2. (4 points) Transform formulas

What is the formula for the Laplace transform of a signal $f(t)$?

$$\int_0^\infty f(t)e^{-st}\,dt$$

3. (15 pts) A system is described by the differential equation

$$(D^2 - D - 12)y(t) = (D+2)f(t)$$

and has initial conditions $y_0(0) = 1, \dot{y}_0(0) = 3$. Using **time-domain** methods:

(a) (4 pts) Determine the zero-input response of the system.

$D^2 - D - 12 = 0$

$(D-4)(D+3) = 0$ roots: $-3, 4$

$y(t) = Ae^{-3t} + Be^{4t}$

$y(0) = Ae^{-3t} + Be^{at} = 1$

$y'(0) = -3Ae^{-3t} + 4Be^{at} = 3$

$A + B = 1$

$-3A + 4B = 3$

$A = 1/7$; $B = 6/7$

$$\boxed{y(t) = \left[\frac{1}{7}e^{-3t} + \frac{6}{7}e^{4t}\right]u(t)}$$

(b) (4 pts) Given that the impulse response of the system is

$$h(t) = [\frac{1}{7}e^{-3t} + \frac{6}{7}e^{4t}]u(t),$$

Determine the zero-state response of the system if the input is $f(t) = 3e^{-5t}u(t)$.

$H(s) = \dfrac{s+2}{(s-4)(s+3)}$ $F(s) = \dfrac{3}{s+5}$

$H(s) \cdot F(s) = \dfrac{3(s+2)}{(s-4)(s+3)(s+5)} \Rightarrow \dfrac{A}{s-4} + \dfrac{B}{s+3} + \dfrac{C}{s+5} \Rightarrow$

$A = \frac{2}{7}$

$B = 3/14$

$C = -1/2$

$$\frac{2/7}{s-4} + \frac{3/14}{s+3} - \frac{1/2}{s+5} \Rightarrow \boxed{\left[\frac{2}{7} e^{4t} + \frac{3}{14} e^{-3t} - \frac{1}{2} e^{-5t}\right] u(t)}$$

(c) (2 pts) Determine the total response of the system.

$$R_{tot} = ZSR + ZIR = \left[\frac{1}{7} e^{-3t} + \frac{6}{7} e^{4t}\right] u(t) + \left[\frac{2}{7} e^{4t} + \frac{3}{14} e^{-3t} - \frac{1}{2} e^{-5t}\right] u(t)$$

$$= \boxed{\left[\frac{8}{7} e^{4t} + \frac{5}{14} e^{-3t} - \frac{1}{2} e^{-5t}\right] u(t)}$$

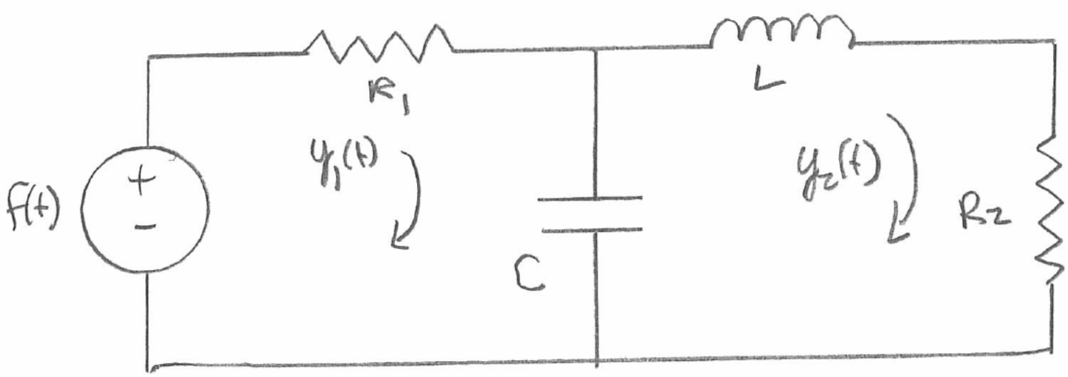
(d) (1 pt) Is the system linear? yes no. (circle the correct answer)

(f) (1 pt) Is the system dynamic? yes no.

(g) (1 pt) Is the system time-invariant? yes no.

(h) (1 pt) Is the system continuous-time? yes no.

4. (8 pts) For the circuit below:



(a) (3 pts) Write the **integro-differential** equations (equations with differentials and integrals) for two loops (one equation for each loop), where the elements $f(t)$, R_1, and C are in the first loop, and L, R_2, and C are in the second loop. Use loop currents $y_1(t)$ and $y_2(t)$ as shown.

$$f(t) = y_1(t) \cdot R_1 + \frac{1}{C}\int_{t_0}^{t} y_1(t)\,d\tau - \frac{1}{C}\int_{t_0}^{t} y_2(t)\,d\tau$$

$$0 = y_2(t) \cdot R_2 + \frac{1}{C}\int_{t_0}^{t} y_2(t)\,d\tau - \frac{1}{C}\int_{t_0}^{t} y_1(t)\,d\tau + L\frac{dy_2(t)}{dt}$$

(b) (3 pts) Express these two equations as a pair of **differential** equations. (Use the D operator in your equations.)

$$\left(R_1 D + \frac{1}{C}\right) y_1(t) = \frac{1}{C} y_2(t) + f(t)$$

$$\frac{1}{C} y_1(t) = \left(L D^2 + R_2 D + \frac{1}{C}\right) y_2(t)$$

(c) (2 pts) What initial conditions would you **measure** in the circuit to find the zero-input solution? (Do **not** compute initial conditions for the solution.)

$$V_C(0^-) \;\&\; I_L(0^-)$$

5. (6 pts) Convolution

(a) (3 pts) The unit impulse response of an LTI system is $h(t) = (1 - 5t)e^{-4t}u(t)$. Find this system's (zero-state) response $y(t)$ if the input is $f(t) = u(t)$.

$$H(s) = \frac{Y(s)}{X(s)} \qquad \mathcal{L}[h(t)] = H(s) = \frac{s-1}{(s+4)^2}$$

$$\mathcal{L}[f(t)] = X(s) = \frac{1}{s}$$

$$ZSR = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s-1}{s(s+4)^2}\right] = \boxed{\frac{1}{16}e^{-4t} + \frac{5}{4}te^{-4t} - \frac{1}{16}}$$

$$\frac{A}{s+4} + \frac{B}{s+4^2} + \frac{C}{s} \qquad \begin{aligned} A &= 1/16 \\ B &= 5/4 \\ C &= -1/16 \end{aligned}$$

(b) (3 pts) The unit impulse response of an LTI system is $h(t) = e^{-3t}u(t)$. Find this system's (zero-state) response $y(t)$ if the input is $f(t) = \cos(2t)u(t)$.

$$\left[\frac{1}{s+3}\right]\left[\frac{s}{(s^2+4)}\right] = \frac{s}{(s+3)(s^2+4)}$$

$$= \frac{A}{s+3} + \frac{B}{s^2+4}$$

$A = -3/13$

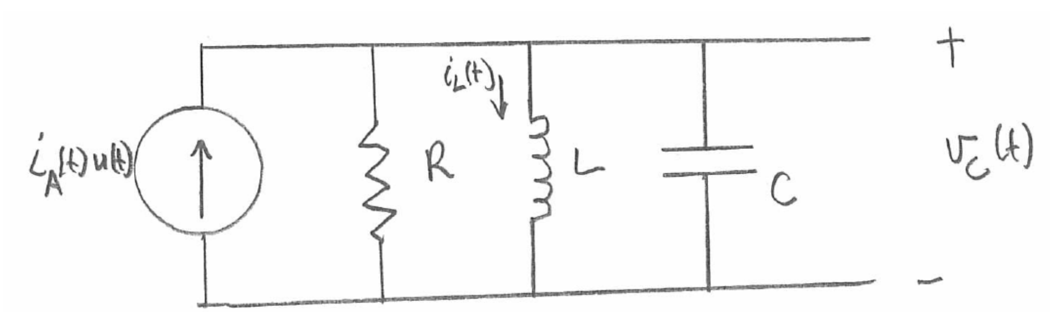
$B = \frac{3}{13}s + \frac{4}{13}$

$$\mathcal{L}^{-1}\left[\frac{\frac{3}{13}s+\frac{4}{13}}{s^2+4} - \frac{3/13}{s+3}\right] =$$

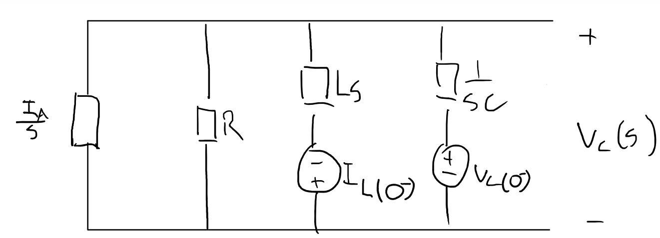
$$\boxed{\left[\frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t) - \frac{3}{13}e^{-3t}\right]u(t)}$$

6. Laplace domain circuit analysis (5 pts.)

For the following circuit, $v_C(0^-) = 20$ and $i_L(0^-) = 5$.



(a) (2 pts) Transform the circuit into the s (Laplace) domain (including initial conditions).



(b) (3 pts) Solve the system for $V_C(s)$. Leave the solution as a function of L, R, and C. (Hint: use admittances.)

$$0 = \frac{V_c}{\frac{I_A}{s}} + \frac{V_c}{R} + \frac{V_c + I_L(0^-)}{Ls} + \frac{V_c - V_c(0^-)}{1/sC}$$

$$0 = \frac{V_C s}{I_A} + \frac{V_C}{R} + \frac{V_C + I_L(0)}{Ls} + (V_C - V_C(0))sC$$

$$0 = \frac{I_A}{V_C s} + \frac{R}{V_C} + \frac{Ls}{V_C} + \frac{Ls}{I_L(0)} + \frac{1}{V_C sC} - \frac{1}{V_C(0) sC}$$

$$0 = \frac{V_C s}{I_A} + \frac{V_C}{R} + \frac{V_C}{Ls} + \frac{I_L(0)}{Ls} + V_C sC - V_C(0) sC$$

$$0 = \left[\frac{1}{I_A} + \frac{1}{R} + \frac{1}{Ls} + sC\right] V_C + \frac{I_L(0)}{Ls} - V_C(0) sC$$

$$-\frac{I_L(0)}{Ls} + V_C(0) sC = \left[\frac{1}{I_A} + \frac{1}{R} + \frac{1}{Ls} + sC\right] V_C$$

$$\frac{-\frac{I_L(0)}{Ls} + V_C(0) sC}{\left[\frac{1}{I_A} + \frac{1}{R} + \frac{1}{Ls} + sC\right]} = V_C$$

$$V_C = \left(-\frac{I_L(0)}{Ls} + V_C(0) sC\right)\left[I_A + R + Ls + \frac{1}{sC}\right]$$

7. Finding the output of a circuit with Laplace transforms (10 pts.)

Given the following equation from circuit analysis,

$$s^2 Y(s) + 8sY(s) + 116Y(s) - sy(0^-) - \dot{y}(0^-) - 10y(0^-) = -20(s + 29)F(s),$$

with $y(0^-) = 2$, $\dot{y}(0^-) = 10.3607$, and $f(t) = u(t)$:

(a) (2 pts) Find the transfer function of the circuit.

$$\frac{-20s - 580}{s^2 + 8s + 116}$$

(b) (3 pts) Find the zero-input solution part of $y(t)$ using Laplace transforms. Express your answer in terms of real functions.

$$s^2 Y + 8sY + 116Y - 2s - 10.3607 - 20 = 0$$

$$s^2 Y + 8Y + 116Y = 2s + 30.3607$$

$Y + 8Y + 116Y = 2s + 30.3607$

$$Y(s) = \frac{2s + 30.3607}{s^2 + 8s + 116} \Rightarrow e^{-4t}\left[2\cos(10t) + 2.2\sin(10t)\right]u(t)$$

(c) (3 pts) Find the zero-state solution part of $y(t)$ using Laplace transforms. Express your answer in terms of real functions. (Hint: math helps.)

$$\frac{-20s - 580}{s^2 + 8s + 116} \cdot \frac{1}{s} = \frac{-20s - 580}{s^3 + 8s^2 + 116s}$$

(d) (2 pts) Find $y(t)$.

$$\left[e^{-4t}\left[2\cos(10t) + 2.2\sin(10t)\right] + \qquad\qquad\right]u(t)$$