Measuring the Gravitational Constant

Overview of the Experiment

The gravitational attraction between a 38.3 gram mass and a 1.5 kg mass when their centers are separated by a dis-tance of approximately 42.2 mm (a situation similar to that of the Gravitational Torsion Balance) is about 7×10^{-10} newtons. If this doesn't seem like a small quantity to measure, consider that the weight of the small mass is more than two hundred million times this amount.

The enormous strength of the Earth's attraction for the small masses, in comparison with their attraction for the large masses, is what originally made the measurement of the gravitational constant such a difficult task. The torsion balance (invented by Charles Coulomb) provides a means of negating the otherwise overwhelming effects of the Earth's attraction in this experiment. It also provides a force delicate enough to counterbalance the tiny gravitational force that exists between the large and small masses. This force is provided by twisting a very thin beryllium copper ribbon.

The large masses are first arranged in Position I, as shown in Figure 12, and the balance is allowed to come to equilibrium. The swivel support that holds the large masses is then rotated, so the large masses are moved to Position II, forcing the system into disequilibrium. The resulting oscillatory rotation of the system is then observed by watching the movement of the light spot on the scale, as the light beam is deflected by the mirror.

> Note: 5% accuracy is possible in Method I if the experiment is set up on a sturdy table in an isolated location where it will not be disturbed by vibration or air movement.
>
> Note: 5% accuracy is possible in Method II if the resting equilibrium points are determined using a graphical analysis program.

Any of three methods can be used to determine the gravitational constant, G, from the motion of the small masses. In Method I, the final deflection method, the motion is allowed to come to resting equilibrium—a process that requires several hours—and the result is accurate to within approximately 5%. In Method II, the equilibrium method, the experiment takes 90 minutes or more and produces an accuracy of approximately 5% when graphical analysis is used in the procedure. In Method III, the acceleration method, the motion is observed for only 5 minutes, and the result is accurate to within approximately 15%.

METHOD I: Measurement by Final Deflection

Setup Time: ~ 45 minutes; Experiment Time: several hours
Accuracy: ~ 5%

Theory

With the large masses in Position I (Figure 13), the gravitational attraction, F, between each small mass (m_2) and its neighboring large mass (m_1) is given by the law of universal gravitation:

$$F = G\frac{m_1 m_2}{b^2} \qquad (1.1)$$

where b is the distance between the centers of the two masses.

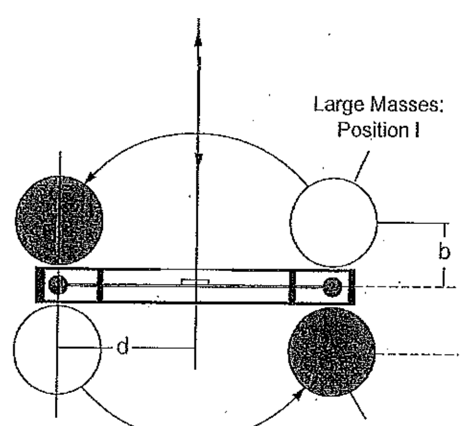

Figure 13: Origin of variables b and d

The gravitational attraction between the two small masses and their neighboring large masses produces a net torque (τ_{grav}) on the system:

$$\tau_{grav} = 2Fd \qquad (1.2)$$

where d is the length of the lever arm of the pendulum bob cross-piece.

Since the system is in equilibrium, the twisted torsion band must be supplying an equal and opposite torque. This torque (τ_{band}) is equal to the torsion constant for the band (κ) times the angle through which it is twisted (θ), or:

$$\tau_{band} = -\kappa\theta \qquad (1.3)$$

Combining equations 1.1, 1.2, and 1.3, and taking into account that $\tau_{grav} = -\tau_{band}$, gives:

$$\kappa\theta = \frac{2dGm_1m_2}{b^2}$$

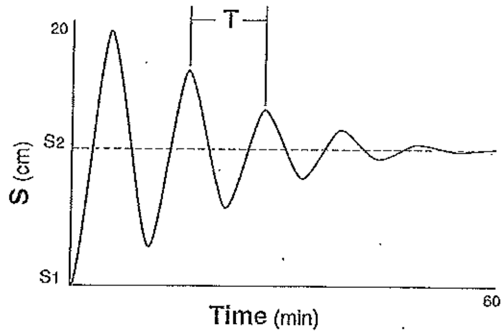

Figure 14: Graph of Small Mass Oscillations

Rearranging this equation gives an expression for G:

$$G = \frac{\kappa\theta b^2}{2dm_1m_2} \qquad (1.4)$$

To determine the values of θ and κ — the only unknowns in equation 1.4 — it is necessary to observe the oscillations of the small mass system when the equilibrium is disturbed. To disturb the equilibrium (from S_1), the swivel support is rotated so the large masses are moved to Position II. The system will then oscillate until it finally slows down and comes to rest at a new equilibrium position (S_2) (Figure 14).

At the new equilibrium position S_2, the torsion wire will still be twisted through an angle θ, but in the opposite direction of its twist in Position I, so the total change in angle is equal to 2θ. Taking into account that the angle is also doubled upon reflection from the mirror (Figure 15):

$$\Delta S = S_2 - S_1,$$

$$4\theta = \frac{\Delta S}{L} \quad \text{or}$$

$$\theta = \frac{\Delta S}{4L} \qquad (1, 5)$$

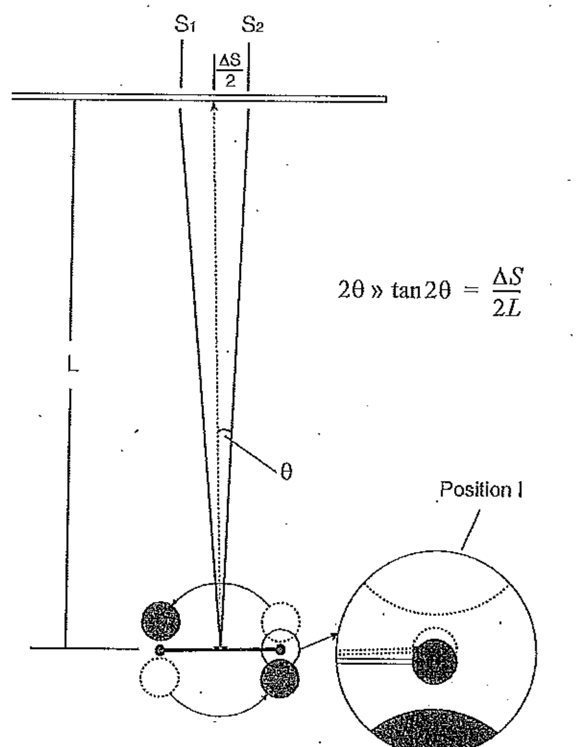

Figure 15: Diagram of the experiment showing the optical lever.s

The torsion constant can be determined by observing the period (T) of the oscillations, and then using the equation:

$$T^2 = \frac{4\pi^2 I}{\kappa} \qquad (1, 6)$$

where I is the moment of inertia of the small mass system.

The moment of inertia for the mirror and support system for the small masses is negligibly small compared to that of the masses themselves, so the total inertia can be expressed as:

$$I = 2m_2\left(d^2 + \frac{2}{5}r^2\right) \qquad (1.7)$$

Therefore:

$$\kappa = 8\pi^2 m_2 \frac{d^2 + \frac{2}{5}r^2}{T^2} \qquad (1.8)$$

Substituting equations 1.5 and 1.8 into equation 1.4 gives:

$$G = \pi^2 \Delta S b^2 \left(\frac{d^2 + \frac{2}{5}r^2}{T^2 m_1 L d}\right) \qquad (1.9)$$

All the variables on the right side of equation 1.9 are known or measurable:

$r = 9.55$ mm

$d = 50$ mm

$b = 42.2$ mm

$m_1 = 1.5$ kg

L = (Measure as in step 1 of the setup.)

By measuring the total deflection of the light spot (ΔS) and the period of oscillation (T), the value of G can therefore be determined.

Procedure

1. Once the steps for leveling, aligning, and setup have been completed (with the large masses in Position I), allow the pendulum to stop oscillating.

2. Turn on the laser and observe the Position I end point of the balance for several minutes to be sure the system is at equilibrium. Record the Position I end point (S_1) as accurately as possible, and indicate any variation over time as part of your margin of error in the measurement.

3. Carefully rotate the swivel support so that the large masses are moved to Position II. The spheres should be just touching the case, but take care to avoid knocking the case and disturbing the system.

Note: You can reduce the amount of time the pendulum requires to move to equilibrium by moving the large masses in a two-step process: first move the large masses and support to an intermediate position that is in the midpoint of the total arc (Figure 16), and wait until the light beam has moved as far as it will go in the period; then move the sphere across the second half of the arc until the large mass support just touches the case. Use a slow, smooth motion, and avoid hitting the case when moving the mass support.

4. Immediately after rotating the swivel support, observe the light spot and record its position (S_1).

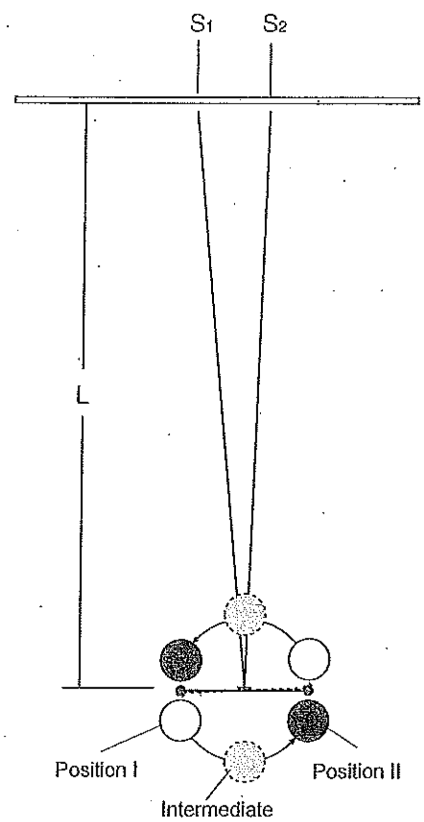

Figure 16: Two-step process of moving the large masses to reduce the time required to stop oscillating.

5. Use a stop watch to determine the time required for one period of oscillation (T). For greater accuracy, include several periods, and then find the average time required for one period of oscillation.

Note: The accuracy of this period value (T) is very important, since the T is squared in the calculation of G.

6. Wait until the oscillations stop, and record the resting equilibrium point (S_2).

Analysis

1. Use your results and equation 1.9 to determine the value of G.

The value calculated in step 2 is subject to the following systematic error. The small sphere is attracted not only to its neighboring large sphere, but also to the more distant large sphere, though with a much smaller force. The geometry for this second force is shown in Figure 17 (the vector arrows shown are not proportional to the actual forces).

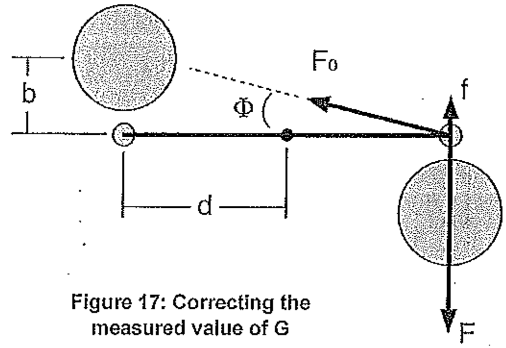

Figure 17: Correcting the measured value of G

From Figure 17,

$$f = F_0 \sin\Phi$$

The force, F_0 is given by the gravitational law, which translates, in this case, to:

$$F_0 = \frac{Gm_2m_1}{(b^2 + 4d^2)}$$

and has a component f that is opposite to the direction of the force F:

$$f = \frac{Gm_2m_1b}{(b^2+4d^2)(b^2+4d^2)^{\frac{1}{2}}} = \beta F$$

This equation defines a dimensionless parameter, β, that is equal to the ratio of the magnitude of f to that of F. Using the equation $F = Gm_1m_2/b^2$, it can be determined that:

$$\beta = \frac{b^3}{(b^2+4d^2)^{\frac{3}{2}}}$$

From Figure 17, $F_{net} = F - f = F - \beta F = F(1 - \beta)$

where F_{net} is the value of the force acting on each small sphere from *both* large masses, and F is the force of attraction to the nearest large mass only.

Similarly, $G = G_0(1 - \beta)$

where G is your experimentally determined value for the gravitational constant, and G_0 is corrected to account for the systematic error.

Finally, $G_0 = G/(1 - \beta)$

Use this equation with equation 1.9 to adjust your measured value.