

2021 UChicago Math REU Notes

Steven Labalme

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Week 1

1.1 Introduction to the Program (May / Rudenko)

- 6/21:
- Mainly given by Peter May.
 - A far broader range of mathematics than any other REU.
 - Things you have to do:
 1. Soak up as much mathematics as you can.
 2. Work with a mentor to write a paper.
 - You can work with people to write a joint paper?
 - This is fairly unique to this REU.
 3. Meet with your mentors at least twice a week.
 - Don't be shy and unwilling to ask questions.
 - Daniil Rudenko is in charge of the apprentice program.
 - Apprentice program:
 - An opportunity particularly early in one's mathematical career to explore mathematics.
 - Asynchronous video lectures.
 - Feel free to share with friends.
 - Problem solving.
 - Problems that are not merely exercises but more difficult, interesting processes.
 - Spend a couple hours a day thinking about these problems.
 - Emphasis on relations between different subjects.
 - They will be organizing social activities.
 - Social meet and greet at 6:00 PM tonight.
 - Breakout rooms:
 1. Apprentice Program.
 2. Probability.
 3. Analysis and Dynamical Systems.
 4. Algebraic Topics.
 5. Main room: Algebraic Topology.
 - More on the apprentice program:

- Daniil wants us to see much more than classical analysis/calculus. He doesn't see dividing lines between fields of mathematics.
- Bijections, binomial coefficients, Catalan numbers, etc. to start.
- Group of permutations, group of isometries of the plane, what a group is, etc.
- We can solve problems individually or in groups.
 - Some problems will say not to collaborate.
- Don't try to solve every problem. Don't try to solve everything fast; it's fine if you fail, if you just think about something for a couple hours that's interesting and don't get everywhere.
- On campus classes option for participants in Chicago.
- This week 10-11 AM Wed/Fri?
- Office hours 10-11 AM on Thursday.
- He will send an email with more information.
- Be consistent in whether you want to be on or off campus.
- You may attend whatever you want, but be careful: The apprentice program is your priority, so don't spend too much time on the other stuff.
 - Follow Piazza groups to get links.
- L^AT_EX one solution each week.

1.2 Introduction to Complex Dynamics (Calegari)

- Main focus: the Mandelbrot Set.
- Let $f_c : \mathbb{C} \rightarrow \mathbb{C}$ be the quadratic polynomial $f_c(z) := z^2 + c$ where $c \in \mathbb{C}$ is a constant and $z \in \mathbb{C}$ is a variable.
 - We study quadratics because they're the simplest nontrivial polynomial, i.e., one that displays the interesting phenomena of higher degree polynomials.
- We want to understand the dynamics of f_c , i.e., what happens as we apply f_c over and over again.
 - In other words $z \rightarrow z^2 + c \rightarrow (z^2 + c)^2 + c \rightarrow ((z^2 + c)^2 + c)^2 + c \rightarrow \dots$
 - Are there any special values of z that have interesting characteristics?
- **Fixed point:** A value z such that $f_c(z) = z$.
 - Fixed points of f_c are equivalent to **roots** of $f_c - z$.
- In this branch of mathematics, we don't care so much about factoring f_c as much as we care about other special entities like fixed points and **critical points**.
- **Critical point:** A point where $df_c/dz = 0$.
- We denote z large by $|z| \gg 1$.
- Note that $z^2 + c$ doesn't change the magnitude of z that much unless z is large.
 - Essentially, if $|z| \gg 1$, then $|f_c(z)| \gg |z|$.
- Introduces composition notation: $z \rightarrow f_c(z) \rightarrow f_c^2(z) \rightarrow f_c^3(z) \rightarrow \dots$ ^[1].
- If z large, then the sequence $z, f_c(z), f_c^2(z), \dots$ converges to infinity.

¹Sometimes, people also use a circled number in the superscript.

- **Riemann Sphere:** The set $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$.
 - Like an open set of complex numbers.
 - In this case, we can think of infinity as a fixed point.
- Any number whose absolute value is sufficiently big will converge to infinity.
- Introduces big N convergence test.
- Infinity is an **attracting fixed point**, i.e. there exists an open neighborhood U containing ∞ such that for all $z \in U$, $f_c^n(z) \rightarrow \infty$ as $n \rightarrow \infty$.
- **Filled Julia set:** The set $\{z : \text{the iterates } f_c^n(z) \text{ do not converge to } \infty\}$. *Also known as $K(f_c)$.*
 - Equivalent to the set $\{z : \exists \text{ a constant } T \text{ s.t. } |f_c^n(z)| \leq T \forall n\}$.
- The points that diverge to infinity are not that interesting; their divergence is their only property.
- Much more interesting are the points that do not diverge to infinity.
- Lemma: $K(f_c)$ is closed and bounded (i.e., compact).
 - Proof: There exists T (depending on c) such that if $|z| > T$, then $z \notin K(f_c)$. Furthermore, $z \in K(f_c)$ if and only if there exists n such that $|f_c^n(z)| > T$. Let $U := \{z : |z| > T\}$. This is open. Thus, $z \in K(f_c) \iff z \text{ iterates } f_c^n(z) \in U$. Therefore, $K(f_c) = \mathbb{C}$, so $\bigcup_n f_c^n(U)$, i.e., $K(f_c)$ is closed.
 - Bounded because numbers are not arbitrarily large. *flesh out details?*
- Calegari's proofs will be somewhat informal throughout the week; he hits the main points and leaves the details as an exercise to the student.
- Question: What other topological properties does the filled Julia set have?
 - Is it possible that $K(f_c) = \emptyset$?
 - No, it is not — as a degree 2 polynomial, $f_c - z$ has at least one root, which will by necessity be a fixed point, i.e., not diverge to infinity, i.e., in the filled Julia set.
 - Could it be a finite set?
 - No — $K(f_c)$ is a **perfect set**.
 - Uncountably infinite, too.
 - Is $K(f_c)$ connected?
 - Sometimes.
- **Perfect set:** A set where every point in the set is a **nontrivial limit point** of the set.
 - Example: A closed interval, *others listed*.
- **Nontrivial limit point:** A point p in a set A such that there is a nontrivial sequence (i.e., not a constant sequence, e.g., p, p, p, \dots) of points in A that converge to p .
- **Not connected:** A set $X \subset \mathbb{C}$ such that there exist disjoint, open sets U, V such that $X \subset U \cup V$, $X \cap U \neq \emptyset$, and $X \cap V \neq \emptyset$.
- **Mandelbrot set:** The set of complex numbers $c \in \mathbb{C}$ such that $K(f_c)$ is connected *Also known as M .*
- We can prove that $K(f_c)$ is connected if and only if the critical point of f_c is an element of $K(f_c)$.
 - Remember that critical points of f_c are equivalent to zeroes of f'_c .
- Note that critical points of $f_c := z^2 + c$ are equal to the roots of $f'_c = 2z$, i.e., the elements of $\{0\}$.

- $K(f_c)$ is connected is equivalent to the sequence $0 \rightarrow c \rightarrow c^2 + c \rightarrow (c^2 + c)^2 + c \rightarrow \dots$ is bounded (an absolute value).
 - Thus, $c \in M$ is equivalent to the sequence $0 \rightarrow c \rightarrow c^2 + c \rightarrow (c^2 + c)^2 + c \rightarrow \dots$ is bounded.
- The Mandelbrot set is compact, too.
- Proposition: $K(f_c)$ is connected if and only if $0 \in K(f_c)$.
 - “Proof”: $\mathbb{C} - K(f_c) = \bigcup_n f_c^{-n}(U)$ where U is an open neighborhood of ∞ , i.e., the set $\{z : |z| > T\}$.
 - Let $X_n := \mathbb{C} - f_c^{-n}(U)$, i.e., $X_0 = \mathbb{C} - U$, so $K(f_c) = \bigcap_n X_n$.
 - Cyclic map? X_n getting “smaller” as n increases? $X_{n+1} \subset X_n$.
 - Assume $X_n = \text{little}$.
 - Two cases: X_n contains 0 and X_n does not contain 0.
 - Either every preimage of X_n is connected or there is a T such that for all $n \geq T$, X_n is not connected.
- Theorem (Douady-Hubbard): M is connected.

1.3 Harmonic Functions, Brownian Motion, and Analysis in the Plane (Lawler)

- These topics will change week to week, so drop in at any point over the summer.
- Schedule:
 - Lectures MWF at 2:30 PM.
 - Group meeting Tuesday at 2:30 PM.
 - Anybody can attend these!
 - No Zoom on Thursday, but there will be an opportunity to talk to Greg Lawler in person at the department of mathematics outside Eckhart when the weather is good.
- Resources:
 - Piazza — look under the resources tab for lecture notes (with some exercises; these are very rough; gives you something to read with the lectures), other materials, etc.
 - There is a 180 page book draft based on his REU lectures last summer.
 - Do not share this.
- This math is at the border of analysis (basically advanced calculus) and probability.
 - Lawler thinks of these as all basically the same subject.
- We will work in \mathbb{R}^2 .
- A lot of what Dr. Lawler does is often called Complex Analysis.
- Complex analysis allows you to get the results quicker even though they encapsulate ideas that are 100% real; we’re going to take a real-function perspective.
- Harmonic function notation:
 - Domains D are connected open sets that are subsets of \mathbb{R}^2 .
 - Mean value: $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ (continuous), or $f : D \rightarrow \mathbb{R}$.

- z, w are points in \mathbb{R}^2 , and we write $z = (x, y)$ where x, y are the one-dimensional components.
- $B(z, \epsilon) = \{w : |z - w| < \epsilon\}$ is an open disk and $\partial B(z, \epsilon)$ is the circle of radius ϵ about z .
- If $B(z, \epsilon) \subset D$, then the (circular) mean value $MV(f; z, \epsilon)$ is the average rate of f on $\partial B(z, \epsilon)$, i.e., the quantity

$$\frac{1}{2\pi\epsilon} \int_{\{|w-z|=\epsilon\}} f(w) |dw|$$

where $|dw|$ is with respect to arc length.

- Let $(\cos \theta, \sin \theta) = e^{i\theta}$.
- **Harmonic function:** $f : D \rightarrow \mathbb{R}$ is harmonic if f is continuous and for all $z \in D$ and every $\epsilon > 0$ with $d(z, \partial D) > \epsilon$, then $f(z) = MV(f; z, \epsilon)$.
- Many applications, notably in physics wrt. heat.
 - Consider D describing a surface with heat. Fix the temperature at the boundary. Let $U(z)$ = temperature at z (in equilibrium).
 - Then U is harmonic on D .
- We're going to understand the mean value in terms of the **Laplacian**.
- If $f : D \rightarrow \mathbb{R}$ is C^2 (the first and second derivatives exist and are continuous [either two derivatives in one variable or one derivative in both variables for \mathbb{R}^2]), then the Laplacian is defined by

$$\Delta f(z) = f_{xx}(z) + f_{yy}(z)$$

- Proposition: If u is C^2 in D , then $\Delta u(z) = \lim_{\epsilon \rightarrow 0} 4 \cdot \frac{MV(u; z, \epsilon) - u(z)}{\epsilon^2}$.
- For ease, let's assume that $z = 0 = (0, 0)$ and $u(z) = 0$.
- Taylor polynomial (in several variables): If $z = (x, y)$, then

$$u(z) = 0 + u_x(0)x + u_y(0)y + \frac{1}{2}u_{xx}(0)x^2 + \frac{1}{2}u_{yy}(0)y^2 + u_{xy}(0)xy + \sigma(|z|^2)^{[2]}$$

- $u_x(0)MV(x; 0, \epsilon) + u_y(0)MV(y; 0, \epsilon) + u_{xy}(0)MV(xy; 0, \epsilon) + \sigma(\epsilon^2) + \frac{1}{2}[u_{xx}(0)x^2 + u_{yy}(0)y^2]$.
- Note that $u_{xx}(0)x^2 = MV(x^2; 0, \epsilon)$ and $u_{yy}(0)y^2 = MV(y^2; 0, \epsilon)$.
- You can use multivariable calculus, or you can observe that $u_{xy}(0)xy = MV(xy; 0, \epsilon)$, thus telling you that $u_{xx}(0)x^2 + u_{yy}(0)y^2 + u_{xy}(0)xy = MV(x^2 + y^2; 0, \epsilon) = \epsilon^2$.
- Since $|z|^2 = \epsilon^2$, we have that $u(z) = \frac{1}{2}[\frac{1}{2}] \dots$
- Proposition: A function $f : D \rightarrow \mathbb{R}$ is harmonic if and only if it is C^2 and $\Delta f(z) = 0$ for all $z \in D$.
 - Proof: Backwards direction first. We want to show that C^2 and $\Delta f(z) = 0$ imply the mean value property. The mean value property clearly holds at $\epsilon = 0$. Consider $MV(f; z, \epsilon)$ as a function of ϵ . The derivative in ϵ ends up looking something like $\frac{1}{2\pi\epsilon} \int_{\text{circle}} \partial_n f(w) |dw|$ where ∂_n is the normal direction.
 - Using the divergence theorem, we have that the above is equal to $\int_{\text{disk}} \Delta f(w) dw$. Note that we sometimes write $\Delta f = \text{div}(\nabla f)$ where $\nabla f = (f_x, f_y)$. Additionally, $\text{div}(\nabla f) = \partial_x(f_x) + \partial_y(f_y)$.
 - Exercise: Show that if u is harmonic, then u is C^2 .
- The notion of probability comes in when we ask, “what is the ‘mean value’ if we are not a disk viewed from the center?”

² σ is pronounced “little oh.”