Stable Matchings and Kidney Exchange
AI5040 Project Presentation

Chintalapudi Abhiroop[1] Nelakuditi Rahul Naga[2]

[1]AI20BTECH11005 [2]AI20BTECH11029

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Introduction

- In this presentation, we will talk about some problems related to stable matchings.
- Namely Marriage Problem and Kidney exchange.
- And their related algorithms - Gale Shapley algorithm and Top Trading Cycles (TTC) Algorithm.

Marriage Problem

In this problem we need to find suitable partners for some men and woman. Both men(m) and women(w) compile their preferences of other partner from best to worst and submit it to us. Using these lists we should find a set of matchings that are divorce-proof.
But first let us talk a little about the terminology we use in this problem:

- **Match:** A *match* is a set of pairs of the form (m,w), (m, m), or (w,w) such that each person has exactly one partner.
- **Unmatched:** Person i is *unmatched* if the match includes (i, i).
- **Acceptable:** i is *acceptable* to j if j prefers i to being unmatched.
- **Blocking:** A pair (m,w) is *blocking* if both prefer each other to the person they're matched with.
- **Unstable:** A match is *unstable* if someone has an unacceptable partner or if there is a blocking pair. Otherwise, it is **stable**.
- **Man-optimal and Woman-optimal:** A match is man-optimal if it is stable and there is no other stable match that some man prefers. Woman-optimal analogously.

Marriage Problem

Now lets talk about the algorithm we used Gale Shapley Algorithm. The main idea of this algorithms to find matches that are all free of instability:

- Each man proposes to the highest ranked woman on his list.
- The women with more than one offer hold onto her most preferred acceptable proposer and rejects all others.
- Each man then rejects the rejecting woman from his list and once again proposes to his highest ranked woman.
- And keep repeating the process until there are no new rejections.
- Once stopping the leftover matching is the most optimal matching with no blocking pairs or unstable pairs.

Marriage Problem

Points to note about this algorithm.

- This algorithm always ends with a stable match.
  - By construction, no person is matched to an unacceptable candidate.
  - No (m,w) can be a blocking pair: if m strictly prefers w to his current match, he must have proposed to her and been rejected in favor of a candidate that w liked better. That is, w finds her match better than m
- The resulting match is man-optimal. But woman-pessimal.
- Strategy proof for men - Men have no incentive to lie, but women does.

Kidney Exchange Problem

Nowadays many people suffer from kidney failure and need a kidney transplant. They can get their get their kidneys from a living person or a dead person donating their organ. A living person has two kidneys but can just survive with one kidney. But willing to donate a kidney isn't enough. Blood and tissue are two primary culprits for incompatibility.

- A a patient with O blood type can only receive a kidney from a donor with the same blood type, and similarly an AB donor can only donate to an AB patient.
- Suppose patient P1 is incompatible with its donor D1 because they have blood types A and B, respectively. Suppose P2 and D2 are in the opposite boat, with blood types B and A, respectively. Even though (P1,D1) may never have met (P2,D2), exchanging donors seems like a pretty good idea — P1 can get its kidney from D2 and P2 from D1. This is called a **Kidney Exchange**.

Kidney Exchange Problem

There are different possible Kidney Exchange methods :

1. **Paired Exchange :** This involves two patient-donor couples, for each of whom a transplant from donor to intended recipient is infeasible, but such that the patient in each couple could feasibly receive a transplant from the donor in the other couple. This pair of couples can then exchange donated kidneys.
2. **Indirect Exchange :** This involves an exchange between one incompatible patient-donor couple, and the cadaver queue. The patient in the couple receives high priority on the cadaver queue, in return for the donation of his donor's kidney to someone on the queue.

Kidney Exchange Problem

TABLE I
U. S. Kidney Transplants

Year	Cadaveric donors	Cadaveric transplants	Live donors	All wait-list patients	New wait-list additions
1992	4,276	7,202	2,535	22,063	15,224
1993	4,609	7,509	2,851	24,765	16,090
1994	4,797	7,638	3,009	27,258	16,538
1995	5,003	7,690	3,377	30,590	17,903
1996	5,038	7,726	3,649	34,000	18,328
1997	5,083	7,769	3,912	37,438	19,067
1998	5,339	8,017	4,361	40,931	20,191
1999	5,386	8,023	4,552	43,867	20,986
2000	5,490	8,089	5,324	47,596	22,269
2001	5,528	8,202	5,924	51,144	22,349
2002	5,630	8,534	6,233	54,844	23,494

The data for years 1992–2001 are constructed from the annual report of UNOS/OPTN, the data for 2002 are constructed from the national database of UNOS/OPTN. National database numbers are slightly higher than the annual report numbers due to continuous updating regarding previous years. Number of registrations may have multiple counts of patients since one patient may have registered in multiple centers for the wait-list.

- From the table we can see that there is a serious shortage of kidneys compared to demand. So they needed to design an algorithm to alleviate this shortage. and improve patient welfare.
- The *goal* of this exchange method is to *thicken the kidney exchange market to enable as many matches as possible*.

Kidney Exchange Problem

Now, we shall discuss one of the early algorithm used to do this kidney matchings on a national level.

Top Trading Cycles(TTC) Algorithm:

- The main idea of this algorithm is to draw a directed graph made of patients and donors and find cycles in them.
- Based on the priority of the patient we can then remove everyone in the cycle from the graph and correspondingly match them so that everyone in the cycle has their own kidney donor.
- Draw the graph again and find the cycles-repeat the above process.
- We can further modify this algorithm to include the cases of Indirect kidney exchange.
- We can also modify it to get chains if there are no more cycles.

Kidney Exchange Problem

There are both advantages and disadvantages to using this method.
The advantages are:

- The TTC outcome is unique core allocation.
- The TTC algorithm is *Dominant Strategy Incentive Compatible*, i.e; no patient or his doctor can misreport information to increase the estimated probability of a successful transplant.

The disadvantages are:

- The cycles along which re-allocations are made can be arbitrarily long. Due to the constraint on the number of surgeons and OR, it is almost impossible to simultaneously perform the surgeries.
- Modeling preferences as a total ordering over the set of living donors is overkill: empirically, patients don't really care which kidney they get as long as it is compatible with them. Binary preferences over donors are therefore more appropriate.

Algorithmic Analysis

Consider the marriage problem where each man and woman has a list of people (in the order of preference) of the opposite sex that he or she would accept as a marriage partner. The lists are submitted to a matchmaker who finds a suitable pairing of the participants. A stable matching μ is one where there is no man m and woman w who are not matched with each other but prefer each other to their mates.

Theorem (Gale and Shapley)

In a standard marriage matching problem described above, for any sets M and W of men and women and any preference pattern submitted by men and women, there is always ***at least one*** *stable matching.*

The proof for the above theorem is **constructive**, thus giving an algorithm for finding the desired stable matching.

Algorithmic Analysis

A preference pattern for a matching problem consists of a triple $(M, W; P)$ where M and W are the men and women and P represents their preferences. A matching μ, is a bijection of $M \cup W$ onto itself if we assume that unmatched persons are self-matched.

Lemma

Suppose $W \subset W'$ and μ is a stable matching for $(M, W; P)$ and μ' for $(M, W'; P')$ where P' agrees with P on W. Let M_μ be all men who prefer μ to μ' and let $W_{\mu'}$ be all women who prefer $\mu^{'}$ to μ. Then μ and μ' are bijections between M_μ and $W_{\mu'}$.

From the above lemma we can show that if μ and μ' are stable matchings for (M, W) then people who are self-matched are the same for both.

Algorithmic Analysis

- If μ and μ' are matchings, we write $\mu \succsim \mu'$ if every man likes his μ-mate at least as well as his μ'-mate This set of partial ordering of stable matchings forms a **lattice**.
- Further, if all men prefer μ to μ', then all women prefer μ' to μ. This can be written mathematically as follows :

$$\mu \underset{M}{\succsim} \mu' \implies \mu' \underset{W}{\succsim} \mu$$

Corollary

Since every finite lattice has a largest and a smallest element, it follows that among all stable matchings there is one which is preferred to all others by the men and another by the women. These will be called the $M(W)$ optimal matchings. The algorithms proposed by Gale and Shapley always arrive at one of these extreme matchings.

Matching Game

Dubins and Freedman considered the question that whether it is possible for an individual or group of individuals to obtain a preferred mate by falsifying their preference lists. This model is known as the **Matching Game**. It was shown by them that :

- No man or coalition of men can ever be made better off by falsifying their true preferences.
- On the other hand, it will almost always pay for some of the women to be **Machiavellian**, that is a woman can do better by falsifying.

The conclusion is that Machiavellian behavior on the part of the men is not profitable and they can do no better than to list their true preferences, regardless of what the women do. Hence, we will assume from here on that the men always submit their true preference list.

Matching Game

Theorem

If there is more than one stable matching, then there is at least one woman who will be better off by falsifying, assuming the others tell the truth.

The above theorem implies that the policy of honest revelation of preference is **unstable** for the women. We will now investigate the set of strategies for each player than form an **equilibrium point**.

Theorem

Let μ be any stable matching for $(M, W; P)$ and suppose each woman in $\mu(M)$ chooses the strategy of listing only $\mu(m)$ on her preference list. This is an equilibrium point.

Matching Game

Theorem (Roth)

Suppose the women choose any set of strategies P'_w (preference lists) that form an equilibrium point for the matching game. Then the corresponding M-optimal matching for $(M, W; P')$ is one of the stable matchings of $(M, W; P)$.

The strategies are said to form a **strong equilibrium point** if no subset of players by changing their strategies can achieve a better payoff for all of its members.

Theorem

Let each woman w submit a preference list in true order of preference but removing all men who are ranked below $\mu_W(w)$. These preferences P' are a strong equilibrium point.

Matching Game

The Dubins-Freedman Theorem shows that for each man revealing true preferences is a **dominant strategy**. For the women there are no dominant strategies except for the special case where $|W| = 1$. If there is only one woman w, then revealing true preferences strictly dominates any other strategy.

Theorem

Any strategy P'_w in which w does not list her true first choice at the head of her list is strictly dominated.

Theorem

Let P'_w be any strategy for w in which w's favourite man is listed first in her list and P'_w contains only men m who are on w's true preference list P_w. Then P'_w is not dominated.

Kidney Exchange

We will now consider algorithmic analysis of an important application of Stable Matchings - **Kidney Exchange** :

- Formally, a (static) kidney exchange problem consists of a set of donor-recipient pairs $\{(k_1, t_1), \ldots, (k_n, t_n)\}$, a set of compatible kidneys $K_i \subset K = [k_1, \ldots, k_n]$ for each patient t_i, and a strict preference relation P_i over $K_i \cup \{k_i, w\}$ for each patient t_i. Here, w denotes the option of entering the waiting list with priority reflecting the donation of his donor's kidney k_i.
- A **kidney exchange mechanism** selects a matching for each kidney exchange problem.

To solve the Kidney Exchange problem, Gale's **TTC** (Top Trading Cycles) algorithm can be used if there are no indirect kidney exchanges involved. But to handle indirect exchanges, we use the **TTCC** (Top Trading Cycles and Chains) mechanism, a generalization of the TTC mechanism.

Kidney Exchange

- In a kidney exchange problem, a matching is **Pareto efficient** if there is no other matching that is weakly preferred by all patients and donors and strictly preferred by at least one patient-donor pair.
- A kidney exchange mechanism is efficient if it always selects a Pareto efficient matching among the participants present at any given time.

Theorem

Consider a chain selection rule such that any w-chain selected at a non-terminal round remains in the procedure, and thus the kidney at its tail remains available for the next round. The TTCC mechanism, implemented with any such chain selection rule, is efficient.

A w-chain is an ordered list of kidneys and patients $(k_1', t_1', \ldots, k_m', t_m')$ such that kidney k_1' points to patient t_1', patient t_1' points to kidney k_2',..., kidney k_m' points to patient t_m' and patient t_m' points to w.

Important Takeaways

- The organization of a kidney exchange faces some of the most stringent constraints, rising from social/legal/ethical concerns, as well as from the practical requirements of kidney transplantation and patient care.
- Substantial gains in the number and match quality of transplanted kidneys might result from adoption of the TTCC mechanism.
- Compared with simple paired and indirect exchanges, the wider exchange implemented by the TTCC mechanism creates additional welfare gains in several ways.
- Increasing kidney exchange among willing donor-recipient pairs offers a way to benefit those pairs who are not well matched. Also, increasing live organ donation reduces competition for cadaver organs which benefits patients who do not have live donors.

References

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