1. $A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} x. \qquad x_0 = [2 \quad 4 \quad 0]^T$

let $A' = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ since x_3 is always 0 and not going into x_1 and x_2.

$$x'(t) = e^{A't} \cdot x_0' \quad \text{where} \quad e^{A't} = \mathcal{L}^{-1}(sI - A')^{-1}$$

$$= \mathcal{L}^{-1}\left(\begin{bmatrix} s+3 & -1 \\ 0 & s+3 \end{bmatrix}^{-1}\right)$$

$$= \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+3)^2} \\ 0 & \frac{1}{(s+3)} \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

So $x'(t) = \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix} x_0'$

So $x(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & \phi \end{bmatrix} x_0 = \begin{bmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & \phi \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-3t} + 4te^{-3t} \\ 4e^{-3t} \\ 0 \end{bmatrix}$

System is internally stable because $\cancel{(-3, -3, 0)}$ $-3, -3 < 0$, $\lambda = 0$ where jordan size is one.

System is not Asymptotically / Exponential stable, since x_3 will not decade to 0.

3.

$\dot{x} = Ax + Bu$ ~~$y = Cx + Du$~~ . $y = Cx$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = [1 \quad 1]$ where $u = -[0 \quad f]x + v$.

$$\dot{x} = Ax + B(-[0 \quad f]x + v)$$

$$= Ax - B \cdot [0 \quad f]x + Bv.$$

$$= Ax - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0 \quad f]x + Bv$$

$$= \left(A - \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}\right)x + Bv.$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -f \end{bmatrix} x + Bv.$$

a. To make the system internal stable, the λ in jordan form size larger than 1 need to be negative, therefore $f > 0$.

$$G(s) = C\left(sI - \begin{bmatrix} 0 & 1 \\ 0 & -f \end{bmatrix}\right)^{-1} B$$

$$= C\left(\cancel{B}\begin{bmatrix} s & 1 \\ 0 & s+f \end{bmatrix}^{-1}\right) B$$

$$= C \begin{bmatrix} s+f & -1 \\ 0 & s \end{bmatrix} \cdot B \cdot \frac{1}{s(s+f)}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+f} & -\frac{1}{s(s+f)} \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} ? & \frac{1}{s} - \frac{1}{s(s+f)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s} - \frac{1}{s(s+f)}$$

$$= \frac{s+f-1}{s(s+f)}$$

b. One of the poles is zero, so it is not BIBO stable .

2. Stability.

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad C = [-3, 2], \quad D = 12.$$

$$G(s) = C(sI - A)^{-1}B + D.$$

$$= [-3, 2] \begin{bmatrix} s+1 & -4 \\ 0 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix} + 12$$

$$= [-3, 2] \begin{bmatrix} s-3 & \cancel{4} \\ 0 & \cancel{s+1} \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cdot \frac{1}{(s+1)(s-3)} + 12$$

$$= [-3, 2] \begin{bmatrix} \frac{1}{s+1} & ? \\ 0 & ? \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} + 12$$

$$= [-3, 2] \begin{bmatrix} \frac{4}{s+1} \\ 0 \end{bmatrix} + 12$$

$$= \frac{-12}{s+1} + 12$$

$s+1=0$ $\quad s=-1<0$ $\quad$ pole <0. so it is BIBO stable.

But A is in jordan form and one of the $\lambda = 3 > 0$, so it is not internally stable.

4.

$$\dot{x} = Ax + Bu, \quad y = Cx$$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

$$G(s) = C(sI - A)^{-1}B$$

$$= C\left(\begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \right) B$$

$$= C \cdot \frac{1}{(s+1)(s+1)(s-1)} \cdot \begin{bmatrix} (s+1)(s-1) & 0 & 0 \\ +(s-1) & (s+1)(s-1) & 0 \\ 0 & 0 & (s+1)^2 \end{bmatrix}^T \cdot B$$

$$= \begin{bmatrix} 1, & 1, & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1, & 1 \end{bmatrix} \begin{bmatrix} \cancel{\frac{1}{s+1}} & \frac{1}{(s+1)^2} & 0 \\ \cancel{0} & \frac{1}{s+1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2} + \frac{1}{s+1}$$

$$= \frac{s+2}{(s+1)^2} \qquad \text{poles} = -1 < 0.$$

a. The system is not internally stable because of one $\lambda = 1 > 0$.

They system is BIBO stable, because poles are negative.

b. Assume the LQR solution is P.

then the $u = -Kx$, where $K = R^{-1}B^TP$.

Then

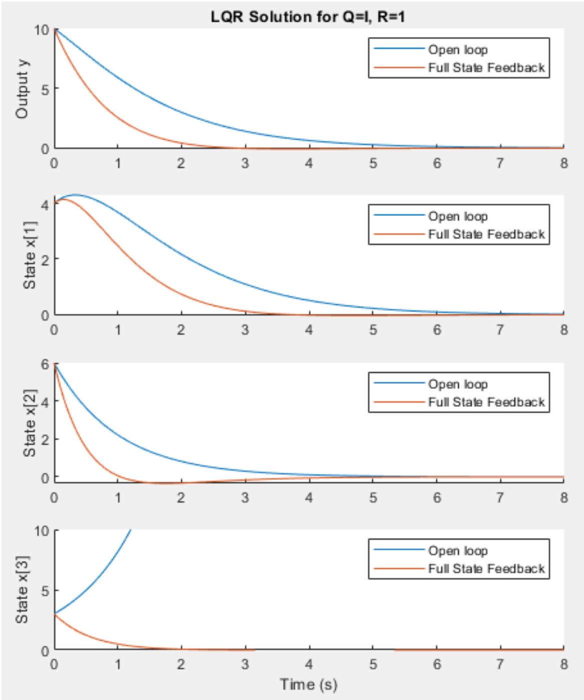
$$\dot{x} = Ax + Bu = Ax + B(-R^{-1}B^TPx)$$

$$= (A - BR^{-1}B^TP)x$$

$$y = Cx.$$

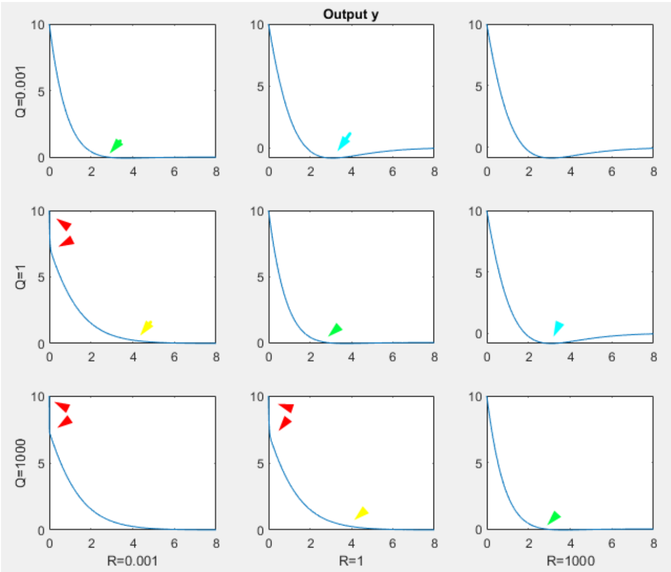
c. Since LQR system has a solution, the system is controllable, indicating that the close loop system is stable. Since the system is under full state feedback control, all the states will go to zero. So the system is both internally & exp stable.

Q4 Part D



We can see the x[3] is bounded with feedback control, which is previously unbounded.

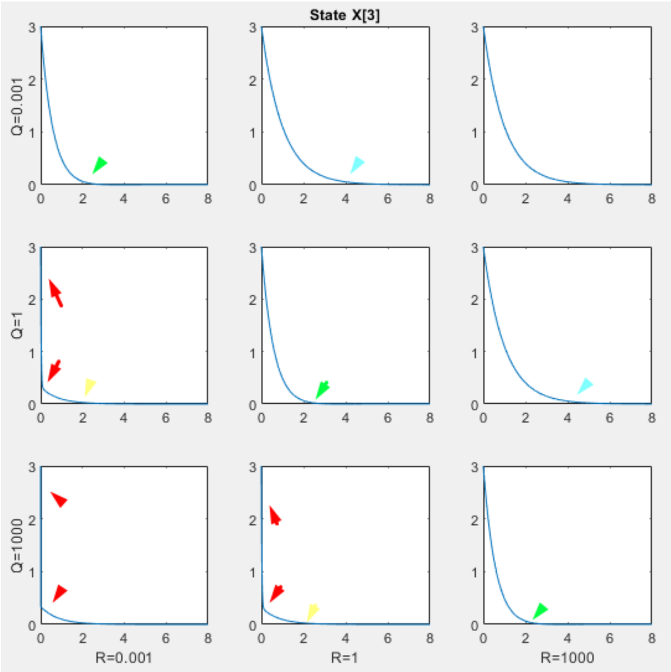
Q4 Part E



When the Q/R ratio is the same, the response is the same, as indicated by yellow/ cyan/ green colors.

- When R>>Q, cyan arrows, we can observe a more rapid response and an overshoot of the output value before decaying to zero. R>>Q, meaning the weighting of LQR equation is mainly put on to control signal u. So, the control signal u is limited and compromised, u<<y. Making the system behave more like an open loop system, where the decay of the states is slow.
- When Q>>R, the decay is flatter and slower. Also, when Q is much larger than R, a very sharp instant response is observed at the beginning, red arrows. This is because when Q>>R, the weight on control signal is small, the control signal can be large to compromise the states, that is, u>>y, making the control more effective and a quick decay in the states.

The same description can be applied to both x[1] and x[2]. But in x[3], some strange behavior is seen:



The sharp drop of output value at the beginning of simulation comes from x[3]. But it is strange since C=[1,1,0], where x[3] is not directly fed into the output.

```
clear all
%% Q4 Part D
A=[-1, 1, 0;0,-1,0;0,0,1];
B=[0;1;1];
C=[1,1,0];
D=0;
xInit=[4;6;3];

sys0 = ss(A,B,C,D);

Q=eye(3);
R=1;

[K,S,P] = lqr(A,B,Q,R);

sys1 = ss(A-B*K,B,C,D);
[y0,tOut,x0] = initial(sys0,xInit);
[y1,tOut,x1] = initial(sys1,xInit,tOut);

figure(1),clf
subplot(4,1,1),hold on
plot(tOut,y0)
plot(tOut,y1)
ylabel("Output y"),xlim([0,8]),legend("Open loop","Full State Feedback")
title("LQR Solution for Q=I, R=1")
subplot(4,1,2),hold on
plot(tOut,x0(:,1))
plot(tOut,x1(:,1))
ylabel("State x[1]"),xlim([0,8]),legend("Open loop","Full State Feedback")
subplot(4,1,3),hold on
plot(tOut,x0(:,2))
plot(tOut,x1(:,2))
ylabel("State x[2]"),xlim([0,8]),legend("Open loop","Full State Feedback")
subplot(4,1,4),hold on
plot(tOut,x0(:,3))
plot(tOut,x1(:,3))
ylabel("State x[3]"),xlim([0,8]),legend("Open loop","Full State Feedback")
ylim([0,10])
xlabel("Time (s)")
%% Q4 Part E
Qs=[0.001, 1, 1000];
Rs=[0.001, 1, 1000];
figure(2),clf
figure(3),clf
figure(4),clf
figure(5),clf
for i=1:3
    for j=1:3
        Q=eye(3);
        R=1;
```

```
        [K,S,P] = lqr(A,B,Q*Qs(i),R*Rs(j));

        sysn = ss(A-B*K,B,C,D);
        [yn,tOut,xn] = initial(sysn,xInit,tOut);

        figure(2)
        subplot(3,3,i*3-3+j)
        plot(tOut,yn)
        xlim([0,8])
        %
        % figure(3)
        % subplot(3,3,i*3-3+j)
        % plot(tOut,xn(:,1))
        % xlim([0,8])
        %
        % figure(4)
        % subplot(3,3,i*3-3+j)
        % plot(tOut,xn(:,2))
        % xlim([0,8])

        figure(5)
        subplot(3,3,i*3-3+j)
        plot(tOut,xn(:,3))
        xlim([0,8])
        % title(sprintf("Q=%f, R=%f",Qs(i),Rs(j)))
    end
end

for i=2:5
figure(i)
subplot(3,3,1), ylabel("Q=0.001")
subplot(3,3,4), ylabel("Q=1")
subplot(3,3,7), ylabel("Q=1000")

subplot(3,3,7), xlabel("R=0.001")
subplot(3,3,8), xlabel("R=1")
subplot(3,3,9), xlabel("R=1000")
end

figure(2)
subplot(3,3,2), title("Output y")

figure(5)
subplot(3,3,2), title("State X[3]")
```