2021 UChicago Math REU Notes

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Weeks

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Week 1

1.1 Introduction to the Program (May / Rudenko)

6/21:

- Mainly given by Peter May.
- A far broader range of mathematics than any other REU.
- Things you have to do:
  1. Soak up as much mathematics as you can.
  2. Work with a mentor to write a paper.
     - You can work with people to write a joint paper?
     - This is fairly unique to this REU.
  3. Meet with your mentors at least twice a week.
- Don't be shy and unwilling to ask questions.
- Daniil Rudenko is in charge of the apprentice program.
- Apprentice program:
  - An opportunity particularly early in one's mathematical career to explore mathematics.
  - Asynchronous video lectures.
    - Feel free to share with friends.
  - Problem solving.
    - Problems that are not merely exercises but more difficult, interesting processes.
    - Spend a couple hours a day thinking about these problems.
- Emphasis on relations between different subjects.
- They will be organizing social activities.
- Social meet and greet at 6:00 PM tonight.
- Breakout rooms:
  1. Apprentice Program.
  2. Probability.
  3. Analysis and Dynamical Systems.
  4. Algebraic Topics.
  5. Main room: Algebraic Topology.
- More on the apprentice program:

- Daniil wants us to see much more than classical analysis/calculus. He doesn't see dividing lines between fields of mathematics.
- Bijections, binomial coefficients, Catalan numbers, etc. to start.
- Group of permutations, group of isometries of the plane, what a group is, etc.
- We can solve problems individually or in groups.
  - Some problems will say not to collaborate.
- Don't try to solve every problem. Don't try to solve everything fast; it's fine if you fail, if you just think about something for a couple hours that's interesting and don't get everywhere.
- On campus classes option for participants in Chicago.
- This week 10-11 AM Wed/Fri?
- Office hours 10-11 AM on Thursday.
- He will send an email with more information.
- Be consistent in whether you want to be on or off campus.
- You may attend whatever you want, but be careful: The apprentice program is your priority, so don't spend too much time on the other stuff.
  - Follow Piazza groups to get links.
- LaTeX one solution each week.

1.2 Introduction to Complex Dynamics 1 (Calegari)

- Main focus: the Mandelbrot Set.
- Let $f_c : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial $f_c(z) := z^2 + c$ where $c \in \mathbb{C}$ is a constant and $z \in \mathbb{C}$ is a variable.
  - We study quadratics because they're the simplest nontrivial polynomial, i.e., one that displays the interesting phenomena of higher degree polynomials.
- We want to understand the dynamics of f_c, i.e., what happens as we apply f_c over and over again.
  - In other words $z \to z^2 + c \to (z^2 + c)^2 + c \to ((z^2 + c)^2 + c)^2 + c \to \cdots$.
  - Are there any special values of z that have interesting characteristics?
- **Fixed point**: A value z such that $f_c(z) = z$.
  - Fixed points of f_c are equivalent to **roots** of $f_c - z$.
- In this branch of mathematics, we don't care so much about factoring f_c as much as we care about other special entities like fixed points and **critical points**.
- **Critical point**: A point where $\mathrm{d}f_c/\mathrm{d}z = 0$.
- We denote z large by $|z| >> 1$.
- Note that $z^2 + c$ doesn't change the magnitude of z that much unless z is large.
  - Essentially, if $|z| >> 1$, then $|f_c(z)| >> |z|$.
- Introduces composition notation: $z \to f_c(z) \to {f_c}^2(z) \to {f_c}^3(z) \to \cdots$[1].
- If z large, then the sequence $z, f_c(z), {f_c}^2(z), \ldots$ converges to infinity.

[1] Sometimes, people also use a circled number in the superscript.

- **Riemann Sphere**: The set $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$.
  - Like an open set of complex numbers.
  - In this case, we can think of infinity as a fixed point.
- Any number whose absolute value is sufficiently big will converge to infinity.
- Introduces big N convergence test.
- Infinity is an **attracting fixed point**, i.e. there exists an open neighborhood U containing ∞ such that for all $z \in U$, ${f_c}^n(z) \to \infty$ as $n \to \infty$.
- **Filled Julia set**: The set $\{z : \text{the iterates } {f_c}^n(z) \text{ do not converge to } \infty\}$. *Also known as* $\boldsymbol{K(f_c)}$.
  - Equivalent to the set $\{z : \exists \text{ a constant } T \text{ s.t. } |{f_c}^n(z)| \leq T \ \forall\ n\}$.
- The points that diverge to infinity are not that interesting; their divergence is their only property.
- Much more interesting are the points that do not diverge to infinity.
- Lemma: $K(f_c)$ is closed and bounded (i.e., compact).
  - Proof: There exists T (depending on c) such that if $|z| > T$, then $z \notin K(f_c)$. Furthermore, $z \in K(f_c)$ if and only if there exists n such that $|{f_c}^n(z)| > T$. Let $U := \{z : |z| > T\}$. This is open. Thus, $z \in K(f_c) \iff z$ iterates ${f_c}^n(z) \in U$. Therefore, $K(f_c) = \mathbb{C}$, so $\bigcup_n {f_c}^n(U)$, i.e., $K(f_c)$ is closed.
  - Bounded because numbers are not arbitrarily large. *flesh out details?*
- Calegari's proofs will be somewhat informal throughout the week; he hits the main points and leaves the details as an exercise to the student.
- Question: What other topological properties does the filled Julia set have?
  - Is it possible that $K(f_c) = \emptyset$?
    - No, it is not — as a degree 2 polynomial, $f_c - z$ has at least one root, which will by necessity be a fixed point, i.e., not diverge to infinity, i.e., in the filled Julia set.
  - Could it be a finite set?
    - No — $K(f_c)$ is a **perfect set**.
    - Uncountably infinite, too.
  - Is $K(f_c)$ connected?
    - Sometimes.
- **Perfect set**: A set where every point in the set is a **nontrivial limit point** of the set.
  - Example: A closed interval, *others listed*.
- **Nontrivial limit point**: A point p in a set A such that there is a nontrivial sequence (i.e., not a constant sequence, e.g., $p, p, p, \dots$) of points in A that converge to p.
- **Not connected**: A set $X \subset \mathbb{C}$ such that there exist disjoint, open sets U, V such that $X \subset U \cup V$, $X \cap U \neq \emptyset$, and $X \cap V \neq \emptyset$.
- **Mandlebrot set**: The set of complex numbers $c \in \mathbb{C}$ such that $K(f_c)$ is connected *Also known as* $\boldsymbol{M}$.
- We can prove that $K(f_c)$ is connected if and only if the critical point of f_c is an element of $K(f_c)$.
  - Remember that critical points of f_c are equivalent to zeroes of f_c'.
- Note that critical points of $f_c := z^2 + c$ are equal to the roots of $f_c' = 2z$, i.e., the elements of $\{0\}$.

- $K(f_c)$ is connected is equivalent to the sequence $0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$ is bounded (an absolute value).
  - Thus, $c \in M$ is equivalent to the sequence $0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$ is bounded.
- The Mandelbrot set is compact, too.
- Proposition: $K(f_c)$ is connected if and only if $0 \in K(f_c)$.
  - "Proof": $\mathbb{C} - K(f_c) = \bigcup_n {f_c}^{-n}(U)$ where U is an open neighborhood of ∞, i.e., the set $\{z : |z| > T\}$.
  - Let $X_n := \mathbb{C} - {f_c}^{-n}(U)$, i.e., $X_0 = \mathbb{C} - U$, so $K(f_c) = \bigcap_n X_n$.
  - Cyclic map? X_n getting "smaller" as n increases? $X_{n+1} \subset X_n$.
  - Assume $X_n =$ little.
  - Two cases: X_n contains 0 and X_n does not contain 0.
  - Either every preimage of X_n is connected or there is a T such that for all $n \geq T$, X_n is not connected.
- Theorem (Douady-Hubbard): M is connected.

1.3 Harmonic Functions, Brownian Motion, and Analysis in the Plane 1 (Lawler)

- These topics will change week to week, so drop in at any point over the summer.
- Schedule:
  - Lectures MWF at 2:30 PM.
  - Group meeting Tuesday at 2:30 PM.
    - Anybody can attend these!
  - No Zoom on Thursday, but there will be an opportunity to talk to Greg Lawler in person at the department of mathematics outside Eckhart when the weather is good.
- Resources:
  - Piazza — look under the resources tab for lecture notes (with some exercises; these are very rough; gives you something to read with the lectures), other materials, etc.
  - There is a 180 page book draft based on his REU lectures last summer.
    - Do not share this.
- This math is at the border of analysis (basically advanced calculus) and probability.
  - Lawler thinks of these as all basically the same subject.
- We will work in $\mathbb{R}^2$.
- A lot of what Dr. Lawler does is often called Complex Analysis.
- Complex analysis allows you to get the results quicker even though they encapsulate ideas that are 100% real; we're going to take a real-function perspective.
- Harmonic function notation:
  - Domains D are connected open sets that are subsets of $\mathbb{R}^2$.
  - Mean value: $f : \mathbb{R}^2 \to \mathbb{R}$ (continuous), or $f : D \to \mathbb{R}$.

- z, w are points in $\mathbb{R}^2$, and we write $z = (x, y)$ where x, y are the one-dimensional components.
- $B(z, \epsilon) = \{w : |z - w| < \epsilon\}$ is an open disk and $\partial B(Z, \epsilon)$ is the circle of radius ϵ about z.

- If $B(z, \epsilon) \subset D$, then the (circular) mean value $MV(f; z, \epsilon)$ is the average rate of f on $\partial B(z, \epsilon)$, i.e., the quantity
$$\frac{1}{2\pi\epsilon} \int_{\{|w-z|=\epsilon\}} f(w) |\,\mathrm{d}w\,|$$
where $|\,\mathrm{d}w\,|$ is with respect to arc length.
- Let $(\cos\theta, \sin\theta) = \mathrm{e}^{i\theta}$.
- **Harmonic function**: $f : D \to \mathbb{R}$ is harmonic if f is continuous and for all $z \in D$ and every $\epsilon > 0$ with $d(z, \partial D) > \epsilon$, then $f(z) = MV(f; z, \epsilon)$.
- Many applications, notably in physics wrt. heat.
  - Consider D describing a surface with heat. Fix the temperature at the boundary. Let $U(z) =$ temperature at z (in equilibrium).
  - Then U is harmonic on D.
- We're going to understand the mean value in terms of the **Laplacian**.
- If $f : D \to \mathbb{R}$ is C^2 (the first and second derivatives exist and are continuous [either two derivatives in one variable or one derivative in both variables for $\mathbb{R}^2$]), then the Laplacian is defined by
$$\Delta f(z) = f_{xx}(z) + f_{yy}(z)$$
- Proposition: If u is C^2 in D, then $\Delta u(z) = \lim_{\epsilon\to 0} 4 \cdot \frac{MV(u;z,\epsilon)-u(z)}{\epsilon^2}$.
- For ease, let's assume that $z = 0 = (0, 0)$ and $u(z) = 0$.
- Taylor polynomial (in several variables): If $z = (x, y)$, then
$$u(z) = 0 + u_x(0)x + u_y(0)y + \frac{1}{2}u_{xx}(0)x^2 + \frac{1}{2}u_{yy}(0)y^2 + u_{xy}(0)xy + \sigma(|z|^2)^{[2]}$$
  - $u_x(0)MV(x; 0, \epsilon) + u_y(0)MV(y; 0, \epsilon) + u_{xy}(0)MV(xy; 0, \epsilon) + \sigma(\epsilon^2) + \frac{1}{2}[u_{xx}(0)x^2 + u_{yy}(0)y^2]$.
  - Note that $u_{xx}(0)x^2 = MV(x^2; 0, \epsilon)$ and $u_{yy}(0)y^2 = MV(y^2; 0, \epsilon)$.
  - You can use multivariable calculus, or you can observe that $MV(x^2; 0, \epsilon) = MV(y^2; 0, \epsilon)$, thus telling you that $MV(x^2; 0, \epsilon) + MV(y^2; 0, \epsilon) = MV(x^2 + y^2; 0, \epsilon) = \epsilon^2$.
  - Since $|z|^2 = \epsilon^2$, we have that $u(z) = \frac{1}{2}[\frac{1}{2}]$...
- Proposition: A function $f : D \to \mathbb{R}$ is harmonic if and only if it is C^2 and $\Delta f(z) = 0$ for all $z \in D$.
  - Proof: Backwards direction first. We want to show that C^2 and $\Delta f(z) = 0$ imply the mean value property. The mean value property clearly holds at $\epsilon = 0$. Consider $MV(f; z, \epsilon)$ as a function of ϵ. The derivative in ϵ ends up looking something like $\frac{1}{2\pi\epsilon}\int_{\text{circle}} \partial_n f(w) |\,\mathrm{d}w\,|$ where ∂_n is the normal direction.
  - Using the divergence theorem, we have that the above is equal to $\int_{\text{disk}} \Delta f(w)\,\mathrm{d}w$. Note that we sometimes write $\Delta f = \mathrm{d}w\,(\nabla f)$ where $\nabla f = (f_x, f_y)$. Additionally, $\operatorname{div}(\nabla f) = \partial_x(f_x) + \partial_y(f_y)$.
  - Exercise: Show that if u is harmonic, then u is C^2.
- The notion of probability comes in when we ask, "what is the 'mean value' if we are not a disk viewed from the center?"

[2] σ is pronounced "little oh."

1.4 The Mathematics of Playing Pool (Mazur)

- Main focus: Billiards in a polygon.
- The ball bounces off a side with the same angle of incidence it struck it with. If the ball hits the corner, it stops (maybe it fell into a pocket).
- **Billiards**: Start with a polygon in the plane. Shoot a billiard ball, thought of as a point mass, ...
- Rectangular tables are fully understood, but other polygons are harder. Curved sides are even more complicated.
- Connection to physics: Ehrenfest windtree model (by Paul and Tatjana Ehrenfest, 1912).
- One thing people study is the diffusion rate of a random particle. This means that if you take a random particle and follow it for a long time t, how far is it from where it started? What people know is that a typical particle is about distance $t^{2/3}$ away.
- Another example: Take two point masses with positions $0 \leq x_1 \leq x_2 \leq 1$ on the unit interval $[0, 1]$. Suppose their masses are m_1, m_2 and they move with velocities v_1, v_2, respectively. They collide with each other and with the barriers at 0 and 1. Momentum and energy are conserved.
  - We can convert this to billiards in a right triangle with the observations that energy and speed are related and momentum and angle of incidence are related.
- Billiards are important examples of **dynamical systems** where one studies behavior of objects under a deterministic system (initial position and velocity define motion for the rest of time).
- Billiards questions:
  - Are there periodic orbits?
  - How does a typical orbit behave in the long term? Is it dense? Is it equidistributed?
  - Illumination problem (can you get from any point to any other?).
- Periodic orbits:
  - There are periodic orbits in acute triangles.
  - Drop perpendiculars; use Euclidean geometry to prove.
  - It is unknown if a general obtuse triangle has a periodic orbit. This is considered to be one of the big unsolved problems in dynamics[3].
- Equidistribution and Ergodicity:
  - ...
- Rational billiards is much more well-defined. Every rational table has periodic orbits.
- Most paths equidistribute.
- Illumination problem:
  - Now imagine you put a light source at a point on your table. The walls are mirrors and a light beam bounces of the mirror with angle of incidence equal to angle of reflection. Is every point illuminated? In other words, can you get from any point to any other? Not in a Penrose unilluminable room (a region is dark in this elliptical room).
  - Polygon example from Tokarski in the 1980s (a zero-dimensional point is unilluminable).

[3] Can we consider the set of all possible initial positions and directions and see what converges to what?

  - Within the last 5 years: For any rational billiard, there are at most a finite number of unilluminable points.

- Unfolding billiards boards.
- If the slope of the line on a torus is rational, it closes up. If the slope of the line on a torus is irrational, it does not close but is equidistributed.
- For a square, when we glue the unfolded version together, we get a genus 1 surface (a torus).
- For a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$, and $\frac{3\pi}{8}$, the unfolded version can be glued together into a genus 2 surface.
- Ergodicity:
  - A common notion in mathematics is that of irreducibility.
  - In our context, irreducible (or ergodic) means you cannot divide your table X nontrivially into 2 pieces $X = X_1 \cup X_2$ so that if you start with a point in X_1 and you move in a straight line, you stay in X_1 and if you start in X_2 you stay there.
  - In other words, there are no invariant sets.
- Proves the Kronecker-Weyl theorem.

1.5 Lecture 1.1: Bijections and Permutations

- Consider the statement $4 = 4$.
  - "Four is an abstraction for any set of four elements."
  - Thus, $4 = 4$ implies that $\{\text{lion}, \text{cat}, \text{tiger}, \text{cheetah}\} = \{\text{burger}, \text{pizza}, \text{pasta}, \text{borsh}\}$, for instance, but in what sense are these sets similar? They are related because there exists a bijection between them. Note, however, that there are $4! = 24$ possible bijections between these two sets.
- Let A, B be sets and $f : A \to B$ be a function (or map).

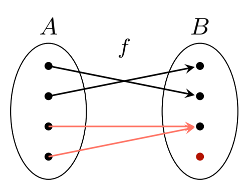

Figure 1.1: A function that is neither injective nor surjective.

  - f is **injective** if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$.
  - f is **surjective** if for every element $b \in B$, there exists an $a \in A$ such that $f(a) = b$.
  - f is **bijective** if it is surjective and injective.
    - These are one-to-one correspondances.
  - The function f in Figure 1.1 is a function because it maps every element of A to an element of B. However, it is not injective because the mappings indicated in light red map two distinct elements of A to the same element of B. Likewise, it is not surjective because the element of B drawn in dark red is not the image of any element of A under f. Because f is neither injective nor surjective, it is not bijective.
- If there exists a bijection between the set A and the nice set $[n] = \{1, 2, \dots, n\}$, we say that $|A| = n$.

- Example:
  - Consider the sets $A = \mathbb{N}$ and $B = \{2k \mid k \in \mathbb{N}\}$.
  - Define $f : A \to B$ by $f(x) = 2x$. f is a bijection.
  - Thus, $|A| = |B|$ despite the intuitive sense that $|B|$ should be half $|A|$.
- Sets in bijection with $\mathbb{N}$ are called **countable**, and we write $|A| = \aleph_0$.
  - We can show that $|\mathbb{Q}| = \aleph_0$ and $|\mathbb{R}| \neq \aleph_0$.
- **Permutation**: A bijection $\sigma : S \to S$, where $S = \{1, \dots, n\}$.
- Examples:
  - $n = 1$:
    - There exists a unique permutation of the set, which is an identity.
  - $n = 2$:
    - There are two different bijections here.
    - They are denoted by the following matrices.

$$\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

  - $n = 3$:
    - Listing[4] the elements of $\mathbb{S}_3$:

$$\mathbb{S}_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

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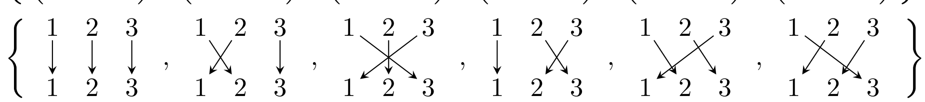


    - There are different important classes of permutations — the first one listed is an identity, the next three listed are identities for one object and **cycles** of length two for the other two, and the last two are **cycles** of length three.
    - $|\mathbb{S}_3| = 6$.
- More generally, the numbers in the first row are the numbers 1 through n and the numbers in the second row are those to which the bijection maps each number:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

- We denote by $\mathbb{S}_n$ the set of permutations of $S = \{1, \dots, n\}$.
- Exercises:
  1. Draw all permutations in $\mathbb{S}_4$.
  2. Prove that $|\mathbb{S}_n| = n!$.
     - Perhaps by induction? Trivial for the base case of 0. Then if true for n, we can map $n+1$ to $n+1$ possible numbers, so we divide into $n+1$ cases. If $\sigma(n+1) = 1$, then we know by the inductive hypothesis that we can permute the remaining n numbers in $n!$ ways. Thus, the number of permutations is $\underbrace{n! + n! + \cdots + n!}_{n+1 \text{ times}} = (n+1) \cdot n! = (n+1)!$.
- Note that in $\S_3$, permutations $\sigma(2,1,3)$, $\sigma(3,2,1)$, and $\sigma(1,3,2)$ are considered odd and the others are considered even.
  - This classification is based on the number of crossings in the function diagrams.

[4] Dr. Rudenko also shows function diagrams and directed graphs (see Figure 1.2).

1.6 Lecture 1.2: The Group of Permutations

- **Composition**: The function $g \circ f : A \to C$ defined by the map $(g \circ f)(a) = g(f(a))$, where A, B, C are sets and $f : A \to B$ and $g : B \to C$ are functions.
- The composition of two bijections is a bijection (Proposition 1.26).
- We can compose permutations to get new (or the same) permutations.
  - Example: If $\sigma = (2, 1, 3)$ and $\tau = (1, 3, 2)$, then $\tau \cdot \sigma = (3, 1, 2)$.
- This new function is known as a **product of permutations**.
- If you do this many times to the same permutation, you construct the power of a permutation.
- Example:
  - Consider $\sigma(3, 5, 1, 2, 4)$.
  - If we draw this out, then we can see that there is a cycle of length two and a cycle of length 3.
  - Thus, $\sigma^{100} = (1, 5, 3, 2, 4)$. $100\%2 = 0$, so $\sigma^{100} = i$ with respect to the $3, 1$ cycle. Additionally, $100\%3 = 1$, so $\sigma^{100} = \sigma$ with respect to the $5, 2, 4$ cycle.
  - Also note that $\sigma^{2 \cdot 3} = \sigma^6 = i$, where i is the identity permutation.
- Multiplication of permutations is associative.
- Thus, we can construct multiplication tables.
- Example:

	e	***x***
e	e	x
x	x	e

Table 1.1: Multiplication table for $\mathbb{S}_2$.

  - This holds where $e = (1, 2)$ and $x = (2, 1)$.
- **Identity**: The function $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) = a$, where A is a set.
  - $\mathrm{id}_{[n]} = (1, 2, \dots, n) = e \in \mathbb{S}_n$.
    - e is also known as the **trivial permutation** or **identity permutation**.
  - For all $\sigma \in \mathbb{S}_n$, $\sigma \cdot e = e \cdot \sigma = \sigma$.
- **Inverse**: The function $f^{-1} : B \to A$ corresponding to the function $f : A \to B$ having the properties that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.
- f^{-1} exists iff f is a bijection (Proposition 1.27).
- Note that it's often easier to check the two inverse properties than to confirm that f is bijective.
- Since permutations are bijections, all permutations have inverses.
  - If $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{smallmatrix}\right) = \sigma \in \mathbb{S}_5$, then $\sigma^{-1} = \left(\begin{smallmatrix} 3 & 4 & 1 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{smallmatrix}\right)$.
  - Note that in this case $\sigma = \sigma^{-1}$, indicating the existence of only 1- and 2-cycles.
- **Group**: A set G along with a map $G \times G \to G$.
- If the following properties hold, then a collection of objects is a group.

1. Associativity: For all $g_1, g_2, g_3 \in G$, $(g_1g_2)g_3 = g_1(g_2g_3)$.
2. Identity: There exists a special element $e \in G$ such that for any $g \in G$, $g \circ e = e \circ g = g$.
3. Inverse: For all $g \in G$, there exists $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

- Examples:
  - $\mathbb{S}_n$ is a group.
  - $\mathbb{Z}$ and addition is a group.
  - $\mathbb{Q}$ and multiplication is a group (notably, $\mathbb{Z}$ and multiplication is not).

1.7 Introduction to Complex Dynamics 2 (Calegari)

6/22:

- Picking up from yesterday on the proof of the proposition, $K(f_c)$ is connected iff $0 \in K(f_c)$.
  - Recall that 0 is the unique critical point.
  - Also recall that U is the set that contains only elements with sufficiently big absolute values, big enough so that $f_c(U) \subset U$.
  - Define $X_0 = \mathbb{C} - U$, $X_{n+1} = {f_c}^{-n}(X_n)$.
    - Each X_n is compact, and $K(f_c)$ is equal to the intersection of all X_n, therefore compact in and of itself.
  - Thus, $K(f_c)$ is connected iff every X_n is connected.
  - Key point: If $0 \in K(f_c)$, then each X_n is a disk; otherwise, some X_n is not a disk.
  - Fact: Every point in $\mathbb{C}$ has exactly 2 preimages under f_c except for the critical value $c = f_c(0)$ since f_c is a degree 2 polynomial.
  - Assume X_n is a disk. If X_n contains the critical value, then X_{n+1} is a disk; otherwise, not (in fact, it will be disconnected).
- Under f_c, the preimage of a circle not containing the critical value is either 2 circles, each of which maps one-to-one, or a single circle mapping two-to-one.
  - Suppose that the preimage of the boundary of the circles is two distinct circles.
  - By continuity, concentric circles narrowing down within the original set narrow down within the other two circles.
  - Each point in X_n has exactly two preimages iff $c \notin X_n$.
  - As we make smaller and smaller circles, than we can split our one-circle preimage into two disconnected subsets.
- Today: The theory of Julia sets for holomorphic functions in general.
- Let f be a **holomorphic** map from $\hat{\mathbb{C}}$ to itself.
- Every such f has finitely many 0s and poles (∞s).
- Therefore, f is a rational function, i.e., a ratio of polynomials. Symbolically, $f(z) = \frac{p(z)}{q(z)}$ for some polynomials p, q.
- To talk about Julia sets, we need some definitions.
- **Normal family**: Let U be an open subset of the Riemann sphere $\hat{\mathbb{C}}$, and let F be a family of holomorphic functions $f : U \to \hat{\mathbb{C}}$. F is **normal** if its closure (in the space of all holomorphic functions from U to $\hat{C}$) is compact.

 - In other words, if ever infinite sequence $f_n \in F$ has a subsequence that converges locally uniformly to some limit $g : U \to \hat{\mathbb{C}}$.

- Normality is local.
- Proposition: Suppose F is a family of holomorphic functions defined on U, and suppose for all $p \in U$, there exists open $p \in V \subset U$ such that $F|_V$ is normal. Then $F|_U$ is normal.
  - Proof 1: Diagonal argument.
  - Proof 2: ???
- **Julia set**: Let f be a holomorphic map from $\hat{\mathbb{C}}$ to itself. Let $\mathcal{F} := \{f^n \mid n \in \mathbb{N}\}$. The Fatou set $\Omega_f :\subset \hat{\mathbb{C}}$ is the open subset whose $\mathcal{F}$ is normal. It is equal to the union of all U where $F|_U$ is normal. Thus, Ω_f is open and $\mathcal{F}|_{\Omega_f}$ is normal. The **Julia set** $J_f :\subset \hat{\mathbb{C}}$ is $\hat{C} - \Omega_f$, i.e., $p \in J_f$ iff for all U containing p, $F|_U$ is not normal on U.
  - Hence, J_f is compact.
- Example: Let's let p be a fixed point for f.
  - p is an attracting fixed point if $|f'(p)| < 1$. p is super attracting if $|f'(p)| = 0$.
  - Example:
    - If f is a polynomial of degree at least 2, then ∞ is a super attracting fixed point.
  - If we take a sufficiently small neighborhood of p, then f shrinks and rotates the neighborhood a little bit.
  - **Basin of attraction** of p.
  - **Immediate basin of attraction** of p is the connected component of the basin of attraction of p.
- If f is a polynomial, then $K(f_c)$ is equal to the complement of the basin of infinity.
- "Most" "typical" f have $\Omega_f = \bigcup$ basins of attraction of attracting periodic orbits.
  - Furthermore: every immediate basin of an attracting periodic orbit contains at least one critical point.
  - A rational map of degree d (the maximum of the degrees of polynomials p, q) has $2d - 2$ critical points.
- Theorem: The closure of the set of repelling periodic orbits (i.e., p with $f^n(p) = p$ and $|(f^n)'(p)| > 1$) is J_f.

1.8 Coambiguous Concepts 1 (May)

- Categories can inscribe things of different types.
- Categories can be interesting mathematical objects in and of themselves much like rings, groups, etc.
- Whenever you define an object, you should define a notion of a morphism (or map) between objects.
- Morphisms have compositions and identities.
- A monoid is a category with one object. A category is a monoid with many objects!
- Today: Category theory.
- **Poset**: A partially ordered set, i.e., one that is transitive, reflexive, and antisymmetric ($A \leq B$ and $A \geq B \iff A = B$).
- **Small category**: A category that has a set of objects as opposed to a class of objects. *Also known as* **kitty category**, **kittygory**.

1.9 Lecture 1.3: Cyclic Structure of a Permutation

- Let $\sigma = \left(\begin{smallmatrix}1&2&3&4&5&6&7&8\\7&3&4&2&6&5&1&8\end{smallmatrix}\right) \in \mathbb{S}_8$.
- Consider the directed graph representation.

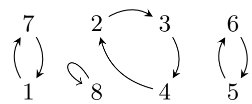


Figure 1.2: Directed graph of a permutation in $\mathbb{S}_8$.

  - Note that this is a graph because it has points connected by edges, it is directed because the edges have a direction (they are arrows/mappings), and it has some loops and some multiple edges (points with more than one edge).
  - There are four cycles in this graph (a 1-cycle, two 2-cycles, and a 3-cycle).
- **Cycle**: A permutation $\sigma = (i_1, \dots, i_k) \in \mathbb{S}_n$, where $1 \leq i_1 \leq \cdots \leq i_k \leq n$, such that $\sigma(i_j) = i_{j+1}$, $\sigma(i_k) = i_1$, and $\sigma(s) = s$ for all $s \in [n] \setminus \{i_1, \dots, i_k\}$.
- For example, $\sigma = (1,3,5) \in \mathbb{S}_5$ is the permutation $\sigma = \left(\begin{smallmatrix}1&2&3&4&5\\3&2&5&4&1\end{smallmatrix}\right)$.
- **Length** (of a cycle): The value k in the above definition.
- **Support** (of a cycle): The set of indices $\{i_1, \dots, i_k\}$ in the above definition.
- **Transposition**: A cycle $(i,j) \in \S_n$, where $i,j \in [n]$ are distinct.
- Thus, $\mathbb{S}_3 = \{e, (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$.
- Exercise:
  - Count the number of transpositions in $\mathbb{S}_n$.
  - Prove that every permutation is a product of transpositions.
    - Consider crossings in the function diagram! Relate to braids from knot theory.
    - For instance, $\left(\begin{smallmatrix}1&2&3&4&5\\3&4&1&2&5\end{smallmatrix}\right) = (1,4)(2,4)(1,3)(2,3)$.
- **Independent** (cycles): Two cycles $\sigma_1, \sigma_2 \in \mathbb{S}_n$ with disjoint supports.
- **Dependent** (cycles): Two cycles that are not independent.
- Propositions:
  1. Every permutation is a product of independent cycles.
     - In the directed graph corresponding to σ, every vertex has on incoming edge.
     - There exists a smallest k such that $\sigma^k(i) = i$.
     - *Return to this?*
  2. Independent cycles commute with each other.
     - Obvious: Consider c_1, c_2 independent. Then

$$c_1c_2(i) = \begin{cases} c_1(i) & i \text{ is in the support of } c_1 \\ c_2(i) & i \text{ is in the support of } c_2 \\ i & i \text{ is in the support of neither} \end{cases}$$

1.10 Lecture 1.4: Binomial Coefficients

- Start with the set $S = \{1, \dots, n\}$. Denote by 2^S the set of subsets of S.
  - If $n = 2$, then $2^S = \{\emptyset, \{1, 2\}, \{1\}, \{2\}\}$ has four elements.
- Proposition: $|2^S| = 2^{|S|}$.
  - Consider the set $S = \{1, 2, 3, 4, 5, 6\}$.
  - Identify subsets of S with a code, exemplified by $\{1, 3, 5\} \mapsto (1, 0, 1, 0, 1, 0)$.
  - Based on this, construct a map f which sends $2^S \to$ the set of sequences of $0, 1$ of length n. In other terms, map each subset $A \in 2^S$ to a sequence $(s_1, \dots, s_n)$ where

$$s_i = \begin{cases} 0 & i \notin A \\ 1 & i \in A \end{cases}$$

  - For example, $f(\emptyset) = (0, 0, 0, 0, 0, 0)$ and $f(S) = (1, 1, 1, 1, 1, 1)$.
  - Note that f is a bijection.
  - Since the set of all sequences of $0, 1$ of length n (more commonly denoted by $\{0, 1\}^n$) clearly has 2^n elements and 2^S is in bijective correspondence with this set, we know that 2^S has $2^{|S|} = 2^n$ elements.
- Note that coding subsets by sequences is very important in computer science and mathematics.
- Geometric model: Vertices of a line segment give you subsets of a set with 1 element, vertices of a square give you subsets of a set with 2 elements, vertices of a cube give you subsets of a set with 3 elements, ...
- If $A \in 2^S$ and $f(A) = (a_1, \dots, a_n)$, then $|A| = \sum_{i=1}^{n} a_i$.
- If $0 \le k \le n$, then $\binom{n}{k}$ is the number of subsets of size k in a set of size n.
- Proposition: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
  - $\binom{n}{k} \cdot k!$ is the number of subsets of size k and the orderings of their elements.
  - But this is the same as choosing the first element from among n elements, the second from among the remaining $n - 1$ elements and so on until $n - k + 1$, i.e.,

$$\begin{aligned}\binom{n}{k} \cdot k! &= n \cdot (n-1) \cdots (n-k+1) \\ &= \frac{n!}{(n-k)!}\end{aligned}$$

- By convention, $\binom{n}{0} = 1 = \binom{n}{n}$.
- Proposition: $\binom{n}{k} = \binom{n}{n-k}$.
  - Follows from the previous proposition.
  - Alternatively, we can seek to show that the number of subsets of size k is equal to the number of subsets of size $n - k$. But since these are inverse maps, they obviously are.
- Proposition: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
  - Follows from the factorial formula.
  - Conceptually:

  - Divide the set of subsets of S of size k into those that contain n and those that do not contain n.
  - The subsets of size k that do not contain n is equal to the subsets of $S \setminus \{n\}$ of size k, i.e., $\binom{n-1}{k}$.
  - The subsets of size k the do contain n are in bijection with the subsets of size $k-1$ in $S \setminus \{n\}$, i.e., $\binom{n-1}{k-1}$.

- Introduces Pascal's triangle.
  - The symmetry of Pascal's triangle follows from the formula $\binom{n}{k} = \binom{n}{n-k}$.
- Theorem (Binomial Formula):

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n \\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\end{aligned}$$

  - Induction proof: Left as an exercise.
  - $(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n\text{ times}}$ = a sum of monomials, e.g., *aabbaba*. We get each monomial by choosing a or b from each parentheses. Thus, $(a+b)^n = \sum_{A\subset\{1,\dots,n\}=S} a^{|A|}b^{|S\setminus A|}$.
- Corollary: If $a = b = 1$, then $2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k = \sum_{k=0}^{n}\binom{n}{k}$.

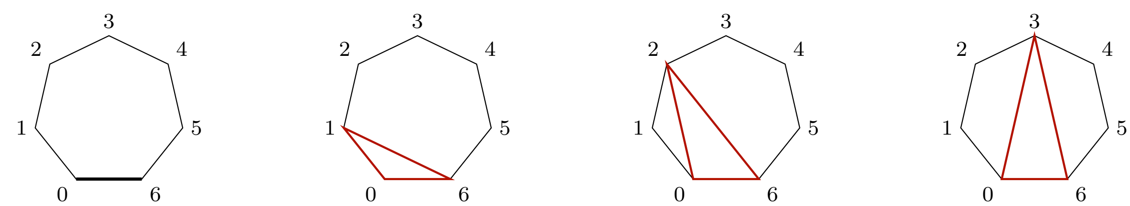
1.11 Problem Session 1

6/23:
- The goal is to get you to think about math all the time. You should let problems sit in your head for a week; no expectation of solving them, but just thinking about it.
- Typo in 5a. Nobody knows what problem 4 means.
- If you're ever lost on a permutation problem, just draw out the connected graph.
- Prove any permutation is a product of adjacencies.
- 5d technically proves 5c, but there is a nicer formula for 5c that he would rather we find.

1.12 Lecture 1.5: Catalan Numbers

6/26:
- The sequence $1, 1, 2, 5, 14, 42, \dots$.
- Discovered by Euler, who was investigating **triangulations** of a polygon.
- Let P be a **convex** polygon with $n+2$ sides.
- **Convex** (polygon): A polygon such that for all chords connecting points, the chord passes entirely through the interior of the polygon.
- **Triangulation**: A decomposition of a polygon P into triangles by nonintersecting diagonals.
- Exercise:
  - Every triangulation of P has n triangles.
- The n^{th} Catalan number C_n is the number of such triangulations.

- $C_0 := 1$.
- $C_1 = 1$.
- $C_2 = 2$.
- $C_3 = 5$ (there are 5 rotations of the "same" triangulation).
- $C_4 = 14$ (there are 6 with all three diagonals coming out of one vertex, 6 with opposite vertices having two diagonals, each, and 2 with a triangle in the center).
- Computing C_5 gets more complicated. Number the vertices of the heptagon 0-6.



(a) Original shape. (b) Term 1. (c) Term 2. (d) Term 3.

Figure 1.3: Computing C_5.

- Observe: There will always be one triangle with $\overline{06}$ as an edge (see Figure 1.3a).
- If we include $\triangle 016$, we know that there are C_4 ways to triangulate the remaining 6-gon and C_0 ways to triangulate the "triangle" to the left of $\triangle 016$, i.e., the edge $\overline{01}$. Thus, we have that $C_5 = C_0 \cdot C_4 + \cdots$ (see Figure 1.3b).
- If we consider $\triangle 026$, there are C_1 ways to triangulate the triangle to the left of $\triangle 026$ and C_3 ways to triangulate the 5-gon to the right of $\triangle 026$ (see Figure 1.3c).
- If we consider $\triangle 036$, there are C_2 ways to triangulate the 4-gons on both sides of $\triangle 026$ (see Figure 1.3d).
- Continuing, we can see that

$$C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0$$

- Proposition: The following recurrence relation holds.

$$C_n = \sum_{i=0}^{n-1} C_iC_{n-1-i}$$

  - Proof: Repeat the same argument.

- A second interpretation of Catalan numbers.
- Consider a sequence of brackets (,) such as ((()))(()))(. A sequence is **admissible** if
  1. The number of opening brackets is equal to the number of closing brackets.
  2. In every initial subsequence, the number of opening brackets is greater than or equal to the number of closing brackets.
- For example, (()))(is not admissible but (())() is.
- Note that admissible sequences are the ones that can be taken to be the parentheses to expressions, e.g., $(a + b(c + d))(e + f)$ makes sense, but we can't fill in anything in (()))(.
- Proposition: The number of admissible sequences with $2n$ brackets equals C_n.
  - $n = 1$: ().

  - $n = 2$: (()), ()().
  - $n = 3$: ((())), (()()), (())(), ()(()), ()()().
  - We have to show that the number of admissible sequences satisfies the same recurrence relation.
  - Consider $n = 3$. For the first two, we have 2 sets of parentheses within the initial subsequence, and 0 outside it (C_2C_0). For the next one, we have 1 set within and 1 set outside (C_1C_1). For the last two, we have 0 sets in and 2 sets outside (C_0C_2). The number is the sum of all three.

- Exercise: Construct an explicit bijection between triangulations and parentheses.

1.13 PSet 1

1. We have discussed three ways to prove identities for binomial coefficients: induction, combinatorial bijection, and binomial formula. Try these three approaches on the following identities.

   (a) $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$.

   (b) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

   Proof. Combinatorial bijection: Divide the $2n$ items into two bins of n items each, which we shall refer to as the red bin and the blue bin. There are as many ways to choose n items from among both bins as there are to choose 0 items from the red bin and n items from the blue bin, plus 1 item from the red bin and $n-1$ items from the blue bin, and so on and so forth. Symbolically, $\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$. We can then apply to this identity the symmetry one (i.e., $\binom{n}{k} = \binom{n}{n-k}$) to yield the final formula. □

   (c) $\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k}$.

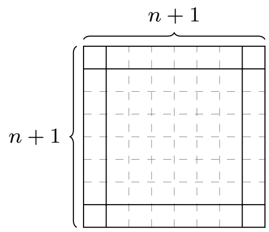
   (d) $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k-1}{k-1} = \binom{n}{k}$.

   Proof. As can be easily seen from Pascal's triangle, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. □

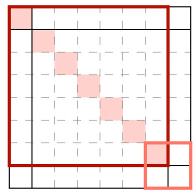
2. We will consider a "disease" that infects cells of a chessboard. Initially, some number of the 64 cells are infected. Subsequently, the infection spreads according to the following rule: If at least two neighbors of a cell are infected, then the cell gets infected. (Neighbors share an edge, so each cell has at most four neighbors.) No cell is ever cured. What is the minimum number of cells one can initially infect so that the whole board is eventually infected? It is easy to see that 8 is sufficient in many ways. Prove that 7 is not enough.

Proof. To prove that 8 is the minimum number of cells one can initially infect so that the whole 8×8 board is eventually infected, it will suffice to show n is the minimum number of cells required to infect an $n \times n$ board for all natural numbers n. We induct on the the edge length of the board n using strong induction. For the base case $n = 1$, it is trivially obvious that the one cell on the board is either infected or not, so to infect the whole board, we need to infect it (i.e., infect 1 cell). Now suppose inductively that we have proven the claim for n; we now seek to prove it for $n + 1$.

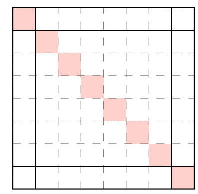
Consider an $(n+1)\times(n+1)$ board, as in Figure 1.4a. As part of infecting the board, it will be necessary to infect the $n \times n$ subgrid in the upper left-hand corner (in dark red in Figure 1.4b, and note that the n shaded cells are the initially infected ones that will infect the whole subgrid). By the inductive hypothesis, it will require a minimum of n infected cells to infect it. However, even after that whole subgrid is infected, there are still healthy cells, none of which has more than one neighbor. Thus, it will be necessary to add in at least one more infected cell. The light red subgrid at the bottom right of Figure 1.4b gives us a hint of where to put it; indeed, for the $n = 2$ case, it is necessary to have at least 2 cells infected by hypothesis, but this subgrid only contains one infected cell (one that overlaps with the other subgrid). Therefore, by placing the $(n + 1)^{\text{st}}$ infected cell in the bottom right-hand corner (see Figure 1.4c), we can infect the whole board. □



(a) An $(n+1) \times (n+1)$ chessboard.



(b) Subgrids.



(c) Infected cells in the beginning.

Figure 1.4: A chessboard to be infected.

3. Generators of $\mathbb{S}_n$

   (a) Prove that
   $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 4 & 5 & 1 \end{pmatrix}$$
   is a product of transpositions.

   Proof. $\sigma = (16)(23)$. □

   (b) Prove that any permutation is a product of transpositions.

   Proof. Let $\sigma = (a_1, a_2, \dots, a_n) \in \mathbb{S}_n$ be an arbitrary permutation. Starting from $e = (1, 2, \dots, n)$, have the first permutation be $(1a_1)$. This will set a_1 in the correct position. The second will then be (xa_2), where x is whatever is in the second position after the first transposition. This will set a_2 in the correct position. Continue down the line, always switching whatever is in the k^{th} spot with what should be there before moving on the the $(k+1)^{\text{th}}$ spot and ending at the n^{th} spot. □

   (c) Prove that any permutation in $\mathbb{S}_n$ can be written as a product of two permutations (12) and $(12 \dots n)$.

4. Two players put 25 cent coins on the round table alternately. They are allowed to put a coin on an empty spot only. The person, who cannot make a move, loses the game. Find a winning strategy for one of the players.

5. Let G be a finite group.

   (a) Prove that for $g_1, g_2, g_3 \in G$, if $g_1 g_2 = g_1 g_3$, then $g_2 = g_3$.

   Proof. By the inverse property of a group, there exists ${g_1}^{-1}$. Therefore, we have that
   $$\begin{aligned} {g_1}^{-1}(g_1 g_2) &= {g_1}^{-1}(g_1 g_3) && \text{Multiplicative POE} \\ ({g_1}^{-1} g_1) g_2 &= ({g_1}^{-1} g_1) g_3 && \text{Associativity} \\ e g_2 &= e g_3 && \text{Inverse} \\ g_2 &= g_3 && \text{Identity} \end{aligned}$$
   as desired. □

   (b) Prove that in the multiplication table of a finite group, every element appears exactly once in each row and once in each column.

   Proof. Let $G = \{g_1, g_2, \dots, g_n\}$ be a finite group of order n. Consider the row/column defined by g_i, where i is an arbitrary natural number between 1 and n, inclusive. Suppose for the sake of contradiction that the object g appears twice in the row, such that $g_i g_j = g$ and $g_i g_k = g$, where $j \neq k$. But by part (a), $g_i g_j = g = g_i g_k$ implies that $g_j = g_k$, a contradiction, since every element of G is only listed once in the row/column headers. □

(c) Prove that for any permutation $\sigma \in \mathbb{S}_n$, there exists $N \in \mathbb{N}$ such that $\sigma^N = e$.

Proof. Decompose σ into its subcycles, i.e., if there are transpositions (2-cycles), 3-cycles, etc., identify all of these. Let A be the set of subcycle lengths. Let N be the least common multiple of every element of A.

Let a_i be the i^{th} element of σ, where i is an arbitrary natural number between 1 and n, inclusive. We want to show that the i^{th} element of σ^N is i. Suppose that i is part of a subcycle of length k. It follows that the i^{th} element of $\sigma^k, \sigma^{2k}, \sigma^{3k}, \dots$ is i. Therefore, since $N = \operatorname{lcm}(\{k, \dots\})$, the i^{th} element of σ^N is i, as desired. $\square$

(d) Prove that for $g \in G$, we have $g^{|G|} = e$.

Proof. Something to do with part (b)? Possibly something with Lagrange's theorem, i.e., elements of subgroups hit e multiple times as the exponent increases while elements not in subgroups cycle through all elements of G before hitting e. $\square$

6. **Erdős-Szekeres theorem**:

Prove that for any $n, m \in \mathbb{N}$, every sequence of $nm + 1$ distinct real numbers contains an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $m + 1$.

Week 2

2.1 Gaussian Curvature (Neves)

6/28: - Plan:
    1. What is a surface?
    2. What is the tangent space?
    3. What are the principal curvatures?
    4. What is the Gaussian curvature?
- In analysis:
    1. What is a function?
    2. What is the derivative?
    3. What is the Hemian of function?
    4. 2nd derivative test (determinant of Hemina).
- **Surface**: A subset $\sigma \subseteq \mathbb{R}^3$ such that for all $p \in \Sigma$, there's a neighborhood B of p in $\mathbb{R}^3$ so that $\Sigma \cap B$ "looks like a disk." More precisely, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ and a map $\varphi : U \to \Sigma \cap B \subseteq \mathbb{R}^3$ such that
    i) φ is continuous and smooth.
    ii) φ is a bijection (with φ^{-1} continuous).
    iii) $\mathrm{d}\varphi_{|x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $x \in U$.
- **Chart**: The quantity (φ, U) near $p \in U$.
- Examples:
    A) Plane. $\Sigma = \mathbb{R}^2 \times |0| \subseteq \mathbb{R}^3$ is a surface with chart $\varphi : \mathbb{R}^2 \to \Sigma$ where $(x, y) \mapsto (x, y, 0)$.
    B) Sphere. $\Sigma = \{\vec{u} \in \mathbb{R}^3 \mid |\vec{u}| = 1\}$.
        - Charts: Consider the sets $U = \{(x_1, x_2) \mid {x_1}^2 + {x_2}^2 < 1\} \subseteq \mathbb{R}^2$. Let $\varphi_1^+ : U \to \Sigma \cap \{(x, y, z) \mid x > 0\}$ be defined by $\varphi_1^+(u_1, u_2) = (\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$, $\varphi_1^- : U \to \Sigma \cap \{(x, y, z) \mid x < 0\}$ be defined by $\varphi_1^-(u_1, u_2) = (-\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$.
        - Same thing for $\varphi_2^\pm, \varphi_3^\pm$.
    C) A cone $\Sigma = \{(x, y, z) \mid z = \sqrt{x^2 + y^2}\}$ is *not* a surface because it fails property (iii).
    D) The closed unit disk $\Sigma = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ is also not a surface.
- **Tangent space**: Let $\Sigma \subseteq \mathbb{R}^3$ be a surface and let $p \in \Sigma$. Then $T_p\Sigma \subseteq \mathbb{R}^3$ is the z-plane so that $p + T_p\Sigma$ is the affine plane that best approximates Σ near p.
    - Best linear approximation near the surface.

  - Very similar/analogous to the derivative.
- If (φ, U) is a chart near p, then $T_p\Sigma = \operatorname{span}\{\frac{\partial\varphi}{\partial x_1}(\bar{x}_1, \bar{x}_2), \frac{\partial\varphi}{\partial x_2}(\bar{x}_1, \bar{x}_2)\}$.
- Proves linear independence of above vectors.
- Principal curvatures.
- Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, $p \in \Sigma$, and $\vec{N}$ be a unit normal vector defined around p (i.e., $\vec{N}(q) \cdot \vec{v} = 0$ for all q near p and $\vec{v} \in T_q\Sigma$).
- Choose $\vec{v} \in T_p\Sigma$ such that $|\vec{v}| = 1$. Set $P_v = \operatorname{span}\{\vec{v}, \vec{N}(p)\}$. Claim: $(\Sigma - p) \cap P_v$ is a curve near the origin.
- **Principal curvature**: The reciprocal of the radius of the circle in P_v that best approximates $(\Sigma - p) \cap P_v$ near the origin. *Also known as* $\boldsymbol{K(\vec{v})}$.
  - The sign is positive if the center of the circle is in the direction of $\vec{N}(p)$ and negative otherwise.
  - If the sign of $\vec{N}(p)$ changes, then $K(\vec{v})$ will change in sign.
- If we change $\vec{N}$ by $-\vec{N}$, then the new $K(\vec{v})$ is the opposite of the old one.
- Given $p \in \Sigma$ and $\vec{N}(p)$ a normal vector at p, we define $K_1(p)$ to have the maximum $K(\vec{v})$ over all unit vectors $\vec{v} \in T_p\Sigma$ and $K_2(p) = \min\{K(\vec{v}) \mid \vec{v} \in T_p\Sigma, |\vec{v}| = 1\}$.
- K_1, K_2 are computable quantities.

2.2 Lecture 1.6: An Explicit Formula for the Catalan Numbers

6/29:
- Theorem (discovered by Euler):

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

- Examples:
  - $C_1 = \frac{1}{2}\binom{2}{1} = 1$.
  - $C_2 = \frac{1}{3}\binom{4}{2} = 2$.
  - $C_3 = \frac{1}{4}\binom{6}{3} = 5$.
- Dyck path:
  - We'll study paths starting with $(0, 0)$, going on each stop either from (x, y) to $(x+1, y+1)$ or from (x, y) to $(x+1, y-1)$.
  - Consider the number of ways to get to each point on the integer grid $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$.
    - Generates a rotated Pascal's triangle.
    - The number of paths from $(0, 0)$ to $(a+b, a-b)$, i.e., a moves up and b moves down is $\binom{a+b}{b}$.
- Proposition: C_n is equal to the number of paths from $(0, 0)$ to $(2n, 0)$ which are contained in the upper half-plane ($y \geq 0$).
  - Proof: C_n is the number of sequences of brackets.
  - Transform a sequence of brackets into a path by ($\mapsto\nearrow$ and) $\mapsto\searrow$.
  - The condition #(= #) implies that the paths start at $(0, 0)$ and end at $(2n, 0)$.
  - The condition that in every initial segment, #(= #) implies that the path lies in the upper half plane.

- Reflection principle:
  - The number of paths from A to B in the upper half plane is equal to the number of paths from A to B minus the number of paths from A to B that intersect the line $y = -1$.
  - Symbolically, $C_n = \binom{2n}{n} - ?$
  - There exists a one-to-one correspondence between two sets: The set of all paths from A to B intersecting ℓ and the set of all paths from A to B', where B' is the reflection of B across ℓ.
- Thus, the number of paths from A to B that intersect the line $y = -1$ is equal to the number of paths from A to $(2n, -2)$.
- Therefore,

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!}\left(1 - \frac{n}{n+1}\right)\binom{2n}{n-1} \\ &= \frac{1}{n+1}\binom{2n}{n}\end{aligned}$$

  - Note that it's not obvious that $\frac{1}{n+1}\binom{2n}{n}$ is an integer unless you present it as the difference of two binomials (i.e., as $\binom{2n}{n} - \binom{2n}{n-1}$).
- Exercise:
  - Take a path from A to B intersecting ℓ and find the closest point of $P \cap \ell$ to B. Reflect the segment of the path after this point.