Week 1

1.1 Introduction to the Program (May / Rudenko)

6/21:
- Mainly given by Peter May.
- A far broader range of mathematics than any other REU.
- Things you have to do:
  1. Soak up as much mathematics as you can.
  2. Work with a mentor to write a paper.
     - You can work with people to write a joint paper?
     - This is fairly unique to this REU.
  3. Meet with your mentors at least twice a week.
- Don't be shy and unwilling to ask questions.
- Daniil Rudenko is in charge of the apprentice program.
- Apprentice program:
  - An opportunity particularly early in one's mathematical career to explore mathematics.
  - Asynchronous video lectures.
    - Feel free to share with friends.
  - Problem solving.
    - Problems that are not merely exercises but more difficult, interesting processes.
    - Spend a couple hours a day thinking about these problems.
- Emphasis on relations between different subjects.
- They will be organizing social activities.
- Social meet and greet at 6:00 PM tonight.
- Breakout rooms:
  1. Apprentice Program.
  2. Probability.
  3. Analysis and Dynamical Systems.
  4. Algebraic Topics.
  5. Main room: Algebraic Topology.
- More on the apprentice program:

  - Daniil wants us to see much more than classical analysis/calculus. He doesn't see dividing lines between fields of mathematics.
  - Bijections, binomial coefficients, Catalan numbers, etc. to start.
  - Group of permutations, group of isometries of the plane, what a group is, etc.
  - We can solve problems individually or in groups.
    - Some problems will say not to collaborate.
  - Don't try to solve every problem. Don't try to solve everything fast; it's fine if you fail, if you just think about something for a couple hours that's interesting and don't get everywhere.
  - On campus classes option for participants in Chicago.
  - This week 10-11 AM Wed/Fri?
  - Office hours 10-11 AM on Thursday.
  - He will send an email with more information.
  - Be consistent in whether you want to be on or off campus.
  - You may attend whatever you want, but be careful: The apprentice program is your priority, so don't spend too much time on the other stuff.
    - Follow Piazza groups to get links.
  - LaTeX one solution each week.

1.2 Introduction to Complex Dynamics 1 (Calegari)

- Main focus: the Mandelbrot Set.
- Let $f_c : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial $f_c(z) := z^2 + c$ where $c \in \mathbb{C}$ is a constant and $z \in \mathbb{C}$ is a variable.
  - We study quadratics because they're the simplest nontrivial polynomial, i.e., one that displays the interesting phenomena of higher degree polynomials.
- We want to understand the dynamics of f_c, i.e., what happens as we apply f_c over and over again.
  - In other words $z \to z^2 + c \to (z^2 + c)^2 + c \to ((z^2 + c)^2 + c)^2 + c \to \cdots$.
  - Are there any special values of z that have interesting characteristics?
- **Fixed point**: A value z such that $f_c(z) = z$.
  - Fixed points of f_c are equivalent to **roots** of $f_c - z$.
- In this branch of mathematics, we don't care so much about factoring f_c as much as we care about other special entities like fixed points and **critical points**.
- **Critical point**: A point where $\mathrm{d}f_c/\mathrm{d}z = 0$.
- We denote z large by $|z| >> 1$.
- Note that $z^2 + c$ doesn't change the magnitude of z that much unless z is large.
  - Essentially, if $|z| >> 1$, then $|f_c(z)| >> |z|$.
- Introduces composition notation: $z \to f_c(z) \to {f_c}^2(z) \to {f_c}^3(z) \to \cdots$[1].
- If z large, then the sequence $z, f_c(z), {f_c}^2(z), \ldots$ converges to infinity.

[1]Sometimes, people also use a circled number in the superscript.

- **Riemann Sphere**: The set $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$.
  - Like an open set of complex numbers.
  - In this case, we can think of infinity as a fixed point.
- Any number whose absolute value is sufficiently big will converge to infinity.
- Introduces big N convergence test.
- Infinity is an **attracting fixed point**, i.e. there exists an open neighborhood U containing ∞ such that for all $z \in U$, ${f_c}^n(z) \to \infty$ as $n \to \infty$.
- **Filled Julia set**: The set $\{z : \text{the iterates } {f_c}^n(z) \text{ do not converge to } \infty\}$. *Also known as* $\boldsymbol{K(f_c)}$.
  - Equivalent to the set $\{z : \exists \text{ a constant } T \text{ s.t. } |{f_c}^n(z)| \leq T \; \forall \; n\}$.
- The points that diverge to infinity are not that interesting; their divergence is their only property.
- Much more interesting are the points that do not diverge to infinity.
- Lemma: $K(f_c)$ is closed and bounded (i.e., compact).
  - Proof: There exists T (depending on c) such that if $|z| > T$, then $z \notin K(f_c)$. Furthermore, $z \in K(f_c)$ if and only if there exists n such that $|{f_c}^n(z)| > T$. Let $U := \{z : |z| > T\}$. This is open. Thus, $z \in K(f_c) \iff z$ iterates ${f_c}^n(z) \in U$. Therefore, $K(f_c) = \mathbb{C}$, so $\bigcup_n {f_c}^n(U)$, i.e., $K(f_c)$ is closed.
  - Bounded because numbers are not arbitrarily large. *flesh out details?*
- Calegari's proofs will be somewhat informal throughout the week; he hits the main points and leaves the details as an exercise to the student.
- Question: What other topological properties does the filled Julia set have?
  - Is it possible that $K(f_c) = \emptyset$?
    - No, it is not — as a degree 2 polynomial, $f_c - z$ has at least one root, which will by necessity be a fixed point, i.e., not diverge to infinity, i.e., in the filled Julia set.
  - Could it be a finite set?
    - No — $K(f_c)$ is a **perfect set**.
    - Uncountably infinite, too.
  - Is $K(f_c)$ connected?
    - Sometimes.
- **Perfect set**: A set where every point in the set is a **nontrivial limit point** of the set.
  - Example: A closed interval, *others listed*.
- **Nontrivial limit point**: A point p in a set A such that there is a nontrivial sequence (i.e., not a constant sequence, e.g., $p, p, p, \dots$) of points in A that converge to p.
- **Not connected**: A set $X \subset \mathbb{C}$ such that there exist disjoint, open sets U, V such that $X \subset U \cup V$, $X \cap U \neq \emptyset$, and $X \cap V \neq \emptyset$.
- **Mandlebrot set**: The set of complex numbers $c \in \mathbb{C}$ such that $K(f_c)$ is connected *Also known as* $\boldsymbol{M}$.
- We can prove that $K(f_c)$ is connected if and only if the critical point of f_c is an element of $K(f_c)$.
  - Remember that critical points of f_c are equivalent to zeroes of f_c'.
- Note that critical points of $f_c := z^2 + c$ are equal to the roots of $f_c' = 2z$, i.e., the elements of $\{0\}$.

- $K(f_c)$ is connected is equivalent to the sequence $0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$ is bounded (an absolute value).
  - Thus, $c \in M$ is equivalent to the sequence $0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$ is bounded.
- The Mandelbrot set is compact, too.
- Proposition: $K(f_c)$ is connected if and only if $0 \in K(f_c)$.
  - "Proof": $\mathbb{C} - K(f_c) = \bigcup_n {f_c}^{-n}(U)$ where U is an open neighborhood of ∞, i.e., the set $\{z : |z| > T\}$.
  - Let $X_n := \mathbb{C} - {f_c}^{-n}(U)$, i.e., $X_0 = \mathbb{C} - U$, so $K(f_c) = \bigcap_n X_n$.
  - Cyclic map? X_n getting "smaller" as n increases? $X_{n+1} \subset X_n$.
  - Assume $X_n =$ little.
  - Two cases: X_n contains 0 and X_n does not contain 0.
  - Either every preimage of X_n is connected or there is a T such that for all $n \geq T$, X_n is not connected.
- Theorem (Douady-Hubbard): M is connected.

1.3 Harmonic Functions, Brownian Motion, and Analysis in the Plane 1 (Lawler)

- These topics will change week to week, so drop in at any point over the summer.
- Schedule:
  - Lectures MWF at 2:30 PM.
  - Group meeting Tuesday at 2:30 PM.
    - Anybody can attend these!
  - No Zoom on Thursday, but there will be an opportunity to talk to Greg Lawler in person at the department of mathematics outside Eckhart when the weather is good.
- Resources:
  - Piazza — look under the resources tab for lecture notes (with some exercises; these are very rough; gives you something to read with the lectures), other materials, etc.
  - There is a 180 page book draft based on his REU lectures last summer.
    - Do not share this.
- This math is at the border of analysis (basically advanced calculus) and probability.
  - Lawler thinks of these as all basically the same subject.
- We will work in $\mathbb{R}^2$.
- A lot of what Dr. Lawler does is often called Complex Analysis.
- Complex analysis allows you to get the results quicker even though they encapsulate ideas that are 100% real; we're going to take a real-function perspective.
- Harmonic function notation:
  - Domains D are connected open sets that are subsets of $\mathbb{R}^2$.
  - Mean value: $f : \mathbb{R}^2 \to \mathbb{R}$ (continuous), or $f : D \to \mathbb{R}$.

  - z, w are points in $\mathbb{R}^2$, and we write $z = (x, y)$ where x, y are the one-dimensional components.
  - $B(z, \epsilon) = \{w : |z - w| < \epsilon\}$ is an open disk and $\partial B(Z, \epsilon)$ is the circle of radius ϵ about z.

- If $B(z, \epsilon) \subset D$, then the (circular) mean value $MV(f; z, \epsilon)$ is the average rate of f on $\partial B(z, \epsilon)$, i.e., the quantity
$$\frac{1}{2\pi\epsilon} \int_{\{|w-z|=\epsilon\}} f(w) |\,\mathrm{d}w\,|$$
where $|\,\mathrm{d}w\,|$ is with respect to arc length.
- Let $(\cos\theta, \sin\theta) = \mathrm{e}^{i\theta}$.
- **Harmonic function**: $f : D \to \mathbb{R}$ is harmonic if f is continuous and for all $z \in D$ and every $\epsilon > 0$ with $d(z, \partial D) > \epsilon$, then $f(z) = MV(f; z, \epsilon)$.
- Many applications, notably in physics wrt. heat.
  - Consider D describing a surface with heat. Fix the temperature at the boundary. Let $U(z) =$ temperature at z (in equilibrium).
  - Then U is harmonic on D.
- We're going to understand the mean value in terms of the **Laplacian**.
- If $f : D \to \mathbb{R}$ is C^2 (the first and second derivatives exist and are continuous [either two derivatives in one variable or one derivative in both variables for $\mathbb{R}^2$]), then the Laplacian is defined by
$$\Delta f(z) = f_{xx}(z) + f_{yy}(z)$$
- Proposition: If u is C^2 in D, then $\Delta u(z) = \lim_{\epsilon \to 0} 4 \cdot \frac{MV(u; z, \epsilon) - u(z)}{\epsilon^2}$.
- For ease, let's assume that $z = 0 = (0, 0)$ and $u(z) = 0$.
- Taylor polynomial (in several variables): If $z = (x, y)$, then
$$u(z) = 0 + u_x(0)x + u_y(0)y + \frac{1}{2}u_{xx}(0)x^2 + \frac{1}{2}u_{yy}(0)y^2 + u_{xy}(0)xy + \sigma(|z|^2)^{[2]}$$

  - $u_x(0)MV(x; 0, \epsilon) + u_y(0)MV(y; 0, \epsilon) + u_{xy}(0)MV(xy; 0, \epsilon) + \sigma(\epsilon^2) + \frac{1}{2}[u_{xx}(0)x^2 + u_{yy}(0)y^2]$.
  - Note that $u_{xx}(0)x^2 = MV(x^2; 0, \epsilon)$ and $u_{yy}(0)y^2 = MV(y^2; 0, \epsilon)$.
  - You can use multivariable calculus, or you can observe that $MV(x^2; 0, \epsilon) = MV(y^2; 0, \epsilon)$, thus telling you that $MV(x^2; 0, \epsilon) + MV(y^2; 0, \epsilon) = MV(x^2 + y^2; 0, \epsilon) = \epsilon^2$.
  - Since $|z|^2 = \epsilon^2$, we have that $u(z) = \frac{1}{2}[\frac{1}{2}]$...
- Proposition: A function $f : D \to \mathbb{R}$ is harmonic if and only if it is C^2 and $\Delta f(z) = 0$ for all $z \in D$.
  - Proof: Backwards direction first. We want to show that C^2 and $\Delta f(z) = 0$ imply the mean value property. The mean value property clearly holds at $\epsilon = 0$. Consider $MV(f; z, \epsilon)$ as a function of ϵ. The derivative in ϵ ends up looking something like $\frac{1}{2\pi\epsilon} \int_{\text{circle}} \partial_n f(w) |\,\mathrm{d}w\,|$ where ∂_n is the normal direction.
  - Using the divergence theorem, we have that the above is equal to $\int_{\text{disk}} \Delta f(w)\,\mathrm{d}w$. Note that we sometimes write $\Delta f = \mathrm{d}w\,(\nabla f)$ where $\nabla f = (f_x, f_y)$. Additionally, $\mathrm{div}\,(\nabla f) = \partial_x(f_x) + \partial_y(f_y)$.
  - Exercise: Show that if u is harmonic, then u is C^2.
- The notion of probability comes in when we ask, "what is the 'mean value' if we are not a disk viewed from the center?"

[2] σ is pronounced "little oh."

1.4 The Mathematics of Playing Pool (Mazur)

- Main focus: Billiards in a polygon.
- The ball bounces off a side with the same angle of incidence it struck it with. If the ball hits the corner, it stops (maybe it fell into a pocket).
- **Billiards**: Start with a polygon in the plane. Shoot a billiard ball, thought of as a point mass, ...
- Rectangular tables are fully understood, but other polygons are harder. Curved sides are even more complicated.
- Connection to physics: Ehrenfest windtree model (by Paul and Tatjana Ehrenfest, 1912).
- One thing people study is the diffusion rate of a random particle. This means that if you take a random particle and follow it for a long time t, how far is it from where it started? What people know is that a typical particle is about distance $t^{2/3}$ away.
- Another example: Take two point masses with positions $0 \leq x_1 \leq x_2 \leq 1$ on the unit interval $[0, 1]$. Suppose their masses are m_1, m_2 and they move with velocities v_1, v_2, respectively. They collide with each other and with the barriers at 0 and 1. Momentum and energy are conserved.
  - We can convert this to billiards in a right triangle with the observations that energy and speed are related and momentum and angle of incidence are related.
- Billiards are important examples of **dynamical systems** where one studies behavior of objects under a deterministic system (initial position and velocity define motion for the rest of time).
- Billiards questions:
  - Are there periodic orbits?
  - How does a typical orbit behave in the long term? Is it dense? Is it equidistributed?
  - Illumination problem (can you get from any point to any other?).
- Periodic orbits:
  - There are periodic orbits in acute triangles.
  - Drop perpendiculars; use Euclidean geometry to prove.
  - It is unknown if a general obtuse triangle has a periodic orbit. This is considered to be one of the big unsolved problems in dynamics[3].
- Equidistribution and Ergodicity:
  - ...
- Rational billiards is much more well-defined. Every rational table has periodic orbits.
- Most paths equidistribute.
- Illumination problem:
  - Now imagine you put a light source at a point on your table. The walls are mirrors and a light beam bounces of the mirror with angle of incidence equal to angle of reflection. Is every point illuminated? In other words, can you get from any point to any other? Not in a Penrose unilluminable room (a region is dark in this elliptical room).
  - Polygon example from Tokarski in the 1980s (a zero-dimensional point is unilluminable).

[3] Can we consider the set of all possible initial positions and directions and see what converges to what?

  - Within the last 5 years: For any rational billiard, there are at most a finite number of unilluminable points.

- Unfolding billiards boards.
- If the slope of the line on a torus is rational, it closes up. If the slope of the line on a torus is irrational, it does not close but is equidistributed.
- For a square, when we glue the unfolded version together, we get a genus 1 surface (a torus).
- For a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$, and $\frac{3\pi}{8}$, the unfolded version can be glued together into a genus 2 surface.
- Ergodicity:
  - A common notion in mathematics is that of irreducibility.
  - In our context, irreducible (or ergodic) means you cannot divide your table X nontrivially into 2 pieces $X = X_1 \cup X_2$ so that if you start with a point in X_1 and you move in a straight line, you stay in X_1 and if you start in X_2 you stay there.
  - In other words, there are no invariant sets.
- Proves the Kronecker-Weyl theorem.

1.5 Introduction to Complex Dynamics 2 (Calegari)

6/22:

- Picking up from yesterday on the proof of the proposition, $K(f_c)$ is connected iff $0 \in K(f_c)$.
  - Recall that 0 is the unique critical point.
  - Also recall that U is the set that contains only elements with sufficiently big absolute values, big enough so that $f_c(U) \subset U$.
  - Define $X_0 = \mathbb{C} - U$, $X_{n+1} = {f_c}^{-n}(X_n)$.
    - Each X_n is compact, and $K(f_c)$ is equal to the intersection of all X_n, therefore compact in and of itself.
  - Thus, $K(f_c)$ is connected iff every X_n is connected.
  - Key point: If $0 \in K(f_c)$, then each X_n is a disk; otherwise, some X_n is not a disk.
  - Fact: Every point in $\mathbb{C}$ has exactly 2 preimages under f_c except for the critical value $c = f_c(0)$ since f_c is a degree 2 polynomial.
  - Assume X_n is a disk. If X_n contains the critical value, then X_{n+1} is a disk; otherwise, not (in fact, it will be disconnected).
- Under f_c, the preimage of a circle not containing the critical value is either 2 circles, each of which maps one-to-one, or a single circle mapping two-to-one.
  - Suppose that the preimage of the boundary of the circles is two distinct circles.
  - By continuity, concentric circles narrowing down within the original set narrow down within the other two circles.
  - Each point in X_n has exactly two preimages iff $c \notin X_n$.
  - As we make smaller and smaller circles, than we can split our one-circle preimage into two disconnected subsets.
- Today: The theory of Julia sets for holomorphic functions in general.
- Let f be a **holomorphic** map from $\hat{\mathbb{C}}$ to itself.

- Every such f has finitely many 0s and poles (∞s).
- Therefore, f is a rational function, i.e., a ratio of polynomials. Symbolically, $f(z) = \frac{p(z)}{q(z)}$ for some polynomials p, q.
- To talk about Julia sets, we need some definitions.
- **Normal family**: Let U be an open subset of the Riemann sphere $\hat{\mathbb{C}}$, and let F be a family of holomorphic functions $f : U \to \hat{\mathbb{C}}$. F is **normal** if its closure (in the space of all holomorphic functions from U to $\hat{C}$) is compact.
  - In other words, if ever infinite sequence $f_n \in F$ has a subsequence that converges locally uniformly to some limit $g : U \to \hat{\mathbb{C}}$.
- Normality is local.
- Proposition: Suppose F is a family of holomorphic functions defined on U, and suppose for all $p \in U$, there exists open $p \in V \subset U$ such that $F|_V$ is normal. Then $F|_U$ is normal.
  - Proof 1: Diagonal argument.
  - Proof 2: ???
- **Julia set**: Let f be a holomorphic map from $\hat{\mathbb{C}}$ to itself. Let $\mathcal{F} := \{f^n \mid n \in \mathbb{N}\}$. The Fatou set $\Omega_f :\subset \hat{\mathbb{C}}$ is the open subset whose $\mathcal{F}$ is normal. It is equal to the union of all U where $F|_U$ is normal. Thus, Ω_f is open and $\mathcal{F}|_{\Omega_f}$ is normal. The **Julia set** $J_f :\subset \hat{\mathbb{C}}$ is $\hat{C} - \Omega_f$, i.e., $p \in J_f$ iff for all U containing p, $F|_U$ is not normal on U.
  - Hence, J_f is compact.
- Example: Let's let p be a fixed point for f.
  - p is an attracting fixed point if $|f'(p)| < 1$. p is super attracting if $|f'(p)| = 0$.
  - Example:
    - If f is a polynomial of degree at least 2, then ∞ is a super attracting fixed point.
  - If we take a sufficiently small neighborhood of p, then f shrinks and rotates the neighborhood a little bit.
  - **Basin of attraction** of p.
  - **Immediate basin of attraction** of p is the connected component of the basin of attraction of p.
- If f is a polynomial, then $K(f_c)$ is equal to the complement of the basin of infinity.
- "Most" "typical" f have $\Omega_f = \bigcup$ basins of attraction of attracting periodic orbits.
  - Furthermore: every immediate basin of an attracting periodic orbit contains at least one critical point.
  - A rational map of degree d (the maximum of the degrees of polynomials p, q) has $2d - 2$ critical points.
- Theorem: The closure of the set of repelling periodic orbits (i.e., p with $f^n(p) = p$ and $|(f^n)'(p)| > 1$) is J_f.