Week 4

4.1 PSet 5

7/14:

1. For a permutation $\sigma \in \mathbb{S}_n$, we denote by $\text{Inv}(\sigma)$ the number of inversions in σ, namely the number of pairs $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$.

   (a) Find permutations in $\mathbb{S}_n$ with the smallest number of inversions and with the biggest number of inversions.

   Proof. $\text{Inv}(\sigma) = 0$ for $\sigma = (1\ 2\ 3\ \dots\ n)$. $\text{Inv}(\sigma) = (n-1)!$ for $\sigma = (n\ (n-1)\ (n-2)\ \dots\ 1)$. $\square$

   (b) Prove that
   $$\sum_{\pi \in \mathbb{S}_n} x^{\text{Inv}(\pi)} = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$$

   (c) Prove that numbers $\text{Inv}(\sigma)$ and $\text{Inv}(\sigma^{-1})$ have the same parity.

2. A bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that there exists $k > 0$ satisfying $|f(A)f(B)| = k|AB|$ is called a **similarity**.

   (a) Prove that similarities form a group. Prove that this group is a subgroup of $\text{Aff}(\mathbb{R}^2)$ and contains a subgroup $\text{Isom}(\mathbb{R}^2)$.

   (b) Prove that similarities send lines to lines, circles to circles, and preserve angles.

   (c) Prove that homothety H_O^λ is a similarity.

   (d) Prove that every similarity is a composition of a homothety and an isometry.

3. (a) Prove that the homothety H_O^λ is a similarity.

   (b) Prove that a composition of two homotheties with coefficients $\lambda_1, \lambda_2 \neq 1$ is a homothety with coefficient $\lambda_1\lambda_2$.

   (c) Prove that if a composition of three homotheties is the identity map, then their centers lie on the same line.

   (d) **Monge's theorem**

   Outer tangent lines to the circles S_1 and S_2, S_2 and S_3, S_3 and S_1 intersect in the points A, B, and C, respectively. Prove that points A, B, and C lie on the same line.

4. Let R_n denote a set of fixed-point-free permutations in $\mathbb{S}_n$ (i.e., $R_n = \{\sigma \in \mathbb{S}_n \mid \sigma(i) \neq i\ \forall\ 1 \leq i \leq n\}$). Prove that
   $$\lim_{n \to \infty} \frac{|R_n|}{n!} = \frac{1}{e}$$