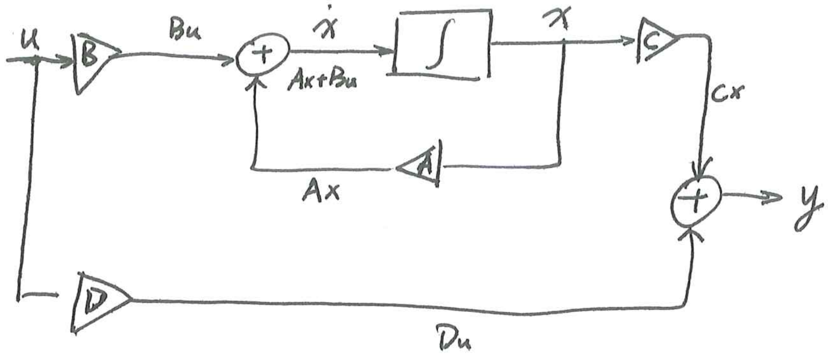
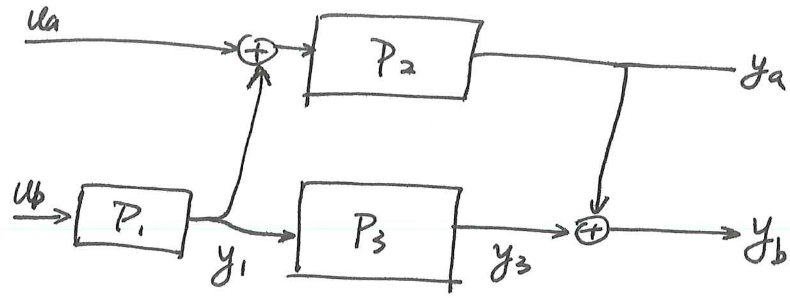
1.



$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

2.



$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_b \\ y_1 = C_1 x_1 + D_1 u_b \end{cases}$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + B_2 (u_a + y_1) \\ y_a = C_2 x_2 + D_2 (u_a + y_1) \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 (u_a + C_1 x_1 + D_1 u_b) \\ y_a = C_2 x_2 + D_2 (u_a + C_1 x_1 + D_1 u_b) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + B_2 C_1 x_1 + B_2 u_a + B_2 D_1 u_b \\ y_a = C_2 x_2 + D_2 C_1 x_1 + D_2 u_a + D_2 D_1 u_b. \end{cases}$$

$$\begin{cases} \dot{x}_3 = A_3 x_3 + B_3 y_1 \\ y_3 = C_3 x_3 + D_3 (y_1) \end{cases}$$

$$y_b = y_a + y_3$$

$$\begin{cases} \dot{x}_3 = A_3 x_3 + B_3 y_1 \\ y_3 = C_3 x_3 + D_3 y_1 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_3 = A_3 x_3 + B_3 (C_1 x_1 + D_1 u_b) \\ y_3 = C_3 x_3 + D_3 (C_1 x_1 + D_1 u_b) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_3 = A_3 x_3 + B_3 C_1 x_1 + B_3 D_1 u_b \\ y_3 = C_3 x_3 + D_3 C_1 x_1 + D_3 D_1 u_b \end{cases}$$

$$\& \; y_b = y_a + y_3 = C_3 x_3 + D_3 C_1 x_1 + D_3 D_1 u_b + C_2 x_2 + D_2 C_1 x_1 + D_2 u_a + D_2 D_1 u_b$$

$$= (D_3 C_1 + D_2 C_1) x_1 + C_2 x_2 + C_3 x_3 + (D_3 D_1 + D_2 D_1) u_b + D_2 u_a$$

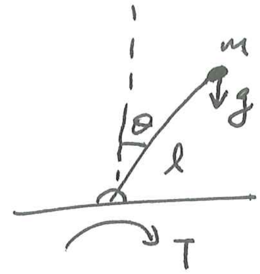
So to summarize:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ B_2 C_1 & A_2 & 0 \\ B_3 C_1 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & B_1 \\ B_2 & B_2 D_1 \\ 0 & B_3 D_1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix}$$

$$\begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} D_2 C_1 & C_2 & 0 \\ D_3 C_1 + D_2 C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} D_2 & D_2 D_1 \\ D_2 & D_3 D_1 + D_2 D_1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix}$$

3.

$$ml^2\ddot{\theta} = mgl\sin\theta - b\dot{\theta} + T.$$



System can be described as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$u = T$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin\theta - \frac{b}{ml^2}\dot{\theta} + \frac{1}{ml^2}T \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}_{g(x)} u.$$

$$y = \theta = \underbrace{x_1}_{h(x)}.$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial y}{\partial x}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ \cdots \end{bmatrix}$$

$$= x_2$$

$$\frac{d^2y}{dt^2} = \frac{\partial x_2}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial x_2}{\partial x}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

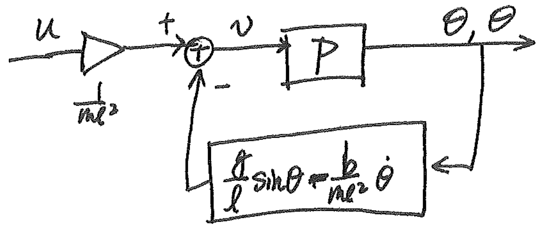
$$= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2} \end{bmatrix}$$

$$= \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2}$$

We define $v = \frac{d^2y}{dt^2} = \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2}$, $u = ml^2 v - mlg\sin x_1 + bx_2$

and system now is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

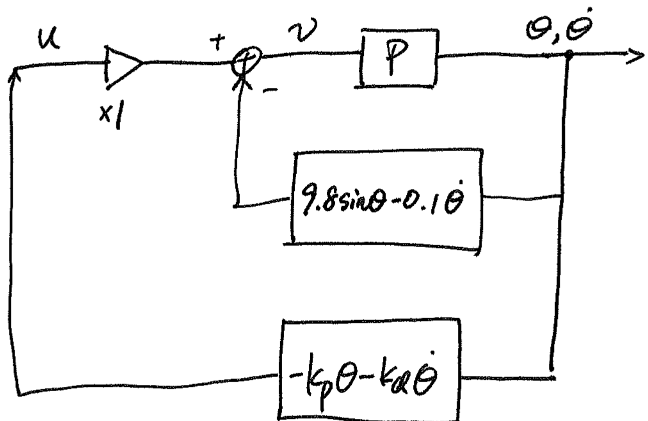


define the feedback control

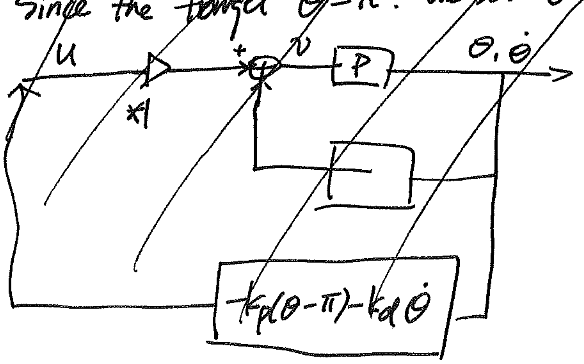
$$u = -\begin{bmatrix} k_p & k_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -k_p\theta - k_d\dot{\theta}$$

The system would be like



~~Since the target $\theta = \pi$. we set $\theta - \pi$ as the difference~~



Sorry $\theta = \pi$ is init condition.
$\theta = 0$ is the target