Week 1

1.1 Gaussian Curvature (Neves)

6/28:

- Plan:
  1. What is a surface?
  2. What is the tangent space?
  3. What are the principal curvatures?
  4. What is the Gaussian curvature?
- In analysis:
  1. What is a function?
  2. What is the derivative?
  3. What is the Hemian of function?
  4. 2nd derivative test (determinant of Hemina).
- **Surface**: A subset $\sigma \subseteq \mathbb{R}^3$ such that for all $p \in \Sigma$, there's a neighborhood B of p in $\mathbb{R}^3$ so that $\Sigma \cap B$ "looks like a disk." More precisely, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ and a map $\varphi : U \to \Sigma \cap B \subseteq \mathbb{R}^3$ such that
  i) φ is continuous and smooth.
  ii) φ is a bijection (with φ^{-1} continuous).
  iii) $\mathrm{d}\varphi_{|x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $x \in U$.
- **Chart**: The quantity (φ, U) near $p \in U$.
- Examples:
  A) Plane. $\Sigma = \mathbb{R}^2 \times |0| \subseteq \mathbb{R}^3$ is a surface with chart $\varphi : \mathbb{R}^2 \to \Sigma$ where $(x, y) \mapsto (x, y, 0)$.
  B) Sphere. $\Sigma = \{\vec{u} \in \mathbb{R}^3 \mid |\vec{u}| = 1\}$.
     - Charts: Consider the sets $U = \{(x_1, x_2) \mid {x_1}^2 + {x_2}^2 < 1\} \subseteq \mathbb{R}^2$. Let $\varphi_1^+ : U \to \Sigma \cap \{(x, y, z) \mid x > 0\}$ be defined by $\varphi_1^+(u_1, u_2) = (\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$, $\varphi_1^- : U \to \Sigma \cap \{(x, y, z) \mid x < 0\}$ be defined by $\varphi_1^-(u_1, u_2) = (-\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$.
     - Same thing for $\varphi_2^\pm, \varphi_3^\pm$.
  C) A cone $\Sigma = \{(x, y, z) \mid z = \sqrt{x^2 + y^2}\}$ is *not* a surface because it fails property (iii).
  D) The closed unit disk $\Sigma = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ is also not a surface.
- **Tangent space**: Let $\Sigma \subseteq \mathbb{R}^3$ be a surface and let $p \in \Sigma$. Then $T_p\Sigma \subseteq \mathbb{R}^3$ is the z-plane so that $p + T_p\Sigma$ is the affine plane that best approximates Σ near p.
  - Best linear approximation near the surface.

  - Very similar/analogous to the derivative.

- If (φ, U) is a chart near p, then $T_p\Sigma = \operatorname{span}\{\frac{\partial\varphi}{\partial x_1}(\bar{x}_1, \bar{x}_2), \frac{\partial\varphi}{\partial x_2}(\bar{x}_1, \bar{x}_2)\}$.
- Proves linear independence of above vectors.
- Principal curvatures.
- Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, $p \in \Sigma$, and $\vec{N}$ be a unit normal vector defined around p (i.e., $\vec{N}(q) \cdot \vec{v} = 0$ for all q near p and $\vec{v} \in T_q\Sigma$).
- Choose $\vec{v} \in T_p\Sigma$ such that $|\vec{v}| = 1$. Set $P_v = \operatorname{span}\{\vec{v}, \vec{N}(p)\}$. Claim: $(\Sigma - p) \cap P_v$ is a curve near the origin.
- **Principal curvature**: The reciprocal of the radius of the circle in P_v that best approximates $(\Sigma - p) \cap P_v$ near the origin. *Also known as* $\boldsymbol{K(\vec{v})}$.
  - The sign is positive if the center of the circle is in the direction of $\vec{N}(p)$ and negative otherwise.
  - If the sign of $\vec{N}(p)$ changes, then $K(\vec{v})$ will change in sign.
- If we change $\vec{N}$ by $-\vec{N}$, then the new $K(\vec{v})$ is the opposite of the old one.
- Given $p \in \Sigma$ and $\vec{N}(p)$ a normal vector at p, we define $K_1(p)$ to have the maximum $K(\vec{v})$ over all unit vectors $\vec{v} \in T_p\Sigma$ and $K_2(p) = \min\{K(\vec{v}) \mid \vec{v} \in T_p\Sigma, |\vec{v}| = 1\}$.
- K_1, K_2 are computable quantities.