HW4

Tuesday, February 15, 2022 2:59 PM

HW 4

- 3-2 Derive a mathematical expression for $H(f)$ and plot $|H(f)|$ and $\angle H(f)$ in Matlab. Draw vertical lines across these plots at each of the frequencies $f_0 = \omega_0/2\pi$ given in this problem. Turn in the plots along with your answers.
- 3-6
- 3-7
- 3-15
- 3-21 Equation (3.30) is labeled (3.29) on page 141.
- 3-26
- 3-33
- 3-39
- 3-40
- 3-41
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- 3-58
- 3-64

Problem 3-2

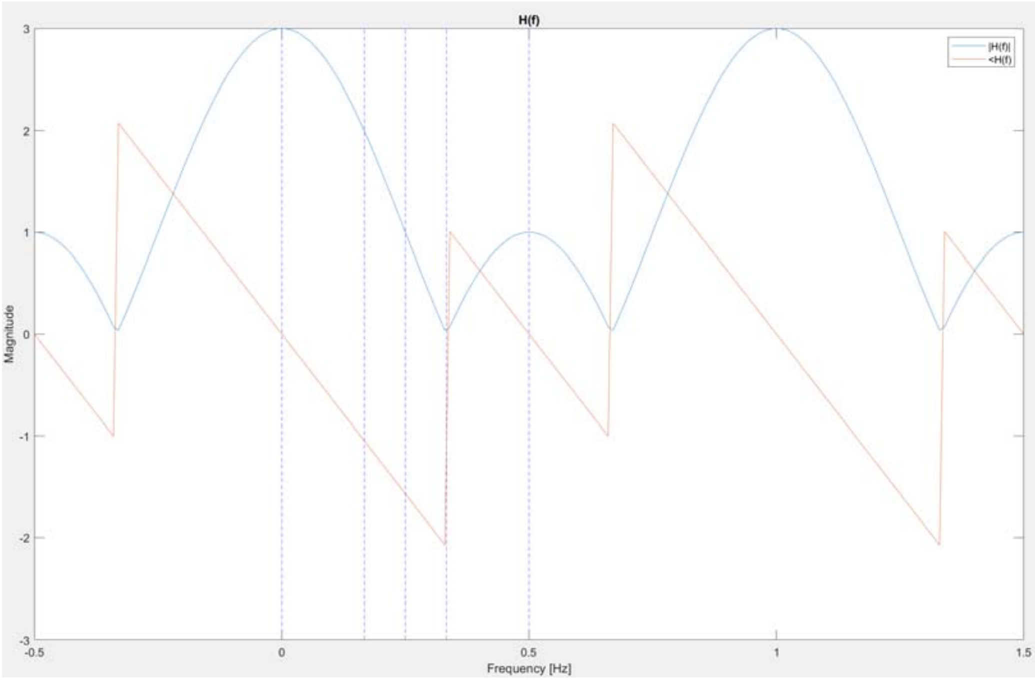
Given $x[n] = \cos\omega_0 n$, with ω_0 specified in each part below and $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$, find $y[n] = x[n] * h[n]$. Express your answer as a single cosine of the form $y[n] = A\cos(\omega_0 n + \theta)$.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$e^{-j\omega(0)} + e^{-j\omega(1)} + e^{-j\omega(2)}$$

$$H(\omega) = 1 + e^{-j\omega} + e^{-j\omega 2}$$

$$H(f) = 1 + e^{-j2\pi f} + e^{-j2\pi f \cdot 2}$$



(a) $\omega_0 = 0$.

$$\left[1 + e^{-j(0)} + e^{-j(0)2}\right]\left[\cos(0n)\right] = 3$$

(b) $\omega_0 = \pi/3$.

$$1 + e^{-j(\frac{\pi}{3})} + e^{-j(\frac{2}{3}\pi)}$$

$$= 1 - j1.732$$

$$\left[1 + e^{-j(\frac{\pi}{3})} + e^{-j(\frac{\pi}{3})2}\right]\left[\cos(\tfrac{\pi}{3}n)\right]$$

$$= .99 - j1.71$$

(c) $\omega_0 = \pi/2$.

$$\left[1 + e^{-j\left(\frac{\pi}{2}\right)} + e^{-j\left(\frac{\pi}{2}\right)2}\right]\left[\cos\left(\frac{\pi}{2}n\right)\right]$$

$$= -.97j$$

(d) $\omega_0 = 2\pi/3$.

$$\left[1 + e^{-j\left(\frac{2}{3}\pi\right)} + e^{-j\left(\frac{2}{3}\pi\right)2}\right]\left[\cos\left(\frac{2}{3}\pi n\right)\right]$$

$$= -2.1E-16 - 2.1E-16j$$

(e) $\omega_0 = \pi$.

$$\left[1 + e^{-j\pi} + e^{-j\pi 2}\right]\left[\cos(\pi n)\right]$$

$$= .88$$

```
function h = p1(f)
h = 1 + exp(-1i*2*pi*f) + exp(-1i*2*pi*f*2);
end

t = [-0.5:0.01:1.5];
plot(t,abs(p1(t))); hold on;
plot(t,angle(p1(t)));
xline([0 (pi/(3*2*pi)) (pi/(2*2*pi)) ((2*pi)/(3*2*pi)) (pi/(2*pi))],'--b'); hold off;
title('H(f)');
legend('|H(f)|','<H(f)');
xlabel('Frequency [Hz]');
ylabel('Magnitude');
shg;
```

Problem 3-6

Find and plot $|H(\omega)|$ and $\measuredangle H(\omega)$, the magnitude and phase of the DTFT of the pulse $h[n]$ shown in **Figure 3.47**. Make sure your plot of $\measuredangle H(\omega)$ is appropriately phase wrapped.

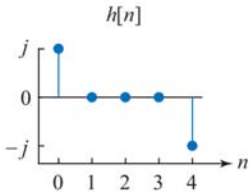
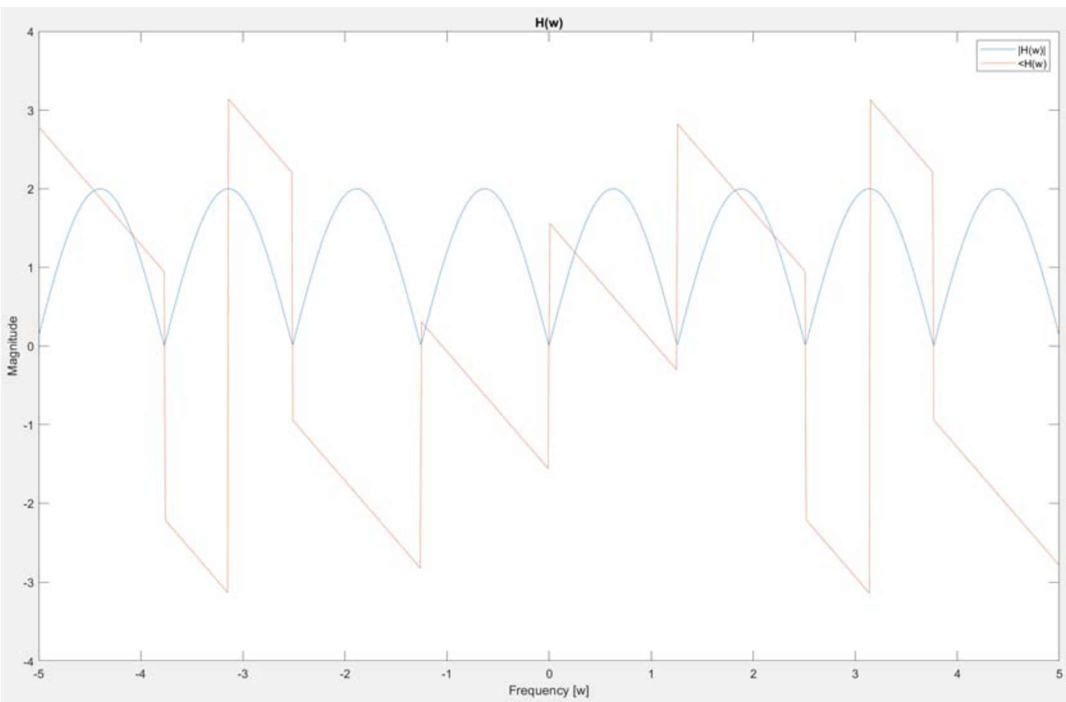


Figure 3.47

$$h[n] = e^{jw} - e^{-jw4}$$



Problem 3-7

A system has a linear constant-coefficient difference equation given by

$$y[n] - y[n-1] = x[n].$$

This system is sometimes called quasi-stable because for certain inputs the output will be bounded, and for others it will explode. Find the output of this system when the input is

(a) $x[n] = \sin \pi n/2$.

$$\sum_{n=-\infty}^{\infty} \left|\sin\left(\tfrac{\pi}{2}n\right)\right| = \infty$$

$$\sum_{n=-\infty} \left|\sin\left(\frac{\pi}{2}n\right)\right| = \infty$$

diverges

(b) $x[n] = 1$ (i.e., $x[n] = \cos(0n)$).

$$\sum_{n=-\infty}^{\infty} \left|\cos(0 \cdot n)\right| = \infty$$

diverges

Problem 3-15

A system has a linear constant-coefficient difference equation given by $y[n] = x[n] - 2x[n-1] + x[n-2]$.

(a) Find $H(\omega)$.

$$y[n] = H(\omega)x[n]$$

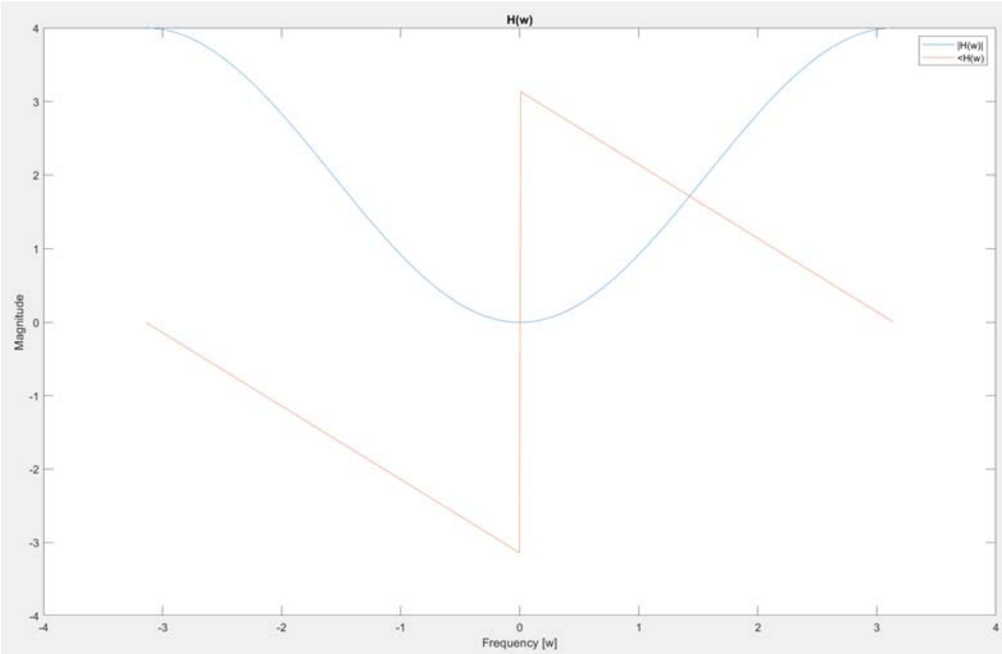
$$y[n] = x[n] - 2x[n-1] + x[n-2]$$

$$Y(\omega) = X(\omega) - 2X(\omega)e^{-j\omega} + X(\omega)e^{-j2\omega}$$

$$Y(\omega) = X(\omega)\left[1 - 2e^{-j\omega} + e^{-j2\omega}\right]$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \boxed{1 - 2e^{-j\omega} + e^{-j2\omega}}$$

(b) Find and make fully labeled plots of $|H(\omega)|$ and $\measuredangle H(\omega)$.



Problem 3-21

Consider the sequence from Example 1.4, $x[n] = (2 + j)\delta[n + 1] + \delta[n] - 3j\delta[n - 1]$.

(a) Find $X(\omega)$, $X_r(\omega)$, $X_i(\omega)$, $X_{re}(\omega)$, $X_{ro}(\omega)$, $X_{ie}(\omega)$ and $X_{io}(\omega)$.

$$X(w) = 2e^{jwn} + je^{jwn} + 1 - 3e^{-jwn} + je^{-jwn}$$

$$X_r(w) = 2e^{jwn}$$

$$X_e(n) = \frac{1}{2}\left(x[n] + x[-n]\right) = \frac{1}{2}\Big((2+j)\delta(n+1) + \delta(n) - 3j\delta(n-1) + (2+j)\delta(-n+1) + \delta(-n) - 3j\delta(-n-1)\Big)$$

$$X_o(n) = \frac{1}{2}\left(x(n) - x(-n)\right) = \frac{1}{2}\Big((2+j)\delta(n+1) + \delta(n) - 3j\delta(n-1) - (2+j)\delta(-n+1) - \delta(-n) + 3j\delta(-n-1)\Big)$$

(b) Show that the four relations of Equation (3.30) apply.

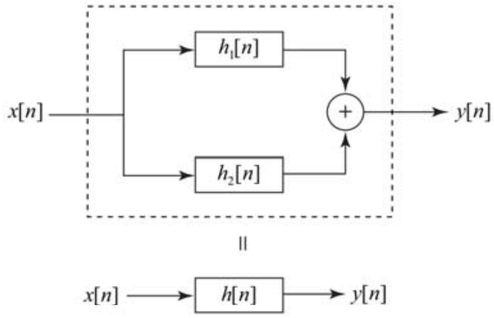


Figure 3.54

Problem 3-26

The system in **Figure 3.54** is defined by the difference equation

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = -\tfrac{1}{4}x[n-1].$$

Given that

$$h_2[n] = -\tfrac{1}{2}^n u[n], \qquad H_2(\omega) = \frac{-1}{1 - \frac{1}{2}e^{-j\omega}}$$

find $h_1[n]$.

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = -\tfrac{1}{4}x[n-1]$$

$$Y(\omega) - \tfrac{3}{4}Y(\omega)e^{-j\omega} + \tfrac{1}{8}Y(\omega)e^{-j\omega 2} = -\tfrac{1}{4}X(\omega)e^{-j\omega}$$

$$Y(\omega)\left[1 - \tfrac{3}{4}e^{-j\omega} + \tfrac{1}{8}e^{-j\omega 2}\right] = X(\omega)\left[-\tfrac{1}{4}e^{-j\omega}\right]$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-\frac{1}{4}e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j\omega 2}} = \frac{-\frac{1}{4}e^{-j\omega}}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

$$H_2(\omega) = \frac{-1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$\left[-1 + \tfrac{1}{4}e^{-j\omega}\right] + B\left[1 - \tfrac{1}{2}e^{-j\omega}\right] = -\tfrac{1}{4}e^{-j\omega}$$

$$1 - \tfrac{1}{2}e^{-j\omega} = B\left[1 - \tfrac{1}{2}e^{-j\omega}\right]$$

$$\frac{-1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$$H(\omega) - H_2(\omega) = H_1(\omega) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$$\boxed{h_1[n] = \tfrac{1}{4}^n u[n]}$$

$$\boxed{h_1[n] = \frac{1}{4}^{n} u[n]}$$

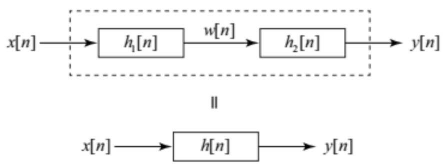


Figure 3.57

Problem 3-33

The system shown in **Figure 3.57** is characterized by the following difference equations:

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-1] = x[n-1]$$

$$y[n] - \tfrac{1}{4}y[n-1] = w[n] - \tfrac{1}{3}w[n-1].$$

Find $h_1[n]$.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n-1]$$

$$Y(\omega) - \frac{3}{4}Y(\omega)e^{-j\omega} + \frac{1}{8}Y(\omega)e^{-j\omega 2} = X(\omega)e^{-j\omega}$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j\omega 2}}$$

$$= \frac{e^{-j\omega}}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)} = H(\omega)$$

$$y[n] - \frac{1}{4}y[n-1] = w[n] - \frac{1}{3}w[n-1]$$

$-j\omega$

$$Y(w) - \frac{1}{4} Y(w) e^{-jw} = W(w) - \frac{1}{3} W(w) e^{-jw}$$

$$\frac{Y(w)}{W(w)} = \frac{1 - \frac{1}{3} e^{-jw}}{1 - \frac{1}{4} e^{-jw}} = H_2(w)$$

$$\frac{1}{1 - \frac{1}{4} e^{-jw}} - \frac{\frac{1}{3} e^{-jw}}{1 - \frac{1}{4} e^{-jw}} \qquad \frac{1}{4}^{n} u[n] - \frac{1}{3} e^{-jw} \cdot \frac{1}{4}^{n} u[n]$$

$$\frac{1}{4}^{n} u[n] \left(1 - \frac{1}{3} e^{-jw}\right)$$

$$\frac{e^{-jw}}{\left(1 - \frac{1}{2} e^{-jw}\right)\left(1 - \frac{1}{4} e^{-jw}\right)} \Big/ \frac{1 - \frac{1}{3} e^{-jw}}{1 - \frac{1}{4} e^{-jw}} = H_1(w)$$

$$\frac{\left(1 - \frac{1}{3} e^{-jw}\right)\left(1 - \frac{1}{2} e^{-jw}\right)}{1 - \frac{1}{4} e^{-jw}} = e^{-jw}$$

$$\frac{1 - \frac{1}{3} e^{-jw}}{1 - \frac{1}{4} e^{-jw}} \cdot \frac{B}{1 - \frac{1}{2} e^{-jw}} = \frac{e^{-jw}}{\left(1 - \frac{1}{2} e^{-jw}\right)\left(1 - \frac{1}{4} e^{-jw}\right)}$$

$$B = \frac{e^{-jw}}{1 - \frac{1}{3} e^{-jw}}$$

$$H_1(w) = \frac{\dfrac{e^{-jw}}{1 - \frac{1}{3} e^{-jw}}}{1 - \frac{1}{2} e^{-jw}}$$

Problem 3-39

For the system shown in **Figure 3.60**, assume that $x[n] = \cos \pi n/8$.

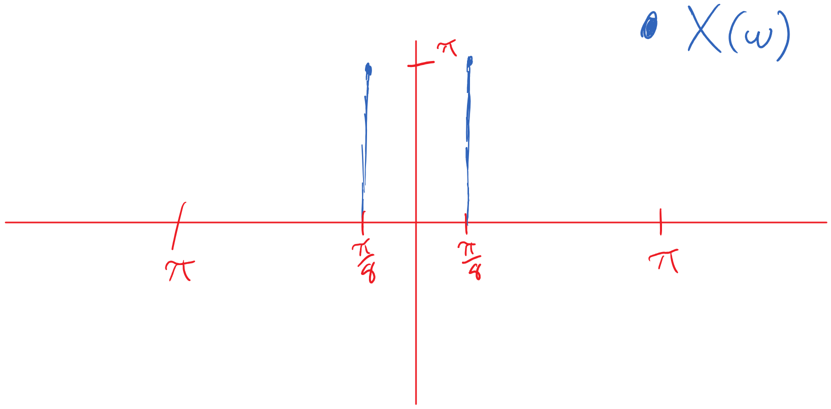
(a) Make a fully labeled sketch of $X(\omega)$, $Q(\omega)$, $R(\omega)$ and $Y(\omega)$.

$$\sum_{n=-\infty}^{\infty}\left[\frac{1}{2}e^{j\frac{\pi}{8}n}+\frac{1}{2}e^{-j\frac{\pi}{8}n}\right]e^{-j\omega n}$$

$$\frac{1}{2}\sum_{-\infty}^{\infty} e^{-j(\omega-\frac{\pi}{8})n} + \sum_{-\infty}^{\infty} e^{-j(\omega+\frac{\pi}{8})n}$$

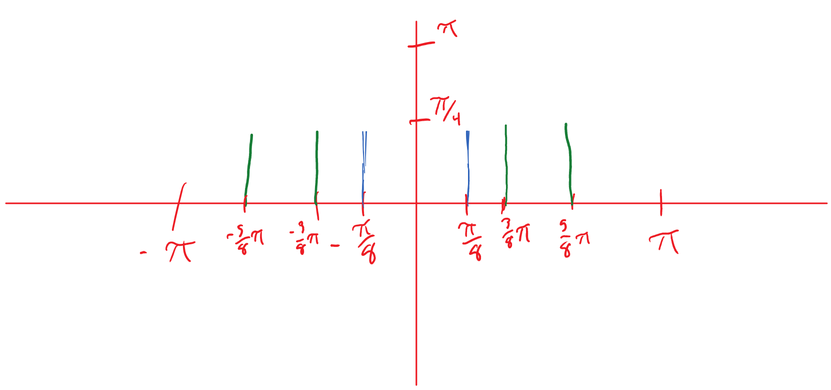
$$X(\omega) = \pi\left[\delta\left(\omega-\frac{\pi}{8}\right)+\delta\left(\omega+\frac{\pi}{8}\right)\right]$$

- $X(\omega)$



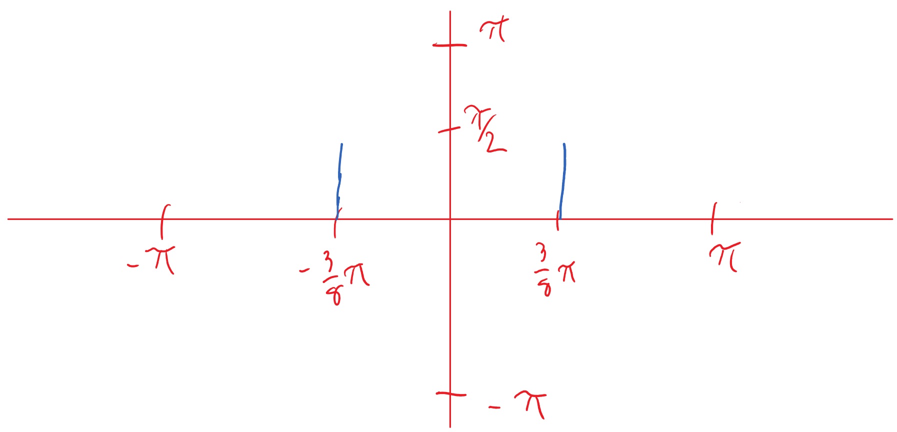
$$Q(\omega) = \frac{\pi}{4}\left(X\left(\omega \pm \frac{\pi}{2}\right)\right)$$

- $X(\omega)$
- $Q(\omega)$



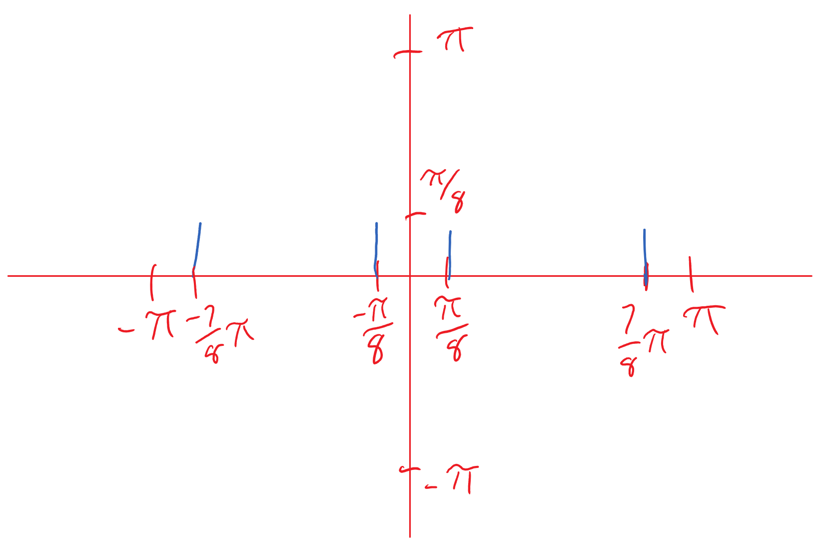
$Q(\omega) H(\omega) = R(\omega)$ $\qquad$ $\left(X(\omega \pm \frac{\pi}{2})\right)\left(\delta(\omega + \frac{\pi}{2}) - \delta(\omega - \frac{\pi}{2})\right)$

- $R(\omega)$



$Y(\omega) = R(\omega \pm \frac{\pi}{2})$

- $Y(\omega)$



$$q[n] = \cos\left(\frac{\pi}{8}n\right)\cos\left(\frac{\pi}{2}n\right)$$

$$= \frac{1}{4}\left[e^{-j\pi\frac{3}{8}n} + e^{j\pi\frac{3}{8}n} + e^{-j\pi\frac{5}{8}n} + e^{j\pi\frac{5}{8}n}\right]$$

$$Q(\omega) = \frac{\pi}{4}\left[\delta\left(\omega - \frac{3\pi}{8}\right) + \delta\left(\omega + \frac{3\pi}{8}\right) + \delta\left(\omega - \frac{5\pi}{8}\right) + \delta\left(\omega + \frac{5\pi}{8}\right)\right]$$

$$R(\omega) = \frac{\pi}{4}\left(\delta\left(\omega \pm \frac{3}{8}\pi\right)\right)$$

$$r[n] = \frac{1}{4}\left[e^{-j\frac{3}{8}\pi} + e^{j\frac{3}{8}\pi}\right]$$

$$y[n] = \frac{1}{8}\left[e^{-j\frac{\pi}{8}} + e^{j\frac{\pi}{8}} + e^{-j\frac{7}{8}\pi} + e^{j\frac{7}{8}\pi}\right]$$

(b) Find $q[n]$, $r[n]$ and $y[n]$.

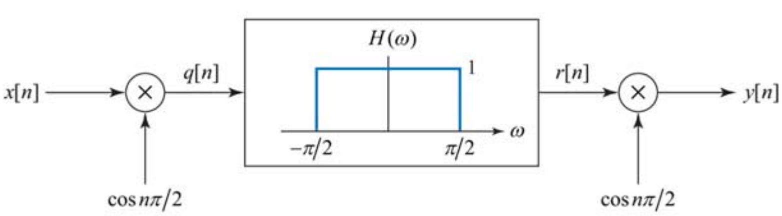


Figure 3.60

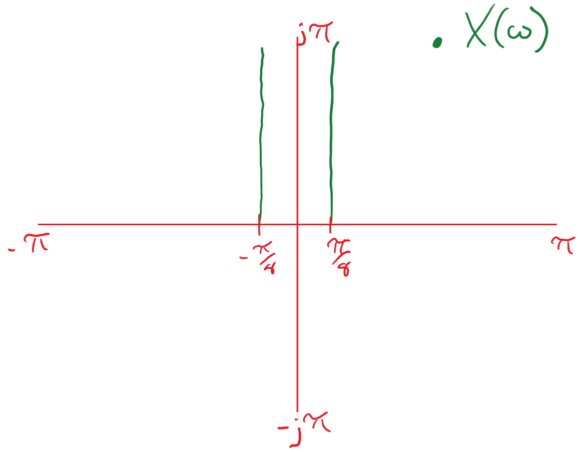
Problem 3-40

Repeat Problem 3-39 assuming that $x[n] = \sin \pi n/8$.

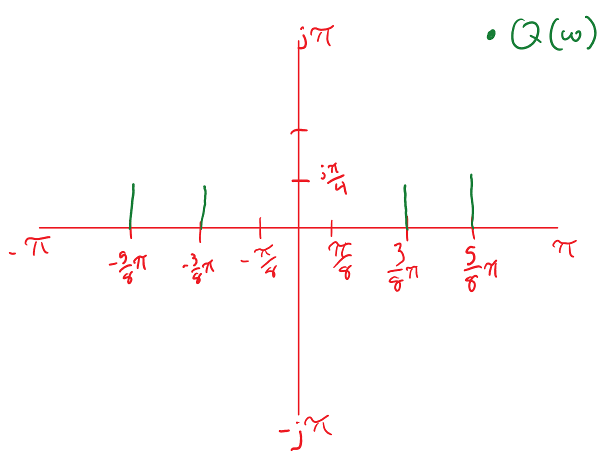
$$\sin\left(\frac{\pi}{8}n\right) = \frac{1}{2i}\left[e^{j\frac{\pi}{8}n} - e^{-j\frac{\pi}{8}n}\right]$$

$$X(\omega) = -\pi j \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{8} - 2\pi k\right) + \pi j \sum_{k=-\infty}^{\infty} \delta\left(\omega + \frac{\pi}{8} - 2\pi k\right)$$

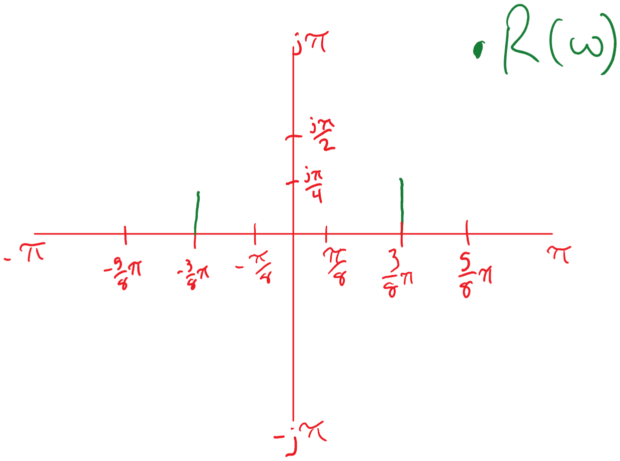
$$= j\pi\left[\delta\left(\omega + \frac{\pi}{8}\right) - \delta\left(\omega - \frac{\pi}{8}\right)\right]$$



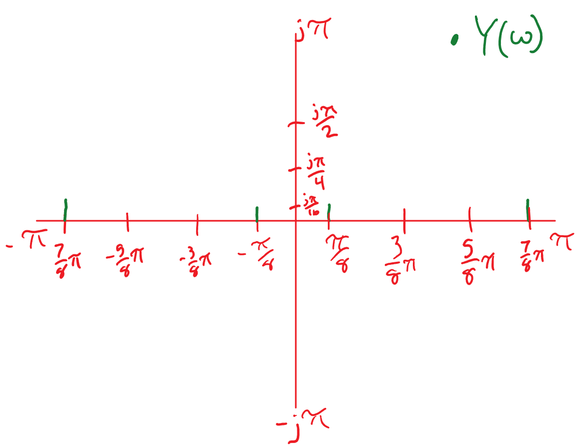
$$Q(\omega) = \frac{j\pi}{4}\left[\delta\left(\omega + \frac{3}{8}\pi\right) + \delta\left(\omega - \frac{3}{8}\pi\right) + \delta\left(\omega + \frac{5}{8}\pi\right) + \delta\left(\omega - \frac{5}{8}\pi\right)\right]$$



$$R(\omega) = \frac{j\pi}{2}\left[\delta\left(\omega + \frac{3}{8}\pi\right) + \delta\left(\omega - \frac{3}{8}\pi\right)\right]$$



$$Y(\omega) = \frac{j\pi}{2}\left[\delta\left(\omega + \frac{3}{8}\pi\right) + \delta\left(\omega - \frac{3}{8}\pi\right) + \delta\left(\omega + \frac{5}{8}\pi\right) + \delta\left(\omega - \frac{5}{8}\pi\right)\right]$$



$$q[n] = \frac{-j}{4}\left[e^{j\frac{3}{8}\pi n} - e^{-j\frac{3}{8}\pi n} + e^{j\frac{5}{8}\pi n} - e^{-j\frac{5}{8}\pi n}\right]$$

$$r[n] = \frac{-j}{4}\left[e^{j\frac{3}{8}\pi n} - e^{-j\frac{3}{8}\pi n}\right]$$

$$y[n] = \frac{j}{4}\left[e^{-j\frac{\pi}{8}n} - e^{j\frac{\pi}{8}n} + e^{j\frac{7}{8}\pi n} - e^{-j\frac{7}{8}\pi n}\right]$$

$$y[n] = \frac{j}{8}\left[e^{-j\frac{\pi}{8}n} - e^{j\frac{\pi}{8}n} + e^{j\frac{5}{8}\pi n} - e^{-j\frac{5}{8}\pi n} \right]$$

(b) Find $q[n]$, $r[n]$ and $y[n]$.

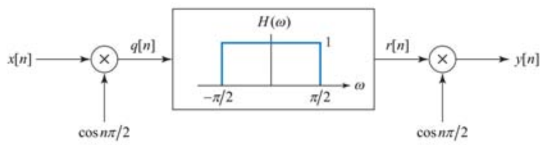


Figure 3.60

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Problem 3-41

(a) Repeat Problem 3-39 assuming that $x[n]$ has the spectrum shown in **Figure 3.61**.

(b) Find $y[n]$ in terms of $x[n]$.

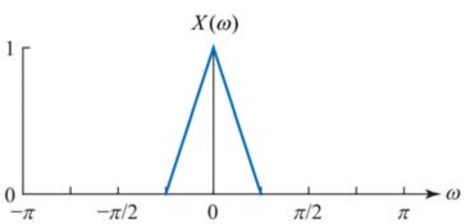


Figure 3.61

$$\frac{4}{\pi}u\left(\omega+\frac{\pi}{4}\right) - \frac{8}{\pi}u(\omega) + \frac{4}{\pi}u\left(\omega-\frac{\pi}{4}\right)$$

$$\int_{-\pi/4}^{0} \frac{4}{\pi}\omega \, d\omega + \int_{0}^{\pi/4} -\frac{4}{\pi}\omega \, d\omega$$

$$X[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi/4}^{0}\frac{4}{\pi}\omega\, d\omega + \int_{0}^{\pi/4} -\frac{4}{\pi}\omega \, d\omega\right) e^{j\omega n} \, d\omega$$

(b) Find $q[n]$, $r[n]$ and $y[n]$.

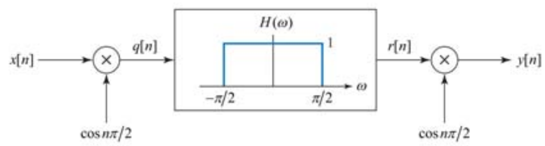


Figure 3.60

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Problem 3-48

(a) Show that if $x[n]$ is conjugate-symmetric (i.e., $x[n] = x^*[-n]$), then $X(\omega)$ is purely real.

$$x[n] = a + jb$$
$$x^*[n] = a - jb$$
$$x^*[-n] = a + jb$$

$$f_r(n) = \cos(w_0 n)$$
$$f_r(-n) = \cos(-w_0 n) = \cos(w_0 n) = f_r(n)$$

$$f_i(n) = \sin(w_0 n)$$
$$f_i(-n) = \sin(-w_0 n) = -\sin(w_0 n) = -f_i(n)$$

imaginary part is odd and thus integrates to 0 over any period

(b) Show that if $x[n]$ is conjugate-antisymmetric (i.e., $x[n] = -x^*[-n]$), then $X(\omega)$ is purely imaginary.

$$re\{x[n]\} = -Re\{x[-n]\}$$
$$im\{x[n]\} = im\{x[-n]\}$$

real: $-\sin(w_0 n)$
imag $\cos(w_0 n)$

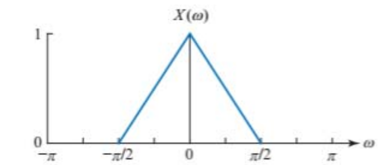
$$f_r[-n] = -\sin(-w_0 n] = \sin(w_0 n) = -f_r[n]$$

$$f_i[-n] = \cos(-w_0 n) = \cos(w_0 n) = f_i[n]$$

thus, real is always odd and integrates to 0 over a given period

Problem 3-58

Suppose that you know $x[n]$, the sequence whose transform is shown in **Figure 3.65**. Given the DTFTs $Y_1(\omega)$, $Y_2(\omega)$, $Y_3(\omega)$ and $Y_4(\omega)$ shown in **Figure 3.66**, find the sequences $y_1[n]$, $y_2[n]$, $y_3[n]$ and $y_4[n]$ in terms of $x[n]$. You should not have to compute explicitly any transforms or inverse transforms.



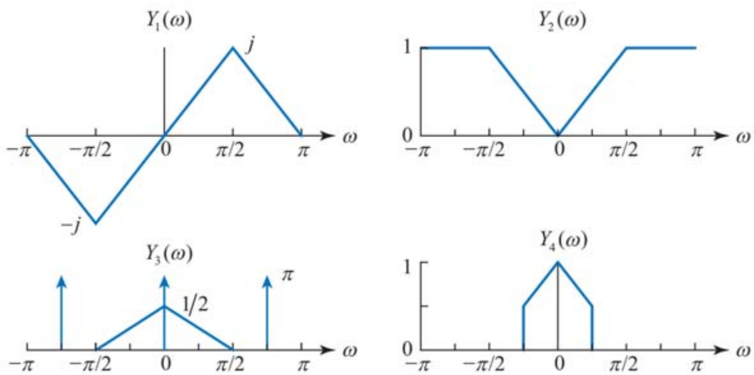


Figure 3.66

$y_1[n] = x[n]e^{j\frac{\pi}{2}n} - x[n]e^{-j\frac{\pi}{2}n}$

$y_2[n] = -x[n] + 1$

$y_3[n] = \cos\left(\frac{3}{4}\pi\right) \cdot x[n]$

$y_4[n] = x[n]\left[u(w_0 + \frac{\pi}{4}) - u(w_0 - \frac{\pi}{4})\right]$

Problem 3-64

For a system characterized by the frequency response $H(\omega)$, the *group delay* is defined as

$$D(\omega) = -\frac{d\measuredangle H(\omega)}{d\omega}.$$

(a) Show that

$$D(\omega) = \mathrm{Re}\left\{ j\frac{\left(\frac{dH(\omega)}{d\omega}\right)}{H(\omega)} \right\}.$$

➤**Hint:** Express $H(\omega) = |H(\omega)|e^{-j\omega n}$ and use the chain rule to take the derivative.

(b) Show that for an FIR system characterized by impulse response $h[n]$,

$$\frac{dH(\omega)}{d\omega} = -j\mathfrak{F}\{nh[n]\},$$

so that

$$D(\omega) = \mathrm{Re}\left\{ \frac{\mathfrak{F}\{nh[n]\}}{\mathfrak{F}\{h[n]\}} \right\}.$$

(c) Show that, for a system characterized by

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{n-0}^{N} b_n e^{-j\omega n}}{\sum_{m-0}^{M} a_n e^{-j\omega n}},$$

we have

$$\begin{aligned}(\omega) &= \mathrm{Re}\left\{ j \left(\frac{\frac{dB(\omega)}{d\omega}}{B(\omega)} \right) \right\} - \mathrm{Re}\left\{ j \left(\frac{\frac{dA(\omega)}{d\omega}}{A(\omega)} \right) \right\} \\ &= \mathrm{Re}\left\{ \frac{\mathfrak{F}\{nb[n]\}}{\mathfrak{F}\{b[n]\}} \right\} - \mathrm{Re}\left\{ \frac{\mathfrak{F}\{na[n]\}}{\mathfrak{F}\{a[n]\}} \right\}.\end{aligned}$$