Part II

Serre

Chapter 1

Generalities on Linear Representations

1.1 Definitions

7/15:

- ***GL(V)***: The group of isomorphisms of V onto itself, where V is a vector space over the field $\mathbb{C}$ of complex numbers.
- If (e_i) is a finite basis of n elements for V, then each linear map $a : V \to V$ is defined by a square matrix (a_{ij}) of order n.
- The coefficients a_{ij} are complex number derived from expressing the images $a(e_j)$ in terms of the basis (e_i) and solving, i.e., we know that each $a(e_j) = \sum_i a_{ij}e_i$.
- a is an isomorphism $\iff \det(a) = \det(a_{ij}) \neq 0$.
- We can thus identify $GL(V)$ with the group of invertible square matrices of order n.
- **Linear representation** (of G in V): A homomorphism $\rho : G \to GL(V)$, where G is a finite group.
- **Representation space** (of G): The vector space V, given a homomorphism ρ. *Also known as* **representation**.
- **Degree** (of a representation V): The dimension of the representation space.
- **Similar** (representations): Two representations ρ, ρ' of the same group G in vector spaces V and V' such that there exists a linear isomorphism $\tau : V \to V'$ which satisfies the identity $\tau \circ \rho(s) = \rho'(s) \circ \tau$ for all $s \in G$. *Also known as* **isomorphic**.
  - When $\rho(s), \rho'(s)$ are given in matrix form by R_s, R'_s, respectively, this means that there exists an invertible matrix T such that $T \cdot R_s = R'_s \cdot T$ for all $s \in G$.

1.2 Basic Examples

- **Unit representation**: The representation ρ of G defined by $\rho(s) = 1$ for all $s \in G$. *Also known as* **trivial representation**.
- **Regular representation**: The representation ρ defined by $\rho(s) = f : V \to V$ where $f : e_t \mapsto e_{st}$, where V is a vector space of dimension $g = |G|$ with basis $(e_t)_{t \in G}$.
- **Permutation representation**: The representation ρ defined by $\rho(s) = f : V \to V$, where $f : e_x \to e_{sx}$, $x \in X$ being the set acted upon by G.