3 Collisionless particle systems

3.1 N-particle ensembles

The state of an N-particle ensemble at time t can be specified by the *exact* particle distribution function, in the form

$$F(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i(t)) \cdot \delta(\mathbf{v} - \mathbf{v}_i(t)). \tag{3.1}$$

This gives effectively the number of particles at phase-space point $(\mathbf{x}, \mathbf{v})$ at time t. Let now

$$p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_N) \, \mathrm{d}\mathbf{x}_1 \, \mathrm{d}\mathbf{x}_2 \cdots \mathrm{d}\mathbf{x}_N \, \mathrm{d}\mathbf{v}_1 \, \mathrm{d}\mathbf{v}_2 \cdots \mathrm{d}\mathbf{v}_N \tag{3.2}$$

be the probability that the system is in the given state at time t. Then a reduced statistical description is obtained by *ensemble averaging*:

$$f_1(\mathbf{x}, \mathbf{v}, t) = \langle F(\mathbf{x}, \mathbf{v}, t) \rangle = \int F \cdot p \cdot \mathrm{d}\mathbf{x}_1 \, \mathrm{d}\mathbf{x}_2 \cdots \mathrm{d}\mathbf{x}_N \, \mathrm{d}\mathbf{v}_1 \, \mathrm{d}\mathbf{v}_2 \cdots \mathrm{d}\mathbf{v}_N. \tag{3.3}$$

We can integrate out one of the Dirac delta-functions in F to obtain

$$f_1(\mathbf{x}, \mathbf{v}, t) = N \int p(\mathbf{x}, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{v}, \mathbf{v}_2, \ldots \mathbf{v}_N) \, \mathrm{d}\mathbf{x}_2 \cdots \mathrm{d}\mathbf{x}_N \, \mathrm{d}\mathbf{v}_2 \cdots \mathrm{d}\mathbf{v}_N. \tag{3.4}$$

Note that we can permute the arguments in p where $\mathbf{x}$ and $\mathbf{v}$ appear. $f_1(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v}$ now gives the *mean number* of particles in a phase-space volume $\mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v}$ around $(\mathbf{x}, \mathbf{v})$.

Similarly, the ensemble-averaged two-particle distribution ("the mean product of the numbers of particles at $(\mathbf{x}, \mathbf{v})$ and $(\mathbf{x}', \mathbf{v}')$") is given by

$$\begin{aligned} f_2(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) &= \langle F(\mathbf{x}, \mathbf{v}, t) F(\mathbf{x}', \mathbf{v}', t) \rangle \\ &= N(N-1) \int p(\mathbf{x}, \mathbf{x}', \mathbf{x}_3, \ldots, \mathbf{x}_N, \mathbf{v}, \mathbf{v}', \mathbf{v}_3, \ldots \mathbf{v}_N) \, \mathrm{d}\mathbf{x}_3 \cdots \mathrm{d}\mathbf{x}_N \, \mathrm{d}\mathbf{v}_3 \cdots \mathrm{d}\mathbf{v}_N. \end{aligned} \tag{3.5}$$

Likewise one may define $f_3, f_4, \ldots$ and so on. This yields the so-called BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) chain.

3.2 Uncorrelated (collisionless) systems

The simplest closure for the BBGKY hierarchy is to assume that particles are *uncorrelated*, i.e. that we have

$$f_2(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) = f_1(\mathbf{x}, \mathbf{v}, t)\, f_1(\mathbf{x}', \mathbf{v}', t). \tag{3.6}$$

Physically, this means that a particle at $(\mathbf{x}, \mathbf{v})$ is completely unaffected by one at $(\mathbf{x}', \mathbf{v}')$. Systems in which this is approximately the case include

- electrons in a plasma
- stars in a galaxy
- dark matter particles in the universe

Such systems are said to be *"collisionless"*, although "collision" in this sense can have a much broader meaning than true collisions such as colliding planets: a "collision" in this definition is also a close encounter that substantially alters the orbit of the two particles. A collisionless system therefore is, in this definition, a system in which two-body interactions play no role. We will later consider in more detail under which conditions a system is collisionless.

Let's now go back to the probability density $p(\mathbf{w})$ which depends on the N-particle phase-space state $\mathbf{w} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_N)$. The conservation of probability in phase-space means that it fulfills a continuity equation

$$\frac{\partial p}{\partial t} + \nabla_{\mathbf{w}} \cdot (p\, \dot{\mathbf{w}}) = 0. \tag{3.7}$$

We can cast this into

$$\frac{\partial p}{\partial t} + \sum_i \left(p \frac{\partial \dot{\mathbf{x}}_i}{\partial \mathbf{x}_i} + \frac{\partial p}{\partial \mathbf{x}_i} \dot{\mathbf{x}}_i + p \frac{\partial \dot{\mathbf{v}}_i}{\partial \mathbf{v}_i} + \frac{\partial p}{\partial \mathbf{v}_i} \dot{\mathbf{v}}_i \right) = 0. \tag{3.8}$$

Now we recall Hamiltonian dynamics with the equations of motion $\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}$ and $\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}$. We can differentiate them to get $\frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} = \frac{\partial^2 H}{\partial \mathbf{x} \partial \mathbf{p}}$, and $\frac{\partial \dot{\mathbf{p}}}{\partial \mathbf{p}} = -\frac{\partial^2 H}{\partial \mathbf{x} \partial \mathbf{p}}$. Hence it follows $\frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} = -\frac{\partial \dot{\mathbf{v}}}{\partial \mathbf{v}}$. Using this we get

$$\frac{\partial p}{\partial t} + \sum_i \left(\mathbf{v}_i \frac{\partial p}{\partial \mathbf{x}_i} + \mathbf{a}_i \frac{\partial p}{\partial \mathbf{v}_i} \right) = 0, \tag{3.9}$$

where $\mathbf{a}_i = \dot{\mathbf{v}}_i = \mathbf{F}_i / m_i$ is the particle acceleration. This is *Liouville's theorem*.

Now, in the collisionless/uncorrelated limit, this directly carries over to the one-point distribution function $f = f_1$, yielding the *Vlasov equation*, also known as collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{3.10}$$

The close relation to Liouville's equation means that also here the phase space-density stays constant along characteristics (i.e. along orbits of individual particles) of the system.

What about the accelerations?

In the limit of a collisionless system, the accelerations cannot be due to a single other particle. However, collective effects, for example from the gravitational or electric field produced by the whole system are still allowed.

For example, the source field of self-gravity can be described as

$$\rho(\mathbf{x}, t) = m \int f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}\mathbf{v}. \tag{3.11}$$

This then produces a gravitational field through Poisson's equation,

$$\nabla^2 \Phi = 4\pi G \rho, \tag{3.12}$$

which yields the accelerations as

$$\mathbf{a} = -\frac{\partial \Phi}{\partial \mathbf{x}}. \tag{3.13}$$

One can also combine these equations to yield the Poisson-Vlasov system, given by

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{3.14}$$

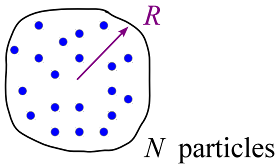
$$\nabla^2 \Phi = 4\pi G m \int f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}\mathbf{v}. \tag{3.15}$$

This holds in an analogous way also for a plasma where the mass density is replaced by a charge density.

It is interesting to note that in this description the particles have basically completely vanished and have been replaced with a continuum fluid description. Later, for the purpose of solving the equations, we will have to reintroduce particles as a means of discretizing the equations (but these are then not the real physical particles any more, rather they are fiducial macro particles that sample the phase-space in a Monte-Carlo fashion).

3.3 When is a system collisionless?

Consider a system of size R containing N particles.



The time for one crossing of a particle through the system is of order

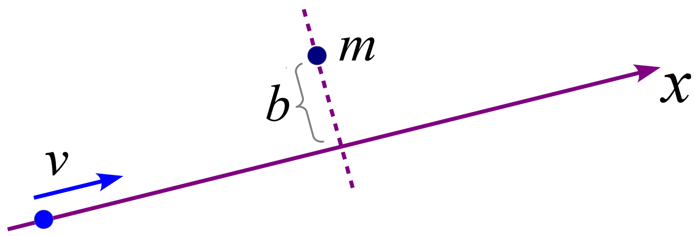
$$t_{\rm cross} = \frac{R}{v}, \tag{3.16}$$

where v is the typical particle velocity. For a self-gravitating system of that size we expect

$$v^2 \simeq \frac{GNm}{R} = \frac{GM}{R}, \tag{3.17}$$

where $M = N\,m$ is the total mass.

We now want to estimate the rate at which a particle experiences weak deflections by other particles, which is the process that violates perfect collisionless behavior and which induces relaxation. We calculate the deflection in the impulse approximation where the particle's orbit is taken as a straight path.



To get the deflection, we compute the transverse momentum acquired by the particle as it flies by the perturber (assumed to be stationary for simplicity):

$$\Delta p = m\Delta v = \int F_\perp {\rm d}t = \int \frac{Gm^2}{x^2+b^2}\frac{b}{\sqrt{x^2+b^2}}\frac{{\rm d}x}{v} = \frac{2Gm^2}{bv}. \tag{3.18}$$

How many encounters do we expect in one crossing? For impact parameters between $[b, b+{\rm d}b]$ we have

$${\rm d}n = N\frac{2\pi b\,{\rm d}b}{\pi R^2} \tag{3.19}$$

targets. The velocity perturbations from each encounter have random orientations, so they add up in quadrature. Per crossing we hence have for the quadratic velocity perturbation:

$$(\Delta v)^2 = \int \left(\frac{2Gm}{bv}\right)^2 {\rm d}n = 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda, \tag{3.20}$$

where

$$\ln \Lambda = \ln \frac{b_{\rm max}}{b_{\rm min}} \tag{3.21}$$

is the so-called Coulomb logarithm. We can now define the relaxation time as

$$t_{\rm relax} \equiv \frac{v^2}{(\Delta v)^2/t_{\rm cross}}, \tag{3.22}$$

i.e. after this time the individual perturbations have reached $\sim 100\%$ of the typical squared velocity, and one certainly not neglect the interactions any more. With our result for $(\Delta v)^2$, and using equation (3.17) this now becomes

$$t_{\rm relax} = \frac{N}{8 \ln \Lambda}\, t_{\rm cross}. \tag{3.23}$$

But we still have to clarify what we can sensibly use for $b_{\rm min}$ and $b_{\rm max}$ in the Coulomb logarithm. For $b_{\rm max}$, we can set the size of the system, i.e. $b_{\rm max} \simeq R$. For $b_{\rm min}$, we can use as a lower limit the b where very strong deflections ensue, which is given by

$$\frac{2Gm}{b_{\rm min} v} \simeq v, \tag{3.24}$$

i.e where the transverse velocity perturbation becomes as large as the velocity itself. This then yields $b_{\rm min} = 2R/N$. We hence get for the Coulomb logarithm $\ln \Lambda \simeq \ln(N/2)$. But a factor of 2 in a logarithm might as well be neglected in this coarse estimate, so that we expect $\ln \Lambda \sim N$. We hence arrive at the final result:

$$t_{\rm relax} = \frac{N}{8 \ln N}\, t_{\rm cross}. \tag{3.25}$$

A system can be viewed as collisionless if $t_{\rm relax} \gg t_{\rm age}$, where $t_{\rm age}$ is the time of interest. We note that $t_{\rm cross}$ depends only on the size and mass of the system, but *not* on the particle number N or the individual masses of the N-body particles. We therefore clearly see that the primary requirement to obtain a collisionless system is to use a sufficiently large N.

Examples

- globular star clusters have $N \sim 10^5$, $t_{\rm cross} \sim \frac{3\,{\rm pc}}{6\,{\rm km/sec}} \simeq 0.5\,{\rm Myr}$. This implies that such systems are strongly affected by collisions over the age of the Universe, $t_{\rm age} = \frac{1}{H_0} \sim 10\,{\rm Gyr}$.

- stars in a typical galaxy: Here we have $N \sim 10^{11}$ and $t_{\rm cross} \sim \frac{1}{100\,H_0}$. This means that these large stellar systems are collisionless over the age of the Universe to extremely good approximation.

- dark matter in a galaxy: Here we have $N \sim 10^{77}$ if the dark matter is composed of a $\sim 100\,{\rm GeV}$ weakly interacting massive particle (WIMP). In addition, the crossing time is longer than for the stars, $t_{\rm cross} \sim \frac{1}{10\,H_0}$, due to the larger size of the 'halo' relative to the embedded stellar system. Clearly, the dark matter represents the mother of all collisionless systems.

3.4 N-body models of collisionless systems

We now reintroduce particles in order to discretize the collisionless fluid described by the Poisson-Vlasov system. We use however *far fewer* particles than in real physical systems, and we correspondingly give them a higher mass (and/or charge). These are hence fiducial macro-particles. Their equations of motions in the case of gravity are written as:

$$\ddot{\mathbf{x}} = -\nabla_i \Phi(\mathbf{x}_i), \tag{3.26}$$

$$\Phi(\mathbf{x}) = -G \sum_{j=1}^{N} \frac{m_j}{[(\mathbf{x}-\mathbf{x}_j)^2 + \epsilon^2]^{1/2}}. \tag{3.27}$$

A few comments are in order here:

- Provided we can ensure $t_{\rm relax} \gg t_{\rm sim}$ despite the smaller N than in the real physical system, the numerical model keeps behaving as a collisionless system over the simulated time-span $t_{\rm sim}$, and the collective gravitational potential is sufficiently smooth.

- The mass of the macro-particles used to discretize the collision system does not enter in the equations of motion. Provided there are enough particles to describe the gravitational potential well, the orbits of the macro-particles will be just as valid as the orbits of the real physical particles.

- The N-body model gives only one (quite noisy) realization of the one-point function. It does not give the ensemble average directly (this would require multiple simulations).

- The equations of motion contain a **softening length** ϵ. The purpose of the force softening is to avoid large angle scatterings and the numerical expense that would be needed to integrate the orbits with sufficient accuracy in singular potentials. Also, we would like to prevent the possibility of the formation of bound particle pairs – they would obviously be highly correlated and hence strongly violate collisionless behaviour. We don't get bound pairs if

  $$\left\langle v^2 \right\rangle \gg \frac{Gm}{\epsilon}, \tag{3.28}$$

  which can be used as a simple condition on reasonable softening settings. The adoption of a softening length also implies the introduction of a smallest resolved length-scale. The specific softening choice one makes ultimately represents a compromise between spatial resolution, discreteness noise in the orbits and the gravitational potential, computational cost, and the relaxation effects that may negatively influence results.