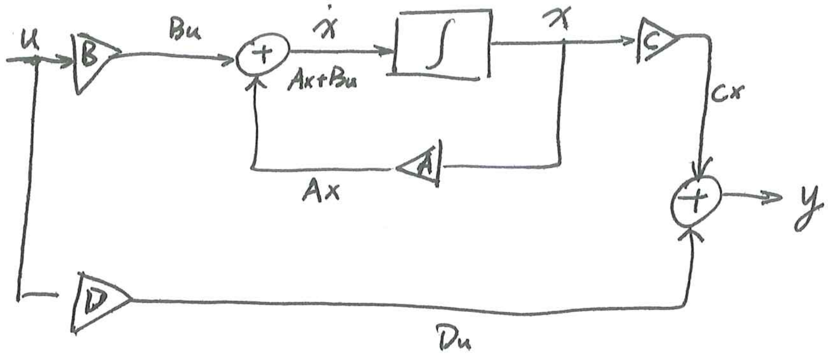
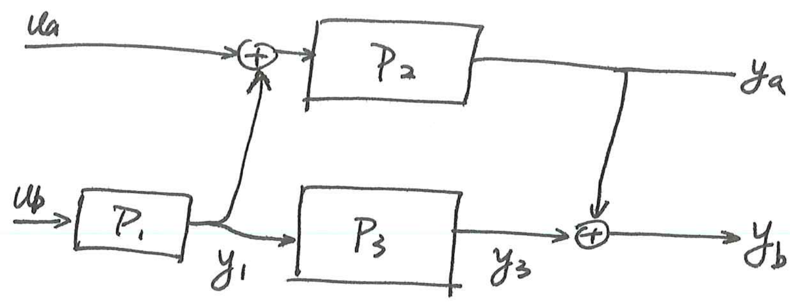
1.



$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

2.



$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_b \\ y_1 = C_1 x_1 + D_1 u_b \end{cases}$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + B_2 (u_a + y_1) \\ y_a = C_2 x_2 + D_2 (u_a + y_1) \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 (u_a + C_1 x_1 + D_1 u_b) \\ y_a = C_2 x_2 + D_2 (u_a + C_1 x_1 + D_1 u_b) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + B_2 C_1 x_1 + B_2 u_a + B_2 D_1 u_b \\ y_a = C_2 x_2 + D_2 C_1 x_1 + D_2 u_a + D_2 D_1 u_b. \end{cases}$$

$$\begin{cases} \dot{x}_3 = A_3 x_3 + B_3 y_1 \\ y_3 = C_3 x_3 + D_3 (y_1) \end{cases}$$

$$y_b = y_a + y_3$$

$$\begin{cases} \dot{x}_3 = A_3 x_3 + B_3 y_1 \\ y_3 = C_3 x_3 + D_3 y_1 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_3 = A_3 x_3 + B_3 (C_1 x_1 + D_1 u_b) \\ y_3 = C_3 x_3 + D_3 (C_1 x_1 + D_1 u_b) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_3 = A_3 x_3 + B_3 C_1 x_1 + B_3 D_1 u_b \\ y_3 = C_3 x_3 + D_3 C_1 x_1 + D_3 D_1 u_b \end{cases}$$

$$\& \; y_b = y_a + y_3 = C_3 x_3 + D_3 C_1 x_1 + D_3 D_1 u_b + C_2 x_2 + D_2 C_1 x_1 + D_2 u_a + D_2 D_1 u_b$$

$$= (D_3 C_1 + D_2 C_1) x_1 + C_2 x_2 + C_3 x_3 + (D_3 D_1 + D_2 D_1) u_b + D_2 u_a$$

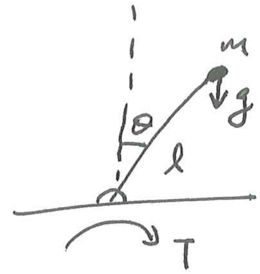
So to summarize:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ B_2 C_1 & A_2 & 0 \\ B_3 C_1 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & B_1 \\ B_2 & B_2 D_1 \\ 0 & B_3 D_1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix}$$

$$\begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} D_2 C_1 & C_2 & 0 \\ D_3 C_1 + D_2 C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} D_2 & D_2 D_1 \\ D_2 & D_3 D_1 + D_2 D_1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix}$$

3.

$$ml^2\ddot{\theta} = mgl\sin\theta - b\dot{\theta} + T.$$



System can be described as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$u = T$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin\theta - \frac{b}{ml^2}\dot{\theta} + \frac{1}{ml^2}T \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}_{g(x)} u.$$

$$y = \theta = \underbrace{x_1}_{h(x)}.$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial y}{\partial x}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ \cdots \end{bmatrix}$$

$$= x_2$$

$$\frac{d^2y}{dt^2} = \frac{\partial x_2}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial x_2}{\partial x}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

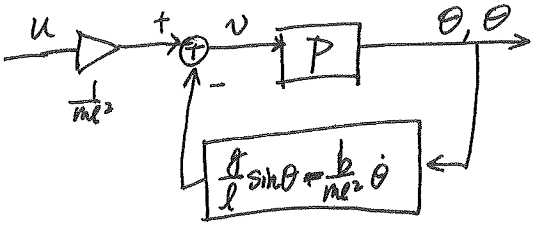
$$= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2} \end{bmatrix}$$

$$= \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2}$$

We define $v = \frac{d^2y}{dt^2} = \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2}$, $u = ml^2 v - mlg\sin x_1 + bx_2$

and system now is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

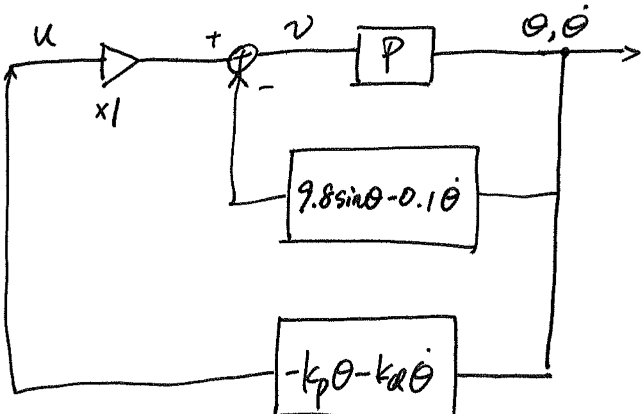


define the feedback control

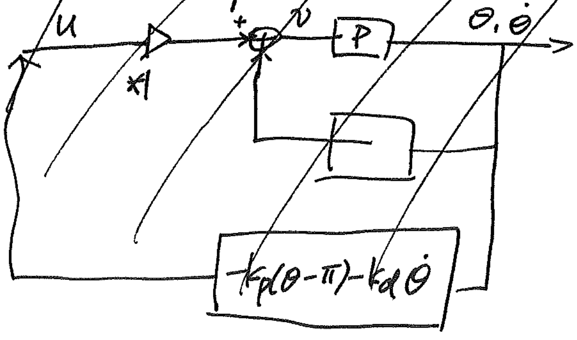
$$u = -\begin{bmatrix} k_p & k_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -k_p\theta - k_d\dot{\theta}$$

The system would be like



~~Since the target θ=π. we set θ−π as the difference~~



Sorry $\theta = \pi$ is init condition.
$\theta = 0$ is the target

Part B, PD Controller Performance

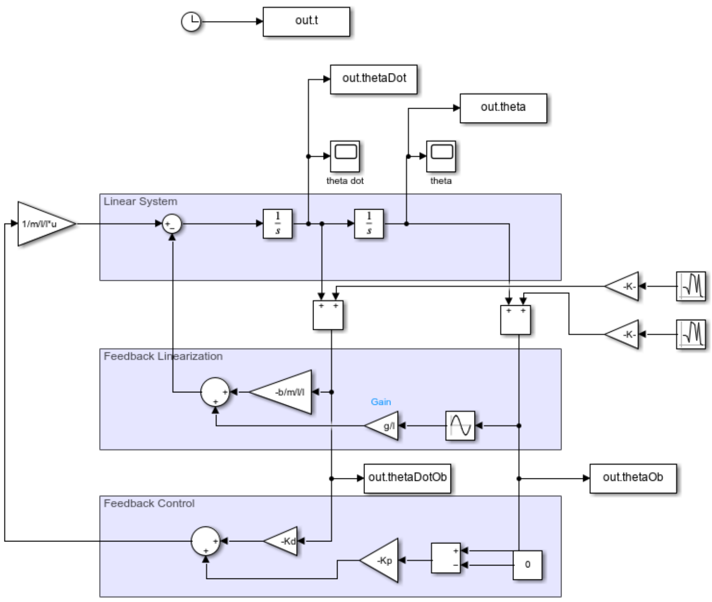


Figure 1 Simulink Design

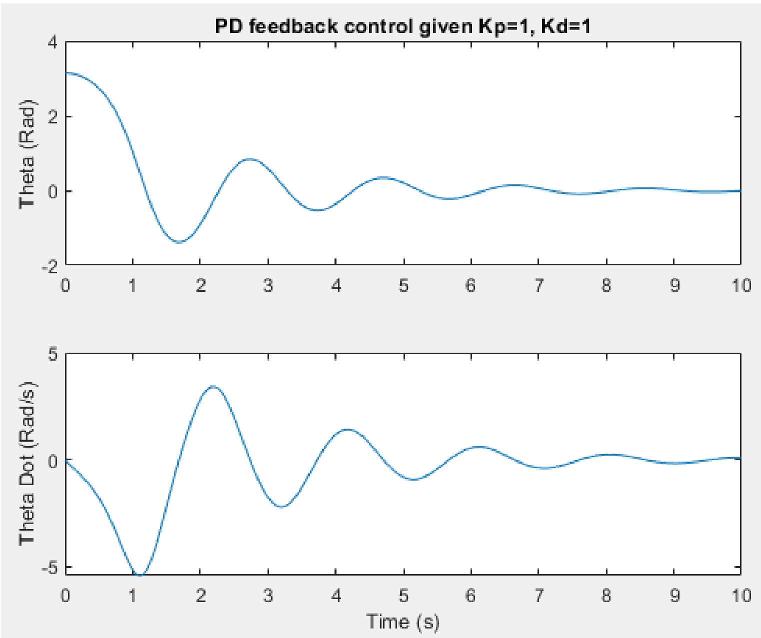


Figure 2 System response given theta=pi, Kp=Kd=1

The PD controller performance can be seen as in Figure 2, the system is Underdamped. It oscillated and gradually approached the zero position.

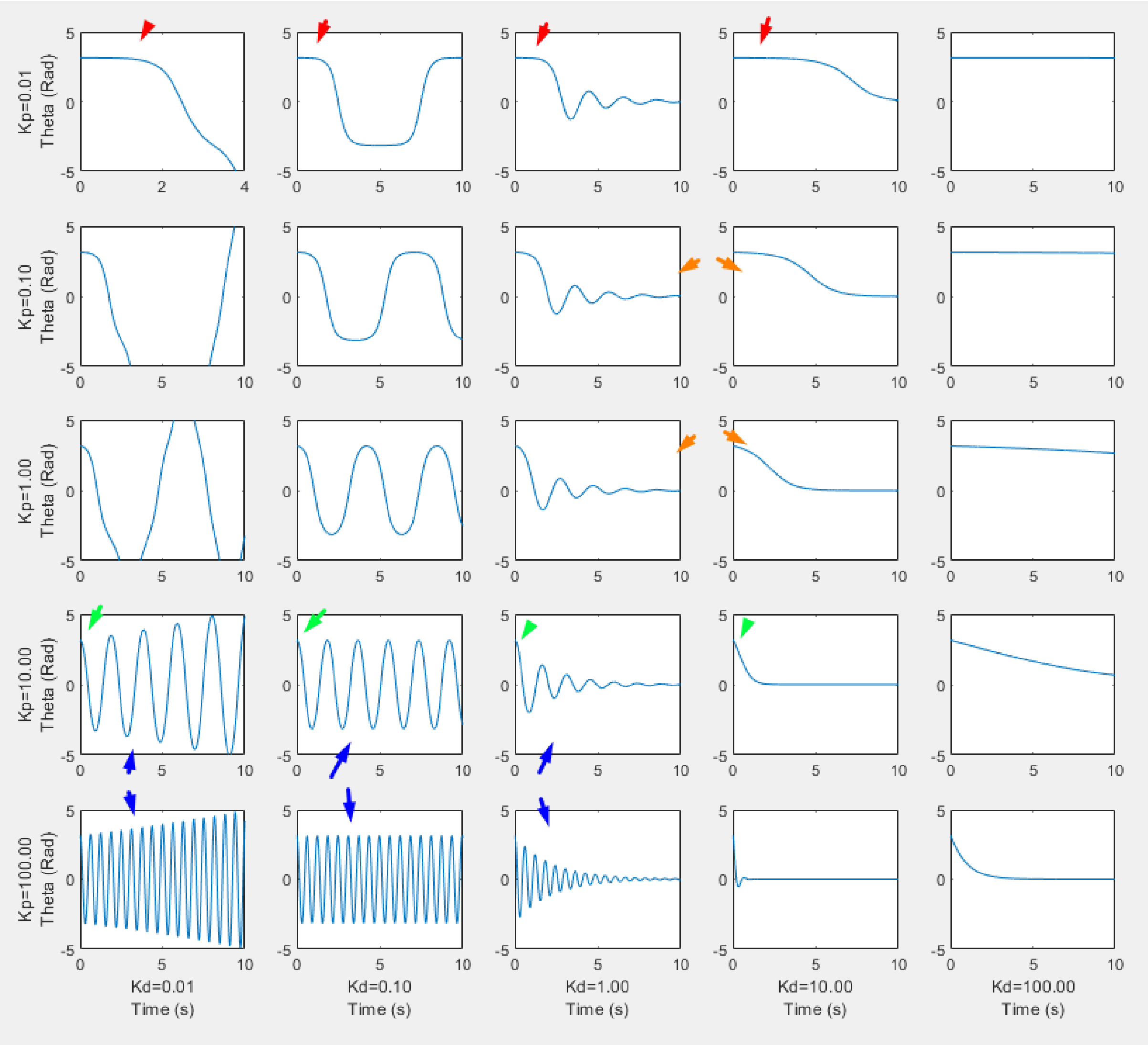


Figure 3 PD Controller Performance

The PD controller performance can be seen as in Figure 3:
1, when Kp is larger, the response of the system starts to move quicker (comparing the red arrows with green arrows)
2, when Kp is larger, the system resident frequency is higher (comparing the blue arrows)
3, when Kd is low, the system is under damped; when Kd is high, the system is over damped (comparing the orange arrows)

Part C Noise

Random noise was applied to both theta and dot theta with a fixed percentage ratio while keeping Kd=1 & Kp=1. The result is as follows:

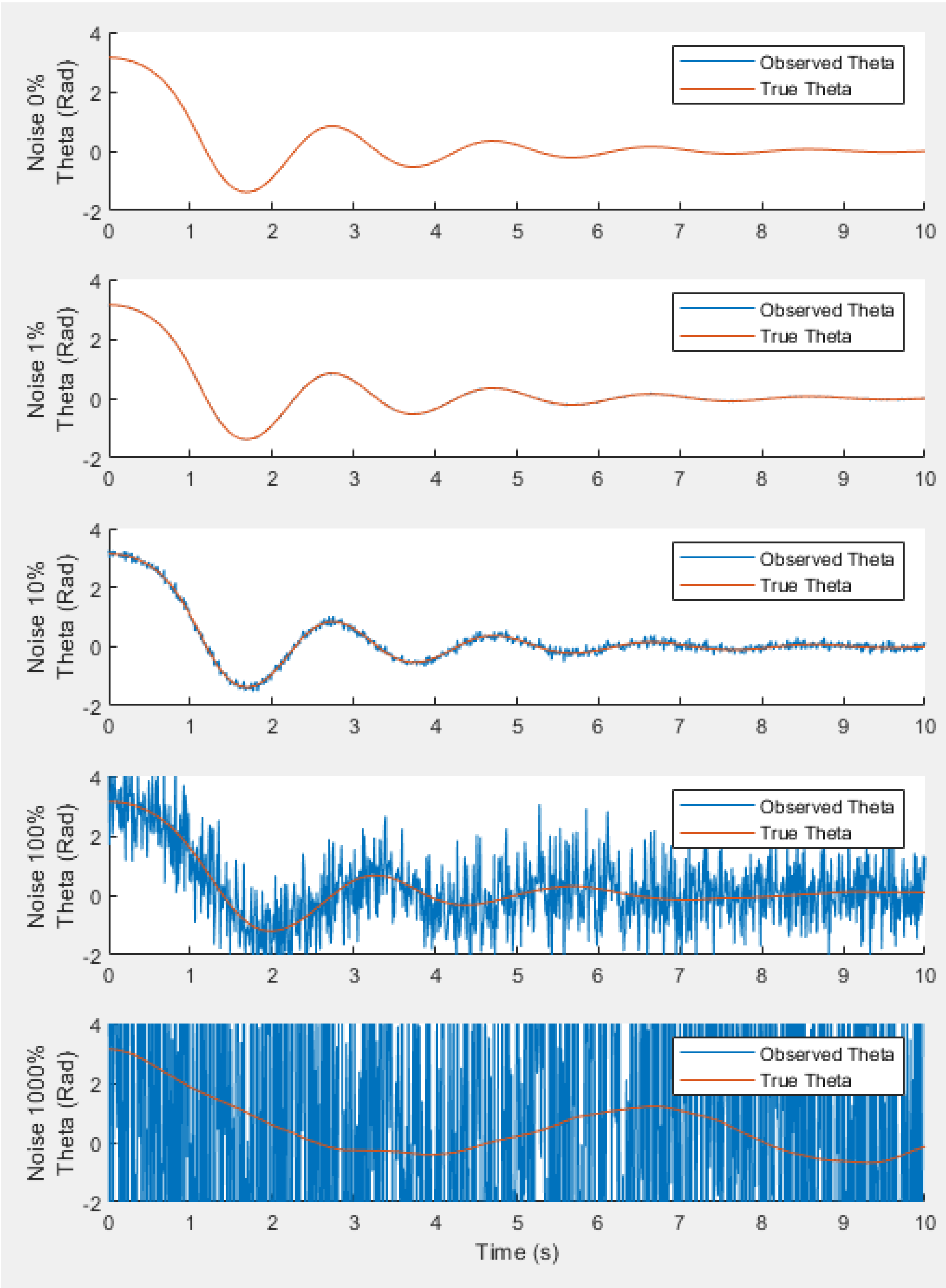


Figure 4 System angle and observed angle under difference noise conditions.

The system performance dropped when noise increased. But it is amazing that it can somehow handle a very noisy input up to 100%.

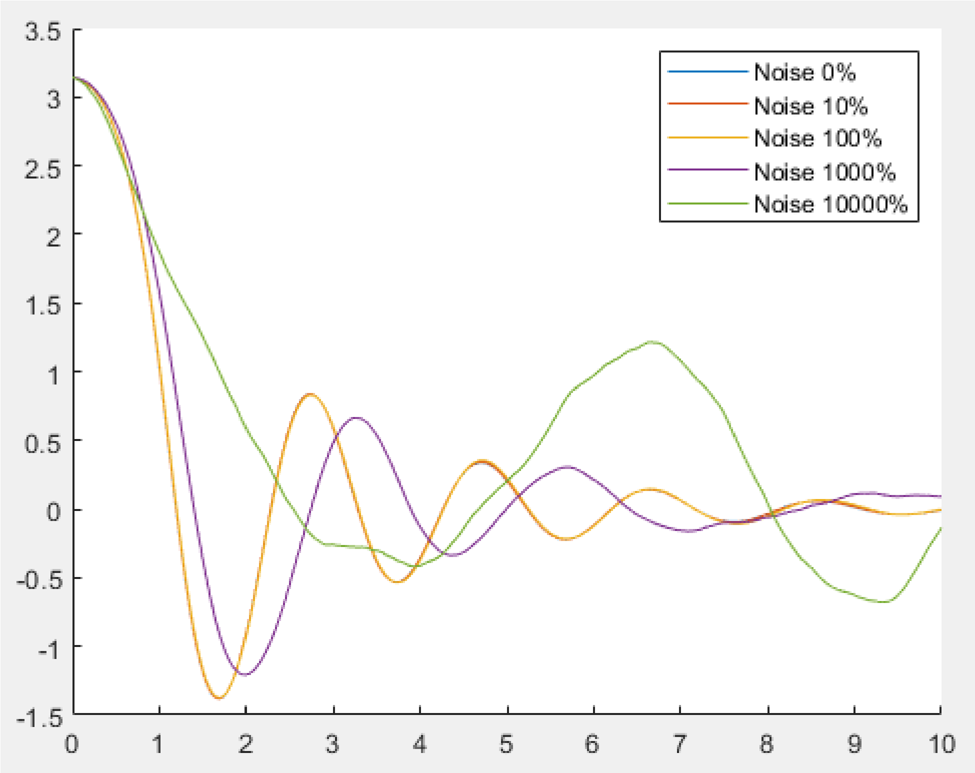


Figure 5 Comparing system angle under difference noise conditions

When the performance of the controller is compared under difference noise level, the result is impressive that, even under 100% noise, the system run as well as no noise. The performance deterioration started to be revealed when the noise was 10 times larger than system variables.

By tuning the PD controller under the condition that 30% of noise applied to both theta and theta dot, the result is as follows:

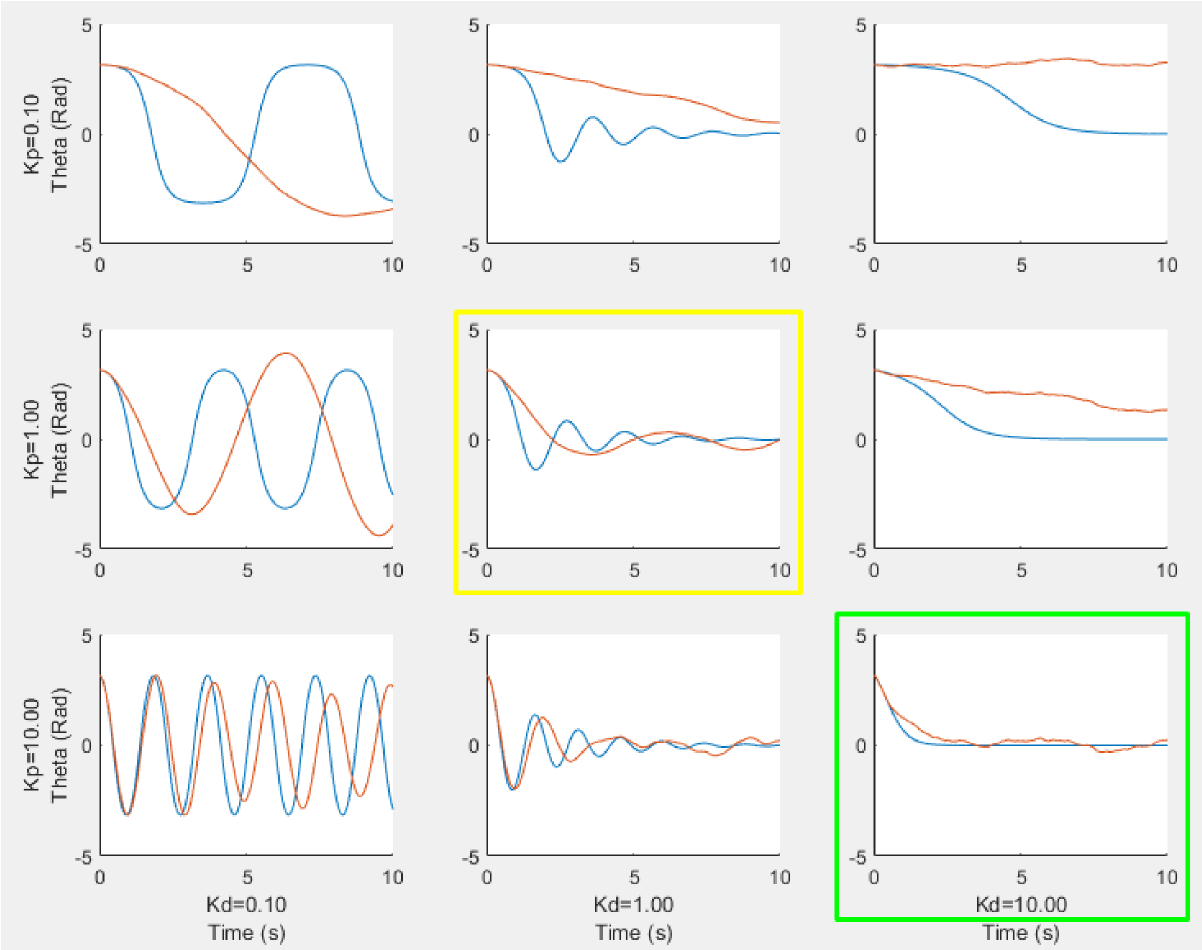


Figure 6 Difference PD controller performance under

It can be seen that the Kd=1, Kp=1 controller is marginally capable of driving the system (yellow box). Yet the strongly under damped system where Kd=10, Kp=10 performed better than other system parameters (green box).