Week 1

1.1 Gaussian Curvature (Neves)

6/28:

- Plan:
  1. What is a surface?
  2. What is the tangent space?
  3. What are the principal curvatures?
  4. What is the Gaussian curvature?
- In analysis:
  1. What is a function?
  2. What is the derivative?
  3. What is the Hemian of function?
  4. 2nd derivative test (determinant of Hemina).
- **Surface**: A subset $\sigma \subseteq \mathbb{R}^3$ such that for all $p \in \Sigma$, there's a neighborhood B of p in $\mathbb{R}^3$ so that $\Sigma \cap B$ "looks like a disk." More precisely, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ and a map $\varphi : U \to \Sigma \cap B \subseteq \mathbb{R}^3$ such that
  - i) φ is continuous and smooth.
  - ii) φ is a bijection (with φ^{-1} continuous).
  - iii) $\mathrm{d}\varphi_{|x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $x \in U$.
- **Chart**: The quantity (φ, U) near $p \in U$.
- Examples:
  - A) Plane. $\Sigma = \mathbb{R}^2 \times |0| \subseteq \mathbb{R}^3$ is a surface with chart $\varphi : \mathbb{R}^2 \to \Sigma$ where $(x, y) \mapsto (x, y, 0)$.
  - B) Sphere. $\Sigma = \{\vec{u} \in \mathbb{R}^3 \mid |\vec{u}| = 1\}$.
    - Charts: Consider the sets $U = \{(x_1, x_2) \mid {x_1}^2 + {x_2}^2 < 1\} \subseteq \mathbb{R}^2$. Let $\varphi_1^+ : U \to \Sigma \cap \{(x, y, z) \mid x > 0\}$ be defined by $\varphi_1^+(u_1, u_2) = (\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$, $\varphi_1^- : U \to \Sigma \cap \{(x, y, z) \mid x < 0\}$ be defined by $\varphi_1^-(u_1, u_2) = (-\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$.
    - Same thing for $\varphi_2^\pm, \varphi_3^\pm$.
  - C) A cone $\Sigma = \{(x, y, z) \mid z = \sqrt{x^2 + y^2}\}$ is *not* a surface because it fails property (iii).
  - D) The closed unit disk $\Sigma = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ is also not a surface.
- **Tangent space**: Let $\Sigma \subseteq \mathbb{R}^3$ be a surface and let $p \in \Sigma$. Then $T_p\Sigma \subseteq \mathbb{R}^3$ is the z-plane so that $p + T_p\Sigma$ is the affine plane that best approximates Σ near p.
  - Best linear approximation near the surface.

  - Very similar/analogous to the derivative.
- If (φ, U) is a chart near p, then $T_p\Sigma = \text{span}\{\frac{\partial\varphi}{\partial x_1}(\bar{x}_1, \bar{x}_2), \frac{\partial\varphi}{\partial x_2}(\bar{x}_1, \bar{x}_2)\}$.
- Proves linear independence of above vectors.
- Principal curvatures.
- Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, $p \in \Sigma$, and $\vec{N}$ be a unit normal vector defined around p (i.e., $\vec{N}(q) \cdot \vec{v} = 0$ for all q near p and $\vec{v} \in T_q\Sigma$).
- Choose $\vec{v} \in T_p\Sigma$ such that $|\vec{v}| = 1$. Set $P_v = \text{span}\{\vec{v}, \vec{N}(p)\}$. Claim: $(\Sigma - p) \cap P_v$ is a curve near the origin.
- **Principal curvature**: The reciprocal of the radius of the circle in P_v that best approximates $(\Sigma - p) \cap P_v$ near the origin. *Also known as* $\boldsymbol{K(\vec{v})}$.
  - The sign is positive if the center of the circle is in the direction of $\vec{N}(p)$ and negative otherwise.
  - If the sign of $\vec{N}(p)$ changes, then $K(\vec{v})$ will change in sign.
- If we change $\vec{N}$ by $-\vec{N}$, then the new $K(\vec{v})$ is the opposite of the old one.
- Given $p \in \Sigma$ and $\vec{N}(p)$ a normal vector at p, we define $K_1(p)$ to have the maximum $K(\vec{v})$ over all unit vectors $\vec{v} \in T_p\Sigma$ and $K_2(p) = \min\{K(\vec{v}) \mid \vec{v} \in T_p\Sigma, |\vec{v}| = 1\}$.
- K_1, K_2 are computable quantities.

1.2 Lecture 1.6: An Explicit Formula for the Catalan Numbers

6/29:

- Theorem (discovered by Euler):

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

- Examples:
  - $C_1 = \frac{1}{2}\binom{2}{1} = 1$.
  - $C_2 = \frac{1}{3}\binom{4}{2} = 2$.
  - $C_3 = \frac{1}{4}\binom{6}{3} = 5$.
- Dyck path:
  - We'll study paths starting with $(0,0)$, going on each stop either from (x, y) to $(x+1, y+1)$ or from (x, y) to $(x+1, y-1)$.
  - Consider the number of ways to get to each point on the integer grid $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$.
    - Generates a rotated Pascal's triangle.
    - The number of paths from $(0,0)$ to $(a+b, a-b)$, i.e., a moves up and b moves down is $\binom{a+b}{b}$.
- Proposition: C_n is equal to the number of paths from $(0,0)$ to $(2n, 0)$ which are contained in the upper half-plane ($y \geq 0$).
  - Proof: C_n is the number of sequences of brackets.
  - Transform a sequence of brackets into a path by ($\mapsto\nearrow$ and) $\mapsto\searrow$.
  - The condition #($=$ #) implies that the paths start at $(0,0)$ and end at $(2n, 0)$.
  - The condition that in every initial segment, #($=$ #) implies that the path lies in the upper half plane.

- Reflection principle:
  - The number of paths from A to B in the upper half plane is equal to the number of paths from A to B minus the number of paths from A to B that intersect the line $y = -1$.
  - Symbolically, $C_n = \binom{2n}{n} - ?$
  - There exists a one-to-one correspondence between two sets: The set of all paths from A to B intersecting ℓ and the set of all paths from A to B', where B' is the reflection of B across ℓ.
- Thus, the number of paths from A to B that intersect the line $y = -1$ is equal to the number of paths from A to $(2n, -2)$.
- Therefore,

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!}\left(1 - \frac{n}{n+1}\right)\binom{2n}{n-1} \\ &= \frac{1}{n+1}\binom{2n}{n}\end{aligned}$$

  - Note that it's not obvious that $\frac{1}{n+1}\binom{2n}{n}$ is an integer unless you present it as the difference of two binomials (i.e., as $\binom{2n}{n} - \binom{2n}{n-1}$).
- Exercise:
  - Take a path from A to B intersecting ℓ and find the closest point of $P \cap \ell$ to B. Reflect the segment of the path after this point.