1. Realization of $G(s)=\left[\dfrac{-12s-6}{3s+34}, \dfrac{22s+23}{3s+34}\right]$

$$D(s)=\lim_{s\to\infty} G(s)=\left[-4, \frac{22}{3}\right]$$

$$G_{sp}(s)=G(s)-D(s)=\left[\frac{130}{3s+34}, \frac{-\frac{679}{3}}{3s+34}\right]$$

$$=\left[\frac{\frac{130}{3}}{s+\frac{34}{3}}, \frac{-679}{s+\frac{34}{3}}\right]$$

$$d(s)=s+\frac{34}{3} \quad \uparrow \alpha$$

$$G_{sp}(s)=\underbrace{\frac{1}{d(s)}\left[\frac{130}{3}, -679\right]}_{N.}$$

$$A=\begin{bmatrix} -\frac{34}{3} & 0 \\ 0 & -\frac{34}{3} \end{bmatrix}$$

$$B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C=\left[\frac{130}{3}, -679\right]$$

$$D=\left[-4, \frac{22}{3}\right]$$

2. $A_1=\begin{bmatrix}2&1&2\\0&2&2\\0&0&1\end{bmatrix}$ $B_1=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $C_1=[1 \quad -1 \quad 0]$

$A_2=\begin{bmatrix}2&1&1\\0&2&1\\0&0&-1\end{bmatrix}$ $B_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $C=[1 \quad -1 \quad 0]$

Eigenvalues of A_1 is $\begin{vmatrix}2-\lambda&1&2\\0&2-\lambda&2\\0&0&1-\lambda\end{vmatrix}=0$

$(2-\lambda)(2-\lambda)(1-\lambda)=0$

$\lambda=2,2,1$

Eigenvalues of A_2 is $\begin{vmatrix}2-\lambda&1&1\\0&2-\lambda&1\\0&0&-1-\lambda\end{vmatrix}=0$

$\lambda=2,2,-1$

Eigenvalues of A_1 & A_2 are different, so they are not algebraically equivalent.

$$G_1(s)=C_1(sI-A_1)^{-1}B_1=[1 \quad -1 \quad 0]\begin{bmatrix}s-2&-1&-2\\0&s-2&-2\\0&0&s-1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$=[1 \quad -1 \quad 0]\frac{1}{(s-2)^2(s-1)}\begin{bmatrix}(s-2)(s-1)&0&0\\(s-1)&(s-2)(s-1)&0\\2&+2(s-2)&(s-2)^2\end{bmatrix}^T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$=\frac{1}{(s-2)^2\cancel{(s-1)}}[1 \quad -1]\begin{bmatrix}(s-2)\cancel{(s-1)}&0\\+\cancel{(s-1)}&(s-2)\cancel{(s-1)}\end{bmatrix}^T\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{(s-2)^2}[1 \quad -1]\begin{bmatrix}s-2&1\\0&s-2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$=\frac{1}{(s-2)^2}$$

$$G_2(s)=C_2(sI-A_2)^{-1}B_2=[1 \quad -1 \quad 0]\begin{bmatrix}s-2&-1&-1\\0&s-2&-1\\0&0&s+1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$=\frac{1}{(s-2)^2(s+1)}[1 \quad -1 \quad 0]\begin{bmatrix}(s-2)(s+1)&0&0\\(s+1)&(s-2)(s+1)&0\\?&?&(s-2)^2\end{bmatrix}^T\begin{bmatrix}1\\1\\0\end{bmatrix}=\frac{1}{(s-2)^2}[1 \quad -1]\begin{bmatrix}s-2&1\\0&s-2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{(s-2)^2}$$

$G_1(s)=G_2(s)$, so they are zero-state equ.

3. $\dot{x} = Ax + Bu \quad y = Cx, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 1] \quad u = -[f_1 \quad f_2]x + v$

a.

$$\dot{x} = Ax + Bu$$

$$= Ax + B(-[f_1 \; f_2]x + v)$$

$$= Ax - B[f_1 \; f_2]x + Bv$$

$$= (A - B[f_1 \; f_2])x + Bv$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \; f_2] \right) x + Bv$$

$$= \begin{bmatrix} 0 & 1 \\ -f_1 & -f_2 \end{bmatrix} x + Bv.$$

$$y = Cx = [1 \quad 1]\, x.$$

b.

$$G(s) = C\left(sI - \begin{bmatrix} 0 & 1 \\ -f_1 & -f_2 \end{bmatrix}\right)^{-1} B$$

$$= [1 \quad 1] \begin{bmatrix} s & -1 \\ f_1 & s+f_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 1] \begin{bmatrix} s+f_2 & 1 \\ -f_1 & s \end{bmatrix} \cdot \cancel{\frac{1}{s(s+f_2)}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s(s+f_2)+f_1}$$

$$= \frac{1}{s(s+f_2)+f_1} [s+f_2-f_1 \quad s+1] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s+1}{s(s+f_2)+f_1}$$

$$= \frac{s+1}{s^2+f_2 s+f_1}$$

c.

Target transfer function is

$$\frac{1}{s+1} = \frac{s+1}{(s+1)(s+1)} = \frac{s+1}{s^2+2s+1} := \frac{s+1}{s^2+f_2 s+f_1}$$

$$f_2 = 2 \quad f_1 = 1$$

d. (a)

$$s^2+f_2 s+f_1 := (s+2)(s+3) = s^2+5s+6$$

$$f_2 = 5, \; f_1 = 6$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \;\; 1] \quad D = 0$$

(b)

$$s^2+f_2 s+f_1 := (s+5)(s+6) = s^2+11s+30$$

$$f_2 = 11, \; f_1 = 30$$

$$A = \begin{bmatrix} 0 & 1 \\ -30 & -11 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \;\; 1] \quad D = 0.$$

e. (a) since we know the poles of transfer function, they are the eigenvalue of A. So

$$(A - \lambda I)v = 0$$

$\lambda = -3$:

$$\begin{bmatrix} +3 & 1 \\ -6 & -5+3 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = -2$:

$$\begin{bmatrix} 2 & 1 \\ -6 & -5+2 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} e^{-3t} & \\ & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3t} & e^{-2t} \\ -3e^{-3t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-3t}+3e^{-2t}, & e^{-3t}+e^{-2t} \\ +6e^{-3t}-6e^{-2t}, & -3e^{-3t}-2e^{-2t} \end{bmatrix}$$

(b) $(A-\lambda I)v=0$

$\lambda=-6$

$$\begin{bmatrix} 6 & \phi \\ -\cancel{\phi}30 & -1\cancel{1}+6 \end{bmatrix} v=0 \qquad v=\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$\lambda=-5$

$$\begin{bmatrix} 5 & \phi \\ -\cancel{\phi}30 & -1\cancel{1}+5 \end{bmatrix} v=0 \qquad v=\begin{bmatrix} \phi \\ -5 \end{bmatrix}$$

$$A=\begin{bmatrix} 1 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -6 & -5 \end{bmatrix}^{-1}$$

$$e^{At}=\begin{bmatrix} 1 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} e^{-6t} & 0 \\ 0 & e^{-5t} \end{bmatrix}\begin{bmatrix} -5 & 1 \\ +6 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} e^{-6t} & e^{-5t} \\ -6e^{-6t} & -5e^{-5t} \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 6 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -5e^{-6t}+6e^{-5t}, & e^{-6t}+e^{-5t} \\ +30e^{-6t}-30e^{-5t}, & -6e^{-6t}+5e^{-5t} \end{bmatrix}$$

4. e^{At}, $A=\begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix}$

a: eigenvalue.

$$\begin{vmatrix} 2-\lambda & 0 \\ -3 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda)=0$$

$$\lambda=2 \text{ or } \lambda=5$$

• $(A-\lambda I)v=0$

For $\lambda=2$

$$\left(\begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)v=0$$

$$\begin{bmatrix} 0 & 0 \\ -3 & 3 \end{bmatrix}v=0 \quad v=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda=5$

$$\left(\begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)v=0$$

$$\begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix}v=0 \quad v \text{ can be } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$

$$=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$e^{At}=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} e^{2t} & 0 \\ e^{2t} & e^{5t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} e^{2t} & 0 \\ e^{2t}-e^{5t} & e^{5t} \end{bmatrix}$$

b: Laplace. Transform.

$$e^{At}=\mathcal{L}^{-1}\left[sI-A\right]^{-1}$$

$$=\mathcal{L}^{-1}\begin{bmatrix} s-2 & 0 \\ 3 & s-5 \end{bmatrix}^{-1}$$

$$=\mathcal{L}^{-1}\frac{1}{(s-5)(s-2)}\begin{bmatrix} s-5 & 0 \\ -3 & s-2 \end{bmatrix}$$

$$=\mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-2} & 0 \\ \frac{-3}{(s-5)(s-2)} & \frac{1}{s-5} \end{bmatrix}$$

$$=\mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-2} & 0 \\ \frac{1}{s-2}-\frac{1}{s-5} & \frac{1}{s-5} \end{bmatrix}$$

$$=\begin{bmatrix} e^{2t} & 0 \\ e^{2t}-e^{5t} & e^{5t} \end{bmatrix}$$

Question 3, Part B

Matlab Code

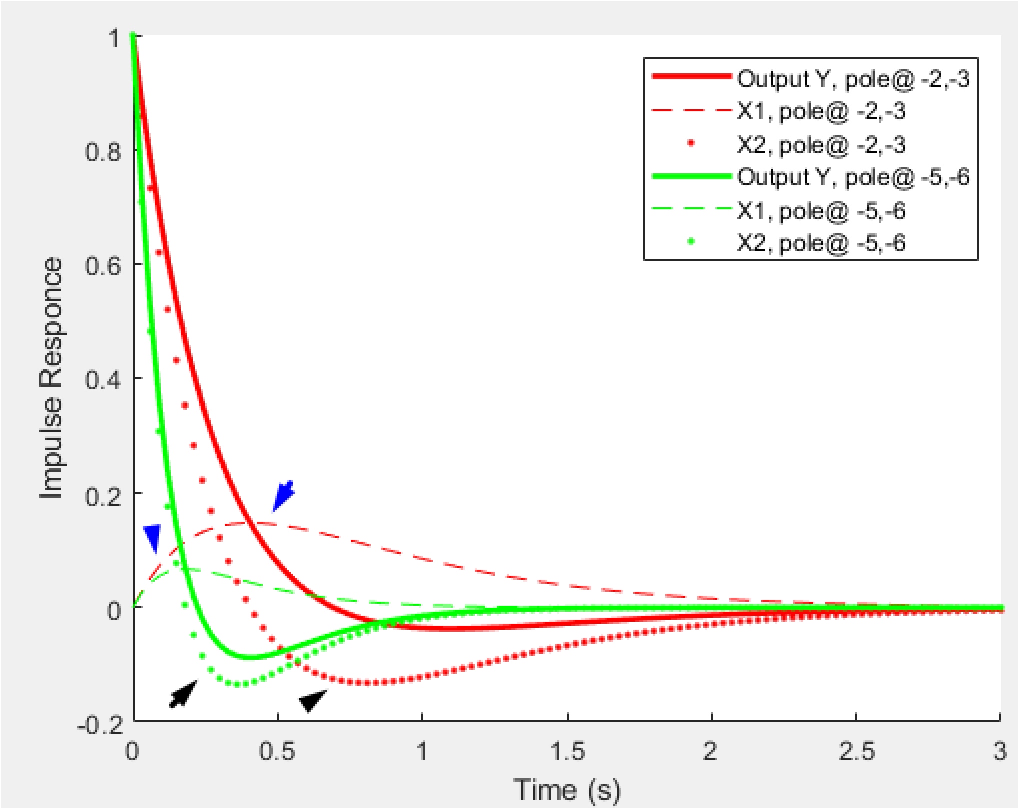
```
A1=[0,1;-6,-5];
A2=[0,1;-30,-11];
B=[0;1];
C=[1,1];
D=[0];
s1=ss(A1,B,C,D);
s2=ss(A2,B,C,D);

[y1,tOut,x1] = impulse(s1,3);
[y2,tOut,x2] = impulse(s2,tOut);
figure(1), clf, hold on

plot(tOut,y1,"r","linewidth",2)
plot(tOut,x1(:,1),"r--")
plot(tOut,x1(:,2),"r.")
plot(tOut,y2,"g","linewidth",2)
plot(tOut,x2(:,1),"g--")
plot(tOut,x2(:,2),"g.")

xlabel("Time (s)")
ylabel("Impulse Responce")

legend("Output Y, pole@ -2,-3","X1, pole@ -2,-3","X2, pole@ -2,-3","Output Y,
pole@ -5,-6","X1, pole@ -5,-6","X2, pole@ -5,-6")
```



The rapid decade can be found in the green X2 where the decade time constant is larger, when compared with the model with smaller pole values, indicated by dark arrows. As a result, the X1 got a larger integrated value in the slow responding model (blue arrows). The overall result is a more lowpass response to the impulse input when the poles have large values.