Week 4

4.1 PSet 5

7/14:

1. For a permutation $\sigma \in \mathbb{S}_n$, we denote by $\text{Inv}(\sigma)$ the number of inversions in σ, namely the number of pairs $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$.
   (a) Find permutations in $\mathbb{S}_n$ with the smallest number of inversions and with the biggest number of inversions.

   Proof. $\text{Inv}(\sigma) = 0$ for $\sigma = (1\ 2\ 3\ \dots\ n)$. $\text{Inv}(\sigma) = (n-1)!$ for $\sigma = (n\ (n-1)\ (n-2)\ \dots\ 1)$. $\square$

   (b) Prove that
   $$\sum_{\pi \in \mathbb{S}_n} x^{\text{Inv}(\pi)} = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$$
   (c) Prove that numbers $\text{Inv}(\sigma)$ and $\text{Inv}(\sigma^{-1})$ have the same parity.
2. A bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that there exists $k > 0$ satisfying $|f(A)f(B)| = k|AB|$ is called a **similarity**.
   (a) Prove that similarities form a group. Prove that this group is a subgroup of $\text{Aff}(\mathbb{R}^2)$ and contains a subgroup $\text{Isom}(\mathbb{R}^2)$.
   (b) Prove that similarities send lines to lines, circles to circles, and preserve angles.
   (c) Prove that homothety H_O^λ is a similarity.
   (d) Prove that every similarity is a composition of a homothety and an isometry.
3. (a) Prove that the homothety H_O^λ is a similarity.
   (b) Prove that a composition of two homotheties with coefficients $\lambda_1, \lambda_2 \neq 1$ is a homothety with coefficient $\lambda_1\lambda_2$.
   (c) Prove that if a composition of three homotheties is the identity map, then their centers lie on the same line.
   (d) **Monge's theorem**
   Outer tangent lines to the circles S_1 and S_2, S_2 and S_3, S_3 and S_1 intersect in the points A, B, and C, respectively. Prove that points A, B, and C lie on the same line.
4. Let R_n denote a set of fixed-point-free permutations in $\mathbb{S}_n$ (i.e., $R_n = \{\sigma \in \mathbb{S}_n \mid \sigma(i) \neq i\ \forall\ 1 \leq i \leq n\}$). Prove that
   $$\lim_{n\to\infty} \frac{|R_n|}{n!} = \frac{1}{e}$$

4.2 Geometric Measure Theory (Jake Fielder)

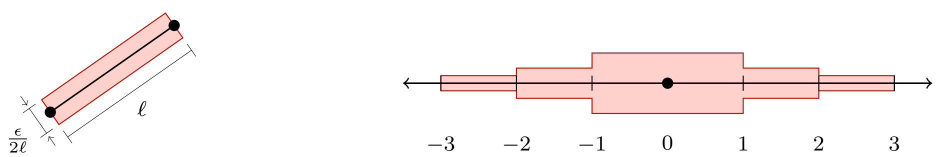
7/16: • Consider the disk of radius $\frac{1}{2}$.

(a) Circle of radius $\frac{1}{2}$. (b) Triangle of height 1.

Figure 4.1: Convex covers of $\{\ell \mid |\ell| = 1\}$.

- One of the disk's less interesting but still curious properties is that it contains (covers) the unit line segments pointing in every direction (the set of its diameters). See Figure 4.1a.
- The same is true with the equilateral triangle of height 1. See Figure 4.1b. However, the triangle does it in less area.
- In fact, Kakeya showed that among all *convex* polygons covering this set of line segments, the triangle with height 1 has the least area.

• How would we go about finding the general shape with the least area that still covers all of the line segments, though? First, we need some geometric measure theory definitions.



(a) Line segment. (b) Line.

Figure 4.2: Covering lines.

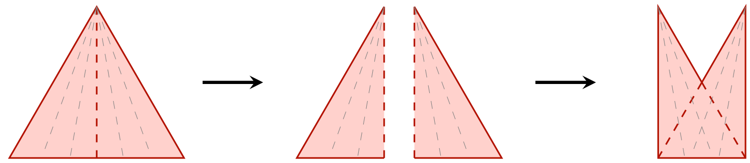
- A shape (set of points) has zero area if it can be covered by arbitrarily small shapes.
- For example, say we want to prove that the line segment of length ℓ has zero area (see Figure 4.2a). In analytical terms, we require that the $A(\ell) < \epsilon$ for all $\epsilon > 0$. Let $\epsilon > 0$ be arbitrary. Consider the rectangle of side lengths ℓ and $\frac{\epsilon}{2\ell}$. Clearly, this rectangle covers the line segment of length ℓ, i.e., $A(\ell) \leq A(R)$. Additionally, however, $A(R) = \ell \cdot \frac{\epsilon}{2\ell} = \frac{\epsilon}{2} < \epsilon$. Therefore, by transitivity, $A(\ell) < \epsilon$, as desired.
- As another example, say we want to prove that a line has zero area (see Figure 4.2b). Let $\epsilon > 0$ be arbitrary. Choose a point on the line. We shall call this point 0. Draw a rectangle of length 1 unit and height $\frac{\epsilon}{2^3}$ units to the right of 0 and to the left of 0. Then on $[1, 2]$ and $[-2, -1]$, draw a rectangle of height $\frac{\epsilon}{2^4}$. Continue on in this fashion forever to $+\infty$ and $-\infty$, always drawing on $[n, n+1]$ and $[-(n+1), -n]$ a rectangle of height $\frac{\epsilon}{2^{n+3}}$. The sequence of areas of rectangles to the right of 0 converges to $\frac{\epsilon}{4}$, as does the sequence of areas of rectangles of those to the left of 0. Thus, the total area of the cover is $\frac{\epsilon}{2}$, and the line falls fully within it, as desired.

• Now back to the original question. Our approach will be thus: If we can prove that $\frac{1}{3}$ of the lines are covered by 0 area, we prove it for all of the lines.

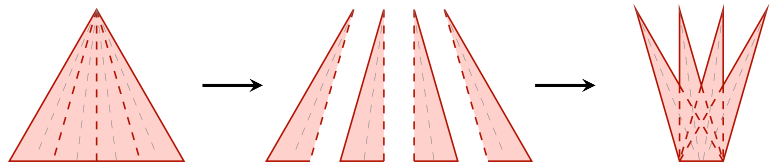
• Consider the $\frac{1}{3}$ of the lines originating from the top of the triangle in Figure 4.1b.

- As we split more and more as per Figure 4.3, we converge to a shape with zero area.

• Other misc. notes:



(a) Splitting once.



(b) Splitting twice.

Figure 4.3: Splitting the triangle.

- Hausdorff dimension: If something is of zero area, this lets us distinguish the size of a set.
- If $B \subseteq \mathbb{R}^n$ is a Besicovitch set, what is $\dim_H(B)$?
- Theorem (1973): If $n = 2$, then $\dim_H(B) = 2$.
- Kakeya conjecture: $\dim_H(B) = n$.