Arithmetization of Boolean Gates

George Fountis

September 10, 2023

1 Arithmetic

1.1 NOT

$$\mathbf{0} \cdot -1 + 1 = \mathbf{1}$$
$$\mathbf{1} \cdot -1 + 1 = \mathbf{0}$$

1.2 AND

$$0 \cdot 0 = 0$$
$$0 \cdot 1 = 0$$
$$1 \cdot 0 = 0$$
$$1 \cdot 1 = 1$$

1.3 NAND

$$(\mathbf{0} \cdot \mathbf{0}) \cdot -1 + 1 = \mathbf{1}$$
$$(\mathbf{0} \cdot \mathbf{1}) \cdot -1 + 1 = \mathbf{1}$$
$$(\mathbf{1} \cdot \mathbf{0}) \cdot -1 + 1 = \mathbf{1}$$
$$(\mathbf{1} \cdot \mathbf{1}) \cdot -1 + 1 = \mathbf{0}$$

1.4 OR

$$((\mathbf{0} \cdot -1 + 1) \cdot (\mathbf{0} \cdot -1 + 1)) \cdot -1 + 1 = \mathbf{0}$$
$$((\mathbf{0} \cdot -1 + 1) \cdot (\mathbf{1} \cdot -1 + 1)) \cdot -1 + 1 = \mathbf{1}$$
$$((\mathbf{1} \cdot -1 + 1) \cdot (\mathbf{0} \cdot -1 + 1)) \cdot -1 + 1 = \mathbf{1}$$
$$((\mathbf{1} \cdot -1 + 1) \cdot (\mathbf{1} \cdot -1 + 1)) \cdot -1 + 1 = \mathbf{1}$$

2 Multi-bit gates as vectors

$$a = (0, 0, 0, 0, 0, 0, 0, 0)$$
$$b = (1, 1, 1, 1, 1, 1, 1, 1)$$

$$k = (-1, -1, -1, -1, -1, -1, -1, -1)$$
$$v = (1, 1, 1, 1, 1, 1, 1, 1)$$

2.1 NOT

$$(a \cdot k) + v$$

2.2 AND

$$a \cdot b$$

2.3 OR

$$((a \cdot k + v) \cdot (b \cdot k + v)) \cdot k + v$$

unchecked

3 Multi-bit gates as matrices

3.1 8-bit NOT Gate

Invert the bitstring 01000001 which is the decimal value 65, corresponding to the ASCII value of 'A'.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3.2 Scalar product representation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2}$$

$$k = -1 \tag{3}$$

$$kA + P \tag{4}$$