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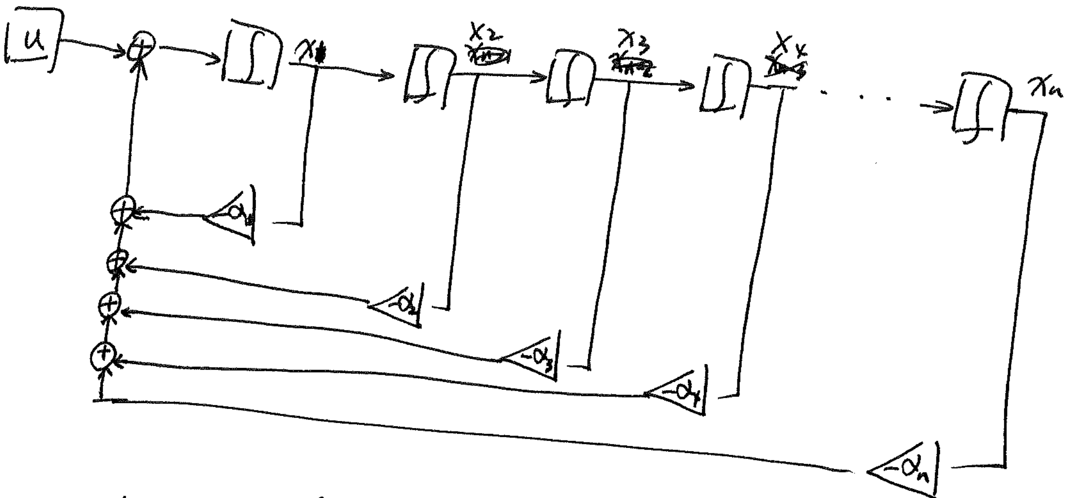
Assignment 4

1. Controllable Canonical Form.

$$A = \begin{bmatrix} -\alpha_1 I & -\alpha_2 I & -\alpha_3 I & \cdots & -\alpha_n I \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The block diagram is :



All the blocks, or to say signals, are connected to u.
That is, uncontrollable ~~sy~~ subspace is null. The system is controllable.

I'm not able to compute the rank of controllability matrix or [sI-A | B]. So this is all that I can do.

2. Satellite.

$$\dot{x} = Ax + Bu. \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad u = [u_1, u_2]$$

a:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \Bigg[\underbrace{\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2w & 1 \end{matrix}}_{\text{sub}\,C^0} \; \cdots \Bigg]$$

$$\text{sub}\,C^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2w & 1 \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 2w \\ 0 & 1 & -2w & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{full rank, regardless of } w \text{ value.}}$$

a: subC is full rank, $\mathcal{C}$ will be full rank, A, B system is controllable.

b: Radial Down $\Rightarrow$ $B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathcal{C}_1 = \begin{bmatrix} B_1 & AB_1 & A^2B_1 & A^3B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2w & 0 & 2w \\ 0 & 0 & 0 & 2w & 0 & 2w & 0 & -2w^3+2w \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1-4w^2 \\ 0 & 1 & 0 & 1 & 0 & 1-4w^2 & 0 & 1-8w^2 \end{bmatrix}$$

These two rows are the same if $w=0$.

So when $w=0$, $\mathcal{C}_1$ is not full rank, system is not controlable.

Tangantal Down $\rightarrow$ $B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathcal{C}_2 = \begin{bmatrix} B_2 & AB_2 & A^2B_2 & A^3B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -w^2 & 0 \\ 1 & 0 & 0 & 0 & -w^2 & 0 & -4w^2 & 0 \\ 0 & 0 & 0 & 0 & -2w & 0 & -2w & 0 \\ 0 & 0 & -2w & 0 & -2w & 0 & 2w(w^2-1) & 0 \end{bmatrix}$$

The same when $w=0$

So when $w=0$, $\mathcal{C}_2$ is not full rank, system is not controllable.

c: $\mathcal{C}\big|_{w=1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 & -1 & 2 \\ 1 & 0 & 0 & 2 & -1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & -2 & -3 \\ 0 & 1 & -2 & 1 & -2 & -3 & 0 & -7 \end{bmatrix}$ Rank $= 4 = n$.

Controllable.

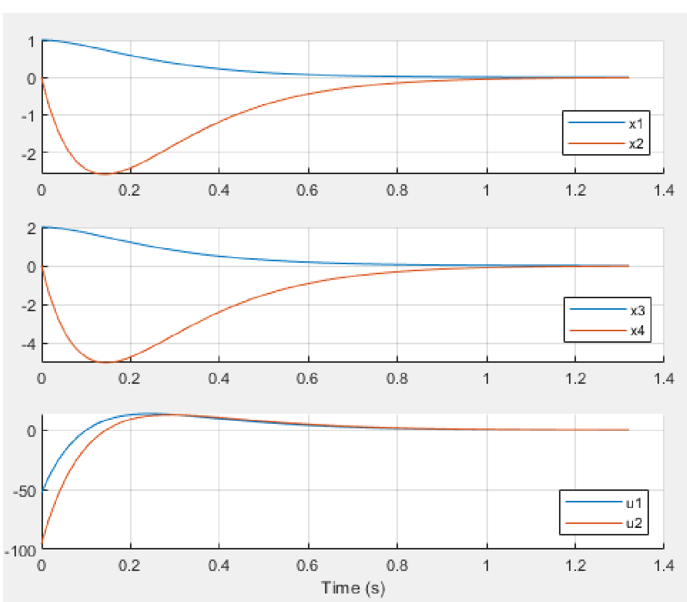
d: $\mathrm{eig}(A\big|_{w=1}) = \begin{bmatrix} 0 \\ -1 \\ 1+\sqrt{2}i \\ 1-\sqrt{2}i \end{bmatrix}$ real part > 0, unstable.

e: $K = \mathrm{place}(A, B, [-5, -6, -7, -8])$

$\Rightarrow K = \begin{bmatrix} 41.12 & 12.5 & 5.8 & 2.9 \\ 5.3 & -1.2 & 44.9 & 14.5 \end{bmatrix}$

$\mathrm{eig}(A - B*K) = [-8 \ -5 \ -7 \ -6]$

f:



The x2 and x4 decay exponentially, but x1 and x3 are not.

A =

$\begin{matrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \end{matrix}$

B =

$\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix}$

when we look into the A and B, we can see:
x1 is the integration of x2 (cyan circles), and
x3 is the integration of x4 (cyan circles), and
we can only directly control x2 and x4 (red arrows)

So the x1 and x3 will not decay as quick as x2 and x4.

3.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

a: $\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$\text{rank}(\mathcal{C}) = 1 < \cancel{\text{rank}(A)}\ 2$

So system is not controllable.

$$\mathcal{C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underset{v_1}{\uparrow}$$

let $P_0^{-1} = \begin{bmatrix} v_1 & \begin{matrix} 1 \\ 0 \end{matrix} \end{bmatrix}$ then $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, then

$$\bar{A} = P \cdot A \cdot P^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\bar{B} = P \cdot B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \text{controllable} \\ \leftarrow \text{uncontrollable} \end{matrix}$$

$$\bar{C} = C \cdot P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\bar{D} = D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

b: Stabilizable? Yes.

$\lambda_1 = \lambda_2 = -1$, so without any control, the system states will all go to zero. Nonetheless to say with the controllable part, the λ_1 can die out even quicker.