

1. Realization of  $G(s) = \left[ \frac{-12s-6}{3s+34}, \frac{22s+23}{3s+34} \right]$

$$D(s) = \lim_{s \rightarrow \infty} G(s) = \left[ -4, \frac{22}{3} \right]$$

$$\begin{aligned} G_p(s) &= G(s) - D(s) = \left[ \frac{130}{3s+34}, \frac{-679}{3s+34} \right] \\ &= \left[ \frac{\frac{130}{3}}{s + \frac{34}{3}}, \frac{-679}{s + \frac{34}{3}} \right] \end{aligned}$$

$$d(s) = s + \frac{34}{3} \quad \uparrow \alpha$$

$$G_{np}(s) = \frac{1}{d(s)} \left[ \frac{130}{3}, -679 \right]$$

N.

$$A = \begin{bmatrix} -\frac{34}{3} & 0 \\ 0 & -\frac{34}{3} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \left[ \frac{130}{3}, -679 \right]$$

$$D = \left[ -4, \frac{22}{3} \right]$$

$$2. \quad A_1 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad C_1 = [1 \quad -1 \quad 0]$$

$$A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad C_2 = [1 \quad -1 \quad 0]$$

Eigenvalues of  $A_1$  is  $\begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$$(2-\lambda)(2-\lambda)(1-\lambda) = 0$$

$$\lambda = 2, 2, 1$$

Eigenvalues of  $A_2$  is  $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$

$$\lambda = 2, 2, -1$$

Eigenvalues of  $A_1$  &  $A_2$  are different, so they are not algebraically equivalent.

$$G_1(s) = C_1(sI - A_1)^{-1}B_1 = [1 \quad -1 \quad 0] \begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= [1 \quad -1 \quad 0] \frac{1}{(s-2)^2(s-1)} \begin{bmatrix} (s-2)(s-1) & 0 & 0 \\ (s-1)(s-2)(s-1) & 0 \\ 2s-2+2(s-2)(s-2)^2 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-2)^2(s-1)} [1 \quad -1] \begin{bmatrix} (s-2)(s-1) & 0 \\ (s-1) & (s-2)(s-1) \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(s-2)^2} [1 \quad -1] \begin{bmatrix} s-2 & 1 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G_2(s) = C_2(sI - A_2)^{-1}B_2 = [1 \quad -1 \quad 0] \begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{(s-2)^2}$$

$$= \frac{1}{(s-2)^2(s+1)} [1 \quad -1] \begin{bmatrix} (s-2)(s+1) & 0 & 0 \\ (s+1) & (s-2)(s+1) & 0 \\ ? & ? & (s-2)^2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{(s-2)^2} [1 \quad -1] \begin{bmatrix} s-2 & 1 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(s-2)^2}$$

$G_1(s) = G_2(s)$ , so they are zero-state eqn.

$$3. \dot{x} = Ax + Bu \quad y = Cx, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 1] \quad u = -[f_1 \quad f_2]x + v$$

a.

$$\dot{x} = Ax + Bu$$

$$= Ax + B(-[f_1 \quad f_2]x + v)$$

$$= Ax - B[f_1 \quad f_2]x + Bv$$

$$= (A - B[f_1 \quad f_2])x + Bv$$

$$= \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \quad f_2] \right) x + Bv$$

$$= \begin{bmatrix} 0 & 1 \\ -f_1 & -f_2 \end{bmatrix} x + Bv.$$

$$y = Cx = [1 \quad 1] x.$$

b.

$$G(s) = C(sI - \begin{bmatrix} 0 & 1 \\ -f_1 & -f_2 \end{bmatrix})^{-1} B$$

$$= [1 \quad 1] \begin{bmatrix} s & -1 \\ f_1 & s+f_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 1] \begin{bmatrix} s+f_2 & 1 \\ -f_1 & s \end{bmatrix} \cdot \frac{1}{s(s+f_2)} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s(s+f_2)+f_1}$$

$$= \frac{1}{s(s+f_2)+f_1} \begin{bmatrix} s+f_2-f_1 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s+1}{s(s+f_2)+f_1}$$

$$= \frac{s+1}{s^2+f_2s+f_1}$$

c.

Target transfer function is

$$\frac{1}{s+1} = \frac{s+1}{(s+1)(s+1)} = \frac{s+1}{s^2+2s+1} := \frac{s+1}{s^2+f_2s+f_1}$$

$$f_2=2 \quad f_1=1$$

d.

(a)  $s^2+f_2s+f_1 := (s+2)(s+3) = s^2+5s+6$   
 $f_2=5, f_1=6$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 1] \quad D=0$$

(b)  $s^2+f_2s+f_1 := (s+5)(s+6) = s^2+11s+30$   
 $f_2=11, f_1=30$

$$A = \begin{bmatrix} 0 & 1 \\ -30 & -11 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 1] \quad D=0.$$

e. since we know the poles of transfer function, they are the eigenvalue of A. So

(a)  $(A - \lambda I)v = 0$

$\lambda = -3:$

$$\begin{bmatrix} +3 & 1 \\ -6 & -5+3 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = -2:$

$$\begin{bmatrix} 2 & 1 \\ -6 & -5+2 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} e^{-3t} & \\ & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-3t} & e^{-2t} \\ 3e^{-3t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & e^{-3t} + e^{-2t} \\ +6e^{-3t} - 2e^{-2t} & -3e^{-3t} - 2e^{-2t} \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}^{-1}$$

$$(b) (A - \lambda I)v = 0$$

$$\lambda = -6$$

$$\begin{bmatrix} 6 & \phi \\ -\phi_{30} & -\phi+6 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\lambda = -5$$

$$\begin{bmatrix} 5 & \phi \\ -\phi_{30} & -\phi+5 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} \phi \\ -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -6 & -5 \end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} e^{-6t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} -5 & 1 \\ +6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-6t} & e^{-5t} \\ -6e^{-6t} & -5e^{-5t} \end{bmatrix} \begin{bmatrix} -5 & 1 \\ 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5e^{-6t} + 6e^{-5t} & e^{-6t} + e^{-5t} \\ +30e^{-6t} - 30e^{-5t} & -6e^{-6t} + 5e^{-5t} \end{bmatrix}$$

$$4. e^{At}, A = \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix}$$

a: eigenvalue.

$$\begin{vmatrix} 2-\lambda & 0 \\ -3 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) = 0$$

$$\lambda = 2 \text{ or } \lambda = 5$$

$$(A - \lambda I)v = 0$$

for  $\lambda = 2$

$$\left( \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) v = 0$$

$$\begin{bmatrix} 0 & 0 \\ -3 & 3 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for  $\lambda = 5$

$$\left( \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) v = 0$$

$$\begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} v = 0 \quad v \text{ can be } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ e^{2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{5t} & e^{5t} \end{bmatrix}$$

b: Laplace Transform.

$$e^{At} = \mathcal{L}^{-1} [sI - A]^{-1}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} s-2 & 0 \\ 3 & s-5 \end{bmatrix}^{-1}$$

$$= \mathcal{L}^{-1} \frac{1}{(s-5)(s-2)} \begin{bmatrix} s-5 & 0 \\ -3 & s-2 \end{bmatrix}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s-2} & 0 \\ \frac{-3}{(s-5)(s-2)} & \frac{1}{s-5} \end{bmatrix}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s-2} & 0 \\ \frac{1}{s-2} - \frac{1}{s-5} & \frac{1}{s-5} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{5t} & e^{5t} \end{bmatrix}$$