Context-Free Grammars

Formalism
Derivations
Backus-Naur Form
Left- and Rightmost Derivations

Informal Comments

- A *context-free grammar* is a notation for describing languages.
- It is more powerful than finite automata or RE's, but still cannot define all possible languages.
- Useful for nested structures, e.g., parentheses in programming languages.

Informal Comments – (2)

- Basic idea is to use "variables" to stand for sets of strings (i.e., languages).
- These variables are defined recursively, in terms of one another.
- Recursive rules ("productions") involve only concatenation.
- Alternative rules for a variable allow union.

Example: CFG for $\{ 0^n1^n \mid n \geq 1\}$

- Productions:

  S -> 01

  S -> 0S1
- Basis: 01 is in the language.
- Induction: if w is in the language, then so is 0w1.

CFG Formalism

- *Terminals* = symbols of the alphabet of the language being defined.
- *Variables* = *nonterminals* = a finite set of other symbols, each of which represents a language.
- *Start symbol* = the variable whose language is the one being defined.

Productions

- A *production* has the form variable -> string of variables and terminals.
- Convention:
  - A, B, C,… are variables.
  - a, b, c,… are terminals.
  - …, X, Y, Z are either terminals or variables.
  - …, w, x, y, z are strings of terminals only.
  - α, β, γ,… are strings of terminals and/or variables.

Example: Formal CFG

- Here is a formal CFG for $\{ 0^n 1^n \mid n \geq 1 \}$.
- Terminals = {0, 1}.
- Variables = {S}.
- Start symbol = S.
- Productions =

  S -> 01

  S -> 0S1

Derivations – Intuition

- We *derive* strings in the language of a CFG by starting with the start symbol, and repeatedly replacing some variable A by the right side of one of its productions.
  - That is, the "productions for A" are those that have A on the left side of the ->.

Derivations – Formalism

- We say $\alpha A \beta$ => $\alpha \gamma \beta$ if A -> γ is a production.
- Example: S -> 01; S -> 0S1.
- S => 0S1 => 00S11 => 000111.

Iterated Derivation

- =>* means "zero or more derivation steps."
- Basis: α =>* α for any string α.
- Induction: if α =>* β and β => γ, then α =>* γ.

Example: Iterated Derivation

- S -> 01; S -> 0S1.
- S => 0S1 => 00S11 => 000111.
- So S =>* S; S =>* 0S1; S =>* 00S11; S =>* 000111.

Sentential Forms

- Any string of variables and/or terminals derived from the start symbol is called a *sentential form*.
- Formally, α is a sentential form iff

$$S \Rightarrow^* \alpha.$$

Language of a Grammar

- If G is a CFG, then L(G), the *language of G*, is {w | S =>* w}.
  - Note: w must be a terminal string, S is the start symbol.
- Example: G has productions S -> ε and S -> 0S1.
- L(G) = $\{0^n 1^n \mid n \geq 0\}$.

Note: ε is a legitimate right side.

Context-Free Languages

- A language that is defined by some CFG is called a *context-free language*.
- There are CFL's that are not regular languages, such as the example just given.
- But not all languages are CFL's.
- Intuitively: CFL's can count two things, not three.

Leftmost and Rightmost Derivations

- Derivations allow us to replace any of the variables in a string.
- Leads to many different derivations of the same string.
- By forcing the leftmost variable (or alternatively, the rightmost variable) to be replaced, we avoid these "distinctions without a difference."

Leftmost Derivations

- Say $wA\alpha \Rightarrow_{lm} w\beta\alpha$ if w is a string of terminals only and $A \rightarrow \beta$ is a production.
- Also, $\alpha \Rightarrow^*_{lm} \beta$ if α becomes β by a sequence of 0 or more $\Rightarrow_{lm}$ steps.

Example: Leftmost Derivations

- Balanced-parentheses grammmar:

  S -> SS | (S) | ()

- $S \Rightarrow_{lm} SS \Rightarrow_{lm} (S)S \Rightarrow_{lm} (())S \Rightarrow_{lm} (())()$
- Thus, $S \Rightarrow^*_{lm} (())()$
- S => SS => S() => (S)() => (())() is a derivation, but not a leftmost derivation.

Rightmost Derivations

- Say $\alpha A w \Rightarrow_{rm} \alpha\beta w$ if w is a string of terminals only and $A \rightarrow \beta$ is a production.
- Also, $\alpha \Rightarrow^*_{rm} \beta$ if α becomes β by a sequence of 0 or more $\Rightarrow_{rm}$ steps.

Example: Rightmost Derivations

- Balanced-parentheses grammmar:

  S -> SS | (S) | ()

- $S \Rightarrow_{rm} SS \Rightarrow_{rm} S() \Rightarrow_{rm} (S)() \Rightarrow_{rm} (())()$
- Thus, $S \Rightarrow^*_{rm} (())()$
- S => SS => SSS => S()S => ()()S => ()()() is neither a rightmost nor a leftmost derivation.

Parse Trees

Definitions

Relationship to Left- and Rightmost Derivations

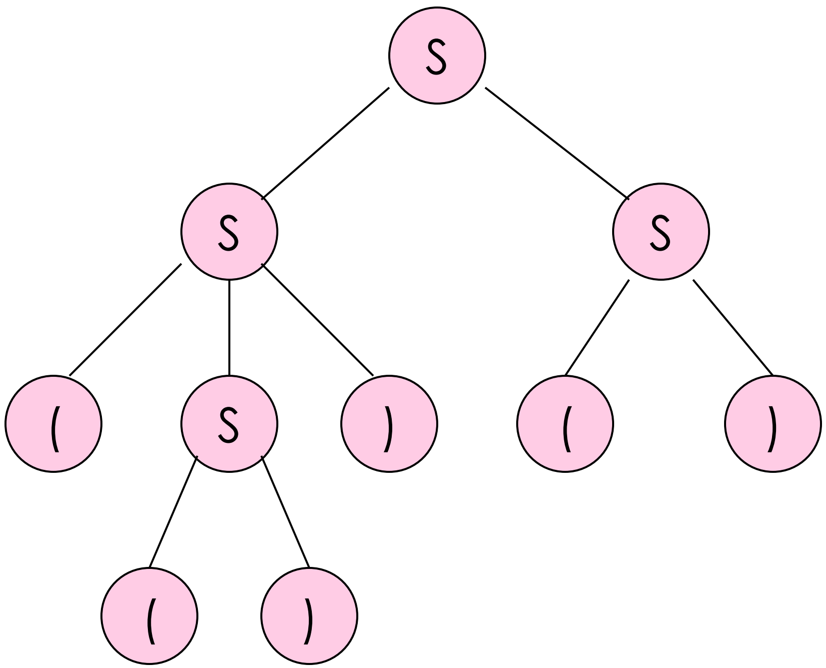
Ambiguity in Grammars

Parse Trees

- *Parse trees* are trees labeled by symbols of a particular CFG.
- Leaves: labeled by a terminal or ε.
- Interior nodes: labeled by a variable.
  - Children are labeled by the right side of a production for the parent.
- Root: must be labeled by the start symbol.

Example: Parse Tree

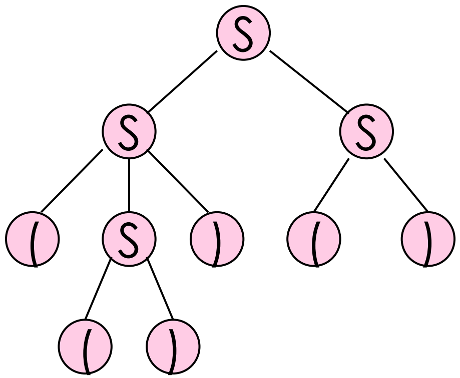
S -> SS | (S) | ()



Yield of a Parse Tree

- The concatenation of the labels of the leaves in left-to-right order
  - That is, in the order of a preorder traversal.

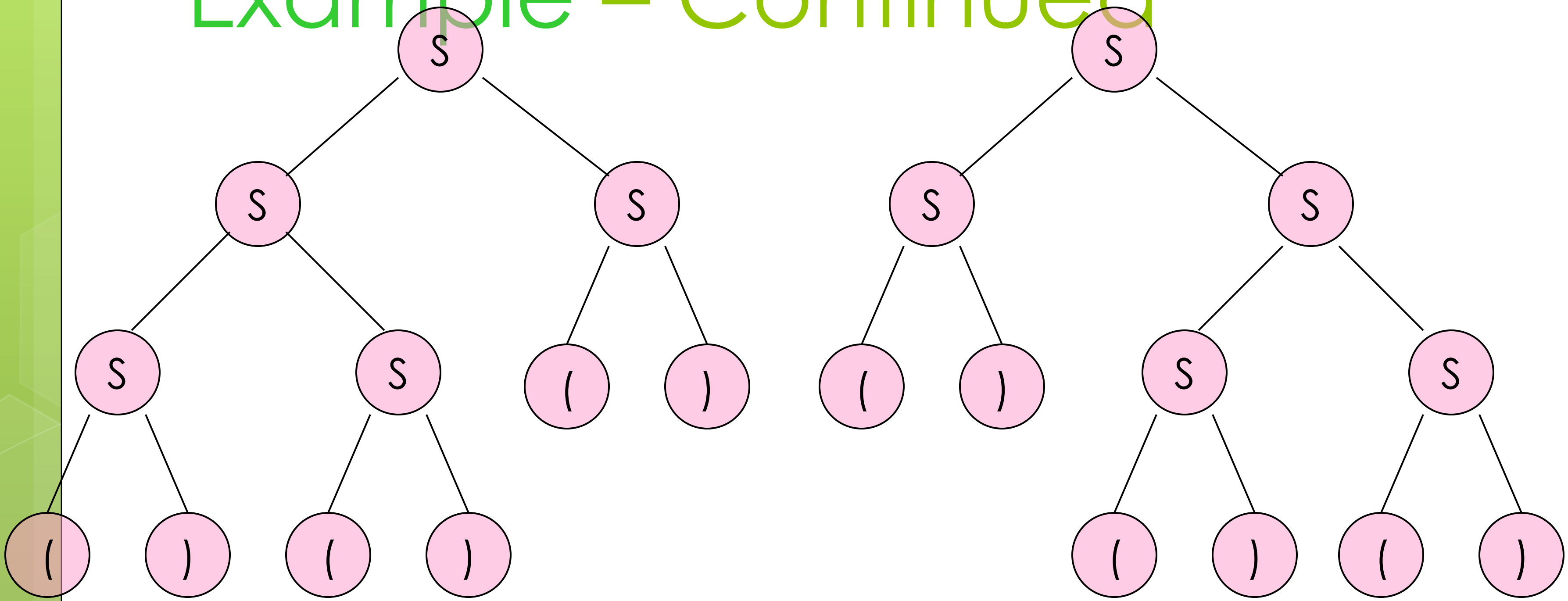
  is called the *yield* of the parse tree.

- Example: yield of  is (())()

Ambiguous Grammars

- A CFG is *ambiguous* if there is a string in the language that is the yield of two or more parse trees.
- Example: S -> SS | (S) | ()
- Two parse trees for ()()() on next slide.

Example – Continued



Ambiguity, Left- and Rightmost Derivations

- If there are two different parse trees, they must produce two different leftmost derivations by the construction given in the proof.
- Conversely, two different leftmost derivations produce different parse trees by the other part of the proof.
- Likewise for rightmost derivations.

Ambiguity, etc. – (2)

- Thus, equivalent definitions of "ambiguous grammar'' are:
  1. There is a string in the language that has two different leftmost derivations.
  2. There is a string in the language that has two different rightmost derivations.

Ambiguity is a Property of Grammars, not Languages

- For the balanced-parentheses language, here is another CFG, which is unambiguous.

B -> (RB | ϵ

R ->) | (RR

B, the start symbol, derives balanced strings.

R generates strings that have one more right paren than left.

Example: Unambiguous Grammar

B -> (RB | ϵ   R ->) | (RR

- Construct a unique leftmost derivation for a given balanced string of parentheses by scanning the string from left to right.
  - If we need to expand B, then use B -> (RB if the next symbol is "(" and ϵ if at the end.
  - If we need to expand R, use R ->) if the next symbol is ")" and (RR if it is "(".

The Parsing Process

Remaining Input:

(())()

Next symbol

Steps of leftmost derivation:

B

B -> (RB | ϵ R ->) | (RR

The Parsing Process

Remaining Input:

())()

Next symbol

Steps of leftmost derivation:

B

(RB

B -> (RB | ϵ R ->) | (RR

The Parsing Process

Remaining Input:

))()

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

)()

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

B -> (RB | ϵ R ->) | (RR

The Parsing Process

Remaining Input:

()

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

)

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

(())(RB

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

(())(RB

(())()B

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

(())(RB

(())()B

(())()

B -> (RB | ϵ   R ->) | (RR

LL(1) Grammars

- As an aside, a grammar such B -> (RB | ϵ R ->) | (RR, where you can always figure out the production to use in a leftmost derivation by scanning the given string left-to-right and looking only at the next one symbol is called LL(1).
  - "Leftmost derivation, left-to-right scan, one symbol of lookahead."

LL(1) Grammars – (2)

- Most programming languages have LL(1) grammars.
- LL(1) grammars are never ambiguous.