Week 1

1.1 Gaussian Curvature (Neves)

6/28:

- Plan:
  1. What is a surface?
  2. What is the tangent space?
  3. What are the principal curvatures?
  4. What is the Gaussian curvature?
- In analysis:
  1. What is a function?
  2. What is the derivative?
  3. What is the Hemian of function?
  4. 2nd derivative test (determinant of Hemina).
- **Surface**: A subset $\sigma \subseteq \mathbb{R}^3$ such that for all $p \in \Sigma$, there's a neighborhood B of p in $\mathbb{R}^3$ so that $\Sigma \cap B$ "looks like a disk." More precisely, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ and a map $\varphi : U \to \Sigma \cap B \subseteq \mathbb{R}^3$ such that
  i) φ is continuous and smooth.
  ii) φ is a bijection (with φ^{-1} continuous).
  iii) $\mathrm{d}\varphi_{|x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $x \in U$.
- **Chart**: The quantity (φ, U) near $p \in U$.
- Examples:
  A) Plane. $\Sigma = \mathbb{R}^2 \times |0| \subseteq \mathbb{R}^3$ is a surface with chart $\varphi : \mathbb{R}^2 \to \Sigma$ where $(x, y) \mapsto (x, y, 0)$.
  B) Sphere. $\Sigma = \{\vec{u} \in \mathbb{R}^3 \mid |\vec{u}| = 1\}$.
    - Charts: Consider the sets $U = \{(x_1, x_2) \mid {x_1}^2 + {x_2}^2 < 1\} \subseteq \mathbb{R}^2$. Let $\varphi_1^+ : U \to \Sigma \cap \{(x, y, z) \mid x > 0\}$ be defined by $\varphi_1^+(u_1, u_2) = (\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$, $\varphi_1^- : U \to \Sigma \cap \{(x, y, z) \mid x < 0\}$ be defined by $\varphi_1^-(u_1, u_2) = (-\sqrt{1 - {x_1}^2 - {x_2}^2}, u_1, u_2)$.
    - Same thing for $\varphi_2^\pm, \varphi_3^\pm$.
  C) A cone $\Sigma = \{(x, y, z) \mid z = \sqrt{x^2 + y^2}\}$ is *not* a surface because it fails property (iii).
  D) The closed unit disk $\Sigma = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ is also not a surface.
- **Tangent space**: Let $\Sigma \subseteq \mathbb{R}^3$ be a surface and let $p \in \Sigma$. Then $T_p\Sigma \subseteq \mathbb{R}^3$ is the z-plane so that $p + T_p\Sigma$ is the affine plane that best approximates Σ near p.
  - Best linear approximation near the surface.

- Very similar/analogous to the derivative.

- If (φ, U) is a chart near p, then $T_p\Sigma = \text{span}\{\frac{\partial\varphi}{\partial x_1}(\bar{x}_1, \bar{x}_2), \frac{\partial\varphi}{\partial x_2}(\bar{x}_1, \bar{x}_2)\}$.
- Proves linear independence of above vectors.
- Principal curvatures.
- Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, $p \in \Sigma$, and $\vec{N}$ be a unit normal vector defined around p (i.e., $\vec{N}(q) \cdot \vec{v} = 0$ for all q near p and $\vec{v} \in T_q\Sigma$).
- Choose $\vec{v} \in T_p\Sigma$ such that $|\vec{v}| = 1$. Set $P_v = \text{span}\{\vec{v}, \vec{N}(p)\}$. Claim: $(\Sigma - p) \cap P_v$ is a curve near the origin.
- **Principal curvature**: The reciprocal of the radius of the circle in P_v that best approximates $(\Sigma - p) \cap P_v$ near the origin. *Also known as* $\boldsymbol{K(\vec{v})}$.
  - The sign is positive if the center of the circle is in the direction of $\vec{N}(p)$ and negative otherwise.
  - If the sign of $\vec{N}(p)$ changes, then $K(\vec{v})$ will change in sign.
- If we change $\vec{N}$ by $-\vec{N}$, then the new $K(\vec{v})$ is the opposite of the old one.
- Given $p \in \Sigma$ and $\vec{N}(p)$ a normal vector at p, we define $K_1(p)$ to have the maximum $K(\vec{v})$ over all unit vectors $\vec{v} \in T_p\Sigma$ and $K_2(p) = \min\{K(\vec{v}) \mid \vec{v} \in T_p\Sigma, |\vec{v}| = 1\}$.
- K_1, K_2 are computable quantities.

1.2 Lecture 1.6: An Explicit Formula for the Catalan Numbers

6/29:
- Theorem (discovered by Euler):

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

- Examples:
  - $C_1 = \frac{1}{2}\binom{2}{1} = 1$.
  - $C_2 = \frac{1}{3}\binom{4}{2} = 2$.
  - $C_3 = \frac{1}{4}\binom{6}{3} = 5$.
- Dyck path:
  - We'll study paths starting with $(0,0)$, going on each stop either from (x,y) to $(x+1, y+1)$ or from (x,y) to $(x+1, y-1)$.
  - Consider the number of ways to get to each point on the integer grid $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$.
    - Generates a rotated Pascal's triangle.
    - The number of paths from $(0,0)$ to $(a+b, a-b)$, i.e., a moves up and b moves down is $\binom{a+b}{b}$.
- Proposition: C_n is equal to the number of paths from $(0,0)$ to $(2n, 0)$ which are contained in the upper half-plane ($y \geq 0$).
  - Proof: C_n is the number of sequences of brackets.
  - Transform a sequence of brackets into a path by ($\mapsto\nearrow$ and) $\mapsto\searrow$.
  - The condition #(= #) implies that the paths start at $(0,0)$ and end at $(2n, 0)$.
  - The condition that in every initial segment, #(= #) implies that the path lies in the upper half plane.

- Reflection principle:
  - The number of paths from A to B in the upper half plane is equal to the number of paths from A to B minus the number of paths from A to B that intersect the line $y = -1$.
  - Symbolically, $C_n = \binom{2n}{n} - ?$
  - There exists a one-to-one correspondence between two sets: The set of all paths from A to B intersecting ℓ and the set of all paths from A to B', where B' is the reflection of B across ℓ.
- Thus, the number of paths from A to B that intersect the line $y = -1$ is equal to the number of paths from A to $(2n, -2)$.
- Therefore,

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!}\left(1 - \frac{n}{n+1}\right)\binom{2n}{n-1} \\ &= \frac{1}{n+1}\binom{2n}{n}\end{aligned}$$

  - Note that it's not obvious that $\frac{1}{n+1}\binom{2n}{n}$ is an integer unless you present it as the difference of two binomials (i.e., as $\binom{2n}{n} - \binom{2n}{n-1}$).
- Exercise:
  - Take a path from A to B intersecting ℓ and find the closest point of $P \cap \ell$ to B. Reflect the segment of the path after this point.

1.3 PSet 2

6/30: 1. Consider n pairs of points on the segment AB, which are symmetric with respect to its center. Half of these points are red, the other are blue. Prove that the sum of distances from the point A to the blue points equals the sum of distances from the point B to the red points.

   Proof. Let C be the midpoint of AB. Clearly, if every point is located at C, the sum of the distances from A to the blue points is $n \cdot \text{len}(AC)$, and similarly for B and the red points. Thus, to prove the claim, it will suffice to show that moving any pair of points located at C to two points that are symmetric with respect to C preserves the equality of the A-to-blue-points and B-to-red-points distances. We divide into three cases (both points are blue, both points are red, and one point is blue and the other red).

   If both points are blue, then moving them will not affect the B-to-red-points distance. As such, we must show that moving them symmetrically similarly does not affect the A-to-blue-points distance. But this is obviously true, as shortening the distance from A to p_1 by ℓ necessitates lengthening the distance from A to p_2 by ℓ, cancelling out any potential change.

   The proof is symmetric if both points are red.

   If one point is blue and the other red, then shortening the distance from A to p_b by ℓ necessitates shortening the distance from B to p_r by ℓ as well, preserving equality. Vice versa is true for lengthening the distance from A to p_b. □

2. Consider a set 2^S of subsets $S = \{1, 2, \dots, n\}$. For two subsets $A, B \in 2^S$, define their symmetric difference by $A \triangle B = (A \cup B) \setminus (A \cap B)$. Prove that 2^S with this operation is a group.

   Proof. To prove tht 2^S with $\triangle$ is a group, we must verify the associativity, identity, and inverse conditions.

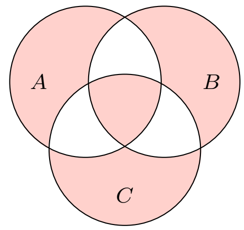

Figure 1.1: Power set group associativity.

To confirm that 2^S is associative, we must show that for all $A, B, C \in G$, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. Let A, B, C be arbitrary elements of 2^S. Then

$$\begin{aligned}
(A \triangle B) \triangle C &= ((A \cup B) \setminus (A \cap B)) \triangle C \\
&= (((A \cup B) \setminus (A \cap B)) \cup C) \setminus (((A \cup B) \setminus (A \cap B)) \cap C) \\
&= (A \cup ((B \cup C) \setminus (B \cap C))) \setminus (A \cap ((B \cup C) \setminus (B \cap C))) \\
&= A \triangle ((B \cup C) \setminus (B \cap C)) \\
&= A \triangle (B \triangle C)
\end{aligned}$$

Note that the two big expressions are equal since we can show that they are both represented by the following picture, where the shaded area may have elements and the unshaded area cannot[1].

To confirm that the identity element is the $\emptyset$, it will suffice to show that $A \triangle \emptyset = \emptyset \triangle A = A$ for all $A \in 2^S$. Let A be an arbitrary element of 2^S. Then

$$\begin{aligned}
A \triangle \emptyset &= (A \cup \emptyset) \setminus (A \cap \emptyset) \\
&= A \setminus \emptyset \\
&= A \\
&= A \setminus \emptyset \\
&= (\emptyset \cup A) \setminus (\emptyset \cap A) \\
&= \emptyset \triangle A
\end{aligned}$$

as desired.

To confirm $A = A^{-1}$, we must show that $A \triangle A = \emptyset$. But we have that

$$\begin{aligned}
A \triangle A &= (A \cup A) \setminus (A \cap A) \\
&= A \setminus A \\
&= \emptyset
\end{aligned}$$

as desired. □

3. Let G be a group.
   (a) Prove that for any $g_1, g_2 \in G$, we have $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$.
   (b) G is called **abelian** if for every two elements $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$. In other words, G is abelian if multiplication is commutative.
   (c) Prove that the group 2^S from the previous problem is abelian.
   (d) Prove that if $g^2 = e$ for any $g \in G$, then G is abelian.

[1]Note that there is an alternate proof of this: Write each set as a permutation of 1s and 0s. Define a bijection f with the property that $f(A \triangle B) = (f(A) + F(B)) \bmod 2$. Then commutativity follows easily.

Proof. To prove that G is abelian, part (b) tells us that it will suffice to show that for all $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$. Let g_1, g_2 be arbitrary elements of G. Then

$$\begin{aligned}
g_1 g_2 &= e g_1 g_2 e && \text{Identity} \\
&= (g_1^{-1} g_1)(g_1 g_2)(g_2 g_2^{-1}) && \text{Inverse} \\
&= g_1^{-1} g_1^2 g_2^2 g_2^{-1} && \text{Associativity} \\
&= g_1^{-1} e e g_2^{-1} && \text{Property} \\
&= g_1^{-1} g_2^{-1} && \text{Identity} \\
&= (g_2 g_1)^{-1} && \text{Part (a)} \\
&= (g_2 g_1)^{-1} e && \text{Identity} \\
&= (g_2 g_1)^{-1} (g_2 g_1)^{-1} (g_2 g_1) && \text{Inverse} \\
&= e g_2 g_1 && \text{Property} \\
&= g_2 g_1 && \text{Identity}
\end{aligned}$$

as desired. □

4. **The Inclusion-Exclusion Principle**

   Consider N objects and some list $P_1, P_2, \dots, P_n$ of their properties. Let N_i be the number of objects satisfying P_i, N_{ij}, the number of objects satisfying P_i and P_j, and so on. Prove that the number of objects satisfying none of these properties is equal to

$$N - \sum N_i + \sum_{i_1 < i_2} N_{i_1 i_2} - \sum_{i_1 < i_2 < i_3} N_{i_1 i_2 i_3} + \cdots + (-1)^n N_{123\dots n}$$

   Proof. Induct on the number of properties. □

5. Prove that if we remove two opposite corners of a chessboard, the board cannot be covered by dominoes (each domino covers two neighboring cells of the chessboard).

   Proof. Opposite corners necessarily have the same color. However, each domino covers two squares of differing color. Therefore, since removing two opposite corners will lead to an excess of two squares of one color, there will be no way to cover every square. □

6. Define a sequence of Fibonacci numbers by formula:

$$F_0 = F_1 = 1$$
$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0$$

   (a) Prove that F_n gives the number of ways to present n as a sum of 1 and 2. For instance, $4 = 1 + 1 + 1 + 1 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 = 2 = 2 + 2$, so $F_5 = 4$.

   Proof. Clearly, F_0 and F_1 trivially equal 1. Then to write F_{n+2}, we can start by fixing 2 and know that there are F_n ways to write $(n+2) - 2$ as a sum of 1 and 2. But 2 can also be written as $1 + 1$, so if we fix a 1, we know that there are F_{n+1} ways to write the sum like this. Thus, the total number of ways to write $n + 2$ are $F_n + F_{n+1}$. □

   (b) Prove the **Cassini identity**:

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^{n+1}$$

   Proof. We induct on n. For the base case $n = 1$, we have

$$\begin{aligned}
F_2 F_0 - F_1^2 &= 2 - 1 \\
&= 1 \\
&= (-1)^{1+1}
\end{aligned}$$

Now suppose inductively that $F_{n+1}F_{n-1}-F_n^2=(-1)^{n+1}$. It follows that

$$\begin{aligned}
F_{n+1}F_{n-1}-F_n^2 &= (-1)^{n+1}\\
F_{n+1}(F_{n+1}-F_n)-F_n^2 &= (-1)^{n+1}\\
F_{n+1}^2-F_nF_{n+1}-F_n^2 &= (-1)^{n+1}\\
-1(F_{n+1}^2-F_n(F_{n+1}+F_n)) &= (-1)^{n+2}\\
F_{(n+1)+1}F_{(n+1)-1}-F_{n+1}^2 &= (-1)^{(n+1)+1}
\end{aligned}$$

as desired[2]. □

(c) Prove that the sum of the elements on any diagonal of the pascal triangle is a Fibonacci number:

$$\sum_{k=0}^{n/2}\binom{n-k}{k}=F_{n+1}$$

(d) Consider a generating function $f(x)=F_0+F_1x+F_2x^2+\cdots$. Prove that $f(x)=\frac{1}{1-x-x^2}$.

(e) Prove the following **Binet's Formula** for Fibonacci numbers:

$$F_n=\frac{\gamma^n-(-\gamma^{-1})^n}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Proof. We induct on n. For the base cases $n=0,1$, we have that

$$\begin{aligned}
F_0 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0-\left(\frac{1-\sqrt{5}}{2}\right)^0\right) & F_1 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1-\left(\frac{1-\sqrt{5}}{2}\right)^1\right)\\
&= \frac{1}{\sqrt{5}}(1-1) & &= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right)\\
&= 0 & &= 1
\end{aligned}$$

Now suppose inductively that $F_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ and $F_{n-1}=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$. Then

$$\begin{aligned}
F_{n+1} &= F_n+F_{n-1}\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)+\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{6+2\sqrt{5}}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right)
\end{aligned}$$

[2] There is an alternate proof using the properties of determinants.

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+2\sqrt{5}+5}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-2\sqrt{5}+5}{4}\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{(1+\sqrt{5})^2}{2^2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{(1-\sqrt{5})^2}{2^2}\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)
\end{aligned}
$$

as desired. $\square$