Week 1

1.1 Introduction to the Program (May / Rudenko)

6/21:

- Mainly given by Peter May.
- A far broader range of mathematics than any other REU.
- Things you have to do:
  1. Soak up as much mathematics as you can.
  2. Work with a mentor to write a paper.
     - You can work with people to write a joint paper?
     - This is fairly unique to this REU.
  3. Meet with your mentors at least twice a week.
- Don't be shy and unwilling to ask questions.
- Daniil Rudenko is in charge of the apprentice program.
- Apprentice program:
  - An opportunity particularly early in one's mathematical career to explore mathematics.
  - Asynchronous video lectures.
    - Feel free to share with friends.
  - Problem solving.
    - Problems that are not merely exercises but more difficult, interesting processes.
    - Spend a couple hours a day thinking about these problems.
- Emphasis on relations between different subjects.
- They will be organizing social activities.
- Social meet and greet at 6:00 PM tonight.
- Breakout rooms:
  1. Apprentice Program.
  2. Probability.
  3. Analysis and Dynamical Systems.
  4. Algebraic Topics.
  5. Main room: Algebraic Topology.
- More on the apprentice program:

- Daniil wants us to see much more than classical analysis/calculus. He doesn't see dividing lines between fields of mathematics.
- Bijections, binomial coefficients, Catalan numbers, etc. to start.
- Group of permutations, group of isometries of the plane, what a group is, etc.
- We can solve problems individually or in groups.
  - Some problems will say not to collaborate.
- Don't try to solve every problem. Don't try to solve everything fast; it's fine if you fail, if you just think about something for a couple hours that's interesting and don't get everywhere.
- On campus classes option for participants in Chicago.
- This week 10-11 AM Wed/Fri?
- Office hours 10-11 AM on Thursday.
- He will send an email with more information.
- Be consistent in whether you want to be on or off campus.
- You may attend whatever you want, but be careful: The apprentice program is your priority, so don't spend too much time on the other stuff.
  - Follow Piazza groups to get links.
- LaTeX one solution each week.

1.2 Introduction to Complex Dynamics (Calegari)

- Main focus: the Mandelbrot Set.
- Let $f_c : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial $f_c(z) := z^2 + c$ where $c \in \mathbb{C}$ is a constant and $z \in \mathbb{C}$ is a variable.
  - We study quadratics because they're the simplest nontrivial polynomial, i.e., one that displays the interesting phenomena of higher degree polynomials.
- We want to understand the dynamics of f_c, i.e., what happens as we apply f_c over and over again.
  - In other words $z \to z^2 + c \to (z^2+c)^2 + c \to ((z^2+c)^2+c)^2 + c \to \cdots$.
  - Are there any special values of z that have interesting characteristics?
- **Fixed point**: A value z such that $f_c(z) = z$.
  - Fixed points of f_c are equivalent to **roots** of $f_c - z$.
- In this branch of mathematics, we don't care so much about factoring f_c as much as we care about other special entities like fixed points and **critical points**.
- **Critical point**: A point where $\mathrm{d}f_c/\mathrm{d}z = 0$.
- We denote z large by $|z| >> 1$.
- Note that $z^2 + c$ doesn't change the magnitude of z that much unless z is large.
  - Essentially, if $|z| >> 1$, then $|f_c(z)| >> |z|$.
- Introduces composition notation: $z \to f_c(z) \to {f_c}^2(z) \to {f_c}^3(z) \to \cdots$[1].
- If z large, then the sequence $z, f_c(z), {f_c}^2(z), \dots$ converges to infinity.

[1] Sometimes, people also use a circled number in the superscript.

- **Riemann Sphere**: The set $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$.
  - Like an open set of complex numbers.
  - In this case, we can think of infinity as a fixed point.
- Any number whose absolute value is sufficiently big will converge to infinity.
- Introduces big N convergence test.
- Infinity is an **attracting fixed point**, i.e. there exists an open neighborhood U containing ∞ such that for all $z \in U$, ${f_c}^n(z) \to \infty$ as $n \to \infty$.
- **Filled Julia set**: The set $\{z : \text{the iterates } {f_c}^n(z) \text{ do not converge to } \infty\}$. *Also known as* $\boldsymbol{K(f_c)}$.
  - Equivalent to the set $\{z : \exists \text{ a constant } T \text{ s.t. } |{f_c}^n(z)| \leq T \ \forall\ n\}$.
- The points that diverge to infinity are not that interesting; their divergence is their only property.
- Much more interesting are the points that do not diverge to infinity.
- Lemma: $K(f_c)$ is closed and bounded (i.e., compact).
  - Proof: There exists T (depending on c) such that if $|z| > T$, then $z \notin K(f_c)$. Furthermore, $z \in K(f_c)$ if and only if there exists n such that $|{f_c}^n(z)| > T$. Let $U := \{z : |z| > T\}$. This is open. Thus, $z \in K(f_c) \iff z$ iterates ${f_c}^n(z) \in U$. Therefore, $K(f_c) = \mathbb{C}$, so $\bigcup_n {f_c}^n(U)$, i.e., $K(f_c)$ is closed.
  - Bounded because numbers are not arbitrarily large. *flesh out details?*
- Calegari's proofs will be somewhat informal throughout the week; he hits the main points and leaves the details as an exercise to the student.
- Question: What other topological properties does the filled Julia set have?
  - Is it possible that $K(f_c) = \emptyset$?
    - No, it is not — as a degree 2 polynomial, $f_c - z$ has at least one root, which will by necessity be a fixed point, i.e., not diverge to infinity, i.e., in the filled Julia set.
  - Could it be a finite set?
    - No — $K(f_c)$ is a **perfect set**.
    - Uncountably infinite, too.
  - Is $K(f_c)$ connected?
    - Sometimes.
- **Perfect set**: A set where every point in the set is a **nontrivial limit point** of the set.
  - Example: A closed interval, *others listed*.
- **Nontrivial limit point**: A point p in a set A such that there is a nontrivial sequence (i.e., not a constant sequence, e.g., $p, p, p, \dots$) of points in A that converge to p.
- **Not connected**: A set $X \subset \mathbb{C}$ such that there exist disjoint, open sets U, V such that $X \subset U \cup V$, $X \cap U \neq \emptyset$, and $X \cap V \neq \emptyset$.
- **Mandlebrot set**: The set of complex numbers $c \in \mathbb{C}$ such that $K(f_c)$ is connected *Also known as* $\boldsymbol{M}$.
- We can prove that $K(f_c)$ is connected if and only if the critical point of f_c is an element of $K(f_c)$.
  - Remember that critical points of f_c are equivalent to zeroes of f_c'.
- Note that critical points of $f_c := z^2 + c$ are equal to the roots of $f_c' = 2z$, i.e., the elements of $\{0\}$.

- $K(f_c)$ is connected is equivalent to the sequence $0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$ is bounded (an absolute value).
  - Thus, $c \in M$ is equivalent to the sequence $0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$ is bounded.
- The Mandelbrot set is compact, too.
- Proposition: $K(f_c)$ is connected if and only if $0 \in K(f_c)$.
  - "Proof": $\mathbb{C} - K(f_c) = \bigcup_n f_c{}^{-n}(U)$ where U is an open neighborhood of ∞, i.e., the set $\{z : |z| > T\}$.
  - Let $X_n := \mathbb{C} - f_c{}^{-n}(U)$, i.e., $X_0 = \mathbb{C} - U$, so $K(f_c) = \bigcap_n X_n$.
  - Cyclic map? X_n getting "smaller" as n increases? $X_{n+1} \subset X_n$.
  - Assume $X_n =$ little.
  - Two cases: X_n contains 0 and X_n does not contain 0.
  - Either every preimage of X_n is connected or there is a T such that for all $n \geq T$, X_n is not connected.
- Theorem (Douady-Hubbard): M is connected.

1.3 Harmonic Functions, Brownian Motion, and Analysis in the Plane (Lawler)

- These topics will change week to week, so drop in at any point over the summer.
- Schedule:
  - Lectures MWF at 2:30 PM.
  - Group meeting Tuesday at 2:30 PM.
    - Anybody can attend these!
  - No Zoom on Thursday, but there will be an opportunity to talk to Greg Lawler in person at the department of mathematics outside Eckhart when the weather is good.
- Resources:
  - Piazza — look under the resources tab for lecture notes (with some exercises; these are very rough; gives you something to read with the lectures), other materials, etc.
  - There is a 180 page book draft based on his REU lectures last summer.
    - Do not share this.
- This math is at the border of analysis (basically advanced calculus) and probability.
  - Lawler thinks of these as all basically the same subject.
- We will work in $\mathbb{R}^2$.
- A lot of what Dr. Lawler does is often called Complex Analysis.
- Complex analysis allows you to get the results quicker even though they encapsulate ideas that are 100% real; we're going to take a real-function perspective.
- Harmonic function notation:
  - Domains D are connected open sets that are subsets of $\mathbb{R}^2$.
  - Mean value: $f : \mathbb{R}^2 \to \mathbb{R}$ (continuous), or $f : D \to \mathbb{R}$.

  - z, w are points in $\mathbb{R}^2$, and we write $z = (x, y)$ where x, y are the one-dimensional components.
  - $B(z, \epsilon) = \{w : |z - w| < \epsilon\}$ is an open disk and $\partial B(Z, \epsilon)$ is the circle of radius ϵ about z.

- If $B(z, \epsilon) \subset D$, then the (circular) mean value $MV(f; z, \epsilon)$ is the average rate of f on $\partial B(z, \epsilon)$, i.e., the quantity

$$\frac{1}{2\pi\epsilon} \int_{\{|w-z|=\epsilon\}} f(w) |\,\mathrm{d}w\,|$$

  where $|\,\mathrm{d}w\,|$ is with respect to arc length.

- Let $(\cos\theta, \sin\theta) = \mathrm{e}^{i\theta}$.
- **Harmonic function**: $f : D \to \mathbb{R}$ is harmonic if f is continuous and for all $z \in D$ and every $\epsilon > 0$ with $d(z, \partial D) > \epsilon$, then $f(z) = MV(f; z, \epsilon)$.
- Many applications, notably in physics wrt. heat.
  - Consider D describing a surface with heat. Fix the temperature at the boundary. Let $U(z) =$ temperature at z (in equilibrium).
  - Then U is harmonic on D.
- We're going to understand the mean value in terms of the **Laplacian**.
- If $f : D \to \mathbb{R}$ is C^2 (the first and second derivatives exist and are continuous [either two derivatives in one variable or one derivative in both variables for $\mathbb{R}^2$]), then the Laplacian is defined by

$$\Delta f(z) = f_{xx}(z) + f_{yy}(z)$$

- Proposition: If u is C^2 in D, then $\Delta u(z) = \lim_{\epsilon\to 0} 4 \cdot \frac{MV(u;z,\epsilon) - u(z)}{\epsilon^2}$.
- For ease, let's assume that $z = 0 = (0, 0)$ and $u(z) = 0$.
- Taylor polynomial (in several variables): If $z = (x, y)$, then

$$u(z) = 0 + u_x(0)x + u_y(0)y + \frac{1}{2}u_{xx}(0)x^2 + \frac{1}{2}u_{yy}(0)y^2 + u_{xy}(0)xy + \sigma(|z|^2)^{[2]}$$

  - $u_x(0)MV(x; 0, \epsilon) + u_y(0)MV(y; 0, \epsilon) + u_{xy}(0)MV(xy; 0, \epsilon) + \sigma(\epsilon^2) + \frac{1}{2}[u_{xx}(0)x^2 + u_{yy}(0)y^2]$.
  - Note that $u_{xx}(0)x^2 = MV(x^2; 0, \epsilon)$ and $u_{yy}(0)y^2 = MV(y^2; 0, \epsilon)$.
  - You can use multivariable calculus, or you can observe that $MV(x^2; 0, \epsilon) = MV(y^2; 0, \epsilon)$, thus telling you that $MV(x^2; 0, \epsilon) + MV(y^2; 0, \epsilon) = MV(x^2 + y^2; 0, \epsilon) = \epsilon^2$.
  - Since $|z|^2 = \epsilon^2$, we have that $u(z) = \frac{1}{2}[\frac{1}{2}]$...
- Proposition: A function $f : D \to \mathbb{R}$ is harmonic if and only if it is C^2 and $\Delta f(z) = 0$ for all $z \in D$.
  - Proof: Backwards direction first. We want to show that C^2 and $\Delta f(z) = 0$ imply the mean value property. The mean value property clearly holds at $\epsilon = 0$. Consider $MV(f; z, \epsilon)$ as a function of ϵ. The derivative in ϵ ends up looking something like $\frac{1}{2\pi\epsilon} \int_{\text{circle}} \partial_n f(w) |\,\mathrm{d}w\,|$ where ∂_n is the normal direction.
  - Using the divergence theorem, we have that the above is equal to $\int_{\text{disk}} \Delta f(w)\,\mathrm{d}w$. Note that we sometimes write $\Delta f = \mathrm{d}w\,(\nabla f)$ where $\nabla f = (f_x, f_y)$. Additionally, $\mathrm{div}\,(\nabla f) = \partial_x(f_x) + \partial_y(f_y)$.
  - Exercise: Show that if u is harmonic, then u is C^2.
- The notion of probability comes in when we ask, "what is the 'mean value' if we are not a disk viewed from the center?"

[2] σ is pronounced "little oh."