Time- domain vs. phasor representation

$$\underset{\text{real}}{V(t)} = V_0\cos(\omega t+\phi) = Re\{V_0 e^{j(\omega t+\phi)}\} = Re\{V_0 e^{j\phi} e^{j\omega t}\} = Re\{\underset{\text{complex}}{V} e^{j\omega t}\} \longrightarrow \underset{\text{time domain}}{V(t)} \to \underset{\text{phasor}}{V}$$

- If $U(t)$ and $V(t)$ can be represented by phasors U and V, then $U(t)+V(t)\to U+V$.
- $\partial V(t)/\partial t \to j\omega V$.
- $V(t)U(t) \not\to VU$.
- Phasor expression can only apply to addition, subtraction and time derivative of time harmonic quantities of the same frequency.
- $\boldsymbol{V}(t) = \hat{x}V_x\cos(\omega t+\phi_x) + \hat{y}V_y\cos(\omega t+\phi_y) + \hat{z}V_z\cos(\omega t+\phi_z)$

  $= Re\left\{\left[\hat{x}V_x e^{j\phi_x} + \hat{y}V_y e^{j\phi_y} + \hat{z}V_z e^{j\phi_z}\right]e^{j\omega t}\right\} \to V = \hat{x}V_x e^{j\phi_x} + \hat{y}V_y e^{j\phi_y} + \hat{z}V_z e^{j\phi_z}$
- $\langle \boldsymbol{A}(t)\times\boldsymbol{B}(t)\rangle = \frac{1}{2}Re\{\boldsymbol{A}\times\boldsymbol{B}^*\}$ where $\boldsymbol{A}(t) = \mathrm{Re}\{\boldsymbol{A}e^{j\omega t}\}$, $\boldsymbol{B}(t) = \mathrm{Re}\{\boldsymbol{B}e^{j\omega t}\}$, $\langle\boldsymbol{V}(t)\rangle = \frac{1}{T}\int_0^T dt\boldsymbol{V}(t)$.
- $\langle \boldsymbol{A}(t)\cdot\boldsymbol{B}(t)\rangle = \frac{1}{2}Re\{\boldsymbol{A}\cdot\boldsymbol{B}^*\}$

Ref [1]

Electromagnetic fields, sources, medium, and equations

Maxwell's equation

(1) $\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{D}} = \rho$

(2) $\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = 0$

(3) $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\,\partial \boldsymbol{\mathcal{B}}/\partial t \longrightarrow \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}) = -\,\partial(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}})/\partial t = 0$

(4) $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \partial \boldsymbol{\mathcal{D}}/\partial t \longrightarrow \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}}) = \boldsymbol{\nabla} \cdot \boldsymbol{J} + \partial(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{D}})/\partial t$

Constitutive relations

$\boldsymbol{\mathcal{D}} = \epsilon \boldsymbol{\mathcal{E}}$

$\boldsymbol{\mathcal{B}} = \mu \boldsymbol{\mathcal{H}}$

ϵ: permittivity, [F/m]

μ: permeability, [H/m]

$\boldsymbol{\mathcal{E}}$: electric field, [V/m]

$\boldsymbol{\mathcal{H}}$: magnetic field, [A/m]

$\boldsymbol{\mathcal{D}}$: electric flux density, [C/m^2]

$\boldsymbol{\mathcal{B}}$: magnetic flux density, [Wb/m^2, T]

$\boldsymbol{\mathcal{J}}$: current density, [A/m^2]

$\boldsymbol{\rho}$: electric charge density, [C/m^3]

$0 = \nabla \cdot \boldsymbol{\mathcal{J}} + \partial \rho\, /\partial t \longleftarrow$ Conservation of electric charge

- If we consider charge conservation law, then (1) is derived from (4).
- (2) can be derived from (3).
- There are 6 scalar equations in Maxwell's equations.
- There are 6 scalar equations in constitutive relations.
- to solve for 12 components of $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{B}}$.

Ref [1]

Time-harmonic Maxwell's equations and wave equations in vacuum

$$\boldsymbol{\mathcal{E}}(x,y,z,t) = Re\{[\hat{x}E_x e^{j\phi_x} + \hat{y}E_y e^{j\phi_y} + \hat{z}E_z e^{j\phi_z}]e^{j\omega t}\} = Re\{\boldsymbol{E}(x,y,z)e^{j\omega t}\} \rightarrow \boldsymbol{E}(x,y,z)$$

(1) $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho$

(2) $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$

(3) $\boldsymbol{\nabla}\times\boldsymbol{E} = -j\omega\boldsymbol{B}$

(4) $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J} + j\omega\boldsymbol{D}$

$$\xrightarrow[\mu=\mu_0,\ \epsilon=\epsilon_0]{J=\rho=0}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = -j\omega\mu_0(\boldsymbol{\nabla}\times\boldsymbol{H})$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \boldsymbol{\nabla}^2\boldsymbol{E} \qquad j\omega\epsilon_0\boldsymbol{E}$$

$$\boldsymbol{\nabla}^2\boldsymbol{E} + \omega^2\epsilon_0\mu_0\boldsymbol{E} = 0$$

$$\boldsymbol{\nabla}^2\boldsymbol{H} + \omega^2\epsilon_0\mu_0\boldsymbol{H} = 0$$

Wave equation
Helmholtz equation

One solution:

Propagation constant, wave number

If $\boldsymbol{E}(x,y,z) = \hat{x}E_{\boldsymbol{x}}(z)$, then $\frac{d^2E_x}{dx^2} + \omega^2\epsilon_0\mu_0 E_x = 0 \longrightarrow E_x = E^+e^{-jkz} + E^-e^{jkz}$ $\quad \omega = k/\sqrt{\epsilon_0\mu_0}$

(3) $\longrightarrow H_y = \frac{1}{\eta_0}(E^+e^{-jkz} - E^-e^{jkz})$ $\quad \eta_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$

Intrinsic impedance

Ref [2]

Plane wave solutions

In time domain, the solutions $E_x = E^+ e^{-jkz} + E^- e^{+jkz}$ can be written as

$$\mathcal{E}_x = Re\{E_x e^{j\omega t}\} = E^+ \cos(\omega t - kz) + E^- \cos(\omega t + kz)$$

- At $z = 0, \mathcal{E}_x$ is periodic in time. Period $T = 2\pi/\omega$.
- At different time, $E^+ \cos(\omega t - kz)$ travels in +z direction with wavelength $\lambda = 2\pi/k$.
- At different time, $E^- \cos(\omega t + kz)$ travels in -z direction with wavelength $\lambda = 2\pi/k$.

Constant phase $\omega t - kz = C$

Phase velocity $v_p = dz/dt = \omega/k = 1/\sqrt{\epsilon\mu}$

Constant phase plane: xy plane with the same z

- The amplitude is a constant throughout the space.
- The constant phase surfaces are parallel planes.

Ref [2]

Standing waves

Superposing two waves of the same frequency travelling in opposite directions

$$E(z) = E^+ e^{-jkz} + E^- e^{+jkz} = (E^+ + E^-)\cos kz - j(E^+ - E^-)\sin kz = E_0 e^{-j\theta}$$

$$E_0 = \sqrt{(E^+)^2 + (E^-)^2 + 2E^+E^-\cos 2kz}$$

$$\tan\theta = \tan kz\left(\frac{E^+ - E^-}{E^+ + E^-}\right)$$

If $E^+ = E^-$, $E(z) = 2E^+\cos kz$

If $E^+ \neq E^-$, $E_{max} = E^+ + E^-$ at $z = m\pi/k$

$E_{min} = |E^+| - |E^-|$ at $z = \left(m + \frac{1}{2}\right)\pi/k$

General plane wave solutions

$$\nabla^2 \boldsymbol{E} + \omega^2 \epsilon_0 \mu_0 \boldsymbol{E} = 0$$

$$\frac{\partial^2 E_i}{dx^2} + \frac{\partial^2 E_i}{dx^2} + \frac{\partial^2 E_i}{dx^2} + k_0^2 E_i = 0$$

$$E_i(x,y,z) = f(x)g(y)h(z)$$

$$\boxed{\begin{gathered} \frac{d^2 f}{dx^2} + k_x^2 f = 0 \\ \frac{d^2 g}{dy^2} + k_y^2 g = 0 \\ \frac{d^2 h}{dz^2} + k_z^2 h = 0 \\ k_x^2 + k_y^2 + k_z^2 = k_0^2 \end{gathered}}$$

$$E_x(x,y,z) = Ae^{-j(k_x x + k_y y + k_z z)} = Ae^{-j\boldsymbol{k_1}\cdot\boldsymbol{r}}$$

$\boldsymbol{k_1} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ wave vector

$\boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}$ position vector

$$E_y(x,y,z) = Be^{-j\boldsymbol{k_2}\cdot\boldsymbol{r}}$$

$$E_z(x,y,z) = Ce^{-j\boldsymbol{k_3}\cdot\boldsymbol{r}}$$

$$\nabla \cdot \boldsymbol{E} = 0 \longrightarrow \boldsymbol{k_1} = \boldsymbol{k_2} = \boldsymbol{k_3} \longrightarrow \boldsymbol{E}(x,y,z) = \boldsymbol{E_0} e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \longrightarrow \boldsymbol{k}\cdot\boldsymbol{E} = 0$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H} \longrightarrow H = \frac{j}{\omega\mu_0}\nabla \times \left(\boldsymbol{E_0} e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\right) = \frac{-j}{\omega\mu_0}\boldsymbol{E_0} \times \nabla e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} = \frac{k_0}{\omega\mu_0}\hat{n} \times \boldsymbol{E_0} e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} = \frac{1}{\eta_0}\hat{n} \times \boldsymbol{E}$$

$$\boldsymbol{\mathcal{E}}(x,y,z,t) = Re\left\{\boldsymbol{E}(x,y,z)e^{j\omega t}\right\} = Re\left\{\boldsymbol{E_0} e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\, e^{j\omega t}\right\} = \boldsymbol{E_0} cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t).$$

Ref [2]

Plane waves in a dissipative media

Assuming the dissipative media is isotropic and ohmic: $\boldsymbol{J} = \boldsymbol{J_0} + \boldsymbol{J_c} = \boldsymbol{J_0} + \sigma \boldsymbol{E}$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + j\omega \boldsymbol{D} = \boldsymbol{J_0} + \sigma \boldsymbol{E} + j\epsilon\omega \boldsymbol{E} = j\omega \underbrace{\left(\epsilon - j\frac{\sigma}{\omega}\right)}_{\epsilon_c = \epsilon' - j\epsilon''} E + \boldsymbol{J_0}$$

complex permittivity

Wave equation in source free region $$\boldsymbol{\nabla}^2 \boldsymbol{E} + \underbrace{\omega^2 \epsilon\mu \left(1 - j\frac{\sigma}{\epsilon\omega}\right)}_{-\gamma^2} \boldsymbol{E} = 0$$

attenuation constant

$$\boxed{\gamma = j\omega\sqrt{\epsilon\mu}\sqrt{1 - j\frac{\sigma}{\epsilon\omega}} = \alpha + j\beta}$$

complex propagation constant

phase constant

If $\boldsymbol{E}$ is a plane wave only in $\hat{x}$ and uniform in x, y

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \Longrightarrow E_x = E^+ e^{-\gamma z} + E^- e^{+\gamma z}$$

$$\boldsymbol{\mathcal{E}}^+(x, y, z, t) = \hat{\boldsymbol{x}} E_0 e^{-\alpha z} cos(\omega t - \beta z). \Longrightarrow v_p = \omega/\beta \qquad \lambda = 2\pi/\beta$$

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = \frac{-j\gamma}{\omega\mu}(E^+ e^{-\gamma z} - E^- e^{+\gamma z}) \Longrightarrow \eta = \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} \Longrightarrow H_y = \frac{1}{\eta}(E^+ e^{-\gamma z} + E^- e^{\gamma z})$$

Ref [2]

Plane waves in lossy media

Non –ideal dielectric
(slightly conducting)

$$\frac{\sigma}{\epsilon\omega} = \frac{\epsilon''}{\epsilon'} \sim 10^{-2} \ll 1 \qquad \gamma = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\epsilon\mu} = \alpha + j\beta \Longrightarrow \beta = \omega\sqrt{\epsilon\mu} \qquad \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

- In a perfect dielectric, $\sigma = 0, \epsilon'' = 0,\ \alpha = 0$, no dissipation.

Good conductors
(Highly conducting)

$$\frac{\sigma}{\epsilon\omega} = \frac{\epsilon''}{\epsilon'} > 10^{-2} \qquad \gamma = (1+j)\sqrt{\frac{\mu\omega\sigma}{2}} = \alpha + j\beta$$

skin depth $\delta_s = 1/\alpha = \sqrt{2/\mu\omega\sigma}$

Intrinsic impedance $\eta = j\omega\mu/\gamma = \underbrace{(1+j)}\sqrt{\omega\mu/2\sigma}$

E and H has a phase difference of $\pi/4$ in a good conductor

Ref [3]

Poynting's Theorem in time domain

$$\boldsymbol{\mathcal{H}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}) = -\boldsymbol{\mathcal{H}} \cdot \partial \boldsymbol{\mathcal{B}}/\partial t \qquad (1)$$

$$\boldsymbol{\mathcal{E}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}}) = \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}} + \boldsymbol{\mathcal{E}} \cdot \partial \boldsymbol{\mathcal{D}}/\partial t \qquad (2)$$

$$\boxed{\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})}$$

Time domain Poynting vector

$$\boldsymbol{S}(\boldsymbol{r},\boldsymbol{t}) = \boldsymbol{E}(\boldsymbol{r},\boldsymbol{t}) \times \boldsymbol{H}(\boldsymbol{r},\boldsymbol{t})$$

(2) – (1)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) = -\boldsymbol{\mathcal{H}} \cdot \partial \boldsymbol{\mathcal{B}}/\partial t - \boldsymbol{\mathcal{E}} \cdot \partial \boldsymbol{\mathcal{D}}/\partial t - \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}} \qquad (3)$$

$$= -\frac{\partial}{\partial t}\left(\frac{1}{2}\,\mu \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{H}}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\,\epsilon \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}\right) - \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}}$$

$$= -\frac{\partial U_E}{\partial t} - \frac{\partial U_H}{\partial t} - \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}}$$

Power supplied to the volume

U_E: stored electric energy density

U_H: stored magnetic energy density

$\boldsymbol{\nabla} \cdot \boldsymbol{S}$: total electromagnetic energy power density flow out of an infinitesimal volume

$\boldsymbol{S} = \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}$: flow of electromagnetic energy power per unit area in the direction of $\boldsymbol{S}$

- Conservation of energy is a natural result of Maxwell's equations.

Ref [1]

Complex Poynting's Theorem

$$\boldsymbol{H}^* \cdot (\nabla \times \boldsymbol{E}) = -j\omega\mu \boldsymbol{H}^* \cdot \boldsymbol{H} \qquad (1)$$

$$\boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}^*) = \boldsymbol{E} \cdot \boldsymbol{J}^* - j\omega\epsilon^* \boldsymbol{E} \cdot \boldsymbol{E}^* \qquad (2)$$

(2) – (1) $$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}^*) = -j\omega\mu \boldsymbol{H}^* \cdot \boldsymbol{H} + j\omega\epsilon^* \boldsymbol{E} \cdot \boldsymbol{E}^* - \boldsymbol{E} \cdot \boldsymbol{J}^* \qquad (3)$$

$$\boldsymbol{J} = \boldsymbol{J_s} + \boldsymbol{J_c} = \boldsymbol{J_s} + \sigma \boldsymbol{E}$$

Complex Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}^*$

Time average Poynting vector

$$\langle \boldsymbol{S}(\boldsymbol{r}) \rangle = \frac{1}{T} \int_{\boldsymbol{0}}^{T} dt\, \boldsymbol{E}(\boldsymbol{r}, \boldsymbol{t}) \times \boldsymbol{H}(\boldsymbol{r}, \boldsymbol{t})$$

$$= \frac{1}{2} Re\{\boldsymbol{E} \times \boldsymbol{H}^*\}$$

$$\int_{\boldsymbol{V}} dv\, \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}^*) = \oint_{\boldsymbol{S}} d\boldsymbol{s} \cdot (\boldsymbol{E} \times \boldsymbol{H}^*) = -\sigma \int_V dv\, |\boldsymbol{E}|^2 + j\omega \int_V dv\, (\epsilon^* |\boldsymbol{E}|^2 - \mu |\boldsymbol{H}|^2) - \int_V dv\, (\boldsymbol{E} \cdot \boldsymbol{J}_S^*)$$

$$\epsilon = \epsilon' - j\epsilon'' \qquad \mu = \mu' - j\mu''$$

$$-\frac{1}{2} \int_V dv\, (\boldsymbol{E} \cdot \boldsymbol{J}_S^*) = \frac{1}{2} \oint_S d\boldsymbol{s} \cdot (\boldsymbol{E} \times \boldsymbol{H}^*) + \frac{\sigma}{2} \int_V dv\, |\boldsymbol{E}|^2 + \frac{\omega}{2} \int_V dv\, (\epsilon'' |\boldsymbol{E}|^2 + \mu'' |\boldsymbol{H}|^2) + j\frac{\omega}{2} \int_V dv\, (\mu' |\boldsymbol{H}|^2 - \epsilon' |\boldsymbol{E}|^2)$$

Complex power supplied to the volume P_s

Complex power flow out of the volume P_o

Power loss by conductivity

Power loss by dielectric and magnetic loss

Stored electric and magnetic energies $2j\omega(U_H - U_E)$

Joule loss P_l

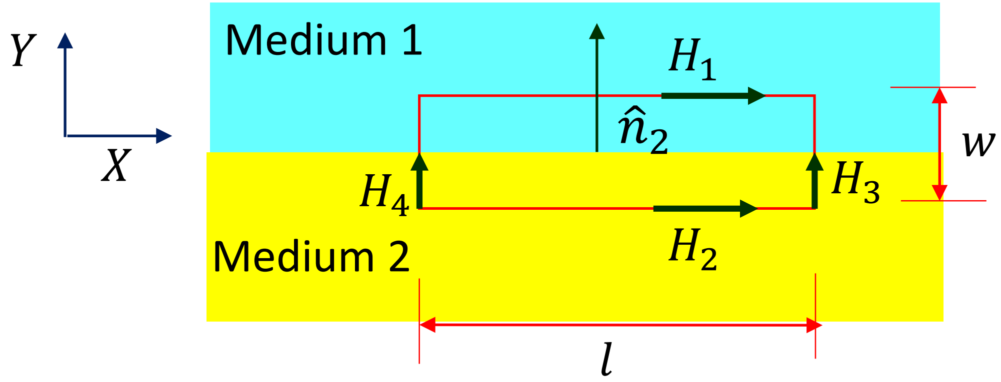
Ref [2]

Boundary conditions (I)

$$\int ds\, \nabla \times \boldsymbol{H} = \int ds\, (\boldsymbol{J} + j\omega \boldsymbol{D})$$

$$\oint d\boldsymbol{l}\, \boldsymbol{H} = (H_2 - H_1)l + (H_3 - H_4)w$$

$$= lw(J_z + j\omega D_z)$$



$$\boldsymbol{J}_s = \lim_{w\to 0} \boldsymbol{J}_v w \quad \Longrightarrow \quad (H_2 - H_1) = J_s$$

In general, $\hat{n}_2 \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$ $\hat{n}_2$: from medium 2 to medium 1

Similarly, $\hat{n}_2 \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0$

- On the surface of perfect conductors, the tangential. The surface current $\boldsymbol{J}_s = \hat{\boldsymbol{n}} \times \mathbf{H}$.
- On finite conductors, $\delta_s \neq 0$, so $\boldsymbol{J}_s = 0$. Both the tangential E and H are continuous across the boundary.

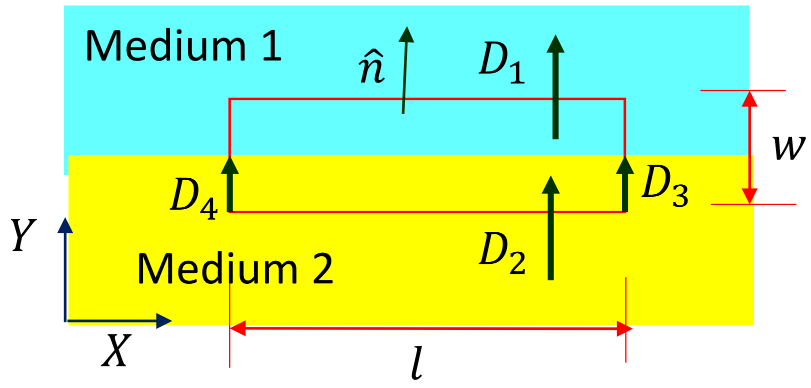
Ref [1]

Boundary conditions (II)

$$\int dv\, \boldsymbol{\nabla} \cdot \boldsymbol{D} = \int dv\, \rho$$

$$\lim_{w \to 0} \int d\boldsymbol{s}\, \boldsymbol{D} = (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \hat{\boldsymbol{n}} A = \lim_{w \to 0} A w \rho_v = A \rho_s$$

$$(D_1 - D_2) = \rho_s$$



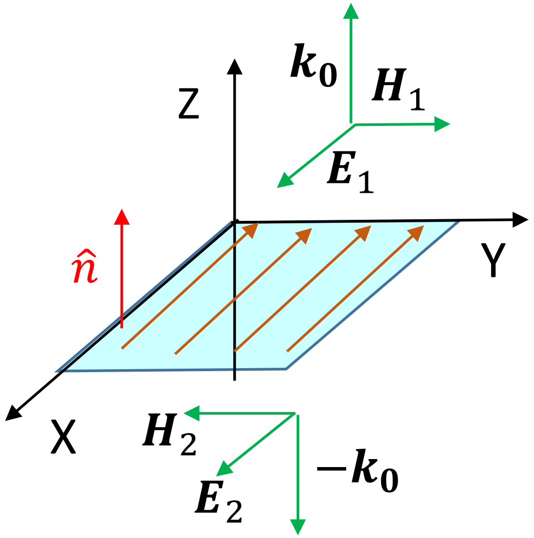
In general, $\hat{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_s$ $\hat{n}$: outward from the box surfaces

Similarly, $\hat{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0$

- The normal component of B is continuous across the boundary surface.
- The discontinuity in the normal component of D is equal to the surface charge density.

Ref [1]

Generate a plane wave



$J_s = -J_0\hat{x}$ only on the z=0 plane in free space

$$\hat{z} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \qquad (1)$$

$$\hat{z} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = -J_0\hat{x} \qquad (2)$$

For $z < 0$ $\quad \boldsymbol{E}_2 = \hat{x}\, \mathrm{A}\eta_0 e^{jk_0 z} \quad \boldsymbol{H}_2 = -\hat{y}\, \mathrm{A}e^{jk_0 z}$

For $z > 0$ $\quad \boldsymbol{E}_1 = \hat{x}\, \mathrm{B}\eta_0 e^{-jk_0 z} \quad \boldsymbol{H}_1 = \hat{y}\, \mathrm{B}e^{-jk_0 z}$

(1) ➡ $\mathrm{A} = \mathrm{B}$

(2) ➡ $B + A = -J_0$

$$B = A = \frac{-J_0}{2}$$

Ref [2]

Spherical wave solutions

$$\boldsymbol{E}(r,\theta,\phi) = \hat{r}E_r(r,\theta,\phi) + \hat{\theta}E_\theta(r,\theta,\phi) + \hat{\phi}E_\phi(r,\theta,\phi)$$

Vector wave equation

$$\boldsymbol{\nabla}^2\boldsymbol{E} + \omega^2\epsilon\mu\boldsymbol{E} = 0$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \boldsymbol{\nabla}^2\boldsymbol{E}$$

$$\longrightarrow \left\{ \begin{aligned} &\nabla^2 E_r - \frac{2}{r^2}\left(E_r + E_\theta cot\theta + \csc\theta\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_\theta}{\partial\theta}\right) + \omega^2\epsilon\mu E_r = 0 \\ &\nabla^2 E_\theta - \frac{1}{r^2}\left(E_\theta csc^2\theta - 2\frac{\partial E_r}{\partial\theta} + 2\,cot\theta\csc\theta\frac{\partial E_\phi}{\partial\phi}\right) + \omega^2\epsilon\mu E_\theta = 0 \\ &\nabla^2 E_\phi - \frac{1}{r^2}\left(E_\phi csc^2\theta - 2\,csc\theta\frac{\partial E_r}{\partial\phi} - 2\,cot\theta\csc\theta\frac{\partial E_\theta}{\partial\phi}\right) + \omega^2\epsilon\mu E_\phi = 0 \end{aligned} \right.$$

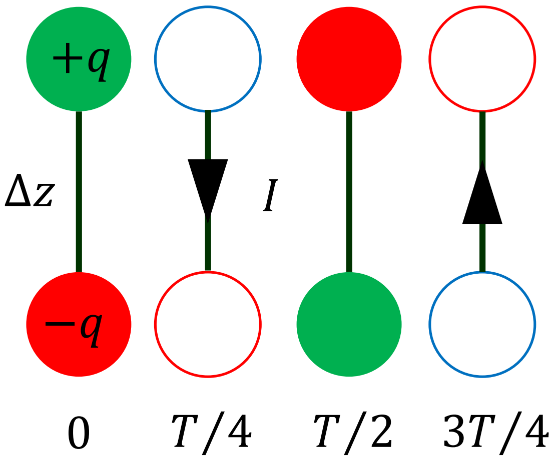
Scalar wave equation

$$\boldsymbol{\nabla}^2\psi + \beta^2\psi = 0$$

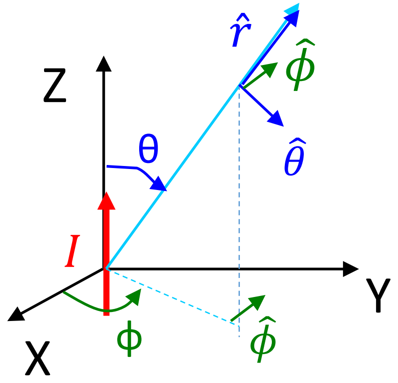
$$\psi\,(r,\theta,\phi) = f(r)g(\theta)h(\phi) \longrightarrow \left\{ \begin{aligned} &\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + [(\beta r)^2 - n(n+1)]f = 0 \\ &\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{dg}{d\theta}\right) + \left[n(n+1) - \left(\frac{m}{sin\theta}\right)^2\right]g = 0 \\ &\frac{d^2h}{d\phi^2} = -m^2h \end{aligned} \right.$$

Ref [4]

Hertzian dipole radiation



$$p = q\Delta z \xrightarrow{\partial_t} I\Delta z = j\omega p$$



$$A = \hat{z}\frac{\mu I\Delta z e^{-jkr}}{4\pi r}$$

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{A} = \hat{\phi}\frac{jkI\Delta z e^{-jkr}}{4\pi r}\left(1+\frac{1}{jkr}\right)sin\theta$$

$$\boldsymbol{E} = \frac{1}{j\omega\epsilon}\boldsymbol{\nabla}\times\boldsymbol{H} = \sqrt{\frac{\mu}{\epsilon}}\frac{jkI\Delta z e^{-jkr}}{4\pi r}\left\{\hat{r}\left[\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]2\,cos\theta + \hat{\theta}\left[1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]sin\theta\right\}$$

In the far field

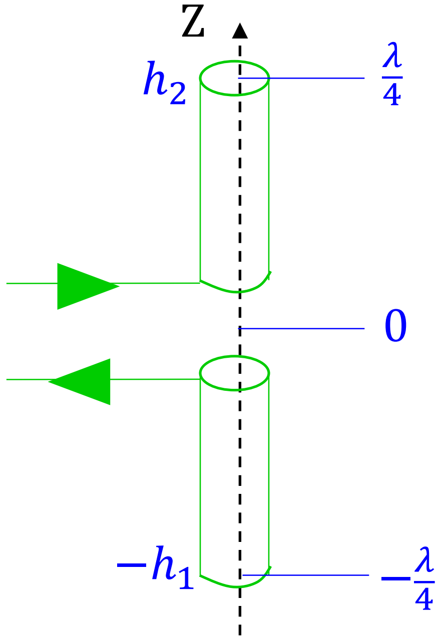
$$\boldsymbol{H} = \hat{\phi}\frac{jkI\Delta z e^{-jkr}}{4\pi r}sin\theta$$

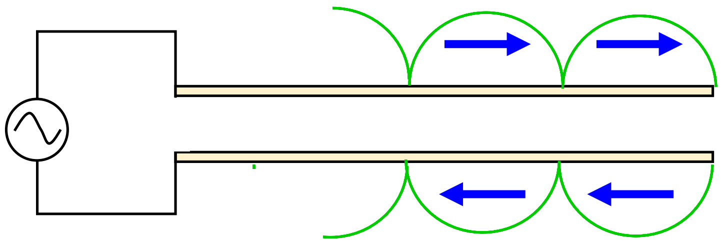
$$\boldsymbol{E} = \hat{\theta}\sqrt{\frac{\mu}{\epsilon}}\frac{jkI\Delta z e^{-jkr}}{4\pi r}sin\theta = \hat{\theta}\eta H_{\phi}$$

$$\langle\boldsymbol{S}\rangle = \frac{1}{2}Re\{\boldsymbol{E}\times\boldsymbol{H}^*\} = \hat{r}\frac{\eta}{2}\left(\frac{k|I|\Delta z}{4\pi r}\right)^2 sin^2\theta$$

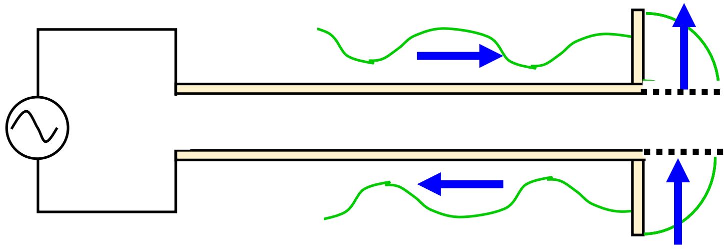
Ref [1]

Half-wave dipole radiation

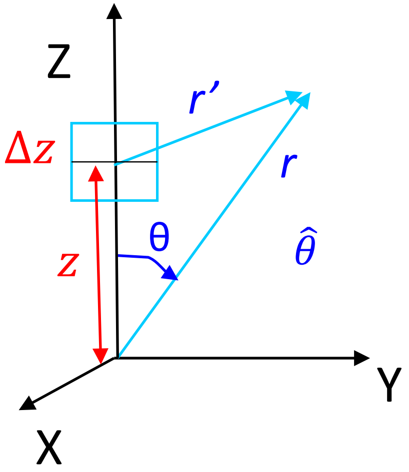




2-wire transmission line



Half-wave dipole antenna



$$A = \hat{z}\frac{\mu I \Delta z e^{-jkr}}{4\pi r}$$

In the far field

$$\boldsymbol{H} = \hat{\phi}\frac{jke^{-jkr}}{4\pi r}U(\theta)sin\theta$$

$$\boldsymbol{E} = \hat{\theta}\frac{jk\eta e^{-jkr}}{4\pi r}U(\theta)sin\theta$$

$$U(\theta) = \int_{-h_1}^{h_2} dz\, I(z)e^{jkzcos\theta}$$

$$I(z) = I_0 cos(kz)$$

$$U(\theta) = \frac{I_0}{k}\int_{-\pi/2}^{\pi/2} d(kz)cos(kz)\, I(z)e^{jkzcos\theta} = \frac{2I_0 cos(\pi cos\theta/2)}{ksin^2\theta}$$

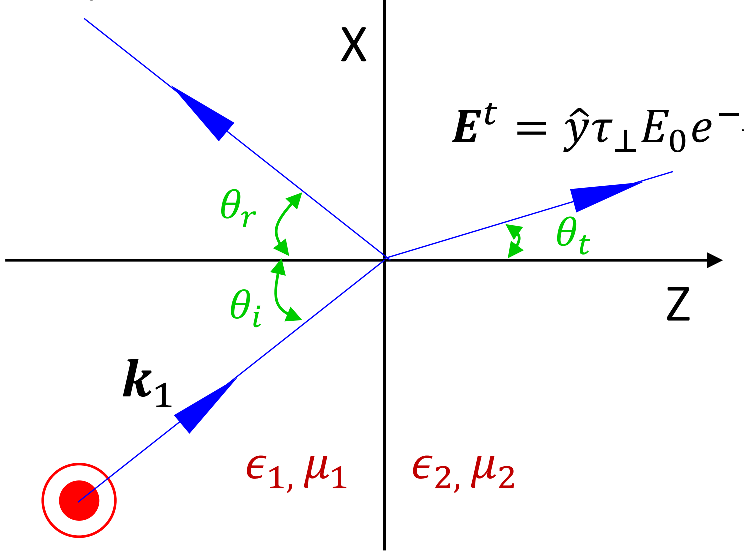
$$\langle \boldsymbol{S} \rangle = \frac{1}{2}Re\{\boldsymbol{E}\times\boldsymbol{H}^*\} = \hat{r}\frac{I_0^2\eta\, cos^2(\pi cos\theta/2)}{8\pi^2 r^2 sin^2\theta}$$

Ref [1]

Reflection and transmission at a dielectric interface – perpendicular polarization

$$\boldsymbol{E}^r = \hat{y}\Gamma_\perp E_0 e^{-j(k_{rx}x - k_{rz}z)}$$

$$\boldsymbol{E}^t = \hat{y}\tau_\perp E_0 e^{-j(k_{tx}x + k_{tz}z)}$$



$$\boldsymbol{E}^i = \hat{y}E_0 e^{-j(k_x x + k_z z)}$$

$$\boldsymbol{H}^i = \frac{j}{\omega\mu}\nabla \times \boldsymbol{E} = (-\hat{x}k_z + \hat{z}k_x)\frac{E_0}{\omega\mu_1}e^{-j(k_x x + k_z z)}$$

$$\boldsymbol{H}^r = (\hat{x}k_{rz} + \hat{z}k_{rx})\frac{\Gamma_\perp E_0}{\omega\mu_1}e^{-j(k_{rx}x - k_{rz}z)}$$

$$\boldsymbol{H}^t = (-\hat{x}k_{tz} + \hat{z}k_{tx})\frac{\tau_\perp E_0}{\omega\mu_2}e^{-j(k_{tx}x + k_{tz}z)}$$

$$E_\parallel^i + E_\parallel^r = E_\parallel^t \xrightarrow{z=0} e^{-jk_x x} + \Gamma_\perp e^{-jk_{rx}x} = \tau_\perp e^{-jk_{tx}x}$$

Since $J_s = 0$ $\quad H_\parallel^i + H_\parallel^r = H_\parallel^t$

$$\frac{-k_z}{\mu_1}e^{-jk_x x} + \frac{k_{rz}}{\mu_1}\Gamma_\perp e^{-jk_{rx}x} = \frac{-k_{tz}}{\mu_2}\tau_\perp e^{-jk_{tx}x}$$

Since the boundary conditions must hold for all x

$$\longrightarrow \boxed{k_x = k_{rx} = k_{tx}}$$ Phase matching conditions

$$(\nabla^2 + \omega^2\mu_1\epsilon_1)\begin{Bmatrix}\boldsymbol{E}^i \\ \boldsymbol{E}^r\end{Bmatrix} = 0 \longrightarrow \begin{matrix} k_x^2 + k_z^2 = \omega^2\mu_1\epsilon_1 = k_1^2 \\ k_{rx}^2 + k_{rz}^2 = \omega^2\mu_1\epsilon_1 = k_1^2 \end{matrix}$$

$$(\nabla^2 + \omega^2\mu_1\epsilon_1)\boldsymbol{E}^t = 0 \longrightarrow k_{tx}^2 + k_{tz}^2 = \omega^2\mu_2\epsilon_2 = k_2^2$$

$$\longrightarrow \boxed{k_z = k_{rz}}$$

$$\boxed{s \equiv \frac{\mu_1 k_{tz}}{\mu_2 k_z}} \longrightarrow \left.\begin{matrix} 1 + \Gamma_\perp = \tau_\perp \\ 1 - \Gamma_\perp = s\tau_\perp \end{matrix}\right\}$$

$\Gamma_\perp = (1-s)/(1+s)$ Fresnel reflection coefficient

$\tau_\perp = 2/(1+s)$ Fresnel transmission coefficient

Ref [1]

Reflection and transmission at a dielectric interface – perpendicular polarization

$$\left.\begin{aligned} k_x &= k_1 \sin\theta_i \\ k_{rx} &= k_1 \sin\theta_r \\ k_{tx} &= k_2 \sin\theta_t \end{aligned}\right\}$$

$\theta_i = \theta_r$ Law of reflection

$k_1 \sin\theta_i = k_2 \sin\theta_t \longrightarrow n \sin\theta_i = n_2 \sin\theta_t$ Snell's Law

$$n = \frac{c}{v_p} = c\sqrt{\mu_1\epsilon_1} = \frac{c}{\omega}k$$

$$k_z = k_1 \cos\theta_i$$

$$k_{tz} = k_2 \cos\theta_t$$

Critical angle θ_c $\sin\theta_c = k_2/k_1 = n_2/n_1$ when $n_1 > n_2$

when $\theta_i > \theta_c$ $k_{tz}^2 = k_2^2 - k_{tx}^2 = k_2^2(1 - sin^2\theta_t) < 0 \longrightarrow k_{tz} = \pm j\alpha$

$\boldsymbol{E}^t = \hat{y}T_\perp E_0\, e^{-\alpha z}e^{-jk_{tx}x}$ Surface wave propagates in the x-direction but decays exponentially in the z-direction.

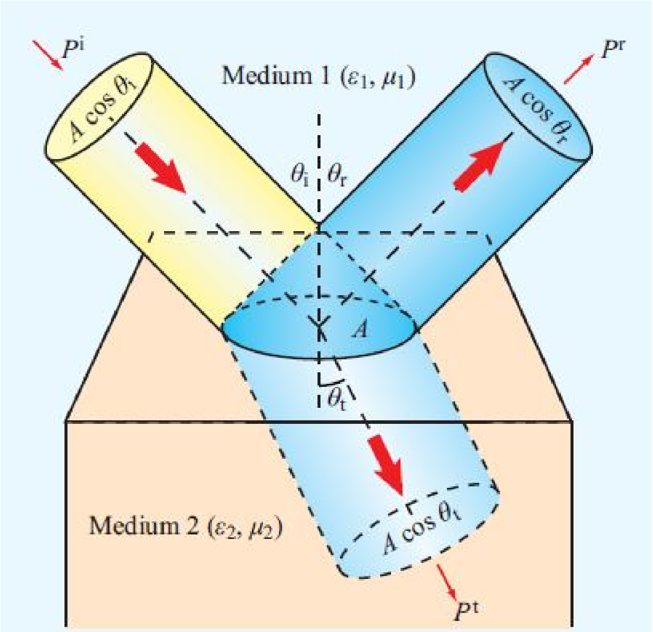
$$\langle \boldsymbol{S}_\perp^i \rangle = \frac{1}{2}Re\left(\boldsymbol{E}^i \times \boldsymbol{H}^{i*}\right) = \frac{|E_0|^2}{2\omega\mu_1}(\hat{x}k_x + \hat{z}k_z) \longrightarrow \langle S_\perp^i \rangle = \frac{|E_0|^2 k_1}{2\omega\mu_1} = \frac{|E_0|^2}{2\eta_1}$$

$$\langle \boldsymbol{S}_\perp^r \rangle = \frac{1}{2}Re(\boldsymbol{E}^r \times \boldsymbol{H}^{r*}) = \frac{|E_0|^2}{2\omega\mu_1}|\Gamma_\perp|^2(\hat{x}k_x - \hat{z}k_z) \longrightarrow \langle S_\perp^r \rangle = \frac{|E_0|^2 k_1}{2\omega\mu_1}|\Gamma_\perp|^2 = \frac{|E_0|^2}{2\eta_1}|\Gamma_\perp|^2$$

$$\langle \boldsymbol{S}_\perp^t \rangle = \frac{1}{2}Re(\boldsymbol{E}^t \times \boldsymbol{H}^{t*}) = \frac{|E_0|^2}{2\omega\mu_2}|\tau_\perp|^2 Re(\hat{x}k_x + \hat{z}k_{tz}^*) \longrightarrow \langle S_\perp^t \rangle = \frac{|E_0|^2 k_2}{2\omega\mu_2}|\tau_\perp|^2 = \frac{|E_0|^2}{2\eta_2}|\tau_\perp|^2$$

Ref [1]

Conservation of power



Averaged incident power $\langle P_\perp^i\rangle = \langle S_\perp^i\rangle A\cos\theta_i = \langle S_z^i\rangle A$

Averaged reflected power $\langle P_\perp^r\rangle = \langle S_\perp^r\rangle A\cos\theta_r = \langle S_z^r\rangle A$

Averaged transmitted power $\langle P_\perp^t\rangle = \langle S_\perp^t\rangle A\cos\theta_t = \langle S_\perp^t\rangle A$

$\boxed{\theta_i = \theta_r}$

Reflectance $R_\perp = \langle P_\perp^r\rangle/\langle P_\perp^i\rangle \longrightarrow |\Gamma_\perp|^2$

Transmittance $T_\perp = \langle P_\perp^t\rangle/\langle P_\perp^i\rangle \longrightarrow |\tau_\perp|^2\left(\dfrac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}\right) = s|\tau_\perp|^2$

$\boxed{s = \mu_1 k_{tz}/\mu_2 k_z}$

Conservation of power $\longrightarrow$ $\langle P_\perp^i\rangle = \langle P_\perp^r\rangle + \langle P_\perp^t\rangle \longrightarrow R_\perp + T_\perp = 1$

- If k_{tz} is imaginary, then $\langle S_z^t\rangle = 0$. $\longrightarrow$ Total reflection
- If k_{tz} is real $\longrightarrow$

$$(\langle S_z^t\rangle - \langle S_z^r\rangle)/\langle S_z^i\rangle = s|\tau_\perp|^2 + |\Gamma_\perp|^2 = R_\perp + T_\perp = 1$$

$$(\langle S_x^t\rangle + \langle S_x^r\rangle)/\langle S_x^i\rangle = \frac{\mu_1}{\mu_2}|\tau_\perp|^2 + |\Gamma_\perp|^2 \quad = 1 \text{ only if } k_{tz} = k_z$$

Ref [1,3]

Complex Poynting vectors at the interface

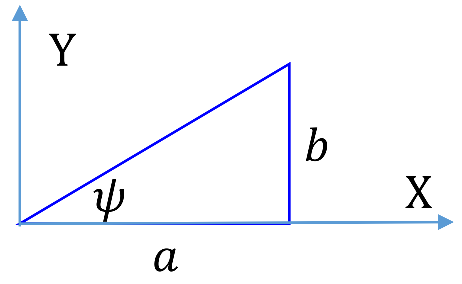
$$\boldsymbol{S}_\perp^- = \left(\boldsymbol{E}^i + \boldsymbol{E}^r\right) \times \left(\boldsymbol{H}^i + \boldsymbol{H}^r\right)^* = \frac{|E_0|^2}{\omega\mu_1}\{\hat{x}k_x[1 + |\Gamma_\perp|^2 + 2\,\mathrm{Re}(\Gamma_\perp e^{2jk_z z})] + \hat{z}k_z[1 - |\Gamma_\perp|^2 + 2j\mathrm{Im}(\Gamma_\perp e^{2jk_z z})]\}$$

$$\boldsymbol{S}_\perp^+ = \boldsymbol{E}^t \times \boldsymbol{H}^{t^*} = (\hat{x}k_{tx} - \hat{z}k_{tz})\frac{|\tau_\perp|^2|E_0|^2}{\omega\mu_2}$$

If $\Gamma_\perp$ and $T_\perp$ are real, at z=0

$$\boxed{s = \mu_1 k_{tz}/\mu_2 k_z}$$

$$S_z^- = \frac{|E_0|^2}{\omega\mu_1}\{k_z[1 - |\Gamma_\perp|^2]\} = \frac{|E_0|^2}{\omega\mu_2}k_{tz}|\tau_\perp|^2 = S_z^+ \quad \longrightarrow \quad s\tau_1^2 + \Gamma_1^2 = 1$$



$$S_x^- = \frac{|E_0|^2}{\omega\mu_1}k_x(1 + |\Gamma_\perp|)^2 = \frac{|E_0|^2}{\omega\mu_1}k_x{\tau_\perp}^2 \quad \text{but} \quad S_x^+ = \frac{|E_0|^2}{\omega\mu_2}k_{tx}{\tau_\perp}^2$$

In the case of internal total reflection, $\Gamma_\perp = e^{j2\psi}$ and $k_{tz} = -j\alpha$

$$S_x^- = \frac{2|E_0|^2 k_x}{\omega\mu_1}(1 + Re\Gamma_\perp) = \frac{4|E_0|^2 k_x}{\omega\mu_1}\frac{a^2}{(a^2 + b^2)}$$

$$S_x^+ = \frac{|E_0|^2 k_x}{\omega\mu_2}|\tau_\perp|^2 = \frac{4|E_0|^2 k_x}{\omega\mu_2}\frac{a^2}{a^2 + b^2}$$

$$S_z^- = j\frac{|E_0|^2\alpha}{\omega\sqrt{\mu_2\epsilon_2}}\frac{4a\cos\theta_i}{(a^2 + b^2)} = S_z^+$$

$$\Gamma_\perp = \frac{a + jb}{a - jb} = e^{j2\psi} \qquad \tau_\perp = \frac{2a}{a - jb}$$

$$a = \eta_2\cos\theta_i \qquad b = \alpha\eta_1/k_2$$

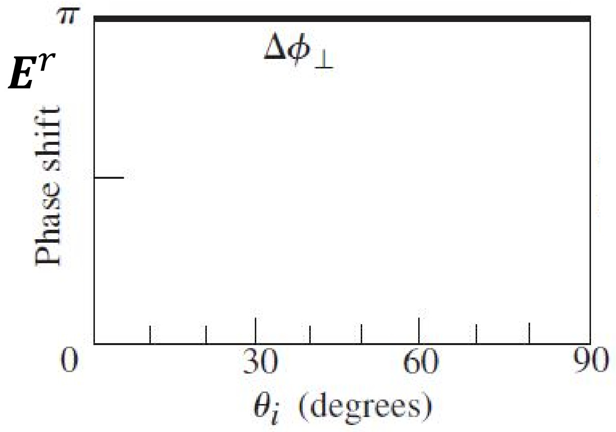
$$|\Gamma_\perp| = 1 \qquad |\tau_\perp| = \frac{2a}{\sqrt{a^2 + b^2}}$$

$$k_{tz} = -j\alpha \qquad \eta_i = \sqrt{\mu_i/\epsilon_i}$$

Phases of reflected waves

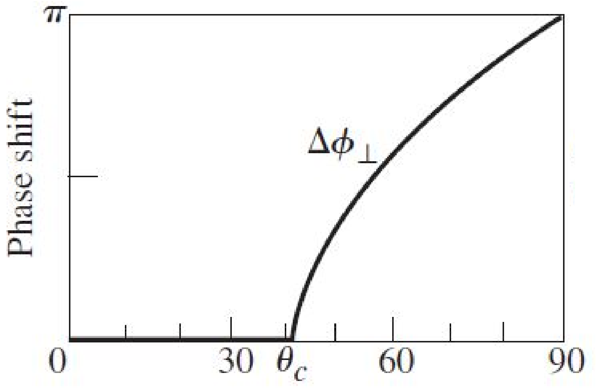
Perpendicular polarization

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \xrightarrow{\mu_1 \sim \mu_2 \sim \mu_0} \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$



$\Gamma_\perp < 0$, if $n_1 < n_2$

$$\frac{n_2}{n_1} = 1.5$$

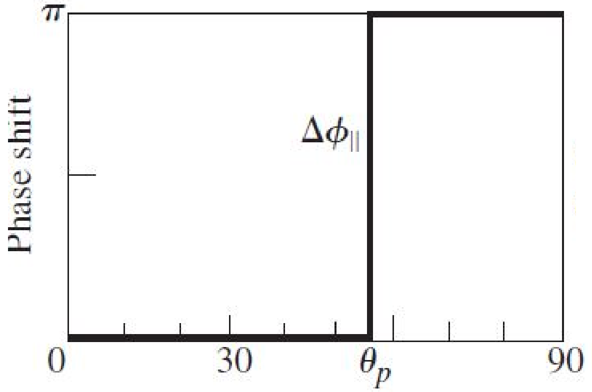


$n_1 > n_2$

$$\frac{n_1}{n_2} = 1.5$$

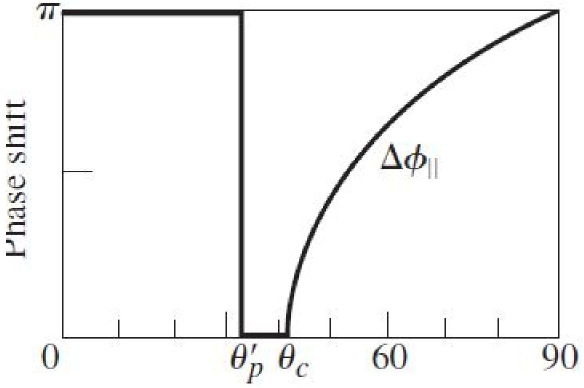
Parallel polarization

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \xrightarrow{\mu_1 \sim \mu_2 \sim \mu_0} \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$



$n_1 < n_2$

$$\frac{n_2}{n_1} = 1.5$$



$n_1 > n_2$

$$\frac{n_1}{n_2} = 1.5$$

Ref [7]

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