will write down the generating function in each case and then use a computer algebra package to find the desired coefficients. As a check, one could carefully enumerate these by hand. In making change, one usually considers order irrelevant.

a) The generating function for the dimes is $1 + x^{10} + x^{20} + x^{30} + \cdots = 1/(1 - x^{10})$, and the generating function for the quarters is $1 + x^{25} + x^{50} + x^{75} + \cdots = 1/(1 - x^{25})$, so the generating function for the whole problem is $1/((1 - x^{10})(1 - x^{25}))$. The coefficient of x^k gives the number of ways to make change for k cents, so we seek the coefficient of x^{100}. If we ask a computer algebra system to find this coefficient (it uses calculus to get the power series), we find that the answer is 3. In fact, this is correct, since we can use four quarters, two quarters, or no quarters (and the number of dimes is uniquely determined by this choice).

b) This is identical to part **(a)** except for a factor for the nickels. Thus we seek the coefficient of x^{100} in $1/((1-x^5)(1-x^{10})(1-x^{25}))$, which turns out to be 29. (If we wanted to list these systematically, we could organize our work by the number of quarters, and within that by the number of dimes.)

c) This is identical to part **(a)** except for a factor for the pennies. Thus we seek the coefficient of x^{100} in $1/((1-x)(1-x^{10})(1-x^{25}))$, which turns out to be 29 again. (In retrospect, this is obvious. The only difference between parts **(b)** and **(c)** is that five pennies are substituted for each nickel.)

d) This is identical to part **(a)** except for factors for the pennies and nickels. Thus we seek the coefficient of x^{100} in $1/((1-x)(1-x^5)(1-x^{10})(1-x^{25}))$, which turns out to be 242.

29. We will write down the generating function in each case and then use a computer algebra package to find the desired coefficients. In making change, one usually considers order irrelevant.

a) The generating function for the \$10 bills is $1+x^{10}+x^{20}+x^{30}+\cdots = 1/(1-x^{10})$, the generating function for the \$20 bills is $1+x^{20}+x^{40}+x^{60}+\cdots = 1/(1-x^{20})$, and the generating function for the \$50 bills is $1+x^{50}+x^{100}+x^{150}+\cdots = 1/(1-x^{50})$, so the generating function for the whole problem is $1/((1-x^{10})(1-x^{20})(1-x^{50}))$. The coefficient of x^k gives the number of ways to make change for k dollars, so we seek the coefficient of x^{100}. If we ask a computer algebra system to find this coefficient (it uses calculus to get the power series), we find that the answer is 10. In fact, this is correct, since there is one way in which we can use two \$50 bills, three ways in which we use one \$50 bill (using either two, one, or no \$20 bills), and six ways to use no \$50 bills (using zero through five \$20's).

b) This is identical to part **(a)** except for a factor for the \$5 bills. Thus we seek the coefficient of x^{100} in $1/((1-x^5)(1-x^{10})(1-x^{20})(1-x^{50}))$, which turns out to be 49.

c) In part **(b)** we saw that the generating function for this problem is $1/((1-x^5)(1-x^{10})(1-x^{20})(1-x^{50}))$. If at least one of each bill must be used, let us assume that this \$50 + \$20 + \$10 + \$5 = \$85 has already been dispersed. Then we seek the coefficient of x^{15}. The computer algebra package tells us that the answer is 2, but it is trivial to see that there are only two ways to make \$15 with these bills.

d) This time the generating function is $(x^5+x^{10}+x^{15}+x^{20})(x^{10}+x^{20}+x^{30}+x^{40})(x^{20}+x^{40}+x^{60}+x^{80})$. When the computer multiplies this out, it tells us that the coefficient of x^{100} is 4, so that is the answer. (In retrospect, we see that the only solutions are $4\cdot\$20+1\cdot\$10+2\cdot\$5$, $3\cdot\$20+3\cdot\$10+2\cdot\$5$, $3\cdot\$20+2\cdot\$10+4\cdot\$5$, and $2\cdot\$20+4\cdot\$10+4\cdot\$5$.)

31. **a)** The terms involving a_0, a_1, and a_2 are missing; $G(x)-a_0-a_1x-a_2x^2 = a_3x^3+a_4x^4+\cdots$. That is the generating function for precisely the sequence we are given. Thus the answer is $G(x)-a_0-a_1x-a_2x^2$.

b) Every other term is missing, and the old coefficient of x^n is now the coefficient of x^{2n}. This suggests that maybe x^2 should be used in place of x. Indeed, this works; the answer is $G(x^2)=a_0+a_1x^2+a_2x^4+\cdots$.

c) If we want a_0 to be the coefficient of x^4 (and similarly for the other powers), we must throw in an extra factor. Thus the answer is $x^4G(x)$. Note that $x^4(a_0+a_1x+a_2x^2+\cdots)=a_0x^4+a_1x^5+a_2x^6+\cdots$.

d) Extra factors of 2 are applied to each term, with the power of 2 matching the subscript (which, of course, gives us the power of x). Thus the answer must be $G(2x)=a_0+a_1(2x)+a_2(2x)^2+a_3(2x)^3\cdots = a_0+2a_1x+4a_2x^2+8a_3x^3\cdots$.

e) Following the hint, we integrate $G(t)=\sum_{n=0}^{\infty}a_nt^n$ from 0 to x, to obtain $\int_0^x G(t)\,dt=\sum_{n=0}^{\infty}a_n\int_0^x t^n\,dt = \sum_{n=0}^{\infty}a_nx^{n+1}/(n+1)$. (If we had tried differentiating first, we'd see that that didn't work. It is a theorem of advanced calculus that it is legal to integrate inside the summation within the open interval of convergence.) This is the series

$$a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\cdots,$$

precisely the sequence we are given (note that the constant term is 0). Thus $\int_0^x G(t)\,dt$ is the generating function for this sequence.

f) If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x)\cdot(1+x+x^2+\cdots)=G(x)/(1-x)$.

33. This problem is like Example 16. First let $G(x)=\sum_{k=0}^{\infty}a_kx^k$. Then $xG(x)=\sum_{k=0}^{\infty}a_kx^{k+1}=\sum_{k=1}^{\infty}a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$). Thus

$$G(x)-3xG(x)=\sum_{k=0}^{\infty}a_kx^k-\sum_{k=1}^{\infty}3a_{k-1}x^k=a_0+\sum_{k=1}^{\infty}(a_k-3a_{k-1})x^k=a_0+\sum_{k=1}^{\infty}2x^k$$
$$=1+\frac{2}{1-x}-2=\frac{1+x}{1-x},$$

because of the given recurrence relation, the initial condition, and the fact from Table 1 that $\sum_{k=0}^{\infty}2x^k=2/(1-x)$. Thus $G(x)(1-3x)=(1+x)/(1-x)$, so $G(x)=(1+x)/((1-3x)(1-x))$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{1+x}{(1-3x)(1-x)}=\frac{A}{1-3x}+\frac{B}{1-x},$$

multiplying through by the common denominator, and equating coefficients, we find that $A=2$ and $B=-1$. Thus

$$G(x)=\frac{2}{1-3x}+\frac{-1}{1-x}=\sum_{k=0}^{\infty}(2\cdot3^k-1)x^k$$

(the last equality came from using Table 1). Therefore $a_k=2\cdot3^k-1$.

35. Let $G(x)=\sum_{k=0}^{\infty}a_kx^k$. Then $xG(x)=\sum_{k=0}^{\infty}a_kx^{k+1}=\sum_{k=1}^{\infty}a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and similarly $x^2G(x)=\sum_{k=0}^{\infty}a_kx^{k+2}=\sum_{k=2}^{\infty}a_{k-2}x^k$. Thus

$$G(x)-5xG(x)+6x^2G(x)=\sum_{k=0}^{\infty}a_kx^k-\sum_{k=1}^{\infty}5a_{k-1}x^k+\sum_{k=2}^{\infty}6a_{k-2}x^k=a_0+a_1x-5a_0x+\sum_{k=2}^{\infty}0\cdot x^k=6,$$

because of the given recurrence relation and the initial conditions. Thus $G(x)(1-5x+6x^2)=6$, so $G(x)=6/((1-3x)(1-2x))$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{6}{(1-3x)(1-2x)}=\frac{A}{1-3x}+\frac{B}{1-2x},$$

multiplying through by the common denominator, and equating coefficients, we find that $A=18$ and $B=-12$. Thus

$$G(x)=\frac{18}{1-3x}+\frac{-12}{1-2x}=\sum_{k=0}^{\infty}(18\cdot3^k-12\cdot2^k)x^k$$

(the last equality came from using Table 1). Therefore $a_k=18\cdot3^k-12\cdot2^k$. Incidentally, it would be wise to check our answers, either with a computer algebra package (see the solution to Exercise 37 for the syntax in *Maple*) or by computing the next term of the sequence from both the recurrence and the formula (here $a_2=114$ both ways).

37. Let $G(x)=\sum_{k=0}^{\infty}a_kx^k$. Then $xG(x)=\sum_{k=0}^{\infty}a_kx^{k+1}=\sum_{k=1}^{\infty}a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and similarly $x^2G(x)=\sum_{k=0}^{\infty}a_kx^{k+2}=\sum_{k=2}^{\infty}a_{k-2}x^k$. Thus

$$G(x)-4xG(x)+4x^2G(x)=\sum_{k=0}^{\infty}a_kx^k-\sum_{k=1}^{\infty}4a_{k-1}x^k+\sum_{k=2}^{\infty}4a_{k-2}x^k=a_0+a_1x-4a_0x+\sum_{k=2}^{\infty}k^2\cdot x^k$$
$$=2-3x+\frac{2}{(1-x)^3}-\frac{3}{(1-x)^2}+\frac{1}{1-x}-x$$
$$=2-4x+\frac{2}{(1-x)^3}-\frac{3}{(1-x)^2}+\frac{1}{1-x},$$

because of the given recurrence relation, the initial conditions, Table 1, and a calculation of the generating function for $\{k^2\}$ (the last "$-x$" comes from the fact that the k^2 sum starts at 2). (To find the generating function for $\{k^2\}$, start with the fact that $1/(1-x)^3$ is the generating function for $\{C(k,2)=(k+2)(k+1)/2\}$, that $1/(1-x)^2$ is the generating function for $\{k+1\}$, and that $1/(1-x)$ is the generating function for $\{1\}$, and take an appropriate linear combination of these to get the generating function for $\{k^2\}$.) Thus

$$G(x)(1-4x+4x^2)=2-4x+\frac{2}{(1-x)^3}-\frac{3}{(1-x)^2}+\frac{1}{1-x},$$

so

$$G(x)=\frac{2-4x}{(1-2x)^2}+\frac{2}{(1-2x)^2(1-x)^3}-\frac{3}{(1-2x)^2(1-x)^2}+\frac{1}{(1-2x)^2(1-x)}.$$

At this point we must use partial fractions to break up the denominators. Setting the previous expression equal to

$$\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{(1-x)^3}+\frac{D}{1-2x}+\frac{E}{(1-2x)^2},$$

multiplying through by the common denominator, and equating coefficients, we find (after a lot of algebra) that $A=13$, $B=5$, $C=2$, $D=-24$, and $E=6$. (Alternatively, one can ask *Maple* to produce the partial fraction decomposition, with the command

$$\texttt{convert}(\textit{expression},\texttt{parfrac},\texttt{x});$$

where the expression is $G(x)$.) Thus

$$\begin{aligned}G(x)&=\frac{13}{1-x}+\frac{5}{(1-x)^2}+\frac{2}{(1-x)^3}+\frac{-24}{1-2x}+\frac{6}{(1-2x)^2}\\&=\sum_{k=0}^{\infty}\left(13+5(k+1)+2(k+2)(k+1)/2-24\cdot 2^k+6(k+1)2^k\right)x^k\end{aligned}$$

(from Table 1). Therefore $a_k=k^2+8k+20+(6k-18)2^k$. Incidentally, it would be most wise to check our answers, either with a computer algebra package, or by computing the next term of the sequence from both the recurrence and the formula (here $a_2=16$ both ways). The command in *Maple* for solving this recurrence is this:

```
rsolve({a(k) = 4 * a(k - 1) - 4 * a(k - 2) + k^2, a(0) = 2, a(1) = 5}, a(k));
```

39. In principle this exercise is similar to the examples and previous exercises. In fact, the algebra is quite a bit messier. We want to solve the recurrence relation $f_k=f_{k-1}+f_{k-2}$, with initial conditions $f_0=0$ and $f_1=1$. Let G be the generating function for f_k, so that $G(x)=\sum_{k=0}^{\infty}f_kx^k$. We look at $G(x)-xG(x)-x^2G(x)$ in order to take advantage of the recurrence relation:

$$\begin{aligned}G(x)-xG(x)-x^2G(x)&=\sum_{k=0}^{\infty}f_kx^k-\sum_{k=1}^{\infty}f_{k-1}x^k-\sum_{k=2}^{\infty}f_{k-2}x^k\\&=f_0+f_1x-f_0x+\sum_{k=2}^{\infty}(f_k-f_{k-1}-f_{k-2})x^k\\&=0+x-0+0=x\end{aligned}$$

Thus G satisfies the equation

$$G(x)=\frac{x}{1-x-x^2}.$$

To write this more usefully, we need to use partial fractions. The roots of the denominator are $r_1=(-1+\sqrt{5})/2$ and $r_2=(-1-\sqrt{5})/2$. We want to find constants A and B such that

$$\frac{x}{1-x-x^2}=\frac{-x}{x^2+x-1}=\frac{A}{x-r_1}+\frac{B}{x-r_2}.$$

This means that A and B have to satisfy the simultaneous equations $A+B=-1$ and $r_2A+r_1B=0$ (multiply the last displayed equation through by the denominator and equate like powers of x). Solving, we obtain $A=(1-\sqrt{5})/(2\sqrt{5})$ and $B=(-1-\sqrt{5})/(2\sqrt{5})$. Now we have

$$\begin{aligned} G(x) &= \frac{A}{x-r_1}+\frac{B}{x-r_2} \\ &= \frac{-A}{r_1}\frac{1}{1-(x/r_1)}+\frac{-B}{r_2}\frac{1}{1-(x/r_2)} \\ &= \frac{-A}{r_1}\sum_{k=0}^{\infty}\left(\frac{1}{r_1}\right)^k x^k+\frac{-B}{r_2}\sum_{k=0}^{\infty}\left(\frac{1}{r_2}\right)^k x^k . \end{aligned}$$

Therefore

$$\begin{aligned} f_k &= -A\left(\frac{1}{r_1}\right)^{k+1}-B\left(\frac{1}{r_2}\right)^{k+1} \\ &= \frac{1}{\sqrt{5}}\left(\frac{2}{-1+\sqrt{5}}\right)^k-\frac{1}{\sqrt{5}}\left(\frac{2}{-1-\sqrt{5}}\right)^k . \end{aligned}$$

We can check our answer by computing the first few terms with a calculator, and indeed we find that $f_2=1$, $f_3=2$, $f_4=3$, $f_5=5$, and so on.

41. a) Let $G(x)=\sum_{n=0}^{\infty}C_nx^n$ be the generating function for the sequence of Catalan numbers. Then by Theorem 1 a change of variable in the middle, and the recurrence relation for the Catalan numbers,

$$G(x)^2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}C_kC_{n-k}\right)x^n=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}C_kC_{n-1-k}\right)x^{n-1}=\sum_{n=1}^{\infty}C_nx^{n-1} .$$

So $xG(x)^2=\sum_{n=1}^{\infty}C_nx^n$. Therefore,

$$xG(x)^2-G(x)+1=\left(\sum_{n=1}^{\infty}C_nx^n\right)-\left(\sum_{n=0}^{\infty}C_nx^n\right)+1=-C_0+1=0 .$$

We now apply the quadratic formula to solve for $G(x)$:

$$G(x)=\frac{1\pm\sqrt{1-4x}}{2x}$$

We must decide whether to use the plus sign or the minus sign. If we use the plus sign, then trying to calculate $G(0)$, which, after all, is supposed to be C_0, gives us the undefined value $2/0$. Therefore we must use the minus sign, and indeed one can find using calculus that the indeterminate form $0/0$ equals 1 here, since $\lim_{x\to 0}G(x)=1$.

b) By Exercise 40 we know that

$$(1-4x)^{-1/2}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n ,$$

so by integrating term by term (which is valid) we have

$$\int_0^x(1-4t)^{-1/2}\,dt=\frac{1-\sqrt{1-4x}}{2}=x\cdot\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}=x\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n .$$

Since $G(x)=(1-\sqrt{1-4x})/(2x)$, equating coefficients of the power series tells us that $C_n=\frac{1}{n+1}\binom{2n}{n}$.

c) It is natural to try a proof by strong induction, because the sequence is defined recursively. We need to check the base cases: $C_1=1\ge 2^{1-1}$, $C_2=2\ge 2^{2-1}$, $C_3=5\ge 2^{3-1}$, $C_4=14\ge 2^{4-1}$, $C_5=42\ge 2^{5-1}$. For the inductive step assume that $C_j\ge 2^{j-1}$ for $1\le j<n$, where $n\ge 6$. Then

$$C_n=\sum_{k=0}^{n-1}C_kC_{n-k-1}\ge\sum_{k=1}^{n-2}C_kC_{n-k-1}\ge(n-2)2^{k-1}2^{n-k-2}=\frac{n-2}{4}\cdot 2^{n-1}\ge 2^{n-1} .$$

43. Following the hint, we note that $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Then applying the binomial theorem, we have

$$\sum_{k=0}^{m+n} C(m+n,r)x^r = \sum_{r=0}^{m} C(m,r)x^r \cdot \sum_{r=0}^{m} C(n,r)x^r = \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} C(m,r-k)C(n,k)\right)x^r$$

by Theorem 1. Comparing coefficients gives us the desired identity.

45. We will make heavy use of the identity $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$.

a) $\sum_{n=0}^{\infty} \frac{2}{n!}x^n = 2\sum_{n=0}^{\infty} \frac{1}{n!}x^n = 2e^x$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = e^{-x}$

c) $\sum_{n=0}^{\infty} \frac{3^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n = e^{3x}$

d) This generating function can be obtained either with calculus or without. To do it without calculus, write

$$\sum_{n=0}^{\infty} \frac{n+1}{n!}x^n = \sum_{n=0}^{\infty} \frac{n}{n!}x^n + \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^n + e^x = x\sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1} + e^x$$
$$= x\sum_{n=0}^{\infty} \frac{1}{n!}x^n + e^x = xe^x + e^x .$$

To do it with calculus, differentiate both sides of $xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$ to obtain $xe^x + e^x = \sum_{n=0}^{\infty}(n+1)\frac{x^n}{n!}$.

e) This generating function can be obtained either with calculus or without. To do it without calculus, write

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \frac{1}{x}(e^x - 1).$$

To do it with calculus, integrate $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ from 0 to x to obtain

$$e^x - 1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \cdot \frac{1}{n!} = x\sum_{n=0}^{\infty} \frac{1}{(n+1)}\frac{x^n}{n!} .$$

Therefore $\sum_{n=0}^{\infty} \frac{1}{(n+1)}\frac{x^n}{n!} = (e^x - 1)/x$.

47. In many of these cases, it's a matter of plugging the exponent of e into the generating function for e^x. We let a_n denote the n^{th} term of the sequence whose generating function is given.

a) The generating function is $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, so the sequence is $a_n = (-1)^n$.

b) The generating function is $3e^{2x} = 3\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}(3 \cdot 2^n)\frac{x^n}{n!}$, so the sequence is $a_n = 3 \cdot 2^n$.

c) The generating function is $e^{3x} - 3e^{2x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - 3\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}(3^n - 3 \cdot 2^n)\frac{x^n}{n!}$, so the sequence is $a_n = 3^n - 3 \cdot 2^n$.

d) The sequence whose exponential generating function is e^{-2x} is clearly $\{(-2)^n\}$, as in the previous parts of this exercise. Since

$$1 - x = \frac{1}{0!}x^0 + \frac{-1}{1!}x^1 + \sum_{n=2}^{\infty} \frac{0}{n!}x^n ,$$

we know that $a_n = (-2)^n$ for $n \ge 2$, with $a_1 = (-2)^1 - 1 = -3$ and $a_0 = (-2)^0 + 1 = 2$.

e) We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n ,$$

so the sequence for which $1/(1-x)$ is the exponential generating function is $\{n!\}$. Combining this with the rest of the function (similar to previous parts of this exercise), we have $a_n = (-2)^n + n!$.

f) This is similar to part **(e)**. The three functions being added here are the exponential generating functions for $\{(-3)^n\}$, $(-1, -1, 0, 0, 0, \ldots)$, and $\{n! \cdot 2^n\}$. Therefore $a_n = (-3)^n + n! \cdot 2^n$ for $n \ge 2$, with $a_0 = (-3)^0 - 1 + 0! \cdot 2^0 = 1$ and $a_1 = (-3)^1 - 1 + 1! \cdot 2^1 = -2$.

g) First we note that

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$
$$= \frac{x^0}{0!} \cdot \frac{0!}{0!} + \frac{x^2}{2!} \cdot \frac{2!}{1!} + \frac{x^4}{4!} \cdot \frac{4!}{2!} + \frac{x^6}{6!} \cdot \frac{6!}{3!} + \cdots .$$

Therefore we see that $a_n = 0$ if n is odd, and $a_n = n!/(n/2)!$ if n is even.

49. a) Let a_n be the number of codewords of length n. There are 8^n strings of length n in all, and only those that contain an even number of 7's are code words. The initial condition is clearly $a_0 = 1$ (the empty string has an even number of 7's); if that seems too obscure, one can take $a_1 = 7$, since one of the eight strings of length 1 (namely the string 7) is disallowed. To write down a recurrence, we observe that a valid string of length n consists either of a valid string of length $n-1$ followed by a digit other than 7 (so that there will still be an even number of 7's), and there are $7a_{n-1}$ of these; or of an invalid string of length $n-1$ followed by a 7 (so that there will still be an even number of 7's), and there are $8^{n-1} - a_{n-1}$ of these. Putting these together, we have the recurrence relation $a_n = 7a_{n-1} + 8^{n-1} - a_{n-1} = 6a_{n-1} + 8^{n-1}$. For example, $a_2 = 6 \cdot 7 + 8 = 50$.

b) Using the techniques of Section 8.2, we note that the general solution to the associated homogeneous recurrence relation is $a_n^{(h)} = \alpha 6^n$, and then we seek a particular solution of the form $a_n = c \cdot 8^n$. Plugging this into the recurrence relation, we have $c \cdot 8^n = 6c \cdot 8^{n-1} + 8^{n-1}$, which is easily solved to yield $c = \frac{1}{2}$ (first divide through by 8^{n-1}). This gives $a_n^{(p)} = \frac{1}{2} \cdot 8^n$, so the general solution is $a_n = \alpha 6^n + \frac{1}{2} \cdot 8^n$. Next we plug in the initial condition and easily find that $\alpha = \frac{1}{2}$. Therefore the solution is $a_n = (6^n + 8^n)/2$. We can check that this gives the correct answer when $n = 2$.

c) We proceed as in Example 17. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$ (by a change of variable). Thus

$$G(x) - 6xG(x) = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 6a_{k-1} x^k = a_0 + \sum_{k=1}^{\infty} (a_k - 6a_{k-1}) x^k = 1 + \sum_{k=1}^{\infty} 8^{k-1} x^k$$
$$= 1 + x \sum_{k=1}^{\infty} 8^{k-1} x^{k-1} = 1 + x \sum_{k=0}^{\infty} 8^k x^k = 1 + x \cdot \frac{1}{1-8x} = \frac{1-7x}{1-8x} .$$

Thus $G(x)(1-6x) = (1-7x)/(1-8x)$, so $G(x) = (1-7x)/((1-6x)(1-8x))$. At this point we need to use partial fractions to break this up (see, for example, Exercise 35):

$$G(x) = \frac{1-7x}{(1-6x)(1-8x)} = \frac{1/2}{(1-6x)} + \frac{1/2}{(1-8x)}$$

Therefore, with the help of Table 1, $a_n = (6^n + 8^n)/2$, as we found in part **(b)**.

51. To form a partition of n, we must choose some 1's, some 2's, some 3's, and so on. The generating function for choosing 1's is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

(the exponent gives the number so obtained). Similarly, the generating function for choosing 2's is

$$1+x^2+x^4+x^6+\cdots=\frac{1}{1-x^2}$$

(again the exponent gives the number so obtained). The other choices have analogous generating functions. Therefore the generating function for the entire problem, so that the coefficient of x^n will give $p(n)$, the number of partitions of n, is the infinite product

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots.$$

53. This is similar to Exercise 51. Since all the parts have to be of different sizes, we can choose only no 1's or one 1; thus the generating function for choosing 1's is $1+x$ (the exponent gives the number so obtained). Similarly the generating function for choosing 2's is $1+x^2$, and analogously for higher choices. Therefore the generating function for the entire problem, so that the coefficient of x^n will give $p_d(n)$, the number of partitions of n into distinct-sized parts, is the infinite product

$$(1+x)(1+x^2)(1+x^3)\cdots.$$

55. It suffices to show that the generating functions obtained in Exercises 53 and 52 are equal, that is, that

$$(1+x)(1+x^2)(1+x^3)\cdots=\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots.$$

Assuming that the symbol-pushing we are about to do with infinite products is valid, we simply rewrite the left-hand side using the trivial algebraic identity $(1-x^{2r})/(1-x^r)=1+x^r$ and cancel common factors:

$$\begin{aligned}(1+x)(1+x^2)(1+x^3)\cdots&=\frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdots\\&=\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots\end{aligned}$$

57. These follow fairly easily from the definitions.

a) $G_X(1)=\sum_{k=0}^{\infty}p(X=k)\cdot 1^k=\sum_{k=0}^{\infty}p(X=k)=1$, since X has to take on some nonnegative integer value. (That the sum of the probabilities is 1 is one of the axioms of a sample space; see Section 7.2.)

b) $G'_X(1)=\frac{d}{dx}\sum_{k=0}^{\infty}p(X=k)\cdot x^k\big|_{x=1}=\sum_{k=0}^{\infty}p(X=k)\cdot k\cdot x^{k-1}\big|_{x=1}=\sum_{k=0}^{\infty}p(X=k)\cdot k=E(X)$, by the definition of expected value from Section 7.4.

c) $G''_X(1)=\frac{d^2}{dx^2}\sum_{k=0}^{\infty}p(X=k)\cdot x^k\big|_{x=1}=\sum_{k=0}^{\infty}p(X=k)\cdot k(k-1)\cdot x^{k-2}\big|_{x=1}=\sum_{k=0}^{\infty}p(X=k)\cdot(k^2-k)=V(X)+E(X)^2-E(X)$, since, by Theorem 6 in Section 7.4, $V(X)=E(X^2)-E(X)^2$. Combining this with the result of part **(b)** gives the desired equality.

59. **a)** In order to have the m^{th} success on the $(m+n)^{\text{th}}$ trial, where $n\geq 0$, we must have $m-1$ successes and n failures in any order among the first $m+n-1$ trials, followed by a success. The probability of each such ordered arrangement is clearly q^np^m, where p is the probability of success and $q=1-p$ is the probability of failure; and there are $C(n+m-1,n)$ such orders. Therefore $p(X=n)=C(n+m-1,n)q^np^m$. (This was Exercise 32 in the Supplementary Exercises for Chapter 7.) Therefore the probability generating function is

$$G(x)=\sum_{n=0}^{\infty}C(n+m-1,n)q^np^mx^n=p^m\sum_{n=0}^{\infty}C(n+m-1,n)(qx)^n=p^m\frac{1}{(1-qx)^m}$$

by Table 1.

b) By Exercise 57, $E(X)$ is the derivative of $G(x)$ at $x = 1$. Here we have

$$G'(x) = \frac{p^m mq}{(1-qx)^{m+1}}, \quad \text{so} \quad G'(1) = \frac{p^m mq}{(1-q)^{m+1}} = \frac{p^m mq}{p^{m+1}} = \frac{mq}{p}.$$

From the same exercise, we know that the variance is $G''(1) + G'(1) - G'(1)^2$; so we compute:

$$G''(x) = \frac{p^m m(m+1)q^2}{(1-qx)^{m+2}}, \quad \text{so} \quad G''(1) = \frac{p^m m(m+1)q^2}{(1-q)^{m+2}} = \frac{m(m+1)q^2}{p^2},$$

and therefore

$$V(X) = G''(1) + G'(1) - G'(1)^2 = \frac{m(m+1)q^2}{p^2} + \frac{mq}{p} - \left(\frac{mq}{p}\right)^2 = \frac{mq}{p^2}.$$

SECTION 8.5 Inclusion–Exclusion

Inclusion–exclusion is not a nice compact formula in practice, but it is often the best that is available. In Exercise 19, for example, the answer contains over 30 terms. The applications in this section are somewhat contrived, but much more interesting applications are presented in Section 8.6. The inclusion–exclusion principle in some sense gives a methodical way to apply common sense. Presumably anyone could solve a problem such as Exercise 9 by trial and error or other ad hoc techniques, given enough time; the inclusion–exclusion principle makes the solution straightforward. Be careful when using the inclusion–exclusion principle to get the signs right—some terms need to be subtracted and others need to be added. In general the sign changes when the size of the expression changes.

1. In all cases we use the fact that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 12 + 18 - |A_1 \cap A_2| = 30 - |A_1 \cap A_2|$.

a) Here $|A_1 \cap A_2| = 0$, so the answer is $30 - 0 = 30$.

b) This time we are told that $|A_1 \cap A_2| = 1$, so the answer is $30 - 1 = 29$.

c) This time we are told that $|A_1 \cap A_2| = 6$, so the answer is $30 - 6 = 24$.

d) If $A_1 \subseteq A_2$, then $A_1 \cap A_2 = A_1$, so $|A_1 \cap A_2| = |A_1| = 12$. Therefore the answer is $30 - 12 = 18$.

3. We may as well treat percentages as if they were cardinalities—as if the population were exactly 100. Let V be the set of households with television sets, and let P be the set of households with phones. Then we are given $|V| = 96$, $|P| = 98$, and $|V \cap P| = 95$. Therefore $|V \cup P| = 96 + 98 - 95 = 99$, so only 1% of the households have neither telephones nor televisions.

5. For all parts we need to use the formula $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.

a) If the sets are pairwise disjoint, then the cardinality of the union is the sum of the cardinalities, namely 300, since all but the first three terms on the right-hand side of the formula are equal to 0.

b) Using the formula, we have $100 + 100 + 100 - 50 - 50 - 50 + 0 = 150$.

c) Using the formula, we have $100 + 100 + 100 - 50 - 50 - 50 + 25 = 175$.

d) In this case the answer is obviously 100. By the formula, the cardinality of each set on the right-hand side is 100, so we can arrive at this answer through the computation $100+100+100-100-100-100+100 = 100$.

7. We need to use the formula $|J \cup L \cup C| = |J| + |L| + |C| - |J \cap L| - |J \cap C| - |L \cap C| + |J \cap L \cap C|$, where, for example, J is the set of students who have taken a course in Java. Thus we have $|J \cup L \cup C| = 1876 + 999 + 345 - 876 - 290 - 231 + 189 = 2012$. Therefore, since there are 2504 students altogether, we know that $2504 - 2012 = 492$ have taken none of these courses.