第一章

考点一定义域

1.

① $\frac{1}{\square}$ $\square \neq 0$. ② $\sqrt{\square}$ $\square \geqslant 0$, $\sqrt[2n]{\square}$ $\square \geqslant 0$ $(n=1,2,3\cdots)$

$\sqrt[3]{\square}$ $\square \in (-\infty,+\infty)$ $\sqrt[2n+1]{\square}$ $\square \in (-\infty,+\infty)$

③ $\begin{cases} \log_a \square & \square > 0 \\ \ln \square & \square > 0 \\ \lg \square & \square > 0 \end{cases}$

④. $\arcsin \square$, $\arccos \square$ $-1 \leq \square \leq 1$ 或 $|\square| \leq 1$

2. 规则
    - **大于取两边小于取中间**
    - **分段函数的定义域：最终结果取并集U**

3.

例1: $y=\sqrt{16-x^2}+\ln(x-2)$ $\quad x^2-16\le 0$

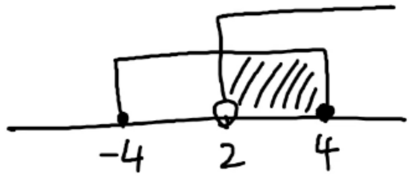
$$\begin{cases}16-x^2\geq 0 \Rightarrow 16\geq x^2 \Rightarrow \underset{\smile}{x^2\leq 16} \Rightarrow -4\leq x\leq 4\\ x-2>0 \Rightarrow x>2\end{cases}$$

$y=x^2-16$

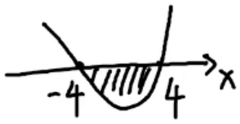
令 $y=0$ $\quad x^2-16=0$

$x^2=16$

$x=\pm 4$



$(2,4]$

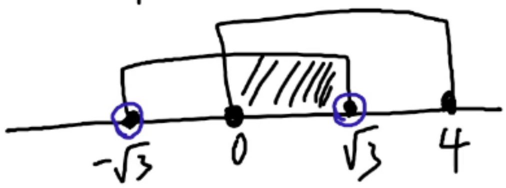


例2: $y=\dfrac{1}{\sqrt{3-x^2}}+\arcsin\left(\dfrac{x}{2}-1\right)$

$$\begin{cases}3-x^2\geq 0 \Rightarrow x^2\leq 3 \Rightarrow -\sqrt{3}\leq x\leq \sqrt{3}\\ \sqrt{3-x^2}\neq 0 \Rightarrow 3-x^2\neq 0 \Rightarrow x^2\neq 3 \Rightarrow x\neq \pm\sqrt{3}\\ -1\leq \dfrac{x}{2}-1\leq 1 \Rightarrow 0\leq x\leq 4\end{cases}$$

$0\leq \dfrac{x}{2}\leq 2$

$0\leq x\leq 4$



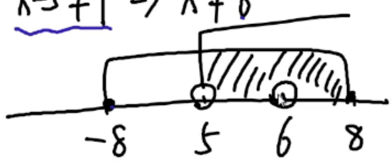
$[0,\sqrt{3})$

例3: $y=\dfrac{\sqrt{64-x^2}}{\ln(x-5)}$ $\qquad y=\log_a x$

$$\begin{cases}64-x^2\geq 0 \Rightarrow x^2\leq 64 \quad -8\leq x\leq 8\\ x-5>0 \Rightarrow x>5\\ \ln(x-5)\neq 0 \Rightarrow \underset{\smile}{x-5\neq 1} \Rightarrow x\neq 6\end{cases}$$

$\log_e x=\ln x$

$\boxed{\begin{matrix}\ln 1=0\\ \ln e=1\end{matrix}}$



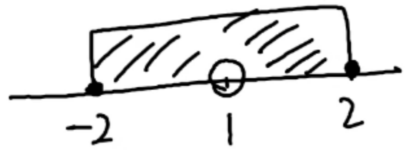
$(5,6)\cup(6,8]$

例4: $y=\dfrac{\sqrt{4-x^2}}{\sqrt[3]{x-1}}$ $\qquad \sqrt{\square}\quad \square\geq 0$

$\sqrt[3]{\square}\quad \square\in(-\infty,+\infty)$

$$\begin{cases}4-x^2\geq 0 \Rightarrow x^2\leq 4 \Rightarrow -2\leq x\leq 2\\ \sqrt[3]{x-1}\neq 0 \Rightarrow x-1\neq 0 \Rightarrow x\neq 1\end{cases}$$

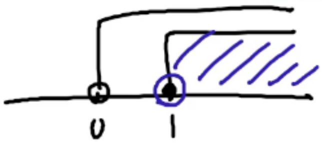
$\sqrt{x-1} \neq 0 \Rightarrow x-1 \neq 0 \Rightarrow x \neq 1$



$[-2, 1) \cup (1, 2]$

2017.11. $y = \dfrac{\sqrt{x-1}}{\ln x}$

$$\begin{cases} x-1 \geq 0 \Rightarrow x \geq 1 \\ x > 0 \\ \ln x \neq 0 \; (=\ln 1) \Rightarrow x \neq 1 \end{cases}$$



$(1, +\infty)$

$\ln 1 = 0$

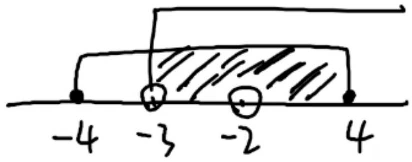
2018.11. $y = \dfrac{\ln(x-1)}{\sqrt{2-x}}$

$$\begin{cases} x-1 > 0 \Rightarrow x > 1 \\ 2-x > 0 \Rightarrow x < 2 \end{cases}$$

$(1, 2)$

2019.11. $y=\dfrac{\sqrt{16-x^2}}{\ln(x+3)}$

$$\begin{cases}16-x^2\geqslant 0 \Rightarrow x^2\leqslant 16 \Rightarrow -4\leqslant x\leqslant 4\\ x+3>0 \Rightarrow x>-3\\ \ln(x+3)\neq 0=\ln 1 \Rightarrow x+3\neq 1\\ \qquad x\neq -2\end{cases}$$



$(-3,-2)\cup(-2,4]$

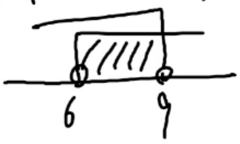
2020.11. $y=\dfrac{\ln(x-1)}{\sqrt{5-x}}$

$$\begin{cases}x-1>0 \Rightarrow x>1\\ 5-x>0 \Rightarrow 5>x\end{cases}$$

$(1,5)$

2021.11. $y=\dfrac{\ln(x-6)}{\sqrt{9-x}}$

$$\begin{cases}x-6>0 \Rightarrow x>6\\ 9-x>0 \Rightarrow 9>x\end{cases}$$



$(6,9)$

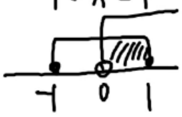
2022.11. $f(x)=\begin{cases}\sqrt{1-x^2} & x>0\\ \ln(1+x) & x\leqslant 0\end{cases}$

分段函数的定义域：最终结果取并集

$\begin{cases}x>0\\ 1-x^2\geqslant 0\end{cases}$ 取交集 $\begin{cases}x\leqslant 0\\ 1+x>0 \Rightarrow x>-1\end{cases}$

$x^2\leqslant 1$

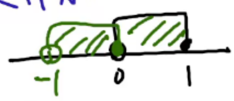
$-1\leqslant x\leqslant 1$



$(0,1]$

$(-1,0]$

取并集



$(-1,1]$

考点二抓大头

例1: $\lim\limits_{n\to\infty}\dfrac{n^2+2n+3}{2n^2+3n+4}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\dfrac{n^2}{2n^2}=\dfrac{1}{2}$

1. 例2: $\lim\limits_{n\to\infty}\dfrac{n^2+2n+3}{n+3}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\dfrac{n^2}{n}=\lim\limits_{n\to\infty}n=\infty$

例3: $\lim\limits_{n\to\infty}\dfrac{n+3}{n^2+2n+3}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\dfrac{n}{n^2}=\lim\limits_{n\to\infty}\dfrac{1}{n}=0$

"$\frac{\infty}{\infty}$" 上大为∞，下大为0，相同则为系数比

例4: $\lim\limits_{x\to\infty}\dfrac{2x^2-2x+1}{x^2+6x+5}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to\infty}\dfrac{2x^2}{x^2}=2$

例5: $\lim\limits_{x\to\infty}\dfrac{4x^2+5x-3}{2x^3+8}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to\infty}\dfrac{4x^2}{2x^3}=0$

例6: $\lim\limits_{x\to\infty}\dfrac{3x^4-2x^2-7}{5x^2+3}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to\infty}\dfrac{3x^4}{5x^2}=\lim\limits_{x\to\infty}\dfrac{3x^2}{5}=\infty$

例7: $\lim\limits_{n\to\infty}\dfrac{(2n\cancel{+1})(n\cancel{+4})(n\cancel{+6})}{5n^3}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\dfrac{2n\cdot n\cdot n}{5n^3}=\lim\limits_{n\to\infty}\dfrac{2\cancel{n^3}}{5\cancel{n^3}}=\dfrac{2}{5}$

例8: $\lim\limits_{n\to\infty}\dfrac{3^n+5^n}{4^n+5^{n+1}}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\dfrac{5^n}{5^{n+1}}=\lim\limits_{n\to\infty}\dfrac{5^n}{5^n\cdot 5}=\dfrac{1}{5}$

2017.12 $\lim\limits_{n\to\infty}\dfrac{2n^2+n-1}{3n^2-5n+7}$

$\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\dfrac{2n^2}{3n^2}$

$=\dfrac{2}{3}$

2020.12. $\lim\limits_{x\to\infty}\dfrac{2x^2+10x-1}{3x^3-5x^2+8}=$

$=\lim\limits_{x\to\infty}\dfrac{2x^2}{3x^3}$

$=\lim\limits_{x\to\infty}\dfrac{2}{3x}$

$=0$

2022.12. $\lim\limits_{x\to\infty}\frac{ax^2}{(x+2)^3-x^3}=2$，则 $a=$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(x+2)^3=x^3+3\cdot x^2\cdot 2+3\cdot x\cdot 2^2+2^3$

$=x^3+6x^2+12x+8$

$=\lim\limits_{x\to\infty}\frac{ax^2}{\cancel{x^3}+6x^2+12x+8-\cancel{x^3}}\overset{\frac{\infty}{\infty}}{=}2$

$=\lim\limits_{x\to\infty}\frac{ax^2}{6x^2}=\frac{a}{6}=2$

$\therefore a=12$

1. $\lim\limits_{n\to\infty}\frac{\sqrt{n^2+1}+n}{\sqrt[3]{n^3+n^2}-1}$

法一：$\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\frac{\sqrt{n^2}+n}{\sqrt[3]{n^3}-1}$

$=\lim\limits_{n\to\infty}\frac{n+n}{n-1}$

$=\lim\limits_{n\to\infty}\frac{2n}{n-1}$

$\overset{\frac{\infty}{\infty}}{=}\lim\limits_{n\to\infty}\frac{2n}{n}$

$=2$

2. $\lim\limits_{x\to\infty}\frac{x^2+2x-\sin x}{2x^2+\sin x}$

$\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to\infty}\frac{x^2}{2x^2}=\frac{1}{2}$

考点三 1^∞

$\frac{1}{\infty}=0 \quad \frac{1}{0}=\infty$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} \overset{1^\infty}{=} 2.71828\cdots = e$

$\lim_{x\to \infty}(1+\frac{1}{x})^{x} \overset{1^\infty}{=} e$

1. 例1: $\lim_{x\to 0}(1+2x)^{\frac{5}{x}} \overset{1^\infty}{=}$

法一: $\lim_{x\to 0}\boxed{(1+2x)^{\frac{1}{2x}}}^{\cdot 10} = e^{10}$

法二: $e^{2x\cdot\frac{5}{x}} = e^{10}$

例2: $\lim_{x\to 0}(1-7x)^{\frac{1}{x}} \overset{1^\infty}{=} e^{-7x\cdot\frac{1}{x}} = e^{-7} = \frac{1}{e^7}$

例3: $\lim_{x\to \infty}(1+\frac{4}{x})^{-2x} \overset{1^\infty}{=} e^{\frac{4}{x}\cdot(-2x)} = e^{-8} = \frac{1}{e^8}$

例4: $\lim_{x\to \infty}(1-\frac{3}{x})^{2x} \overset{1^\infty}{=} e^{-\frac{3}{x}\cdot 2x} = e^{-6} = \frac{1}{e^6}$

$\frac{1}{0}=\infty \quad \frac{1}{\infty}=0$

2017.3. $\lim_{x\to \infty}(1-\frac{2}{x})^{x} \overset{1^\infty}{=} e^{-2} = \frac{1}{e^2}$

$\frac{1}{\infty}=0 \quad \frac{1}{0}=\infty$

2018.3 $\lim_{x\to 0}(1+2x)^{\frac{1}{x}} \overset{1^\infty}{=} e^{2}$

2019.3. $\lim_{x\to \infty}(1-\frac{1}{3x})^{x} \overset{1^\infty}{=} e^{-\frac{1}{3}} = \frac{1}{e^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{e}}$

2020.3. $\lim_{x\to 0}(1+x)^{\frac{3}{x}} \overset{1^\infty}{=} e^{3}$

2021.3. $\lim_{x\to \infty}(1+\frac{1}{x})^{\frac{x}{3}} \overset{1^\infty}{=} e^{\frac{1}{3}}$

2022.3 $\lim_{x\to 0}(1+2x)^{\frac{1}{6x}} \overset{1^\infty}{=} e^{\frac{1}{3}}$

例1：$\lim\limits_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{2x}$

$\overset{1^\infty}{=} e^{\lim\limits_{x\to\infty}\frac{2x}{x+1}}$

$= e^2$

例2：$\lim\limits_{x\to\infty}\left(1-\frac{1}{2x+3}\right)^{(x+2)}$

$\overset{1^\infty}{=} e^{\lim\limits_{x\to\infty}\frac{-1}{2x+3}(x+2)}$

$= e^{\lim\limits_{x\to\infty}\frac{-x-2}{2x+3}}$

$= e^{-\frac{1}{2}}$

$= \frac{1}{e^{\frac{1}{2}}} = \frac{1}{\sqrt{e}}$

例3：$\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}=$

$\overset{1^\infty}{=} e^{\lim\limits_{x\to 0}\frac{\sin x}{x}}$

$= e^{\lim\limits_{x\to 0}\frac{x}{x}}$

$= e^1$

$= e$

例4：$\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^{(x+2)}=$

$\overset{1^\infty}{=} e^{\lim\limits_{x\to\infty}\frac{2(x+2)}{x}}$

$= e^{\lim\limits_{x\to\infty}\frac{2x+4}{x}}$

$= e^2$

例5：$\lim\limits_{x\to\infty}\left(\frac{x+2}{x+1}\right)^x$

$\overset{1^\infty}{=} \lim\limits_{x\to\infty}\left(\frac{x+1+1}{x+1}\right)^x$

$= \lim\limits_{x\to\infty}\left(1+\frac{1}{x+1}\right)^x$

$= e^{\lim\limits_{x\to\infty}\frac{x}{x+1}}$

$= e$

例6：$\lim\limits_{x\to\infty}\left(\frac{x+3}{x-1}\right)^x=$

$\overset{1^\infty}{=} \lim\limits_{x\to\infty}\left(\frac{x-1+4}{x-1}\right)^x$

$= \lim\limits_{x\to\infty}\left(1+\frac{4}{x-1}\right)^x$

$= e^{\lim\limits_{x\to\infty}\frac{4x}{x-1}}$

$= e^4$

考点四无穷小的比较

定义：$\lim\limits_{x\to\square} f(x) = \lim\limits_{x\to\square} g(x) = 0$

$x\to\square \begin{cases} x\to x_0 \quad x\to x_0^+ \quad x\to x_0^- \\ x\to\infty \quad x\to+\infty \quad x\to-\infty \end{cases}$

1. $\lim\limits_{x\to\square} \dfrac{f(x)}{g(x)} \overset{\frac{0}{0}}{=} \begin{cases} 0 & f(x)是g(x)的高阶无穷小 \\ \infty & f(x)是g(x)的低阶无穷小 \\ C\ (C\neq 0且C\neq 1) & f(x)是g(x)的同阶无穷小 \\ 1 & f(x)是g(x)等价无穷小. \end{cases}$

等价无穷小：

$x\to 0$ $\left.\begin{array}{l} \sin x \\ \tan x \\ \arcsin x \\ \arctan x \\ e^x-1 \\ \ln(1+x) \end{array}\right\} \sim x$

$1-\cos x \sim \frac{1}{2}x^2$

$(1+x)^m-1 \sim mx$

$\sqrt{1+x}-1 \sim \frac{1}{2}x$

$x\to 0$ $\ln(1-x) = \ln[1+(-x)] \sim -x$

狗.

狗$\to 0$ $\left.\begin{array}{l} \sin 狗 \\ \tan 狗 \\ \arcsin 狗 \\ \arctan 狗 \\ e^{狗}-1 \\ \ln(1+狗) \end{array}\right\} \sim 狗$

$1-\cos 狗 \sim \frac{1}{2}狗^2$.

$(1+狗)^m-1 \sim m狗$

$\sqrt{1+狗}-1 \sim \frac{1}{2}狗$