Chapter 18 (AIAMA)

Learning From Examples-02

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Model

input output

train

trained model

inference

predictions

unknown input

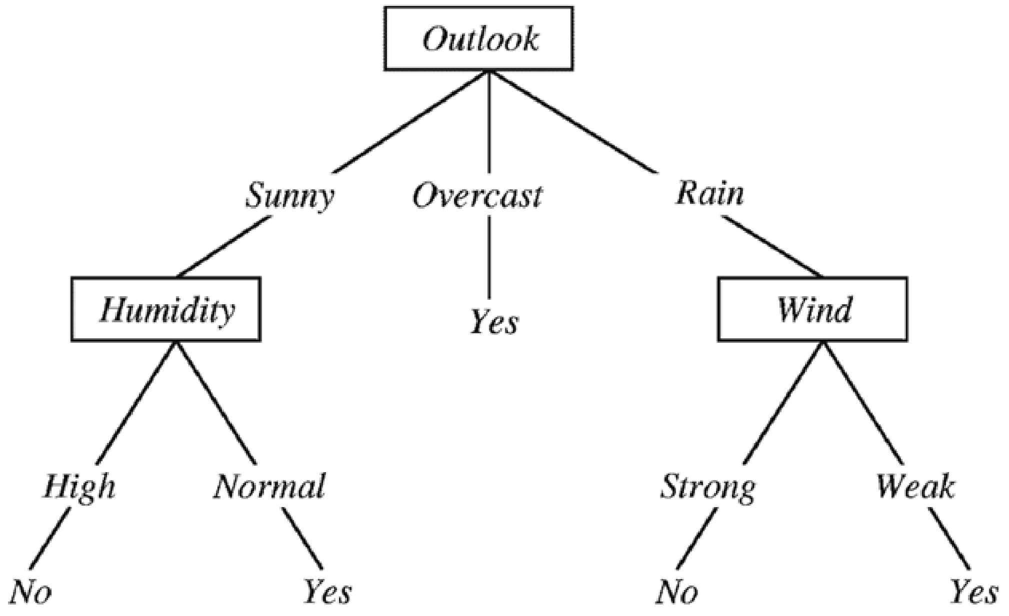
Decision Tree

Interpretable

- **A decision tree:**
  - Represents a function that takes as input a vector of attribute values
  - Returns a "decision"—a single output value.
  - The input and output values can be discrete or continuous.
  - For now we will concentrate on problems where
    - Inputs have discrete values
    - Output has exactly two possible values; this is Boolean classification, where each example input will be classified as *true* (a **positive** example) or *false* (a **negative** example).

Decision Tree

- A decision tree for rain forecasting.
  - Input: {Outlook=Sunny, Humidity=Normal, Wind=Strong}
  - Output: Yes [Rainy]



Decision Tree

- **In a decision tree:**
  - Each **internal node** in the tree corresponds to a test of the value of one of the input attributes, A_i
  - The **branches** from the node are labeled with the possible values of the attribute, $A_i = v_{ik}$.
  - Each **leaf node** in the tree is marked with a outcome to be returned by the function.

Decision Tree: Example

- **Problem:** Build a decision tree to decide whether to wait at a restaurant.

- **Learning Goal**:
  - WillWait - to decide whether to wait for a table at a restaurant.
  - Goal is binary valued (i.e., binary classification task)
    - Values: {Yes, No}

Decision Tree: Example

- **Input Attributes**

1. *Alternate*: whether there is a suitable alternative restaurant nearby.

2. *Bar* : whether the restaurant has a comfortable bar area to wait in.

3. *Fri/Sat*: *true* on Fridays and Saturdays, *false* otherwise.

4. *Hungry*: whether we are hungry.

5. *Patrons*: how many people are in the restaurant (values are None, Some, and Full).

6. *Price: the restaurant's price range ($, $$, $$$).*

Decision Tree: Example

- **Input Attributes**

  7. *Raining*: whether it is raining outside.
  8. *Reservation*: whether we made a reservation.
  9. *Type*: the kind of restaurant (French, Italian, Thai, or burger).
  10. *WaitEstimate*: the wait estimated by the host (0–10 minutes, 10–30, 30–60, or >60).

- Note that every variable has a small set of possible values; the value of *WaitEstimate*, for example, is not an integer, rather it is one of the four discrete values 0–10, 10–30, 30–60, or >60.

Decision Tree: Example

- An example decision tree for the restaurant problem

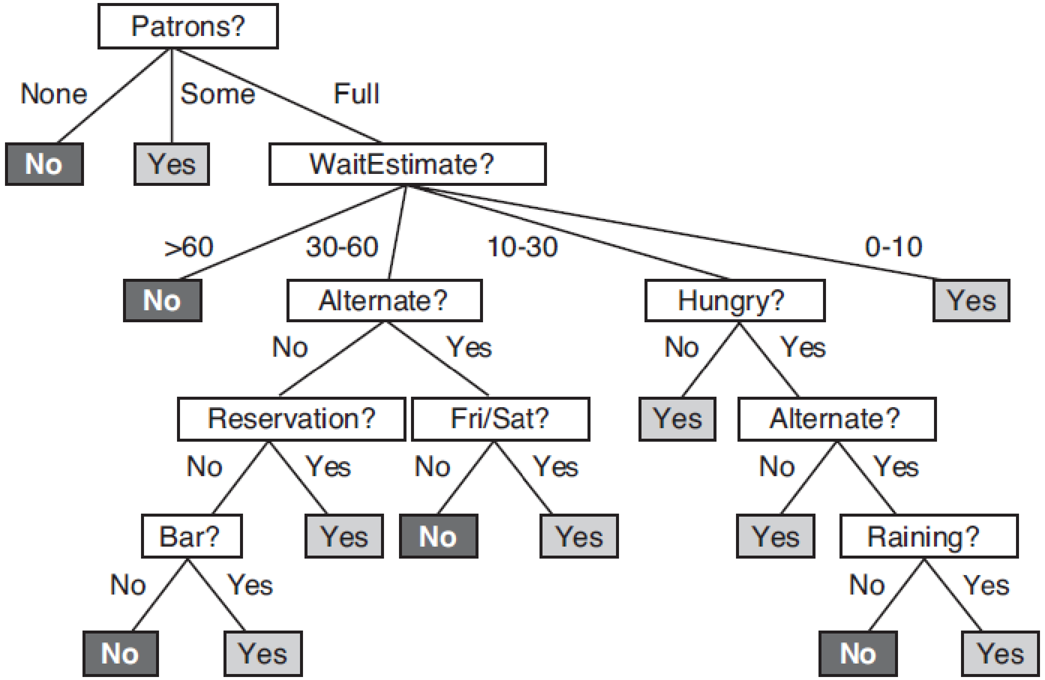


Figure 18.2 A decision tree for deciding whether to wait for a table.

Expressiveness of Decision Trees

- In a Boolean decision tree, the goal attribute is true if and only if the input attributes satisfy one of the paths leading to a leaf with value true.
- $Goal \Leftrightarrow (Path1 \vee Path2 \vee \cdot\cdot\cdot)$, where each $Path$ is a conjunction of attribute-value tests required to follow that path.
- Thus, the whole expression is equivalent to *disjunctive normal form*, which means that any function in propositional logic can be expressed as a decision tree.

Expressiveness of Decision Trees

- Any Boolean function can be represented by a decision tree.
  - Consider the following Boolean function: $f(A, B) = A + B$.
  - Draw a decision tree for the function:

Expressiveness of Decision Trees

- For a wide variety of problems, the decision tree format yields a nice, concise result.
- <u>But some functions cannot be represented concisely:</u>
  - For example, the majority function, which returns true if and only if more than half of the inputs are true, requires an exponentially large decision tree.
- Decision trees are good for some kinds of functions and bad for others.

Expressiveness of Decision Trees

- **How many different decision trees can be obtained for a Boolean function with $\boldsymbol{n}$ variables?**
  - A truth table over n attributes has 2^n rows, one for each combination of values of the attributes.
  - There are 2^{2^n} different functions.
  - Each function can be represented by a decision tree. Hence, number of decision trees at least 2^{2^n} [why more?]

Expressiveness of Decision Trees

- **How many different decision trees can be obtained for a Boolean function with n variables?**
  - For 10 Boolean attributes of our restaurant problem there are 2^{1024} or about 10^{308} different functions to choose from.
  - Number of possible decision trees $\geq 10^{308}$

Inducing Decision Trees

- **Training data**: A set of examples where each example a pair $(\boldsymbol{x}, y)$ pair where $\boldsymbol{x}$ is a vector of values for the input attributes, y is a single Boolean output value.

Example	Input Attributes										Goal
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
$\mathbf{x}_1$	*Yes*	*No*	*No*	*Yes*	*Some*	*$$$*	*No*	*Yes*	*French*	*0–10*	$y_1 = Yes$
$\mathbf{x}_2$	*Yes*	*No*	*No*	*Yes*	*Full*	*$*	*No*	*No*	*Thai*	*30–60*	$y_2 = No$
$\mathbf{x}_3$	*No*	*Yes*	*No*	*No*	*Some*	*$*	*No*	*No*	*Burger*	*0–10*	$y_3 = Yes$
$\mathbf{x}_4$	*Yes*	*No*	*Yes*	*Yes*	*Full*	*$*	*Yes*	*No*	*Thai*	*10–30*	$y_4 = Yes$
$\mathbf{x}_5$	*Yes*	*No*	*Yes*	*No*	*Full*	*$$$*	*No*	*Yes*	*French*	*>60*	$y_5 = No$
$\mathbf{x}_6$	*No*	*Yes*	*No*	*Yes*	*Some*	*$$*	*Yes*	*Yes*	*Italian*	*0–10*	$y_6 = Yes$
$\mathbf{x}_7$	*No*	*Yes*	*No*	*No*	*None*	*$*	*Yes*	*No*	*Burger*	*0–10*	$y_7 = No$
$\mathbf{x}_8$	*No*	*No*	*No*	*Yes*	*Some*	*$$*	*Yes*	*Yes*	*Thai*	*0–10*	$y_8 = Yes$
$\mathbf{x}_9$	*No*	*Yes*	*Yes*	*No*	*Full*	*$*	*Yes*	*No*	*Burger*	*>60*	$y_9 = No$
$\mathbf{x}_{10}$	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$$$*	*No*	*Yes*	*Italian*	*10–30*	$y_{10} = No$
$\mathbf{x}_{11}$	*No*	*No*	*No*	*No*	*None*	*$*	*No*	*No*	*Thai*	*0–10*	$y_{11} = No$
$\mathbf{x}_{12}$	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$*	*No*	*No*	*Burger*	*30–60*	$y_{12} = Yes$

Figure 18.3 Examples for the restaurant domain.

Inducing Decision Trees

- **Build a decision tree that is -**
  - Consistent with the examples [*Not always expected though, generalization may suffer*]
  - Is as small as possible. [*Occam's razor*]

Inducing Decision Trees

- **Question**: Can we always find a consistent decision tree given a set of training examples?.
- **Answer**: Yes.
  - However, if two training examples $(\boldsymbol{x}, y)$ and $(\boldsymbol{x}', y')$ have different outputs but same input attributes $(\boldsymbol{x} = \boldsymbol{x}')$, then a consistent decision tree is not possible.

Inducing Decision Trees

- Unfortunately, it is an intractable problem to find the smallest consistent tree; there is no way to efficiently search through *hypotheses space*.
    - An NP-hard problem!
- What can we do if cannot find the smallest decision tree?
- **Solution**: Use heuristics to find a closest one.
    - With some simple heuristics, we can find a good approximate solution: a small (*but not the smallest*) consistent tree.
    - This is a greedy approach. [Remember what is *greedy*]

Decision Tree Learning

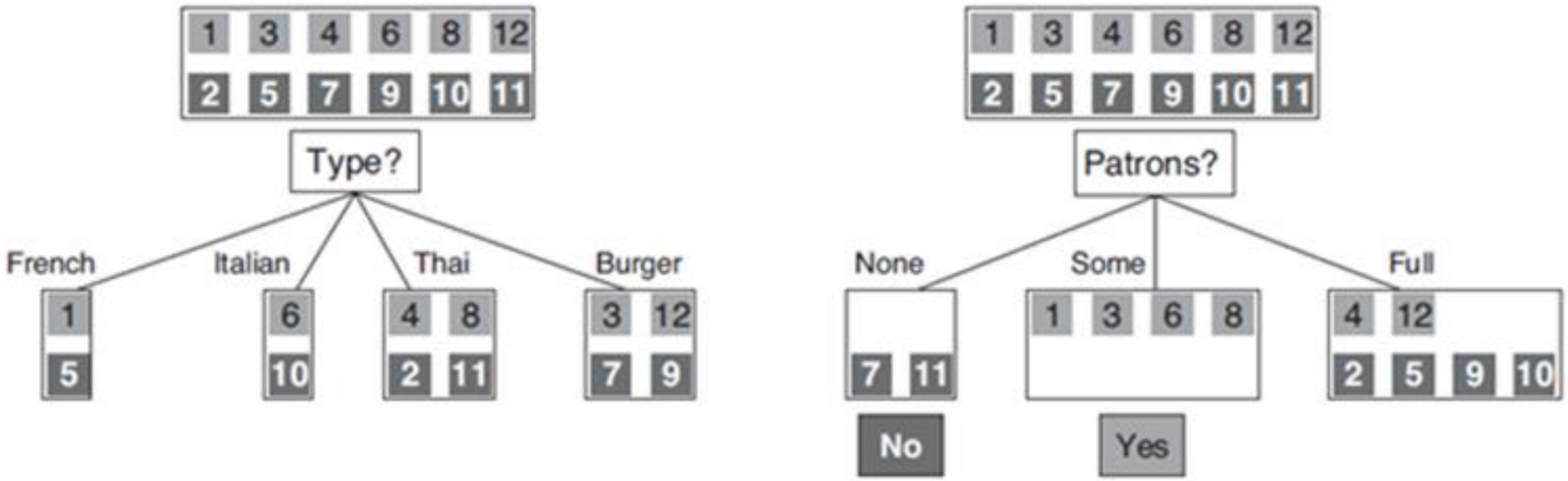
- **Greedy approach to build a decision tree:**
  - Start with empty decision tree.
  - Select an attribute to test at the next level [node]:
    - *Always select the most important attribute to test first. [greedy strategy]*
  - The test creates new branches and divides the problem into smaller subproblems.
  - Recurse on each child (created for each branch)

Decision Tree Learning

- **Greedy strategy:** *Always select the most important attribute to test first.*
  - *Most important attribute* implies the one that makes the most difference to the classification of an example. [*Get leaves as early as possible*]
    - Get correct classification with a small number of test
    - All paths in the tree will be short
    - Tree as a whole will be shallow.
  - *Above greedy strategy is a local optimal choice, may not necessarily leads to the globally smallest tree!*

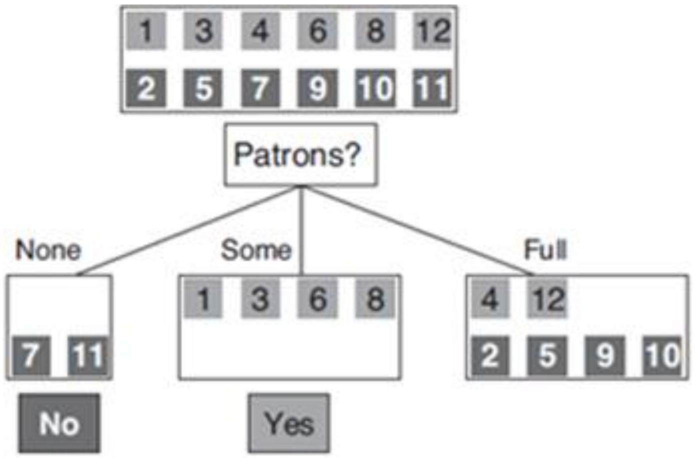
Decision Tree Learning

- Which attribute to test at root? Type vs Patron?
  - Type: all subsets (i.e., branches) needs further exploration.
  - Patron: two branches become leaves, only one needs further exploration.



Decision Tree Learning Algorithm

- **Decision tree construction:**
  - Step 1: Test an attribute at each node.
  - Step 2: Partition the examples according to values and create child nodes with relevant examples.
  - Step 3: Now consider child node for further tests of attributes except the one which have already been tested in the hierarchy (*recursive operation*)



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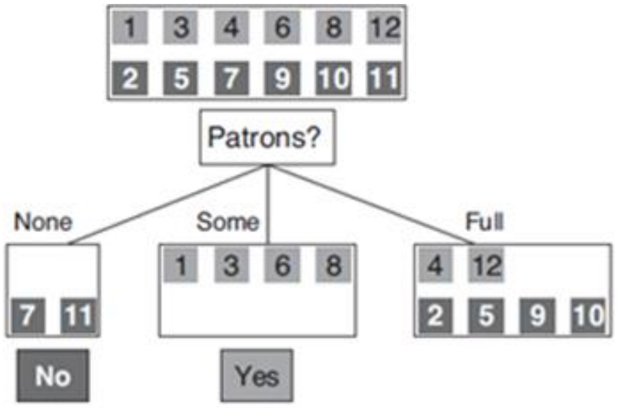
Decision Tree Learning Algorithm

- **Four cases to consider at a child node:**
  - Case 1: All examples are positive (or all negative) → we are done; we can answer Yes or No. [e.g., *Patron=None*]
  - Case 2: Some positive and some negative examples → then choose the best attribute to split them recursively. [e.g., *Patron=Full*]



Decision Tree Learning Algorithm

- **Four cases to consider at a child node:**
  - <u>Case 3:</u> No examples left → No example has been observed for this combination of attribute values, and we return a default value (e.g., *plurality of parent node*)

  - <u>Case 4:</u> No attributes left, but both positive and negative examples remain → These examples have exactly the same description, but different classifications. This can happen because there is an error or **noise** in the data.
    - *In this case return the plurality classification of the remaining examples*. [tree will not be consistent]

Decision Tree Learning Algorithm

- **Algorithm pseudocode**

function DECISION-TREE-LEARNING(*examples*, *attributes*, *parent_examples*) **returns** a tree

```
if examples is empty then return PLURALITY-VALUE(parent_examples)
else if all examples have the same classification then return the classification
else if attributes is empty then return PLURALITY-VALUE(examples)
else
    A ← argmax_{a ∈ attributes} IMPORTANCE(a, examples)
    tree ← a new decision tree with root test A
    for each value v_k of A do
        exs ← {e : e ∈ examples and e.A = v_k}
        subtree ← DECISION-TREE-LEARNING(exs, attributes − A, examples)
        add a branch to tree with label (A = v_k) and subtree subtree
    return tree
```

- Function PLURAITY-VALUE selects the most common class/output among the examples

Choosing the Most Important Attribute

- **Which attribute is the most important now?**
  - **Perfect attribute**: One that splits into subsets where each subset contain either all positive or all negative examples. [*all branches become leaf nodes*]
  - **Useless attribute**: One that splits into subsets where each subset contain fairly equal mix of positive and negative examples. [*all branches need recursive exploration*]

Choosing the Most Important Attribute

- **Perfect vs. useless? How to measure?**
  - A formal measure of perfect vs useless**: Entropy**
  - The fundamental quantity in information theory (*Shannon and Weaver, 1949*).
  - Entropy is a measure of the uncertainty of a random variable.
  - Acquisition of information corresponds to a reduction in entropy.
  - A random variable with only one value—a coin that always comes up heads—has no uncertainty and thus its entropy is defined as zero; thus, we gain no information by observing its value.

Choosing the Most Important Attribute

- **Entropy:** Average number of bits per symbol to encode information.
  - The roll of a fair *four*-sided die has 2 bits of entropy, because it takes two bits to describe one of four equally probable choices.
  - An unfair coin that comes up heads 99% of the time.
    - This coin has less uncertainty than the fair coin—if we guess heads we'll be wrong only 1% of the time—it's entropy measure should be close to zero, but positive.

Entropy Measure

- In general, the entropy $H(V)$ of a random variable V with values v_k, each with probability $P(v_k)$, is defined as :

$$H(V) = \sum_k P(v_k) \log_2 \frac{1}{P(v_k)} = -\sum_k P(v_k) \log_2 P(v_k)$$

Entropy Measure

- Verify that entropies measures are correct.

We can check that the entropy of a fair coin flip is indeed 1 bit:

$$H(Fair) = -(0.5 \log_2 0.5 + 0.5 \log_2 0.5) = 1 \,.$$

If the coin is loaded to give 99% heads, we get

$$H(Loaded) = -(0.99 \log_2 0.99 + 0.01 \log_2 0.01) \approx 0.08 \text{ bits.}$$

Entropy Measure

- Define $B(q)$ as the entropy of a Boolean random variable that is true with probability q:

$$B(q) = -(q \log_2 q + (1-q)\log_2(1-q))$$

- If a training set contains p positive examples and n negative examples, then the entropy of the goal attribute on the whole set is:

$$H(Goal) = B\left(\frac{p}{p+n}\right).$$

Entropy Measure in Decision Tree

- **Decision tree contest**: Entropy represents an impurity measure of the set.
  - A set with 5 positive and 5 negative examples: Most impure, entropy should be highest.
  - A set with 10 positive and 0 negative examples: Purest, entropy should be the lowest.

Entropy Before Split

- A set with p positive and n negative examples.
  - Entropy of the set is: $B\left(\frac{p}{p+n}\right)$

Entropy After Splitting

- An attribute A with d distinct values divides the training set E into subsets E_1, E_2, . . . ,E_d.
- Each subset E_k has p_k positive examples and n_k negative examples, with entropy of $B(p_k/(p_k + n_k))$ bits of information. [E_k contains examples with $A = kth$ value]
- A randomly chosen example from the training set has the kth value for the attribute with probability $(p_k + n_k)/(p + n)$, so the expected entropy (*weighted average)* of the d subsets after splitting on attribute A is:

$$Remainder(A) = \sum_{k=1}^{d} \frac{p_k+n_k}{p+n} B(\frac{p_k}{p_k+n_k})$$

Information Gain

- **Information Gain:** The amount of information gain is the amount of entropy reduction after the split on attribute A.
- Assume E_{BS} = Entropy before split, E_{AS}= Entropy after split [*weighted average*]
  - Hence, information gain (= *reduction of entropy*):

$$Gain = E_{BS} - E_{AS} \text{ [reduction of entropy]}$$

  - Hence, this is simply:

$$Gain(A) = B(\tfrac{p}{p+n}) - Remainder(A)$$

Information Gain

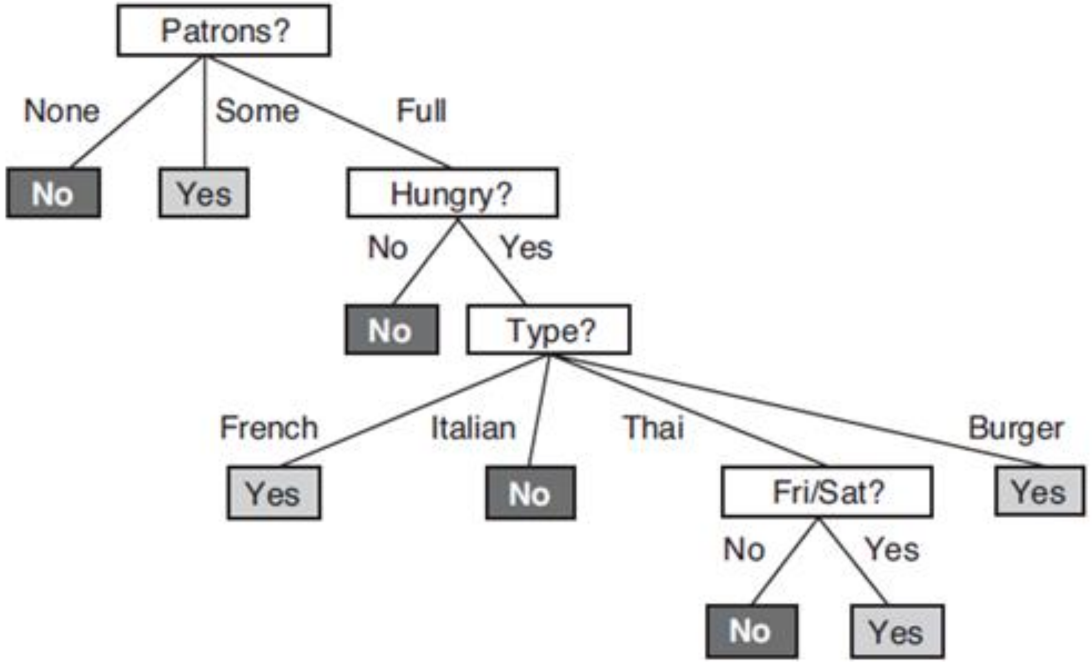
- **Information Gain:** $Gain(A) = B(\frac{p}{p+n}) - Remainder(A)$
- Compute the information gain for the attributes Patron and Type:

$$Gain(Patrons) = 1 - [\tfrac{2}{12}B(\tfrac{0}{2}) + \tfrac{4}{12}B(\tfrac{4}{4}) + \tfrac{6}{12}B(\tfrac{2}{6})] \approx 0.541 \text{ bits},$$
$$Gain(Type) = 1 - [\tfrac{2}{12}B(\tfrac{1}{2}) + \tfrac{2}{12}B(\tfrac{1}{2}) + \tfrac{4}{12}B(\tfrac{2}{4}) + \tfrac{4}{12}B(\tfrac{2}{4})] = 0 \text{ bits},$$

- Patron is a better attribute than Gain! Hence, choose Patron over Type.
- *Choose the attribute which gives the highest information gain!*

Final Decision Tree

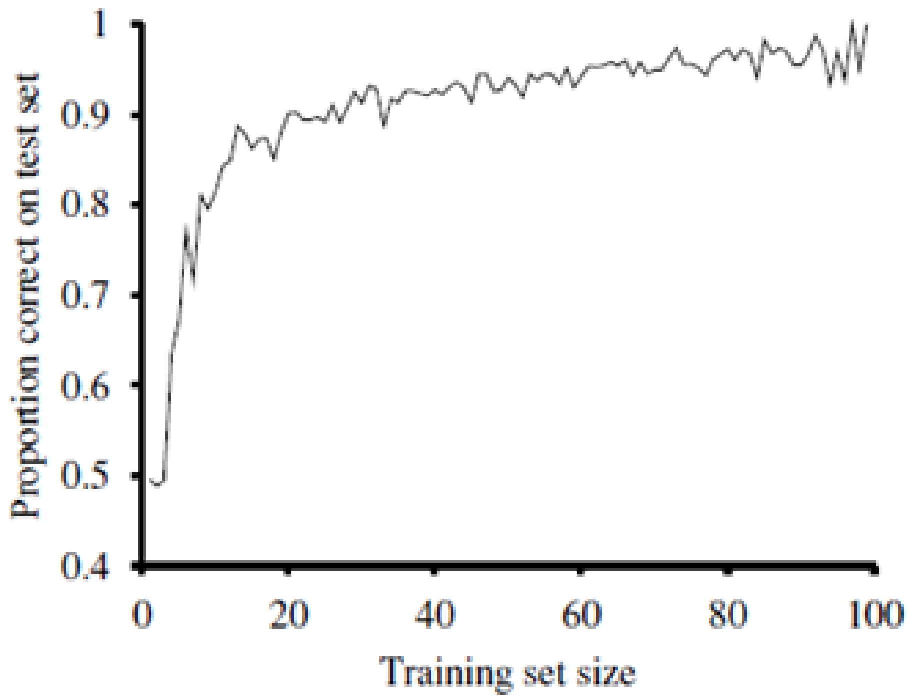
- Final decision tree constructed from given examples.



- **Question**: What is the information gain for Type attribute at level 3?

Decision Tree Learning Curve

- Experiment with 100 examples: Construct decision trees
  - Split into train and test (e.g., 1 and 99, 2 and 98, etc.)
  - Random split 20 times and report average accuracy on test set
  - Note: As the training size grows, accuracy increases.



Overfitting

- **Overfitting**: Too much importance on every training example
  - Complex tree
  - Suffers generalization: very low training error, but very high test error.
  - Misses important concepts
- A consistent decision tree over training data may result in a complex tree
  - Noisy data may also induce complexity in the tree

Overfitting Solution

- **Pre-prune**: Prune the tree before it gets large
  - Early stopping: limit depth during tree construction
- **Post-pruning**: Prune the tree after construction
  - Remove irrelevant nodes
  - Replace internal nodes with most common class [*Only replace if test error do not increase*]

Decision Tree Issues

- **Missing values**: Some examples have missing value in some attributes
  - Solution: Replace missing value with mean of the attribute over the entire dataset
- **Continuous-valued attribute**: For example, height is continuous-valued.
  - Convert to categorical attribute:
    - Height > 40cm: *Tall* and Height <40cm: *Short*

8. (a) Consider the following data set comprised of three binary input attributes (A_1,A_2, and A_3) and one binary output y: (1

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_2	1	0	1	0
x_3	0	1	0	0
x_4	1	1	1	1
x_5	1	1	0	1

Construct a decision tree for these data. Show the entropy and information-gain computations made to determine the attribute to split at each node.

Suppose a bank wants to decide whether to approve a small loan (output y). They use three simple binary attributes about an applicant:

- **A1** = Has a stable job? (1 = Yes, 0 = No)
- **A2** = Has no unpaid loans? (1 = Yes, 0 = No)
- **A3** = Owns a house? (1 = Yes, 0 = No)
- **Output y** = Loan approved? (1 = Yes, 0 = No)

train the dataset

$A_1 A_2 A_3$