Week 1

1.1 PSet 4

7/7:

1. Consider two lines in the plane with the angle γ between them and suppose a grasshopper is jumping from one line to the other. Every jump is exactly 30 inches long, and the grasshopper jumps backwards whenever it has no other options. Prove that the sequence of its jumps is periodic if and only if $\frac{\gamma}{\pi}$ is a rational number.

2. Let $ABCD$ be a convex 4-gon and consider four squares constructed on the outside of each of its edges. Prove that the segments connecting the centers of the opposite squares are mutually perpendicular and equal in length.

3. Prove that a composition of three symmetries is a sliding symmetry.

4. The points $A_1, \ldots, A_n$ form a regular polygon, inscribed in a circle with the center O. A point X lies on the same circle. Prove that the images of the point X under the symmetries with axes $OA_1, OA_2, \ldots, OA_n$ form a regular polygon.

5. Remove a corner from a 101×101 chessboard. Prove that the rest cannot be covered by triominoes. A triomino is like a domino except that it consists of three squares in a row; each cell can cover one cell on a chessboard. Each triomino can either "stand" or "lie."

6. Consider a finite collection of segments on a line so that every two of them intersect. Prove that all segments have a common point.

7. Let S be a set of $n + 1$ integers from 1 to $2n$. Prove that at least two elements in S are coprime.