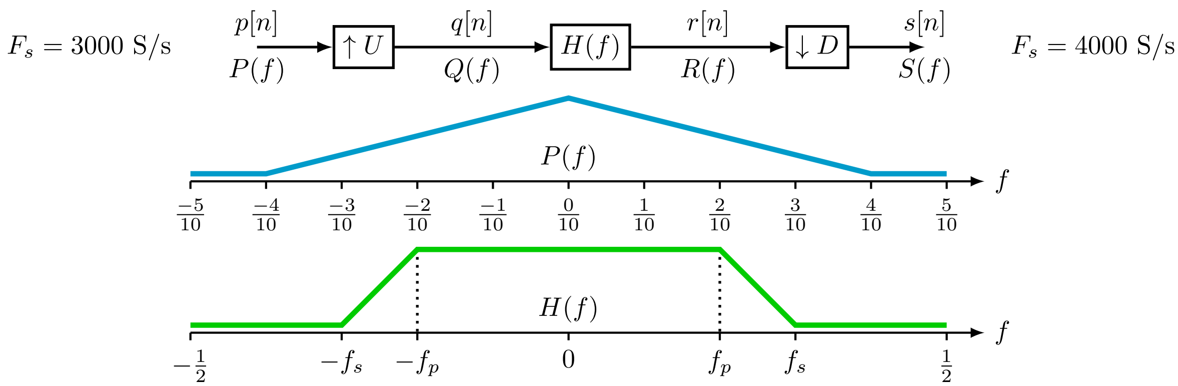
ECE 3640 - Discrete-Time Signals and Systems
Midterm 1 - Spring 2019

Name:	Due: Friday, 8 March 2019 at 11 PM in the homework box.

Instructions:

1. Allowed resources: Your text book, your homework, your notes, the course web site, and Matlab.
2. Do not talk to anyone about this exam or get help from any source on this exam.
3. Write your answers in the spaces provided. Draw a box around your answers.
4. By signing in the name box above, you verify that you have complied with these instructions.

1. Consider the sample rate conversion system depicted below comprised by a U-fold exapander, a low pass filter $H(f)$, and a D-fold compressor.



This problem is to choose U and D and design the low pass filter $H(f)$ by selecting f_p and f_s to perform the sample rate conversion from $F_s = 3000$ samples/second at the input to $F_s = 4000$ samples/second at the output. Use the smallest integer factors U and D to perform the conversion. Choose f_p and f_s giving $H(f)$ the widest possible transition band. Write the parameters you choose in the space below.

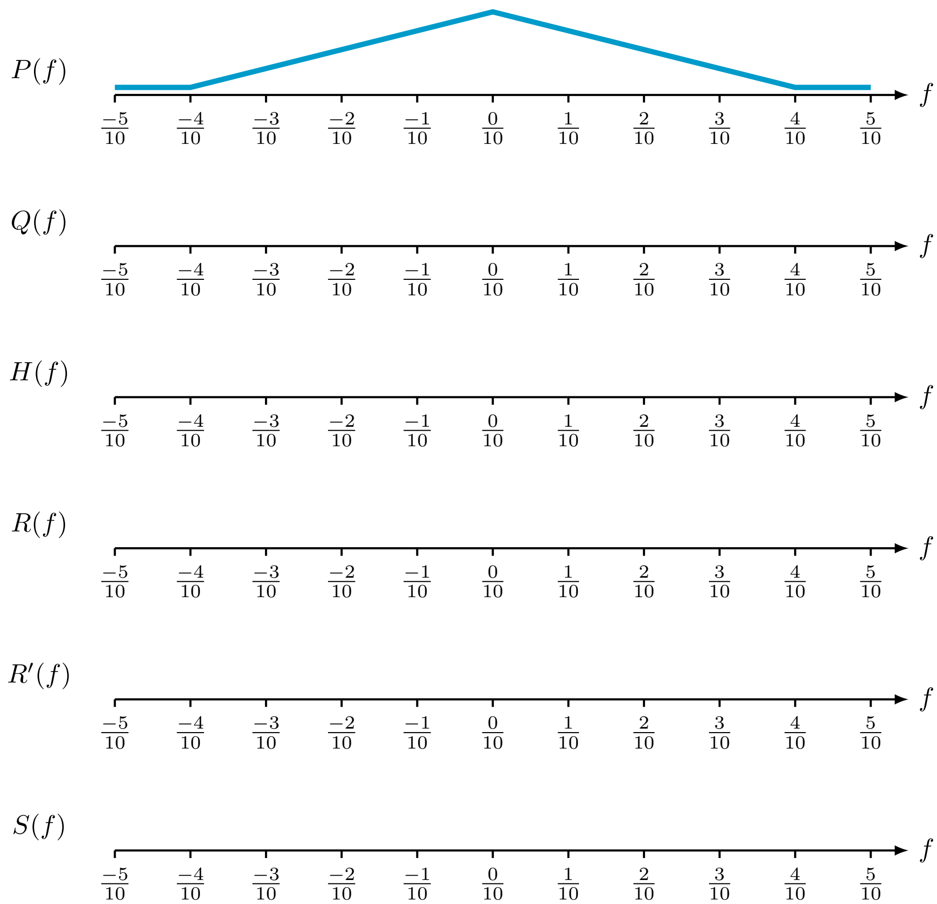
$U =$

$D =$

$f_p =$

$f_s =$

Sketch the spectra $Q(f), H(f), R(f), S(f)$ on the axes below. An extra axis has been provided to sketch $R'(f)$, the aliased version of $R(f)$. Label important frequencies in each plot. Important frequencies are the start, end, and center frequency of each "image" or "replica". Only the frequencies matter. Don't worry about the amplitudes in these plots.



Solution:

First we find the expand and compression factors by writing down an equation for the output sample rate,

$$3000 \cdot \frac{U}{D} = 4000 \qquad \Longrightarrow \qquad \begin{array}{l} U = 4, \\ D = 3. \end{array}$$

Now we sketch the spectrum $Q(f)$ at the output of the expander and notice the gap between the image centered at $f = 0$ and the image centered at $f = \frac{1}{U} = \frac{1}{4}$. The low pass filter $H(f)$ is designed to pass the image at $f = 0$ and remove the other images. The maximum frequency of the image at $f = 0$ is 0.1 and we let this be the passband edge frequency for $H(f)$: $f_p = 0.1$. The minimum frequency of teh image at $f = \frac{1}{4}$ is $0.25 - 0.1 = 0.15$ and we let this be the stop band edge frequency for $H(f)$: $f_s = 0.15$. A low pass filter $H(f)$ with these critical frequencies is sketched on the axis below. The spectrum $R(f)$ of the filter output is also shown. The aliased spectrum

$$R'(f) = \frac{1}{3}\left(R(f) + R(f - 1/3)R(f - 2/3)\right)$$

and the downsampled spectrum

$$S(f) = R'(f/3) = \frac{1}{3}\left(R(f/3) + R([f - 1]/3)R([f - 2]/3)\right)$$

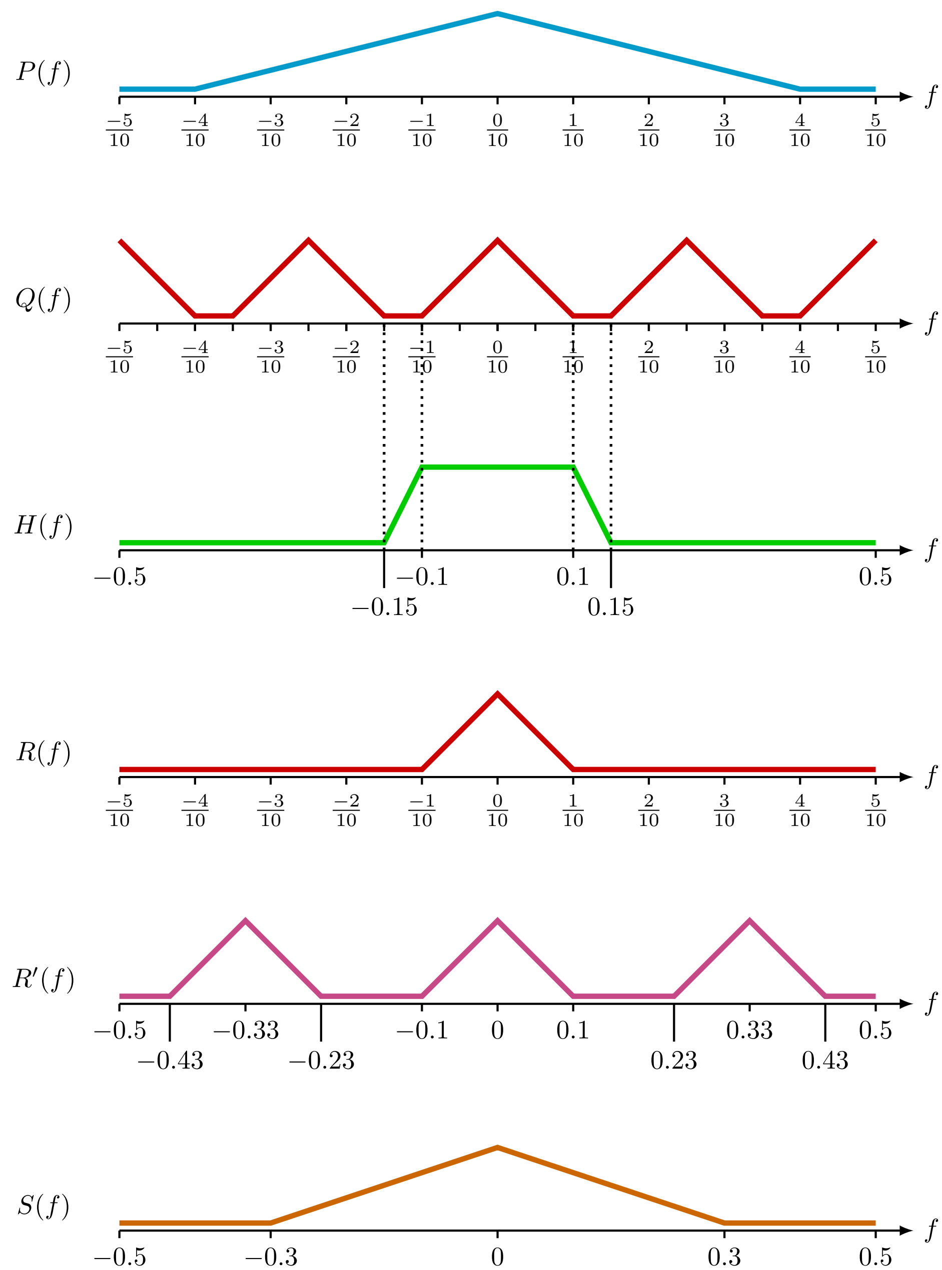
are also shown.

$U = 4$

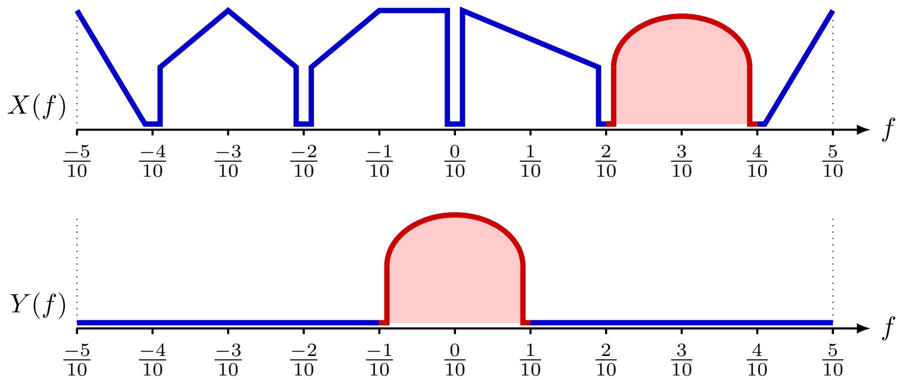
$D = 3$

$f_p = 0.1$

$f_s = 0.15$

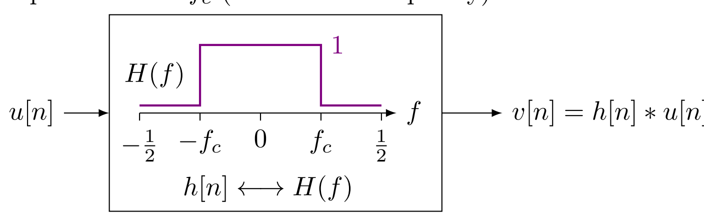


2. Consider the signal $x[n]$ having a spectrum $X(f)$ shown in the figure below. Your job is to isolate at baseband the red-shaded component occupying $\frac{2}{10} \leq f \leq \frac{4}{10}$ cycles/sample. The resulting signal $y[n]$ has the spectrum $Y(f)$ shown below.

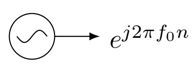


To accomplish this task, you are allowed to use the three different parts as shown below. Using these parts, design a system to process $x[n]$ into $y[n]$. Illustrate your design by drawing a block diagram. On the block diagram indicate the configuration parameters of each part used. On a piece of graph paper (such as the one provided for HW 5), neatly sketch the spectrum of the signal at the input(s) and output for each part in your diagram being careful to label the frequency axes and indicate any important frequencies.

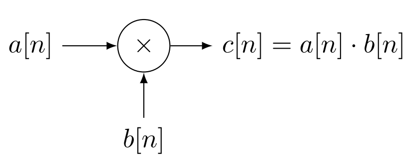
part name: low pass filter
parameter: f_c (filter cutoff frequency)



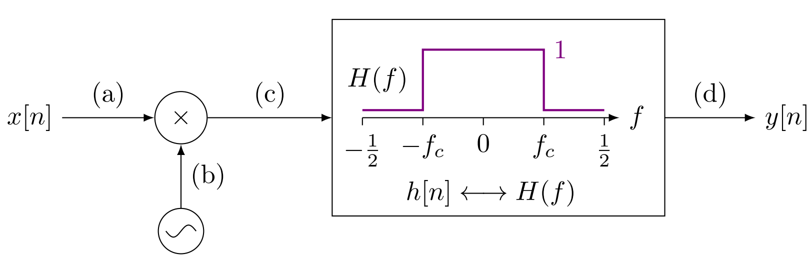
part name: oscillator
parameter: f_0 (oscillator frequency)



part name: mixer
parameter: none



Solution: A system that performs the required operations is illustrated below.



Time domain expressions for the signals at the labeled points are as follows:

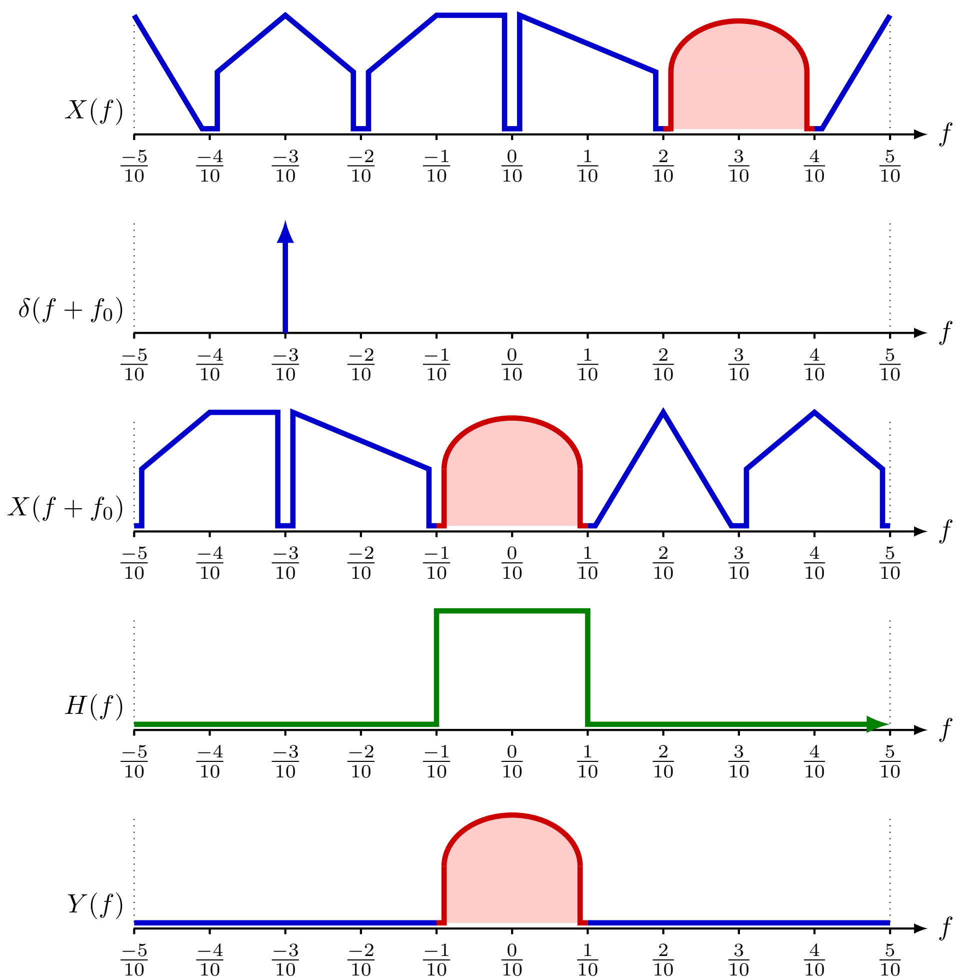
(a) the signal is $x[n]$,
(b) the signal is $e^{j2\pi f_0 n}$ where $f_0 = -0.3$ cycles/sample,
(c) the signal is $x[n]e^{j2\pi f_0 n}$, and
(d) the signal is $y[n] = h[n] * x[n]e^{j2\pi f_0 n}$, where $f_c = 0.1$ cycles/sample.

Frequency domain expressions for the signals at the labeled points are as follows:

(a) the spectrum is $X(f)$,
(b) the spectrum is $\delta(f + f_0)$ for $|f| \leq 0.5$ where $f_0 = -0.3$ cycles/sample,
(c) the spectrum is $X(f + f_0)$, and
(d) the spectrum $Y(f)$ is given by

$$Y(f) = \begin{cases} X(f+f_0), & \text{for } |f| \leq 0.1 = f_c \\ 0, & 0.1 < |f| \leq 0.5 \end{cases}$$

These spectra are illustrated in the graphs below.

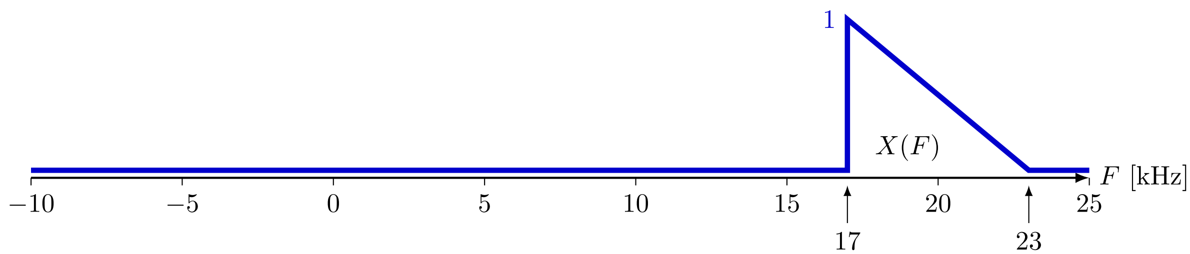


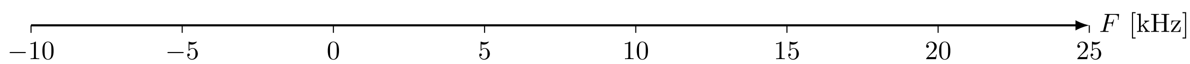
Scoring rubric:

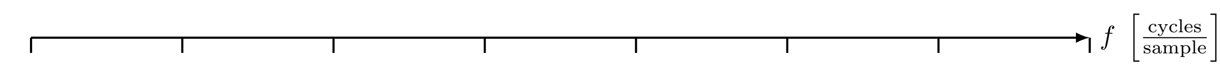
- 6 points for properly setting the part configuration parameters.
- 5 points for the correct block diagram.
- 12 points for the correct spectral sketches.

23 points total

3. Use the principles of sampling to find the minimum sample rate F_s that avoids aliasing when the complex-valued signal $x(t)$ is sampled. The CTFT $X(F)$ of $x(t)$ is pictured below. Note that $X(F) = 0$ at all frequencies not included in the picture. (Hint: The highest frequency in $x(t)$ is 23 kHz. Choosing $F_s > 2 \cdot 23 = 46$ kHz does avoid aliasing, but aliasing can be avoided at lower sample rates as well. This question asks you to find the minimum.)





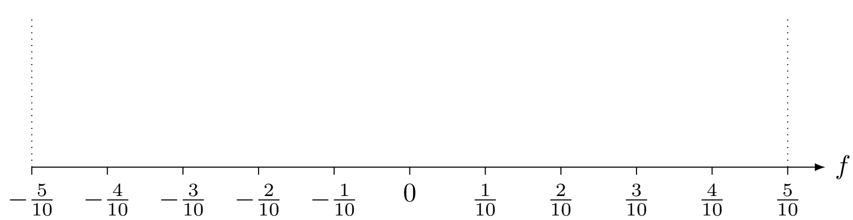


On the axis in the middle, sketch the result of applying the aliasing formula, and on the bottom axis sketch the result of applying the sampling formula and label the frequency axis. In these sketches, use the minimum sampling rate. What is the minimum sampling rate?

$$\text{aliasing formula:} \qquad F_s \sum_{k=-\infty}^{\infty} X(F - kF_s) \tag{1}$$

$$\text{sampling formula:} \qquad F_s \sum_{k=-\infty}^{\infty} X([f - k]F_s) \tag{2}$$

On the axis below, sketch the spectrum of the sampled signal for frequencies $-\frac{1}{2} \le f < \frac{1}{2}$. Label important frequencies and amplitudes.



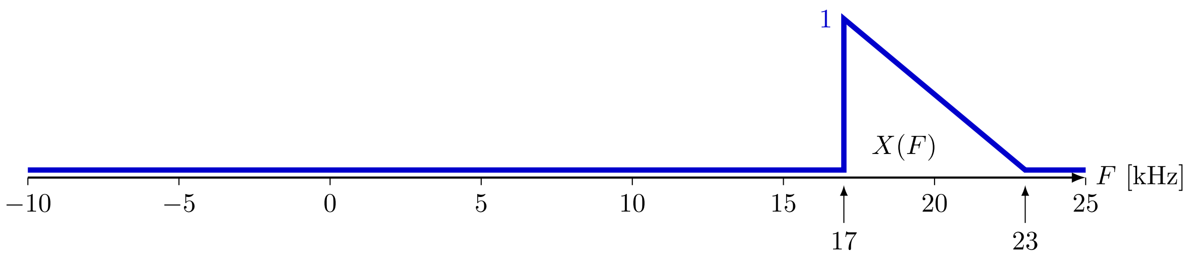
Scoring rubric:

- 3 points for identifying the correct minimum sampling rate

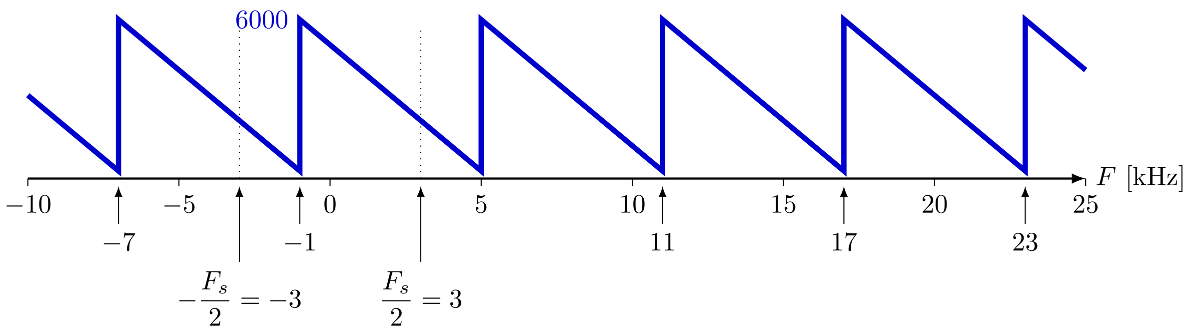
- 9 points for the proper sketches in the middle of the page
- 3 points for the final sketch at the bottom of the page

15 points total

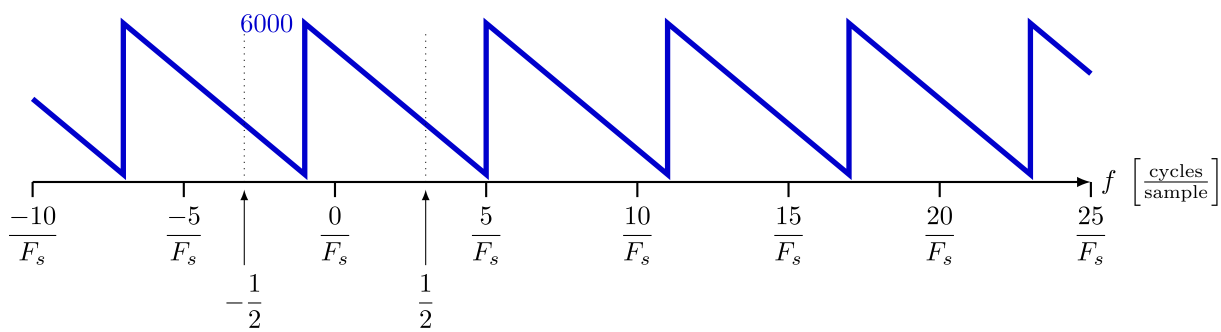
Solution: The minimum sample rate that avoids aliasingn is $F_s = 6000$ samples/second. The spectrum $X(F)$ of the original signal $x(t)$ looks like this.



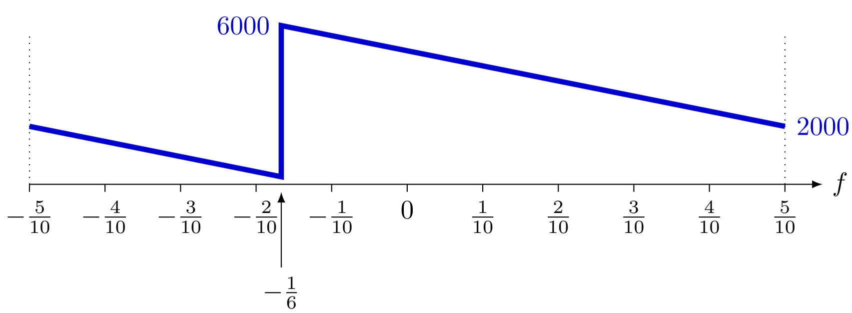
The aliased spectrum is given by $F_s \sum_{k=-\infty}^{\infty} X(F - kF_s)$ and with the minimum sample rate $F_s = 6000$ samples/second, the aliased spectrum looks like this.



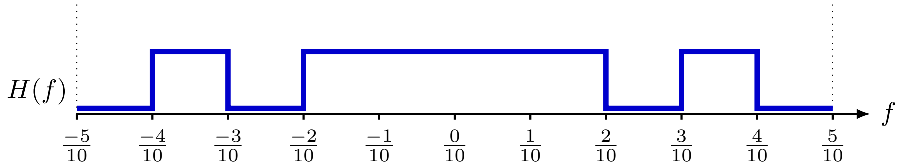
The sampled spectrum is given by $F_s \sum_{k=-\infty}^{\infty} X([f - k]F_s)$ and with the minimum sample rate $F_s = 6000$ samples/second, the sampled spectrum looks like this.

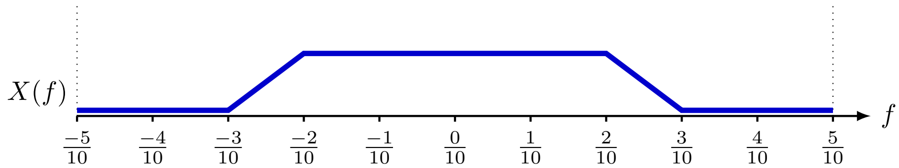


For frequencies in the range $-\frac{1}{2} \leq f \leq \frac{1}{2}$, the sampled spectrum looks like this.

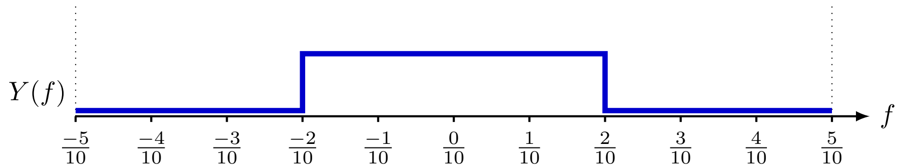


4. Consider a filter with frequency response $H(f)$ having multiple pass bands as shown below. Let $h[n]$ be the impulse response of this filter. Also consider the signal $x[n]$ having the spectrum $X(f)$ shown below. Calculate $y[n] = h[n] * x[n]$. Assume that the peak magnitudes in $H(f)$ and $X(f)$ are 1.





Solution: The result of convolution $y[n] = h[n] * x[n]$ may be computed by starting in the frequency domain and calculating $Y(f) = H(f) \cdot X(f)$. This is easily done for the given signals. The result is pictured below.



Now $y[n]$ may be computed from $Y(f)$ by inverse DTF as follows:

$$
\begin{aligned}
y[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} Y(f) e^{j2\pi f n} df \\
&= \int_{-b}^{b} e^{j2\pi f n} df, \qquad b = \frac{2}{10} = \frac{1}{5} \\
&= \frac{e^{j2\pi \frac{1}{5} n} - e^{-j2\pi \frac{1}{5} n}}{j2 \quad \pi n} \\
&= \frac{\sin\left(\pi \frac{2}{5} n\right)}{\pi n}, \qquad -\infty < n < \infty.
\end{aligned}
$$

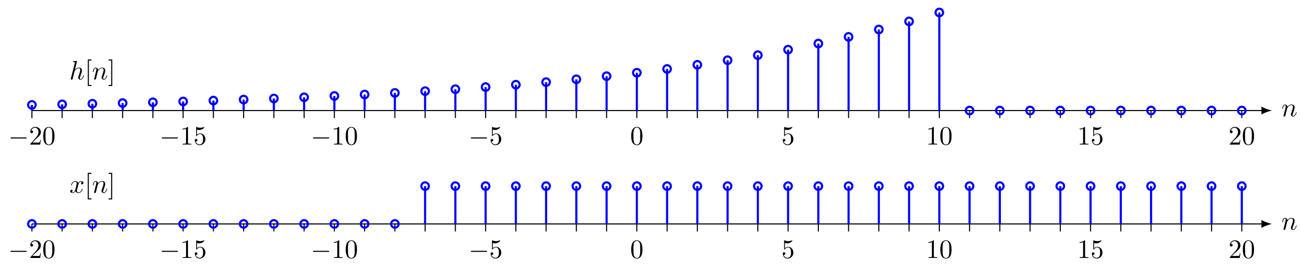
Thus, whatever the strange expressions for $x[n]$ and $h[n]$ may be, their convolution is the sinc function given above.

Scoring rubric:

- 3 points for observing that the problem should be solved in the frequency domain
- 5 points for getting the correct function $y[n]$

8 points total

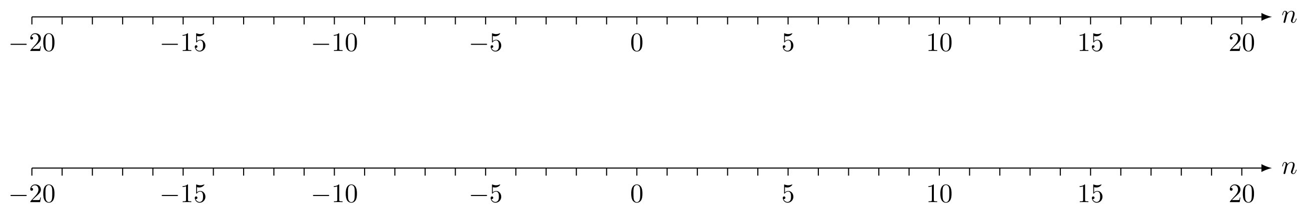
5. Consider a non-causal system with impulse response $h[n] = a^n u[10-n]$ which his plotted below for $a = 1.1$. Let the input to this system be $x[n] = u[n+7]$.



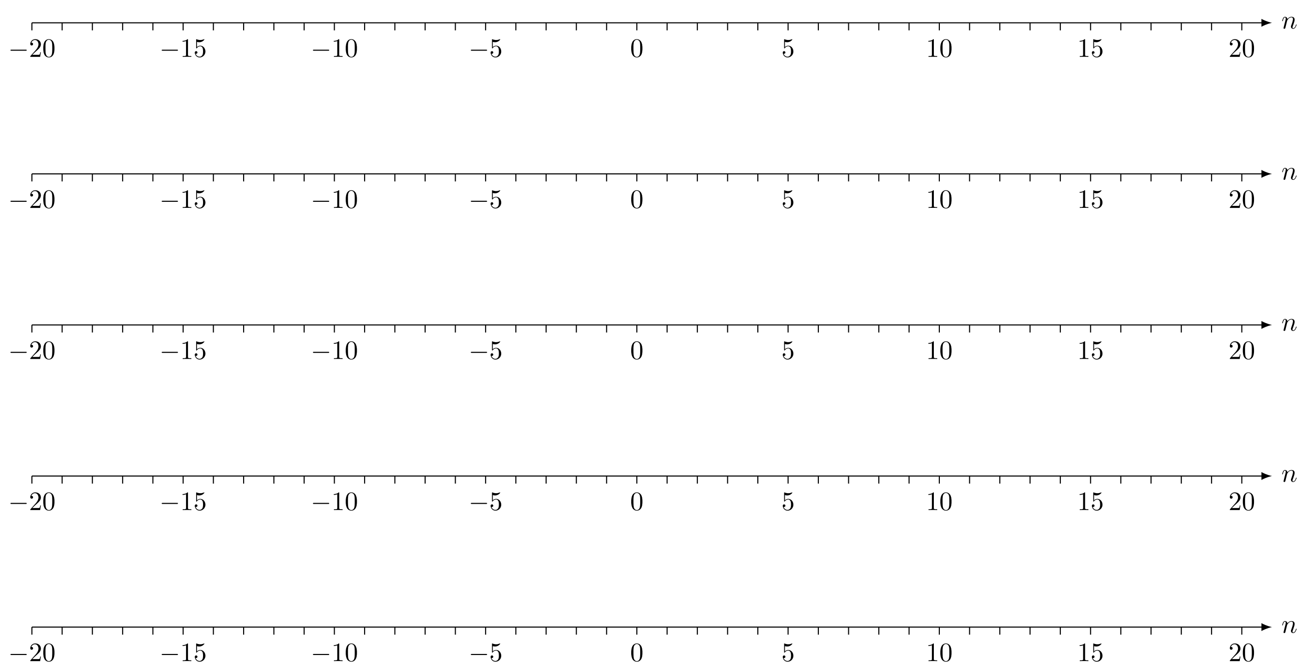
Convolution evaluates the sum

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

for each n. On the axes below, sketch $x[k]$ and $h[-k]$.



Considering $x[k]$ and $h[n-k]$, how many different overlap cases are there? Sketch $h[n-k]$ on the axes below for each of the cases and indicate the range of n values for which each case holds.



For each of the distinct cases identified above, what are the simplified limits on the convolution sum? Put your answers on the sums below and include the time limits from above.

$$y[n] = \sum_{k=\boxed{}}^{\boxed{}} x[k]h[n-k] = \sum_{k=\boxed{}}^{\boxed{}} a^{n-k}, \qquad \boxed{}$$

$$y[n] = \sum_{k=\boxed{}}^{\boxed{}} x[k]h[n-k] = \sum_{k=\boxed{}}^{\boxed{}} a^{n-k}, \qquad \boxed{}$$

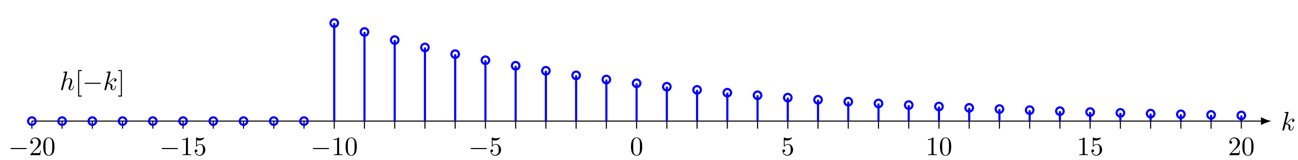
$$y[n] = \sum_{k=\boxed{}}^{\boxed{}} x[k]h[n-k] = \sum_{k=\boxed{}}^{\boxed{}} a^{n-k}, \qquad \boxed{}$$

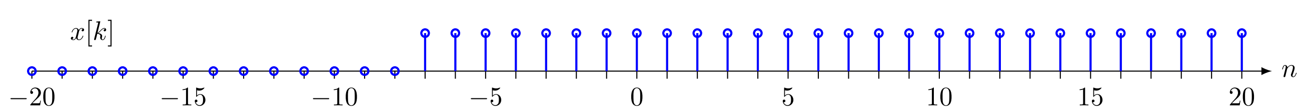
$$y[n] = \sum_{k=\boxed{}}^{\boxed{}} x[k]h[n-k] = \sum_{k=\boxed{}}^{\boxed{}} a^{n-k}, \qquad \boxed{}$$

$$y[n] = \sum_{k=\boxed{}}^{\boxed{}} x[k]h[n-k] = \sum_{k=\boxed{}}^{\boxed{}} a^{n-k}, \qquad \boxed{}$$

Calculate the value of the output $y[n]$ at time $n = -2$ and the value at time $n = 5$.

Solution:



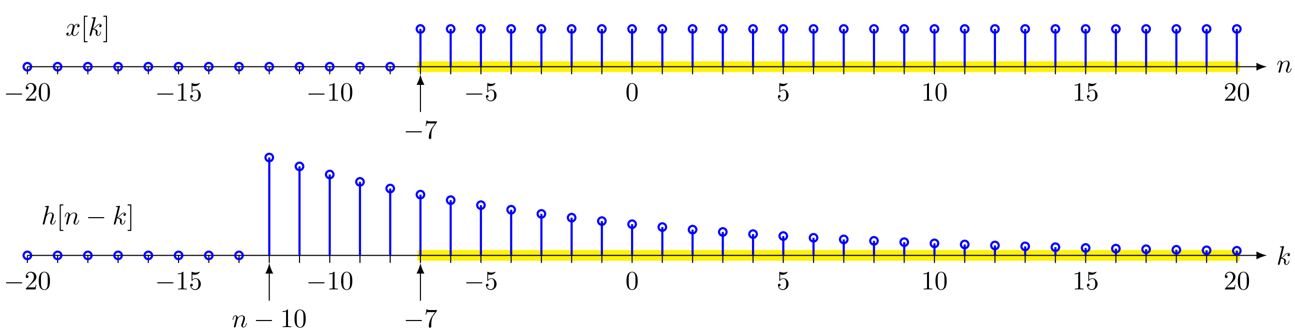


For these two signals, there are exactly two cases for the convolution sum

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

For both cases, there is partial overlap between the two signals.

Case 1. The first case occurs for $n - 10 \leq -7$ or when $n \leq 3$. Note that I have drawn the picture for $n = -2$.

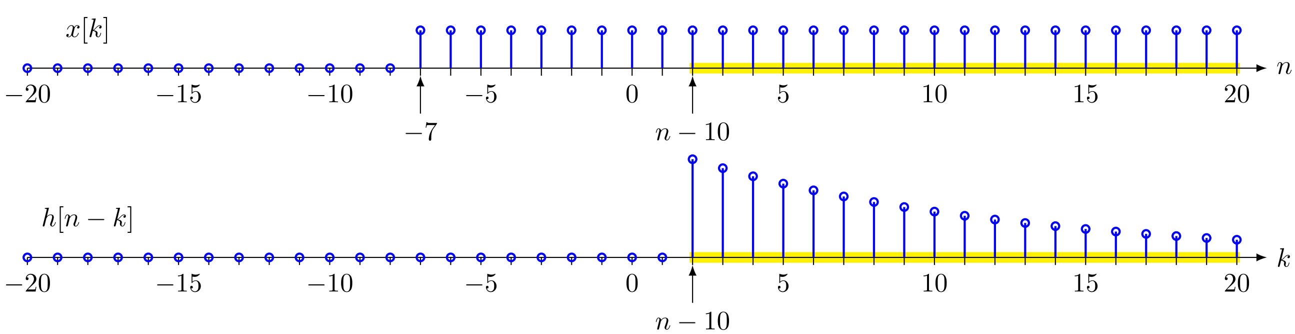


Yellow shading has been used to highlight the overlapping regions which define the limits on the convolution sum,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-7}^{\infty} a^{n-k} = \frac{a^{n+7}}{1-a^{-1}}, \qquad n \leq 3.$$

The geometric series formula may be applied becasue $a^{-1} = (1.1)^{-1} < 1$.

Case 2. The second case occurs for $n - 10 \geq -7$ or when $n \geq 3$. Note that I have drawn the picture for $n = 12$.



Yellow shading has been used to highlight the overlapping regions which define the limits on the convolution sum,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=n-10}^{\infty} a^{n-k} = \frac{a^{n-n+10}}{1-a} = \frac{a^{10}}{1-a^{-1}} \approx 28.5312, \qquad n \geq 3.$$

The geometric series formula may be applied becasue $a^{-1} = (1.1)^{-1} < 1$. Note that the two cases agree for $n = 3$.

The output at time $y[-2] \approx 17.7156$ and $y[5] \approx 28.5312$.

Scoring rubric:

- 5 points for the sketch of $x[k]$ and $h[-k]$
- 5 points for the cases and the associated sketches and ranges for n
- 5 points for getting the limits right on the convolution sums
- 4 points of extra credit for correctly computing $y[-2]$ and $y[5]$ (2 points each).

15 points total

6. The everlasting complex exponential signal $x[n] = e^{j2\pi 0.2n}$ is input to a LTI system having frequency response given by

$$H(f) = \frac{2 - 1.5371e^{-j2\pi f}}{1 - 1.5371e^{-j2\pi f} + 0.9025e^{-j4\pi f}}.$$

Write an expression for the output signal $y[n]$.

Plot the magnitude $|H(f)|$ and phase $\angle H(f)$ in Matlab for `f=[-0.5:0.001:0.5]`. (Include your code and plots on a separate page.) Given the symmetry of $H(f)$, what can you say about the impulse response $h[n]$? Given the continuity of $H(f)$, what can you say about the impulse response $h[n]$? (Is it absolutely summable, square summable (energy), or mean-square summable (power)?)

Solution: Because this is an everlasting complex exponential signal, the output signal has the form

$$y[n] = H(0.2)x[n] = H(0.2)e^{j2\pi 0.2n}$$

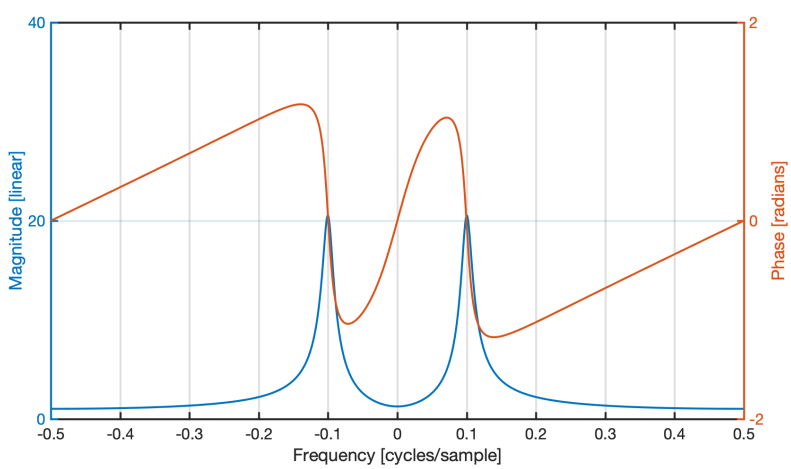
for all n. Using the given formula, $H(f = 0.2) = 1.1530 - j1.8913 = 2.2150e^{-j1.0233} = 2.2150e^{-j2\pi 0.1629}$. With these values, the output signal may be expressed as

$$y[n] = 2.2150e^{j[2\pi 0.2n - 1.0233]} = 2.2150e^{j2\pi[0.2n - 0.1629]}$$

for all n.

A plot of the magnitude and phase responses is given below.

```
H = @(f) (2-1.5371*exp(-1j*2*pi*f))./(1-1.5371*exp(-1j*2*pi*f)+0.9025*exp(-1j*4*pi*f));
f = [-0.5:0.001:0.5];
plotyy(f,abs(H(f)),f,angle(H(f)));
```



The frequency response $H(f)$ exhibhts Hermitian symmetry (even magnitude, odd phase). This implies that the impulse response $h[n]$ must be purely real valued, i.e. Imag$\{y[n]\} = 0$. Also beause $H(f)$ is a continuous (smooth) function, $h[n]$ must be absolutely summable. Therefore this is a stable LTI system.

Scoring rubric:

- 5 points for writing down an appropriate expression for $y[n]$.
- 5 points for the magnitude and phase plots.
- 5 points for correctly identifying and using the symmetry and continuity of $H(f)$.

15 points total

7. What are the frequencies and periods of the following signals. Give all frequencies in the interval $-\frac{1}{2} \leq f < \frac{1}{2}$.

   (a) $x[n] = \cos\left(2\pi \frac{430}{45} n\right), \quad -\infty < n < \infty.$

   Solution: The frequency is $430/45 = 86/9$ cycles/sample. This frequency aliases to $86/9 - 90/9 = -4/9$. Becasue the frequency is rational, the sinusoidal sequence is periodic. The period is 9.

   (b) $x[n] = \sin\left(2\pi \sqrt{\frac{430}{45}} n + \frac{4\pi}{3}\right), \quad -\infty < n < \infty.$

   Solution: The frequency $\sqrt{430/45} = \sqrt{86/9} = \sqrt{2}\sqrt{43}/3$ cycles/sample. This frequency is irrational. Therefore, the sinusoidal sequence is not periodic. The period is undefined.

   Scoring rubric:

   - 5 points for getting the correct frequency and period in the first part
   - 5 points for getting the correct frequency and period in the second part

   10 points total