Week 5

5.1 Measure Theory

7/21:

- **Metric space**: A space X with a metric d satisfying
  1. For all $x \in X$, $d(x,x) = 0$.
  2. If $d(x,y) = 0$, then $x = y$.
  3. $d(x,y) = d(y,x)$.
  4. $d(x,z) \leq d(x,y) + d(y,z)$.
- For example, on $\mathbb{R}$, the metric d is defined by $d(x,y) = |x-y|$ for all $x,y \in \mathbb{R}$. One can check that d satisfies the four properties under this definition.
- There is a topology induced by a metric.
  - Let $B_r(x) = \{y \in X : d(x,y) < r\}$.
- **Cauchy sequence**: A sequence (x_i) such that for all $\epsilon > 0$, there exists an N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.
  - If (x_i) is Cauchy, then there exists y such that $\lim_{i\to\infty} d(x_i, y) = 0$.
- **Complete** (metric space): A metric space X such that all Cauchy sequences converge to some $x \in X$.
  - Informally, a complete metric space is a metric space X such that all convergent sequences of elements of X converge to an element of X.
  - For example, $(0,1)$ under $d(x,y) = |x-y|$ has contains convergent sequences that converge to 1, so it is not complete.
- Completeness and compactness are not identical, but they are quite analogous.
- **Baire Category Theorem** (informal): Open subsets of a complete metric space are big.
  - Big: Not made out of countably many small things.
  - Fact: If $X_1, X_2, \dots$ are all measure 0, then $\bigcup_{i\in\mathbb{N}} X_i$ is measure 0.
  - $\overline{\mathbb{Q}} = \mathbb{R}$.
- **First category** (of small sets): A countable union of nowhere dense sets.
- **Second category**: Not a first category.
- Prove the theorem by showing that open subsets of the metric space are second category.