Mechanics

Physics 151

Lecture 20

Canonical Transformations

(Chapter 9)

What We Did Last Time

- Hamilton's Principle in the Hamiltonian formalism
  - Derivation was simple $\delta I \equiv \delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t) \right) dt = 0$
  - Additional end-point constraints

    $$\delta q(t_1) = \delta q(t_2) = \delta p(t_1) = \delta p(t_2) = 0$$

    - Not strictly needed, but adds flexibility to the definition of the action integral
  - This connects to: Canonical Transformations
- Principle of Least Action $\Delta \int_{t_1}^{t_2} p_i \dot{q}_i dt = 0$

Got into this a bit

Canonical Transformation

- Goal: To find transformations

$$Q_i = Q_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \qquad P_i = P_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$$

that satisfy Hamilton's equation of motion

$$\dot{q}_i = \frac{\partial H}{dp_i} \qquad \dot{p}_i = -\frac{\partial H}{dq_i} \quad \Longrightarrow \quad \dot{Q}_i = \frac{\partial K}{dP_i} \qquad \dot{P}_i = -\frac{\partial K}{dQ_i}$$

  - K is the transformed Hamiltonian $K = K(Q, P, t)$

- Hamilton's principle requires

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t) \right) dt = 0 \quad \text{and} \quad \delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q, P, t) \right) dt = 0$$

General Transformation

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t) \right) dt = 0$$ and $$\delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q, P, t) \right) dt = 0$$

- Two types of transformations are possible
  - $P_i \dot{Q}_i - K = \lambda (p_i \dot{q}_i - H)$ ⇐ Scale transformation
  - $P_i \dot{Q}_i - K + \dfrac{dF}{dt} = p_i \dot{q}_i - H$ ⇐ Canonical transformation
  - Both satisfy Hamilton's principle
- Combined, we find

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = \lambda (p_i \dot{q}_i - H)$$ ⇐ Extended Canonical transformation

Scale Transformation

- We can always change the scale of (or unit we use to measure) coordinates and momenta

  $P_i = \nu p_i \qquad Q_i = \mu q_i$

- To satisfy Hamilton's principle, we can define

  $K(P,Q,t) = \mu\nu H(p,q,t)$

  $\Rightarrow P_i\dot{Q}_i - K = \mu\nu(p_i\dot{q}_i - H)$ ⇐ Scale transformation

- This is trivial

- We now concentrate on – Canonical transformations

Canonical Transformation

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = p_i \dot{q}_i - H$$

- Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K \right) dt = \delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H - \frac{dF}{dt} \right) dt = -\delta \left[F \right]_{t_1}^{t_2} = 0$$

  - Satisfied if $\delta p = \delta q = \delta P = \delta Q = 0$ at t_1 and t_2

- F can be any function of p_i, q_i, P_i, Q_i and t
  - It defines a canonical transformation
  - Call it the generating function of the transformation

or generator

Simple Example [1]

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- Try a generating function: $F = q_i P_i - Q_i P_i$
  - Canonical transformation generated by F is

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = -K + (q_i - Q_i)\dot{P}_i + P_i\dot{q}_i = p_i\dot{q}_i - H$$

$\Rightarrow$ $Q_i = q_i$ $P_i = p_i$ $\Leftarrow$ Identity transformation

$K = H$

  - OK, that was too simple
    - Let's push this one step further…

Simple Example [2]

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- Let's try this one: $F = f_i(q_1,\ldots,q_n,t)P_i - Q_iP_i$
  - f_i are arbitrary functions of $q_1 \ldots q_n$ and t

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = -K + (f_i - Q_i)\dot{P}_i + P_i\frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial t}P_i = p_i\dot{q}_i - H$$

$\Rightarrow$ $Q_i = f_i(q_1,\ldots,q_n,t)$

All "point transformations" of generalized coordinates are covered

$$p_i = \frac{\partial f_j}{\partial q_i}P_j$$

Must invert these n equations to get P_i

$$K = H + \frac{\partial f_i}{\partial t}P_i$$

- We can do all what we could do before

Arbitrarity

- Generating function F → a canonical transformation
  - Opposite mapping is not unique
    - There are many possible Fs for each transformation
  - e.g. add an arbitrary function of time $g(t)$ to F

$$P_i\dot{Q}_i - K + \frac{dF}{dt} \to P_i\dot{Q}_i - K + \frac{dF}{dt} + \boxed{\frac{dg(t)}{dt}}$$

Does not affect the action integral

$$\Rightarrow \quad K \to K + \frac{dg(t)}{dt}$$

Just modifies the Hamiltonian without affecting physics

- F is arbitrary up to any function of time only
  - So is the Hamiltonian

Finding the Generator

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- Let's look for a generating function
  - Suppose $K(Q,P,t) = H(q,p,t)$ for simplicity

$$\Rightarrow \quad \frac{dF}{dt} = p_i\dot{q}_i - P_i\dot{Q}_i$$

  - Easiest way to satisfy this would be

$$F = F(q,Q) \qquad \frac{\partial F}{\partial q_i} = p_i \qquad \frac{\partial F}{\partial Q_i} = -P_i$$

- Trivial example: $F(q,Q) = q_i Q_i$

$$\Rightarrow \quad p_i = Q_i \qquad P_i = -q_i$$

In the Hamiltonian formalism, you can freely swap the coordinates and the momenta

Type-1 Generator

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- $F = F(q, Q)$ is not very general
  - It does not allow *t*-dependent transformation
  - Fix this by extending to $F = F_1(q, Q, t)$ Call it Type-1

$$p_i = \frac{\partial F_1(q, Q, t)}{\partial q_i} \qquad P_i = -\frac{\partial F_1(q, Q, t)}{\partial Q_i}$$

  - This affects the Hamiltonian

$$\frac{dF}{dt} = \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} = p_i\dot{q}_i - P_i\dot{Q}_i + K - H$$

$$\Rightarrow \quad K = H + \frac{\partial F_1}{\partial t}$$

Harmonic Oscillator

- Consider a 1-dimensional harmonic oscillator

$$H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \qquad \omega^2 \equiv \frac{k}{m}$$

  - Sum of squares → Can we make them sine and cosine?
  - Suppose $p = f(P)\cos Q$ $\quad q = \dfrac{f(P)}{m\omega}\sin Q$

$$\Rightarrow \quad K = H = \frac{\{f(P)\}^2}{2m}$$

Q is cyclic → P is constant

  - Trick is to find $f(P)$ so that the transformation is canonical
    - How?

Harmonic Oscillator

- Let's try a Type-1 generator

$$F_1(q, Q, t) \qquad p = \frac{\partial F_1}{\partial q} \qquad P = -\frac{\partial F_1}{\partial Q}$$

  - Express p as a function of q and Q

$$p = f(P)\cos Q \qquad q = \frac{f(P)}{m\omega}\sin Q \Rightarrow p = m\omega q \cot Q$$

  - Integrate with q $\Rightarrow$ $F_1 = \frac{m\omega q^2}{2}\cot Q$

$$\Rightarrow P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

We are getting somewhere

Harmonic Oscillator

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

- We need to turn $H(q, p)$ into $K(Q, P)$
- Solve the above equations for q and p

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \qquad p = \sqrt{2Pm\omega} \cos Q$$

- Now work out the Hamiltonian

$$K = H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \omega P$$

  - Things don’t get much simpler than this…

Harmonic Oscillator

$$K = \omega P = E$$

- Solving the problem is trivial

$$P = \text{const} = \frac{E}{\omega} \qquad \dot{Q} = \frac{\partial K}{\partial P} = \omega \qquad Q = \omega t + \alpha$$

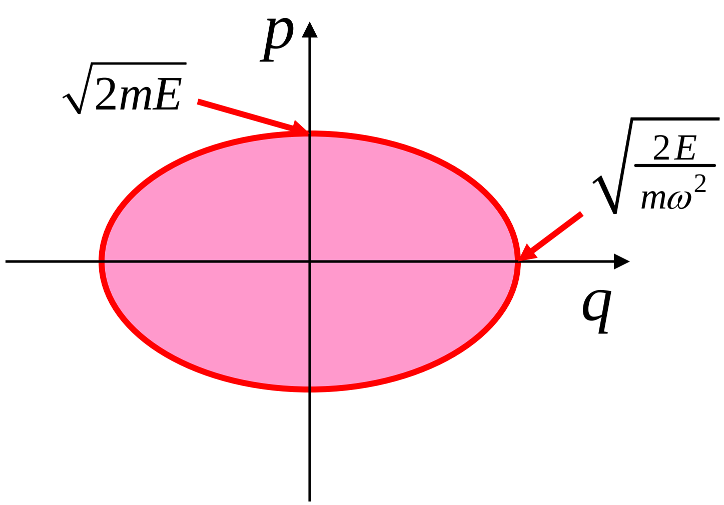
Finally

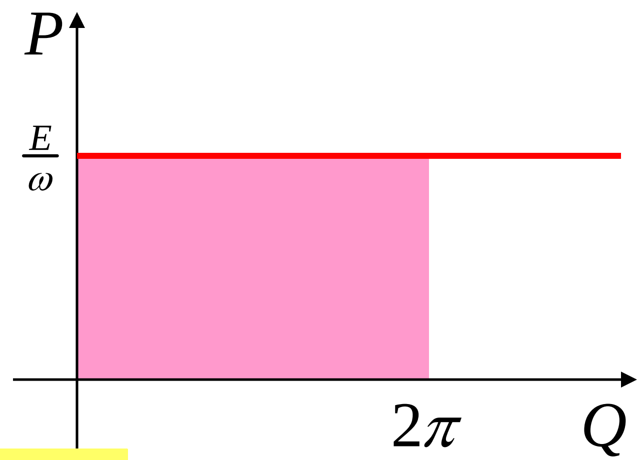
$$p = \sqrt{2Pm\omega}\cos Q = \sqrt{2mE}\cos(\omega t + \alpha)$$

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha)$$

Phase Space

- Oscillator moves in the *p*-*q* and *P*-*Q* phase spaces





  - One cycle draws the same area $\dfrac{2\pi E}{\omega}$ in both spaces

- The area swept by a cyclic system in the phase space is invariant

Will come back to this in Lecture 23

Other Types of Generators

- Type-1 generator $F = F_1(q, Q, t)$ is still not so general
  - Just try to find a generator for $Q_i = q_i$ $P_i = p_i$
- We need generating functions of different set of independent variables
  - In fact, we may have 4 basic types of them

    $F_1(q, Q, t)$ $F_2(q, P, t)$ $F_3(p, Q, t)$ $F_4(p, P, t)$
- We can derive them using the now-familiar rule
  - i.e. we can add any *dF/dt* inside the action integral

Type-2 Generator

- In the last lecture, I used $F = -q_i p_i$ to convert

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t) \right) dt = 0 \Rightarrow \delta \int_{t_1}^{t_2} \left(-\dot{p}_i q_i - H(q, p, t) \right) dt = 0$$

  - Switch the definition of canonical transformations

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = p_i \dot{q}_i - H \Rightarrow -\dot{P}_i Q_i - K + \frac{dF}{dt} = p_i \dot{q}_i - H$$

$$\Rightarrow \frac{dF}{dt} = p_i \dot{q}_i + Q_i \dot{P}_i + K - H$$

  - To satisfy this

$$F = F_2(q, P, t) \qquad \frac{\partial F_2}{\partial q_i} = p_i \qquad \frac{\partial F_2}{\partial P_i} = Q_i \qquad K = H + \frac{\partial F_2}{\partial t}$$

Type-2 Generator

- If we go back to the original definition of generating function $P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$

$F = F_2(q, P, t) - Q_i P_i$ $\quad \frac{\partial F_2}{\partial q_i} = p_i$ $\quad \frac{\partial F_2}{\partial P_i} = Q_i$ $\quad K = H + \frac{\partial F_2}{\partial t}$

  - Trivial case: $F_2 = q_i P_i$

  $\Rightarrow$ $p_i = P_i$ $\quad Q_i = q_i$ $\Leftarrow$ Identity transformation

- We push the same idea to define the other 2 types

Four Basic Generators

Generator	Derivatives	Trivial Case
$F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i}$ $P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$ $Q_i = p_i$ $P_i = -q_i$
$F_2(q,P,t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$ $Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$ $Q_i = q_i$ $P_i = p_i$
$F_3(p,Q,t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$ $Q_i = -q_i$ $P_i = -p_i$
$F_4(p,P,t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$ $Q_i = p_i$ $P_i = -q_i$

Four Basic Generators

- The 4 types of generators are almost equivalent
  - It may look as if F_1 is special, but it isn't

$$P_i \dot{Q}_i - K + \frac{dF_1}{dt} = p_i \dot{q}_i - H$$

$$-\dot{P}_i Q_i - K + \frac{dF_2}{dt} = p_i \dot{q}_i - H$$

$$P_i \dot{Q}_i - K + \frac{dF_3}{dt} = -\dot{p}_i q_i - H$$

$$-\dot{P}_i Q_i - K + \frac{dF_4}{dt} = -\dot{p}_i q_i - H$$

There is no reason to consider any of these 4 definitions to be more fundamental than the others

We arbitrarily chose the first form (which happens to be the Lagrangian form) to write the generating functions in the table

Four Basic Generators

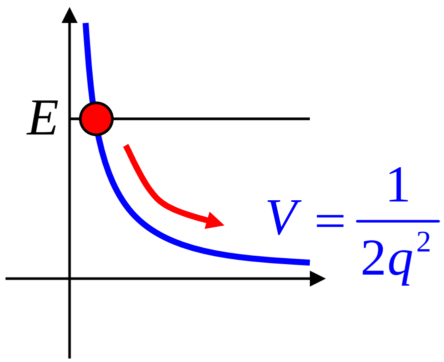
- Some canonical transformations cannot be generated by all 4 types
  - e.g. identity transf. is generated only by F_2 or F_3
- This does not present a fundamental problem
  - One can always swap coordinate and momentum

    $Q_i = p_i$ $\quad P_i = -q_i$
  - One can always change sign by scale transformation

    $Q_i = \pm q_i$ $\quad P_i = \pm p_i$
- These transformations make the 4 types practically equivalent

One More Example

- 1-dim system with $H = \dfrac{p^2}{2} + \dfrac{1}{2q^2}$
  - Try $P = pq$
  - Let's use Type-2 $F = F_2(q, P, t)$ $\quad \dfrac{\partial F_2}{\partial q} = p \quad \dfrac{\partial F_2}{\partial P} = Q$



  - Step 1: Express *p* with *q* and *P* $\Rightarrow$ $p = \dfrac{P}{q}$
  - Step 2: Integrate with *q* to get $F_2 = P \log q$ $\Leftarrow$ assuming $q > 0$
  - Step 3: Differentiate to get $Q = \log q$ $\Rightarrow$ $q = e^Q$
  - Now we have a canonical transformation

One More Example

$$F_2 = P\log q \qquad q = e^Q \qquad p = \frac{P}{q} = Pe^{-Q}$$

- Now rewrite the Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2q^2} = \frac{P^2+1}{2}e^{-2Q} = E$$

constant

- Equation of motion: $\dot{P} = (P^2+1)e^{-2Q} = 2E$

$$\Rightarrow \quad P = 2Et + C$$

$$\Rightarrow \quad q = e^Q = \sqrt{\frac{P^2+1}{2E}} = \sqrt{2Et^2 + 2Ct + \frac{C^2+1}{2E}}$$

Summary

- Canonical transformations

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

  - Hamiltonian formalism is invariant under canonical + scale transformations
  - Generating functions define canonical transformations
  - Four basic types of generating functions

$F_1(q,Q,t)$ $F_2(q,P,t)$ $F_3(p,Q,t)$ $F_4(p,P,t)$

    - They are all practically equivalent
- Used it to simplify a harmonic oscillator
  - Invariance of phase space area