1. $A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} x.$ $\quad x_0 = [2 \;\; 4 \;\; 0]^T$

let $A' = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ since x_3 is always 0 and not going into x_1 and x_2.

$x'(t) = e^{A't} \cdot x_0'$ where $e^{A't} = \mathcal{L}^{-1}(sI - A')^{-1}$

$$= \mathcal{L}^{-1}\left(\begin{bmatrix} s+3 & -1 \\ 0 & s+3 \end{bmatrix}^{-1}\right)$$

$$= \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+3)^2} \\ 0 & \frac{1}{(s+3)} \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

So $x'(t) = \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix} x_0'$

So $x(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & \phi \end{bmatrix} x_0 = \begin{bmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & \phi \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-3t} + 4te^{-3t} \\ 4e^{-3t} \\ 0 \end{bmatrix}$

System is internally stable because ~~(-3, -3, 0)~~ $-3, -3 < 0$, $\lambda = 0$ where jordan size is one.

System is not Asymptotically/Exponential stable, since x_3 will not decade to 0.

3.

$\dot{x} = Ax + Bu$ $\quad \cancel{y = Cx + Du}$. $y = Cx$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ where $u = -[0 \;\; f]x + v.$

$$\dot{x} = Ax + B(-[0 \;\; f]x + v)$$

$$= Ax - B \cdot [0 \;\; f]x + Bv.$$

$$= Ax - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0 \;\; f]x + Bv$$

$$= \left(A - \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}\right)x + Bv.$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -f \end{bmatrix} x + Bv.$$

a. To make the system internal stable, the λ in jordan form size larger than 1 need to be negative, therefore $f > 0$.

$$G(s) = C\left(sI - \begin{bmatrix} 0 & 1 \\ 0 & -f \end{bmatrix}\right)^{-1} B$$

$$= C\left(\begin{bmatrix} s & 1 \\ 0 & s+f \end{bmatrix}^{-1}\right) B$$

$$= C \begin{bmatrix} s\cancel{+f} & -1 \\ 0 & s+f \end{bmatrix} \cdot B \cdot \frac{1}{s(s+f)}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+f} & -\frac{1}{s(s+f)} \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} ? & \frac{1}{s} - \frac{1}{s(s+f)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s} - \frac{1}{s(s+f)}$$

$$= \frac{s+f-1}{s(s+f)}$$

b. One of the poles is zero, so it is not BIBO stable .

2. Stability.

$\dot{x} = Ax + Bu, \quad y = Cx + Du.$

$A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad C = [-3, 2], \quad D = 12.$

$G(s) = C(sI - A)^{-1}B + D.$

$= [-3, 2] \begin{bmatrix} s+1 & -4 \\ 0 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix} + 12$

$= [-3, 2] \begin{bmatrix} s-3 & 4 \\ 0 & s+1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cdot \frac{1}{(s+1)(s-3)} + 12$

$= [-3, 2] \begin{bmatrix} \frac{1}{s+1} & ? \\ 0 & ? \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} + 12$

$= [-3, 2] \begin{bmatrix} \frac{4}{s+1} \\ 0 \end{bmatrix} + 12$

$= \frac{-12}{s+1} + 12$

$s+1 = 0 \quad s = -1 < 0 \quad$ pole < 0. so it is BIBO stable.

But A is in jordan form and one of the $\lambda = 3 > 0$, so it is not internally stable.

4.

$$\dot{x} = Ax + Bu\ ,\ y = Cx$$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

$$G(s) = C(sI - A)^{-1}B$$

$$= C\left(\begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s-1 \end{bmatrix}^{-1}\right)B$$

$$= C \cdot \frac{1}{(s+1)(s+1)(s-1)} \cdot \begin{bmatrix} (s+1)(s-1) & 0 & 0 \\ +(s-1) & (s+1)(s-1) & 0 \\ 0 & 0 & (s+1)^2 \end{bmatrix}^T \cdot B$$

$$= \begin{bmatrix} 1, & 1, & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1, & 1 \end{bmatrix}\begin{bmatrix} \cancel{\frac{1}{s+1}} & \frac{1}{(s+1)^2} & 0 \\ \cancel{0} & \frac{1}{s+1} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2} + \frac{1}{s+1}$$

$$= \frac{s+2}{(s+1)^2} \qquad \text{poles} = -1 < 0.$$

a. The system is not internally stable because of one $\lambda = 1 > 0$.

They system is BIBO stable, because poles are negative.

b. Assume the LQR solution is P.

then the $u = -Kx$, where $K = R^{-1}B^TP$.

Then

$$\dot{x} = Ax + Bu = Ax + B(-R^{-1}B^TPx)$$

$$= (A - BR^{-1}B^TP)x$$

$$y = Cx.$$

c. Since LQR system has a solution, the system is controllable, indicating that the close loop system is stable. Since the system is under full state feedback control, all the states will go to zero. So the system is both internally & exp stable.