

Topic 3

Dilute Solutions

3.1 Intrinsic Viscosity - 1

10/7:

- Announcements.
 - Grade our HWs on Canvas.
 - The profs are grading the first quiz.
 - PSet 3 will be posted tomorrow morning.
 - After today's lecture, we'll have a lot of fodder to get going on it!
- Overview of Topic 3.
 - We've built up a lot of theory, but now we want to discuss how we measure the parameters we've introduced.
 - Example: Measuring polymer size and conformation.
 - We'll also touch on why such parameters are important for various material properties.
 - Examples of where polymer size and conformation are important: Non-entangled rubber elasticity, shear thickeners, elastic modulus of crosslinked networks, and electrical conductivity.
- Upcoming lectures by day.
 - Today: Viscometry. What is the viscosity, intrinsic viscosity, etc. of a polymer? How rheology tells us stuff about a polymer sample.
 - Specifically, we'll relate the viscosity of a dilute polymer-solvent solution to the polymer molecular weight.
 - Thursday: Standard through state-of-the-art ways to measure the viscosity of a polymer solution.
 - Next Tuesday: Fractionation (e.g., via GPC), and how it works (based on our theory). Also osmotic pressure.
 - Fractionation techniques can provide us the full molecular weight distribution of a polymer sample.
 - Osmotic pressure can provide information on molecular size and polymer-polymer interactions.
 - Next Thursday: A high-level discussion of light scattering. Different power sources (e.g., NMR), and how scattering tells us something about the polymer size.
- Outline of today's lecture.
 - Drag coefficient of a polymer.
 - Also referred to (e.g., by the textbook) as the "friction factor."

- This number is related to a polymer's conformation.
- Draining and free draining models.
- The diffusion coefficient, and how molecules randomly move in a solution due to thermal energy.
- Intrinsic viscosity.
- This is key, and the first three topics all build up to it.
- The intrinsic viscosity has a scaling with the diffusion coefficient?? Did we cover this today?
- Mark-Houwink-Sakurada model.
- Before we get to the viscosity of dilute solutions of polymers, let's introduce some basic concepts around just plain viscosity. This will basically serve as a "fluid dynamics 101."
- To help visualize the following definitions, consider this thought experiment.

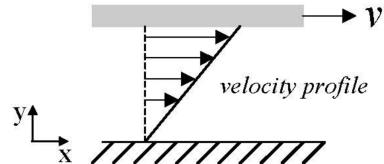


Figure 3.1: Viscosity visualization.

- Put a liquid between two plates, and move the top one.
- You can show (via the fluid equation or **Navier-Stokes equations**) that you will quickly develop a velocity profile. Molecular friction exerts a force, and the force is related to the **viscosity**.
- Consideration of the viscosity leads us to the other definitions below.
- **Navier-Stokes equations:** The governing equations of fluids.
- **Viscosity:** A relationship which tells us something about when we exert stresses on surfaces as a result of flow, and the relation to how fast we're moving a fluid. *Also known as coefficient of viscosity, shear viscosity. Units Pas. Denoted by η . Given by*
$$\eta := \frac{\text{shear stress}}{\text{rate of shear}}$$
- **Shear rate:** How fast we move the plates relative to each other, normalized by the distance between the plates. *Also known as velocity gradient, rate of shear. Units s⁻¹. Denoted by $\dot{\gamma}$.^[1] Given by*
$$\dot{\gamma} := \frac{U}{H}$$
- U is the difference in velocity between the top and bottom plates.
- H is the distance between the top and bottom plates.
- **Shear stress:** The pressure resulting from the viscous force applied over the area of the moving plate. *Units Pa. Denoted by τ , τ_{yx} . Given by*
$$\tau := \frac{\text{viscous force}}{\text{area}} = \eta \dot{\gamma}$$
- The τ_{yx} notation comes from the fact that stress is a tensor, and the yx element of this tensor is the shear stress.

¹"gamma dot"

- However, we are not concerned with tensors in this class, so we may think of shear stress as nothing more than a scalar with units of pressure.

- **Newtonian** (fluid): A fluid that satisfies the following condition. *Constraint*

$$\eta \neq \eta(\gamma_{\text{tot}}, \dot{\gamma})$$

- The above constraint states that for a Newtonian fluid, the viscosity η is a constant. In particular, viscosity is *not* a function of the shear strain γ or the shear rate $\dot{\gamma}$ (as is the case for some non-Newtonian fluids).

- **Non-Newtonian** (fluid): Any fluid that is not Newtonian.

- This encompasses a *broad* range of fluid properties.
- Polymers are very non-Newtonian.

- This concludes our introduction to the terminology of fluid mechanics.

- We now address today's focus, the behavior of dilute ($\phi_2 \ll 1$) polymer-solvent solutions at small shear rates ($\dot{\gamma} \ll 1$).

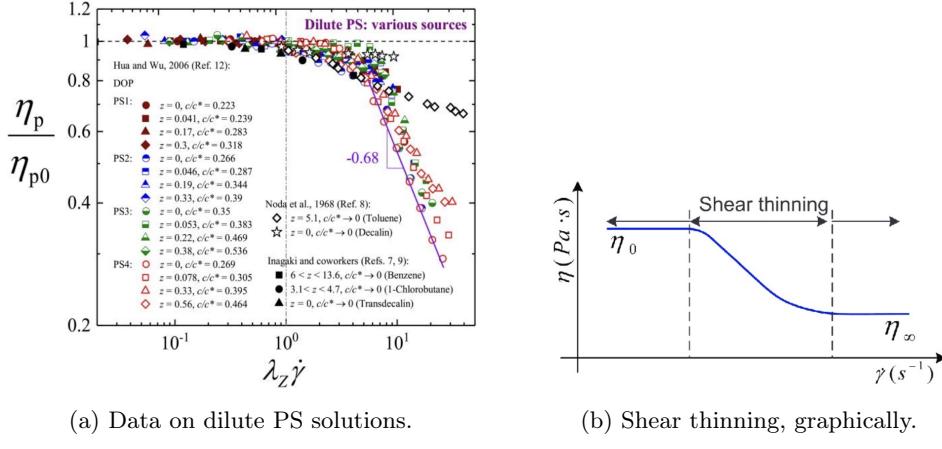


Figure 3.2: Polymer viscosity at varying shear rates.

- Figure 3.2a compiles a bunch of data from the literature on viscosity vs. shear rates.
 - The *y*-axis is the viscosity η_p of the polymer-solvent solution, normalized by said solution's asymptotic viscosity η_{p0} as you go to very small shear rates.
 - The *x*-axis is the shear rate, again normalized by some constant λ_Z . Here, we have picked a scaling λ_Z that makes our data dimensionless (i.e., 1 on the *x*-axis) and reveals the two important trends that...
 - Faster shearing ($\lambda_Z \dot{\gamma} > 1$) leads to a decrease in viscosity known as **shear thinning**;
 - Lower shearing ($\lambda_Z \dot{\gamma} < 1$) leads to the asymptotic viscosity.
 - Below a certain shear rate, we observe Newtonian behavior and “zero shear viscosity.”
- Figure 3.2b sketches the idea of shear thinning.
 - **Zero shear viscosity** occurs at left.
 - Note that this regime would more aptly be termed the “low shear viscosity” regime, since we're only asymptotically approaching zero shear.
 - However, “zero shear viscosity” is what's in the lexicon.
 - Shear thinning occurs in the middle.

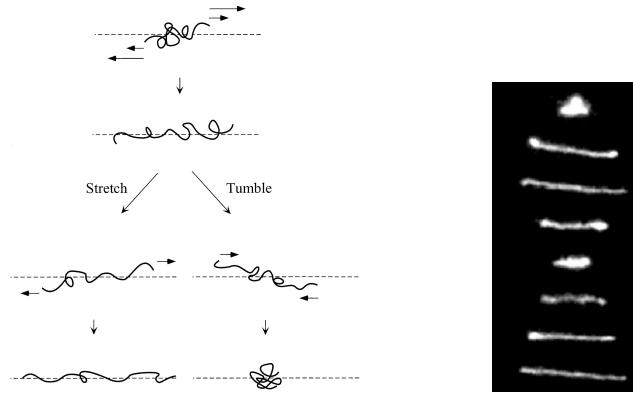
- At right, we enter the asymptotic regime wherein most viscosity comes from the solvent, not the polymer.
- Reference: Pan et al. (2018).
- **Shear thinning:** A property of certain fluids wherein viscosity decreases the faster the fluid is sheared (i.e., as the shear rate $\dot{\gamma}$ increases).
- **Zero shear viscosity:** The viscosity under a shear rate sufficiently low that the fluid behaves as if its Newtonian. Denoted by η_0 . Given by

$$\eta_0 := \lim_{\dot{\gamma} \rightarrow 0} \eta$$

- **Weissenberg number:** A dimensionless shear rate. Denoted by Wi . Given by

$$Wi := \dot{\gamma} \tau_{\text{polymer}}$$

- Before we dive deeper, we should state an important assumption about shear rate underpinning our analysis.



(a) Terminology definitions.

(b) DNA being sheared.

Figure 3.3: Stretching and tumbling upon shearing.

- In this class, we will assume that polymer coils are *not* deformed by the flow (i.e., $Wi < 1$).
- Typically, polymers **stretch** and **tumble** when $Wi > 0$ (Figure 3.3a).
- For example, DNA has been micrographed stretching and tumbling at higher shear rates and Weissenberg numbers (Figure 3.3b).
- Reference: Smith et al. (1999).
- **Stretch:** A behavior of a sheared polymer in which a usually flexible coil is elongated by shear forces.
- **Tumble:** A behavior of a sheared polymer in which a stretched coil is compressed with rotation by shear forces.
- We now describe how the Navier-Stokes equations treat viscosity.

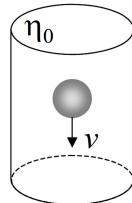


Figure 3.4: Drag coefficient on a hard sphere in a viscous fluid.

- Consider a hard sphere of radius r_s falling through a fluid of viscosity η_0 with velocity v .
- By Newton's laws, the fluid will exert a viscous force F_{viscous} on the sphere. Moreover, this force will be given by the following.

$$F_{\text{viscous}} = fv$$

- v is the velocity with which the sphere is moving.
- f is the drag coefficient or friction factor. It can be thought of as the proportionality factor between the velocity and viscous force.
- Solving the Navier-Stokes equations for this scenario, we obtain **Stokes' law**.
- How, then, can we treat a polymer molecule moving through a viscous fluid? There are two limiting cases.
 1. The polymer coil acts as an impenetrable sphere with dimensions given by the pervaded volume.
 2. The polymer coil is entirely penetrable, with only individual monomers interacting with the viscous fluid.

- **Stokes' law:** The Navier-Stokes equations' relation between the drag coefficient f on a sphere of radius r_s falling through a fluid, and the fluid's viscosity η_0 . *Given by*

$$f = 6\pi\eta_0 r_s$$

- The 6π falls out of the math when solving the Navier-Stokes equations.
- It makes intuitive sense that $f \propto \eta_0$ and $f \propto r_s$, so it is not hard to rationalize or visualize that such proportionality manifests itself as a bilinear relation.
- Let's now extend our investigation into the possible behaviors of a polymer molecule in shear flow.

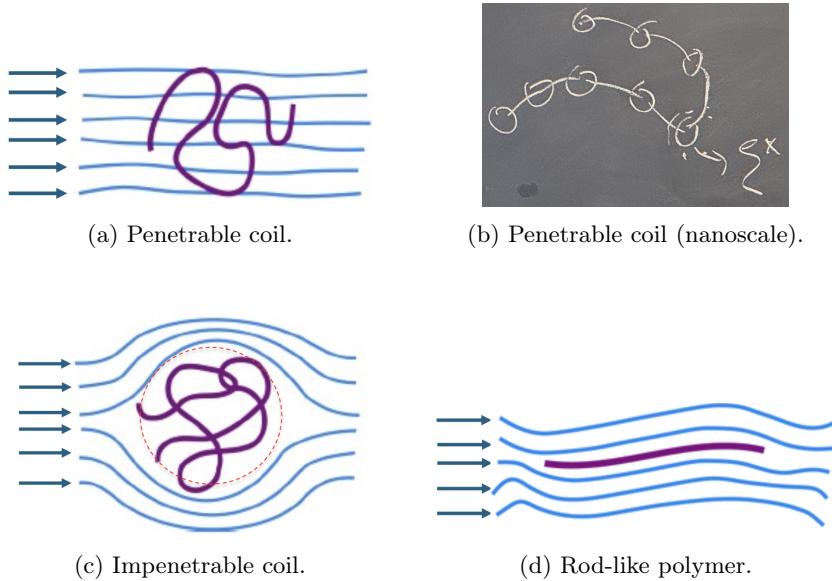


Figure 3.5: Modeling a polymer in a viscous fluid.

1. Penetrable coil, i.e., Rouse's free draining model.

$$f \propto N_2 \zeta^*$$

- Notation.

- We still let N_2 be the number of segments per chain (e.g., number of Kuhn steps).
 - $\xi^* = \zeta$ is the **monomeric friction factor**.
 - Globally, the fluid passes through the coil freely (Figure 3.5a).
 - However, very locally (Figure 3.5b), there is no slip. In other words, each monomer contributes a small amount ξ^* to the total friction factor. It follows that in this case, the friction factor scales linearly with the number of units N_2 in the polymer!
 - 2. Impenetrable sphere, i.e., Zimm's non-free draining model (Figure 3.5c).
- $$f = 6\pi\eta_0 R_H \quad R_H \propto \langle R_g^2 \rangle^{1/2} \quad f \propto N_2^\nu$$
- R_H denotes the sphere's **hydrodynamic radius**.
 - The rightmost relation follows from transitivity and the scaling of R_g .
 - ν is the same measure of solvent quality (e.g., 1/2, 3/5, or 1/3) discussed previously.
 - There is a strong **hydrodynamic interaction** (HI), or coupling, of the motion of monomers in dilute solution.
 - Essentially, when one monomer moves somewhere along a polymer chain, it yanks others in the chain along with it.
 - This interaction decreases the farther out along the chain you go from the monomer that is moved. Symbolically, $HI \propto 1/r$, where r is the relative separation of monomers.
 - Additional equation for $\mathbf{v}(\mathbf{r})$??
 - This model is appropriate for a (1) high molecular weight and (2) flexible polymer.

- 3. A rod-like molecule (Figure 3.5d).

$$f = 6\pi\eta_0 R_H \quad R_H \propto R_{\max} = L \quad f \propto N_2$$

- This is one more commonly encountered possibility.
- The above relations should be fairly self-explanatory.
- When choosing which of the above three models to use for a given situation, remember that the one that is most appropriate depends on the size and concentration of the polymers.
- How might we measure the friction factor f ?

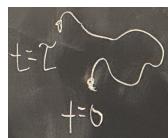


Figure 3.6: A diffusing molecule.

- Consider the Stokes-Einstein diffusivity / diffusion coefficient

$$D_t = \frac{k_B T}{\xi}$$

- $\xi = f$ is the drag coefficient / friction factor.
- We are taking this to be of a tracer molecule, i.e., some molecule in solution far from everything else.
- $k_B T$ is an energy (i.e., a force times a length), and the friction factor has been defined as the quotient of (the viscous) force divided by a velocity. Thus, the units cancel out to length squared per time for D_t .
- We have a flux J satisfying **Fick's law**.

- This is a continuum view of diffusion.
 - The average squared displacement of a molecule as it moves around over a period of time τ is
$$\langle \Delta x^2 + \Delta y^2 + \Delta z^2 \rangle_\tau = 6D_t\tau$$
 - This is a molecular view of diffusion, from time $t = 0$ to time $t = \tau$ (Figure 3.6).
 - D_t has units of length squared over time here, as well!
 - Thus, to actually get the diffusion coefficient, we could track molecules, e.g., by taking movies of them.
 - Take movies of everywhere it goes, break up the path into segments like a random walk (recall that Brownian motion is a continuous random walk), and then take the ensemble or time average over a period of time of length τ .
$$\langle \Delta r^2 \rangle = 6D_t\tau$$
 - It follows by combining several of the above equations that
$$R_H = \frac{k_B T}{6\pi\eta_s D_t}$$
 - Additionally, it is important that
$$R_H \propto R_g \propto R$$
- **Fick's law:** A relationship between flux J , diffusivity D_m , concentration c , and distance x . *Given by*

$$J = -D_m \frac{\partial c}{\partial x}$$
 - How long does a polymer have to be to obey Zimm scaling?
 - For most systems in which we are interested, we will use Zimm's non-draining sphere.
 - However, there is still significant debate in the literature about when this non-draining sphere assumption is ok to use.
 - Smith et al. (1996) posits that once $N > 10$, we can use this model.
 - Wang et al. (2023) — published just two years ago — posits that we need to wait until $N > 30$.
 - Many polymers we have are well into the hundreds or thousands, though. Above $N = 100$ is a generally solid cutoff for this behavior, so we can think of these as very good and very non-draining.
 - Scaling behavior of the friction factor.
 - ν values reviewed again.
 - For high MW flexible chains, the correct model is non-draining impenetrable units (Zimm-like).
 - As discussed above, this model obeys $\xi = f = 6\pi\eta_s R_H$, where $R_H \propto R_g$.
 - R_H vs. R_g vs. R .

Polymer structure	Solvent	R_g/R_h Zimm theory	R_g/R_h experiment
Randomly branched	Good	—	2.0
Linear monodisperse	Good	1.6	1.5
Randomly branched	θ	1.7	—
Linear $M_w/M_n = 2$	θ	1.7	—
Linear monodisperse	θ	1.5	1.3
3-Arm star	θ	1.4	1.2
4-Arm star	θ	1.3	1.05
12-Arm star	θ	1.17	0.93
18-Arm star	θ	1.14	0.82
270-Arm star	θ	1.08	0.77
Hard sphere	—	0.77	0.77

Table 3.1: Ratio of radius of gyration and hydrodynamic radius for different polymer architectures.

- The prefactor γ in

$$R_H = \gamma R_g$$

is on the order of 1.

- The above equation is really important! We will use it later today.
- Recall that R_H is related to a hydrodynamic interaction tensor??
- Note that $R_g/R_H = \gamma^{-1}$ converges to 1 (middle-right column in Table 3.1) as the number of arms on a polymer increases because, as you get more arms, you do just start to look like a sphere.
- Rubinstein and Colby (2003) has some good explanations on this; Table 3.1 is also its “Table 8.4.”
- For the rest of class, we focus on viscosity. We’ll get as far into this topic as we can today, and then we’ll carry on next time.
- Einstein model for viscosity.
 - Simulating viscosity in full is a computationally intensive process.
 - On pen and paper, we will focus on what we can do for idealized hard spheres in a Newtonian fluid. This is the setup Einstein considered.
 - Let the hard spheres be of volume fraction ϕ_{hs} in the solution.
 - Einstein did his calculation in the dilute limit, obtaining

$$\eta(\phi_{hs}) = \eta_s(1 + 2.5\phi_{hs} + \dots)$$

- Einstein’s detailed calculation got 2.5 as the linear correction.
- The power series continues, but in the dilute limit ($\phi_2 \ll 1$ and $c_2 \ll c_2^*$, where c_2^* is the overlap concentration), we only need the linear correction.
- This solution will have a viscosity η , composed of a linear contribution from the solvent η_s , and then an added contribution from the polymer in the form of a **specific viscosity** η_{sp} . This yields

$$\eta = \eta_s + \eta_s \eta_{sp}$$

- Thus, from the above by transitivity,

$$\frac{\eta}{\eta_s} - 1 = \eta_{sp} = \underbrace{2.5\phi_{hs}}_{\text{for hard spheres}} + \dots$$

- From the specific viscosity, we can define the **intrinsic viscosity** $[\eta]$.
- It follows that in general, our solution has a viscosity

$$\eta = \eta_s (1 + c \cdot [\eta] + K_H c^2 [\eta]^2 + \dots)$$

- This equation represents viscosity as a perturbation in concentration c .
- The first prefactor is the intrinsic viscosity.
- K_H denotes the **Huggins coefficient**. As in Flory-Huggins theory, the 2nd term reflects pairwise interactions of solute molecules.
- We now want to relate the intrinsic viscosity to our polymer models.
 - Let’s go back to hard spheres.
 - The volume fraction of hard spheres is the number of spheres times the sphere volume V_{hs} divided by the total volume.
 - To get it in terms of concentration so as to bridge this model over to our new concentration-based model, we obtain

$$\phi_{hs} = \frac{(\# \text{ spheres})(\text{vol sphere})}{\text{total volume}} = \frac{c N_A V_{hs}}{M}$$

- We measure concentration in g/L.
- We measure molar mass in g/mol.
- The units of Avogadro's number N_A is mol⁻¹.
- Thus, in the above expression, the units do work out correctly to number times volume divided by volume!
- Thus, we can now put everything together.

- The Einstein model's correction is

$$2.5\phi_{hs} = \frac{2.5cN_A V_{hs}}{M}$$

- We can relate the hard sphere volume back to polymer radius with

$$V_{hs} = \frac{4}{3}\pi R_{hs}^3$$

- The specific viscosity is then equal to $2.5\phi_{hs}$ by definition, so

$$\eta_{sp} = \frac{2.5 \cdot 4\pi R_H^3 c N_A}{3M}$$

where we have replaced the hard sphere radius with the polymer's hydrodynamic radius.

- Now $R_H = \gamma R_g$, as mentioned earlier today. Additionally, we can divide through by concentration to turn the specific viscosity into an intrinsic viscosity. Therefore,

$$[\eta] = \frac{10\pi N_A \gamma^3 \langle R_g^2 \rangle^{3/2}}{3M}$$

- The above result is our final theory — a very commonly used theory.
- It follows that for our polymer-solvent solution, we have the following scaling.

$$[\eta] \propto \frac{R_g^3}{M} \propto M^{3\nu-1} \propto N^{3\nu-1}$$

- We still have ν dependent on the solvent type exactly as discussed repeatedly above.
- Thus, for a θ solvent, scaling of the intrinsic viscosity is $3(1/2) - 1 = 1/2$. For a good solvent, $3(3/5) - 1 = 4/5$.
- Note that we get the scaling because $R_g \propto R \propto N^\nu$, and $N \propto M$.
- **Specific viscosity:** A dimensionless quantity describing how much more viscous a solution becomes as polymers are introduced. *Denoted by η_{sp} . Given by*

$$\eta_{sp} := \frac{\eta}{\eta_s} - 1$$

- From the definition, we can see that for a pure solution (only solvent), the specific viscosity is zero. As the viscosity of the solution increases with increasing solute, η_{sp} will obtain a value greater than zero (and likewise for decreases in viscosity).
- **Intrinsic viscosity:** A measure of a solute's contribution to the viscosity of a solution. *Units conc⁻¹, mL/g. Denoted by $[\eta]$. Given by*

$$[\eta] := \lim_{c \rightarrow 0} \left(\frac{\eta_{sp}}{c} \right) = \lim_{c \rightarrow 0} \left(\frac{\eta - \eta_s}{c\eta_s} \right)$$

- **Huggins coefficient:** The viscometric equivalent of a Virial coefficient. *Denoted by K_H .*
- Implicit assumptions used today.

- We are in a dilute solution.
- If we use this viscosity in a real flow, our Weissenberg number must be much less than one ($Wi \ll 1$, where $Wi := \tau_p \dot{\gamma}$).
- The Zimm/non-draining argument is essential.
- Do we need to know anything else about the radius of gyration?
 - We will not be doing any detailed calculations, so no.
 - Just know that all of the different radii are applicable in different circumstances, and all related by order 1 coefficients. Thus, we can convert between all of them as needed.
- **Mark-Houwink-Sakurada model:** A scaling of the intrinsic viscosity derived above. *Also known as Mark-Houwink model. Given by*

$$[\eta] = KM^a$$
 - K, a — the **Mark-Houwink parameters** — depend on the polymer and solvent quality, respectively.
 - Exponent scaling.
 - $a < 1/2$ implies a poor solvent.
 - $a = 1/2$ implies a θ solvent.
 - $4/5 > a > 1/2$ implies a good solvent.
 - There's lots of data for various polymer types.

3.2 Intrinsic Viscosity - 2

- 10/9:
- Review of last lecture (various terms introduced last time).
 - To begin, we'll wrap up some concepts from last time.
 - More types of viscosity.
 - **Relative, reduced, and inherent** viscosity.
 - These exist, but we don't need to know much about them.

- **Relative viscosity:** The total viscosity normalized by that of the solvent. *Denoted by η_r . Given by*

$$\eta_r := \frac{\eta}{\eta_s}$$

- **Reduced viscosity:** The increase in viscosity, normalized by the concentration of the added solute. *Denoted by η_{red} . Given by*

$$\eta_{\text{red}} := \frac{1}{c} \left(\frac{\eta}{\eta_s} - 1 \right)$$

- **Inherent viscosity:** The measure of viscosity defined as follows. *Denoted by η_{inh} . Given by*

$$\eta_{\text{inh}} := \frac{1}{c} \ln \left(\frac{\eta}{\eta_s} \right)$$

- On the scaling of the intrinsic viscosity with the polymer molecular weight.
 - Doyle provides the justification I thought up for the $3\nu - 1$ proportionality discussed last time!
 - The Mark-Houwink-Sakurada model's benefit is that it allows us to fit our data to a more general law that doesn't depend on "good" or "bad" solvents so explicitly, but rather allows you to interpolate between them.

- Doyle skips the slide on Mark-Houwink parameters, but know that these are tabulated.
 - Last class, we built a lot of theory around viscosity. But is that theory consistent with our theory from Topics 1-2?
 - Let's show that it is in one example: Intrinsic viscosity and c^* .
 - Recall Figure 1.10.
 - Recall the geometric argument that

$$c^* = \left(\frac{M}{N_A} \right) \frac{1}{V_p} = \frac{3M}{4\pi N_A R_q^3}$$

- M is the mass of 1 mole of polymers.
 - Thus, M/N_A is the mass of a single polymer and $M/N_A V_p$ is the concentration of polymer within the excluded volume.
 - But this is equal to the concentration of n polymers within n of their excluded volumes, or of a whole solution at the critical concentration!
 - V_p is related to the radius of gyration by the equation for the volume of a sphere.

– Alternatively, the critical concentration occurs when the effective volume fraction $\phi_{\text{eff}} = \phi_{hs} = 1$. Thus, substituting in the values at the critical moment,

$$\begin{aligned} [\eta] &= \frac{2.5\phi_{hs}}{c_2} \\ [\eta]_c &= \frac{2.5 \cdot 1}{c^*} \\ c^* &= \frac{2.5}{[\eta]} = \frac{3M}{4\pi N A R_q^3 \gamma^3} \end{aligned}$$

- The second line above is a *very* famous equation that shows up in many research articles.
 - The two definitions of c^* are very similar!
 - This provides a good “sanity check” for our theory.
 - We can rest easy about the γ^3 difference because γ is an order 1 term, so it doesn’t change the value of the expression much.
 - Determination of the intrinsic viscosity.

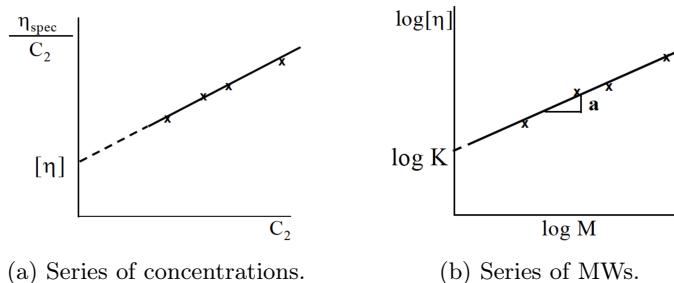


Figure 3.7: Determination of intrinsic viscosity.

- Let's start with a derivation.

- Recall that for $c < c^*$,

$$\eta \approx \eta_s (1 + c \cdot [\eta] + K_H c^2 [\eta]^2) \quad \eta_{sp} = \frac{\eta}{\eta_s} - 1$$

- Thus, we can divide the right equation through by concentration and the solvent viscosity, rearrange, and relate it to the intrinsic viscosity, as follows.

$$\frac{\eta}{\eta_s c} = \frac{1}{c} + [\eta] + K_H[\eta]^2 c$$

$$\underbrace{\frac{\eta/\eta_s - 1}{c}}_{\eta_{sp}/c} = [\eta] + \underbrace{K_H[\eta]^2 c}_{\text{constant}}$$

- The equation above tells us that given a series of concentrations, we can plot η_{sp}/c vs. c and get a linear plot, the y -intercept of which will be the intrinsic viscosity and the slope of which could give us the Huggins coefficient (Figure 3.7a).
- Notice that in the limit of $c \rightarrow 0$, the above equation reduces to the definition of intrinsic viscosity!
- Now that we have the intrinsic viscosity, we can do even more.
 - The Mark-Houwink model tells us that given a series of molecular weights and their intrinsic viscosities, we can make a log-log plot to measure K and a (Figure 3.7b).
 - The data on the next slide shows agreement with the theoretical expectation.
- Doyle is skipping the Intrinsic Viscosity - Key Idea & Viscosity-Average Molecular Weight derivations.
 - Key result: Each of our species i in solution obeys its own Mark-Houwink equation.
 - The **viscosity-averaged molecular weight** tends to lie between the number- and weight-averaged MW; it's what you use for M in the Mark-Houwink plot analysis.
- Example of the above method of measuring intrinsic velocity, from a recent article in Nature.
 - The authors were interested in the degradation of cellulose.
 - They measured the decrease in molecular weight by Mark-Houwink analysis, but in reverse! They knew K, a , measured $[\eta]$, and found M for each sample.
 - Reference: Maraghechi et al. (2023).
- We now move into today's lecture content in earnest: Measuring intrinsic viscosity, from classic methods to the state-of-the-art ones people are currently researching.
- Challenges to the measurement of intrinsic viscosity.
 - Low concentrations of polymer.
 - This implies that the viscosity of the polymer solution is very close to that of the solvent, so we need very accurate instruments.
 - Need to operate in the $Wi < 1$ regime.
 - This is because we need the polymer to not be deformed, so we can't shear *too* fast (or we'll get into stretching and tumbling regimes that throw off our calculations, as in Figure 3.3).
 - Since the shear stress $\tau = \eta \dot{\gamma}$,
 - $Wi = \tau \dot{\gamma} = \eta \dot{\gamma}^2$
 - and so we need $\eta \dot{\gamma}^2$ less than one.
 - We may not have much of our sample to commit to this experiment, especially in the case of precious materials.
 - This is especially challenging because relatively large volumes of solutions are typically needed for viscosity measurements.
 - Classic method: Capillary viscometer.

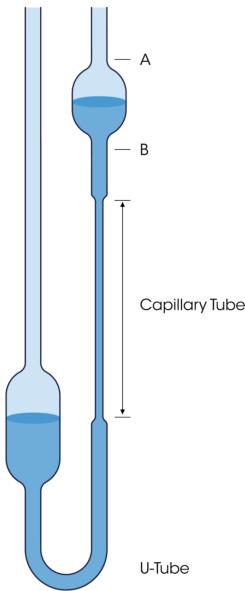


Figure 3.8: Capillary viscometer.

- From the 1800s.
- We suck some liquid up into a straw.
- Gravity will then want to equilibrate the chambers, and we measure the time that the equilibration takes.
- All you need is the set up, a means to draw your liquid up, and a stopwatch.
- Basic (without Navier-Stokes) physics for the why a capillary viscometer works.

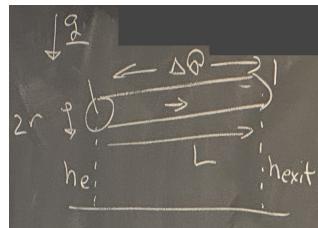


Figure 3.9: Hagen-Poiseuille derivation.

- Hagen and Poiseuille^[2] solved a nice problem for a Newtonian fluid (constant viscosity η) traveling through a straight tube of diameter $2r$ and length $L = h_{\text{entrance}} - h_{\text{exit}}$ at pressure ΔP .
 - The flow Q is assumed to be sufficiently slow so as not to be **turbulent**. In other words, the flow is **laminar**.
 - Specifically, they wanted the flow to have **Reynolds number** $\text{Re} \leq 2000$.
 - Note that Q is measured in units of volume per second (e.g., m^3/s).
 - The result of the derivation is that the flow is

$$Q = \frac{\pi r^4}{8\eta L} \Delta P$$

²“POY-say”

➤ $\Delta P = P + \rho gh$ is a dynamic pressure, which is equal to the regular pressure P plus ρgh . It is the pressure difference between the two ends of the pipe.

- It follows that — in the context of a capillary viscometer — the time to drain a volume V is going to be

$$\frac{V}{Q} \propto \underbrace{\frac{VL}{r^4 g \Delta h}}_{\text{constant}} \cdot \frac{\eta}{\rho}$$

- Thus, we have a relation between our observable (V/Q), the desired quantity (η), and parameters baked into our system.
- Note that the parameters baked into the system can be broken down into two classes. The first is the contents of the constant, which are all specific to a given capillary viscometer. The second is the density of the fluid, which we can experimentally measure independently of our capillary viscometer (e.g., with a micropipette and a balance) for any fluid whose viscosity we seek to determine.
- A convenient rewrite of the above equation (in the specific case of water, a common solvent) is

$$t_{\text{H}_2\text{O}} = \text{constant} \cdot \frac{\eta_{\text{H}_2\text{O}}}{\rho_{\text{H}_2\text{O}}}$$

- Water is very extensively studied, so we can look up the right terms above in a handbook.
- Doing the same for our polymer solution and combining the two equations, we get the following result. This result will give us our material properties.

$$\frac{t_{\text{H}_2\text{O}}}{t_{\text{polymer}}} = \frac{\eta_{\text{H}_2\text{O}}}{\rho_{\text{H}_2\text{O}}} \cdot \frac{\rho_{\text{polymer}}}{\eta_{\text{polymer}}}$$

- We measure $t_{\text{H}_2\text{O}}$ in our specific viscometer, either measure or look up $\eta_{\text{H}_2\text{O}}$ and $\rho_{\text{H}_2\text{O}}$, determine the density of our polymer-water solution, and then can obtain the viscosity of our polymer!
 - This equation eliminates the need to measure the constant for our specific viscometer.
 - Basically, it replaces measuring a constant with calibrating the viscometer against a known solvent.
- Note that every term subscripted “polymer” refers to a property of the polymer-solvent solution. This is because we want to measure quantities at the conditions under which we’re running the experiment.
- We can also use the approximation for $c \ll c^*$ that $\rho_{\text{polymer}} \approx \rho_{\text{solvent}}$.
- We need about 5-10 mL of solution to run this experiment.
- These instruments are \$200-\$400 from Sigma today!

- **Reynolds number:** A dimensionless quantity that helps predict fluid flow patterns (e.g., laminar vs. turbulent). Denoted by **Re**. Given by

$$\text{Re} := \frac{2rU\rho}{\eta}$$

- **Kinematic viscosity:** A measure of the viscosity of a solution that is directly proportional to the time the solution takes to flow under certain conditions. Given by

$$\frac{\eta}{\rho}$$

- We now begin discussing **rheology**.
- **Rheology:** The study of the flow of matter. *Etymology* from Greek “rhei” (to flow or stream) and “logos” (the study of).

- History of rheology.
 - Plato (400 BC): “All things move and nothing stands still.”
 - Simplicius (500 AD): “Panta rhei.”
 - Translation: Everything flows.
 - This is true! Some things just appear not to because we’re looking at them at too short of a timescale.
 - Prof. Bingham (1920s): Coins the term “rheology.”
 - Influence of the idea that everything flows: The **Deborah number**.
 - Named after Deborah in the Bible’s Book of Judges, who famously said, “the mountains flowed before the Lord.”
 - The idea is that on the scale of observation of God (which is infinite), even mountains will flow.
- **Deborah number:** A dimensionless number used to characterize the fluidity of materials over a given timescale. *Denoted by De. Given by*

$$\text{De} := \frac{\tau}{\tau_{\text{flow}}}$$
 - τ is the time it takes for a material to react to a stimulus.
 - τ_{flow} is the time over which the change is observed.
 - Thus, low Deborah numbers correspond to fast-flowing substances, and high Deborah numbers correspond to more immutable substances (e.g., the proverbial mountains).
- What is a rheometer?
 - A rheometer applies a controlled flow or stress via a specific geometry.
 - Many geometries to do this.
 - A spinaret is common. It’s a moving boundary, you put your sample inside, you get your force.
- Intrinsic viscosity is measured via a co-centric cylinder rheometer.

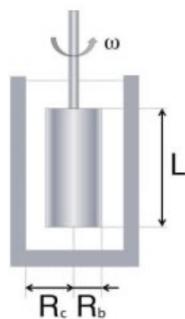


Figure 3.10: Co-centric cylinder rheometer.

- The gap is very small relative to the size of the central bob. Thus, the surface looks flat locally.
- A bit of geometry gets you your shear rate and shear stress, from which we know viscosity $\eta = \tau/\dot{\gamma}$.

$$\dot{\gamma} = \frac{2\omega R_c^2}{R_c^2 - R_b^2} \quad \tau = \frac{M}{2\pi R_b^2 L}$$

- Typical sample volume: 1-50 mL.
- People like these because you put your sample in, press a button, and get your result.

- Aside: Just how sensitive are rheometers?
 - Gareth McKinley (in MechE) will tell you that rheometers can typically measure 6 orders of magnitude dynamic range.
 - Modern rheometers are about an order of magnitude more sensitive than the capillary rheometer.
- Microfluidics approaches to rheometry.
 - Still pretty similar to Hagen-Poiseuille.
 - Advantages.
 - Help when you only have very little sample (as is often the case in pharma).
 - Subject fluids to variety of flows in a single device.
 - Etc.
 - Challenges.
 - No moving surface, so no direct analog to Figure 3.10.
 - Thus more similar to the capillary viscometer, fundamentally speaking.
 - Key challenge: We have to extrapolate to zero concentration to take the limit in the definition of intrinsic viscosity.
- Microrheometry overview.
 - Many types.
 - Rheometry on a chip (a device you can buy now).
 - Microrheology, also a thing we'll discuss.
- Measuring intrinsic viscosity on a chip.
 - More benefits and challenges.
- Theory for an intrinsic viscometer.

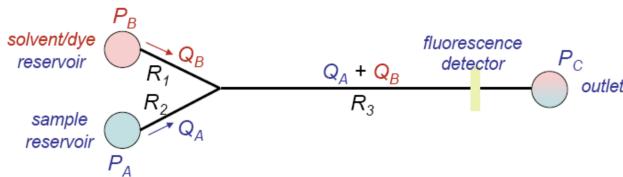


Figure 3.11: Intrinsic viscometer schematic.

- Black lines are very small; width on the order of our hair ($100 \mu\text{m}$).
- The rightmost circle is just an outlet; that's trash.
- One reservoir has our sample, of which we'd like to take a measurement.
- The other reservoir has just the solvent in which the sample is dissolved, and a dye!
- Flow rates are given by volumetric resistances. Then the flows combine with conservation of mass.
- You create a dilution series with optically detectable concentration.
- How an intrinsic viscometer works.
 - Recall that $\Delta P = QR_H$, where $R_H = 8\eta L/\pi r^4$ is the **hydrodynamic resistance**.
 - Some of R_H is geometric, and some is the viscosity.
 - The geometric terms we denote $\tilde{R}_H = 8L/\pi r^4$.
 - Hydrodynamic resistance is analogous to electrical resistance and Ohm's law $V = IR$.

- Four equations.

$$\Delta P = P_A - P^* = \tilde{R}_1 \eta_p Q_1 \quad (1)$$

$$P_B - P^* = \tilde{R}_2 \eta_s Q_2 \quad (2)$$

$$P^* - P^{**} = \tilde{R}_3 \eta (Q_1 + Q_2) \quad (3)$$

$$P^{**} - P^* = \tilde{R}_4 \eta (Q_1 + Q_2) \quad (4)$$

- P^* is the pressure right where the two flows meet.
- P^{**} is the pressure once the two flows have equilibrated.
 - We don't know what either of these are, hence why we use placeholder variables for them.
- η_p is the viscosity of the polymer solution.
- Suppose $P_A = P_B$.
 - Then the lefthand sides of equations 1 and 2 are the same, so

$$\tilde{R}_1 \eta_p Q_1 = \tilde{R}_2 \eta_s Q_2$$

$$\eta_p = \underbrace{\frac{\tilde{R}_2 \eta_s}{\tilde{R}_1}}_{\text{known}} \cdot \frac{Q_2}{Q_1}$$

- Pressure and flow are related in a dependent/independent relation; we can control one of them or the other, but not both. Thus, the one that we're not controlling will have to be measured.
- In this case, we're applying pressures, so we don't know the flows Q_1, Q_2 .
- The detector signal divided by the max signal gives

$$Y = \frac{Q_2}{Q_1 + Q_2}$$

- The remaining math is hard and not worth our time. Using all four allows us to get rid of the unknowns P^*, P^{**} and fully analyze the system.
- Intrinsic viscometer: Device performance.
 - Lee and Tripathi (2005) showed that it worked pretty well, even with a tiny sample volume of 5 μL .
 - Recall that a traditional viscometer requires closer to 5 mL.
 - The authors did both kinds of measurements in Figure 3.7 successfully.
 - Design calculations: The authors wanted this to be a device that everyone could use, so they defined an operating range.
 - The Deborah or Weissenberg number must be less than 1.
 - Assuming a good solvent, λ should scale as molecular weight to the 1.8 power.
 - We need very fast diffusion laterally across the channel so that the two fluids merge into one. This is the Pe calculation. The ratio cancelling out with fast diffusion time implies that **Peclet number** is greater than 1.
- Commercial realization: Formulaction.
 - Dye might interact with polymer; initial device was too complicated for commercialization.
 - Now we let the fluids *not* mix quickly and look at the index of refraction for them flowing next to each other.
 - Hydrodynamic resistance is related to the width.

- Microrheology.
 - Spherical bead probes act like constant stress rheometers.
 - Look at variations in Brownian motion!
 - We can ensemble average the trajectories (as in Figure 3.6), plot against τ , and get $2D$ as the parameter.
 - By Einstein, $D = k_B T / 6\pi a \eta$, so we can back out viscosity!
 - Can do this with a few microliters, can do this inside living objects, etc.
 - Developed by Doyle's lab!
 - Here, τ is time, not the shear stress.
 - $k_B T$ divided by a length is a force (the way the units work out). Thus, $k_B T$ divided by a volume (L^3) is a force over length squared, which is a pressure. This pressure is what we call the **thermal stress**.
- High throughput microrheology.
 - By one of Doyle's old UROPs who's now doing a PhD w/ Eric Furst.
 - Each drop is a microliter experiment.
- More on Figure 1.1.
 - Doyle recapitulates what he said when he originally showed the figure.
- Expectations for PSet 3 and the Quiz: We're not gonna have to solve a pipe flow problem or anything, but the diffusivity relation to mean squared displacement is well within our wheelhouse.
- Class next Tuesday confirmed!
- Exam.
 - Average 80, standard deviation of 10.

3.3 Office Hours (Doyle)

10/14:

- Zimm's non-free draining model: The following equation from the slides?

$$\mathbf{v}(\mathbf{r}) = \frac{1}{8\pi\eta} \left(\frac{\delta}{\mathbf{r}} + \frac{\mathbf{r}\mathbf{r}}{\mathbf{r}^3} \right) \cdot \mathbf{F}$$

- This is an equation from an advanced fluid dynamics textbook, and we do not need to know it.
- Basically, this equation gives the vector field \mathbf{v} induced by displacing one particle along the polymer by \mathbf{r} .
- The only relevance it has to us is that it shows that there is a generic scaling of $1/r$ in both terms to the hydrodynamic interaction $\text{HI} = \mathbf{v}$.
- PSet 3: General approach to the questions?
 - It is basically a plug and chug PSet.
 - I have the correct scalings for N in PSet 3, Q1a-b! All of the rest of the factors are less important.
 - PSet 3, Q2 is just a cheap way of teaching us something (sedimentation) that we don't have time to cover in class, so that we recognize it if we ever come across it. It is as simple as it seems.
 - PSet 3, Q3 is purely plug and chug. Moreover, for PSet 3, Q3d, I did have it right that all we need is a qualitative recognition of what the scaling between intrinsic velocity and the mass tells us about solvent quality; we do not have to do anything quantitative with the numbers here.

- What do we need to know about Fick's law?
 - For anyone who has taken Transport Phenomena, Fick's law is a touchstone. Doyle just wanted to show all of the students in class that the tracer diffusion coefficient we discussed in class is the “same” as the mutual diffusion coefficient in Fick's law.
 - Now, as the textbook would imply, there are subtle differences between D_t and D_m , but we don't need to worry about them at our level.
- What is the Weissenberg number?
 - The Weissenberg number (sometimes used interchangeably with the Deborah number) is part of something called **dimensionless groups**.
 - The core idea here is relative scaling. Dimensionless groups address questions like “how fast is fast” by setting a reference point and talking about how fast something is *relatively (dimensionlessly)* compared to something else.
 - With the Weissenberg number, we are concerned with the timescale of polymer relaxation (e.g., from being stretched) relative to the timescale of fluid flow.
 - Since the shear rate is measured in units of reciprocal seconds, an approximate relaxation time scale τ_{flow} for the flow is $\tau_{\text{flow}} = 1/\dot{\gamma}$. It follows that the Weissenberg number Wi , as a dimensionless group, should be equal to the ratio

$$\frac{\tau_{\text{polymer}}}{\tau_{\text{flow}}} = \frac{\tau_{\text{polymer}}}{1/\dot{\gamma}} = \dot{\gamma}\tau_{\text{polymer}}$$

as it is.

- From the leftmost definition above, we can see that for $Wi < 1$, the polymer relaxes faster than the flow. This means that stretching and tumbling can be neglected, which is the regime we'd like to analyze.
- This is why we set $Wi < 1$ in this class.
- Do you have any intuition for the shear rate $\dot{\gamma}$?
 - In the context of Figure 3.1, $\dot{\gamma}$ is a constant equal to the slope of the velocity profile.
 - This is because $\dot{\gamma} = dU/dH$, and Figure 3.1's velocity profile plots the speed U as a function of height H (or y).
 - Note that the calculus definition is important because in some scenarios, the velocity profile may not be constant. For instance, in the Hagen-Poiseuille capillary, the velocity profile is parabolic.

3.4 Osmometry and GPC

- Announcements.
 - Exam solutions posted soon.
 - Thursday's lecture will almost certainly be an asynchronous video.
- Lecture outline.
 - Gel permeation chromatography.
 - Osmometry.
 - Derivations will be abbreviated, but overviewed.
- “GPC” and “SEC” will be used interchangeably in this class; the difference has to do with the material packed inside the column.
- A broad overview of how GPC works.

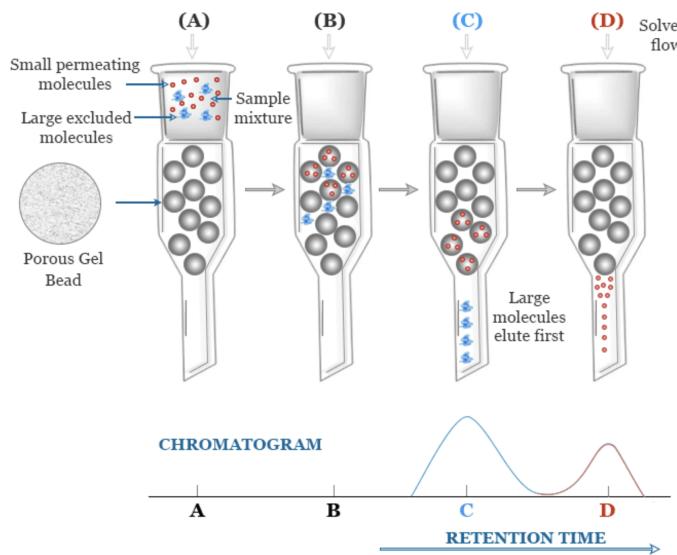


Figure 3.12: Gel permeation chromatography overview.

- Porous resin beads are packed within the column.
 - Pores are similar in size to the radius of gyration of the molecules you want to chromatograph.
- Put your polymer mixture in at the top, and elute it with pressure.
- Measure the chromatogram with some kind of detector (e.g., light scattering or refractive index).
- Bigger polymers move through the interstitial space first because its entropically unfavorable for them to fit in pores.
 - Little polymers sample both the interstitial and bead volume, and thus take more time to elute.
- GPC-selling companies (e.g., Waters or Agilent) sell various types of beads.
 - Sometimes, you go through multiple columns in series to separate different molecular weights better and get better resolution overall. Doyle has seen up to 6 columns wired in series.
- GPC theory.
 - There are two important volumes.
 - V_i is the total volume inside the porous beads' pores.
 - V_o is the volume outside the beads.
 - The elution time of a given species depends on the volume explored. This volume can be V_i , V_o , or a combination of the two. Symbolically,
 - $t_{\text{elute}} = \frac{V_{\text{explored}}}{Q_{\text{injection}}}$
- $Q_{\text{injection}}$ is the **volumetric flow rate**, i.e., the cubic meters per second your pump is flowing through the column.
 - We commonly think of elution time in terms of an alternative measure, the **retention volume**.
- **Retention volume:** The volume explored by a molecule as it passes through a GPC column. Denoted by V_r . Given by

$$V_r := t_{\text{elute}} \cdot Q_{\text{injection}}$$

- Let's look at a typical response curve.

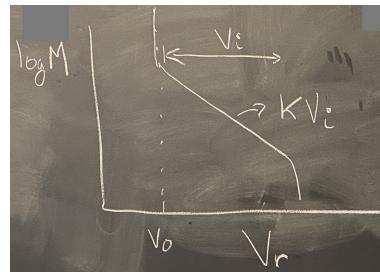


Figure 3.13: GPC response curve.

- We plot the log of the molar mass of whatever's coming out vs. V_r .
 - For big enough molecules (less volume is explored than V_o), everything will come out.
 - Then there's a region where we get separation.
 - The width of this interesting region is equal to the volume inside the pores, V_i .
 - A **partition coefficient** $K \in [0, 1]$ interpolates between how much volume a given polymer can sample. Specifically,
- $$V_r = V_o + V_i K$$
- In other words, K determines the concentration inside and outside the pores for a given polymer.
 - Let's consider the extrema of K .
 - When $K = 0$, M is large and the corresponding particles do not enter the pores at all.
 - When $K = 1$, M is small and the correspondign particles sample all the space inside the pores *and* outside in solution.
 - K will be a function of R_g divided by the diameter of our pores.
 - Then the molecules are so small ($V_r > V_o + V_i$) that they practically don't come out at all / all come out at once.

- Partition coefficient:** A real number between 0 and 1 describing the extent to which polymers enter the pores. *Also known as size exclusion equilibrium constant. Denoted by K . Given by*

$$K = \frac{c_{p,\text{inside bead}}}{c_{p,\text{outside bead}}}$$

- In words, K is equal to the concentration of the polymer inside vs. outside a bead.
- Since being in a pore is equivalent to a polymer being in a higher energy state, K is governed by the Boltzmann distribution

$$K = \exp\left(-\frac{\Delta\hat{G}_{pp}}{RT}\right)$$

- $\Delta\hat{G}_{pp}$ is the change in Gibbs free energy upon 1 mole of polymer entering pores.
- Note that this Boltzmann distribution — like all of them — describes a *dynamic* equilibrium under which polymers are constantly going in and out of various pores.
- Rearranging the above Boltzmann distribution and invoking the definition of the Gibbs free energy affords

$$\begin{aligned}\Delta\hat{G}_{pp} &= -RT \ln K \\ &= \Delta\hat{H}_{pp} - T\Delta\hat{S}_{pp}\end{aligned}$$

- We typically choose systems (i.e., column material and solvent) so that there is no enthalpic change upon entering or leaving a pore.
 - This implies that $\Delta\hat{H}_{pp} = 0$ in systems of interest to us.
 - Thus,
- $$K = \exp\left(\frac{\Delta\hat{S}_{pp}}{R}\right)$$
- Sanity check: If there's no change in entropy (because our polymer is so small it doesn't even notice it's in the pore), $K \rightarrow 1$.
 - Sanity check: If entropy is decreasing substantially (because the polymer is so large that it must significantly restrict its conformational freedom to enter a pore), $K \rightarrow 0$.
 - The polymer has some entropy loss when it enters a pore.

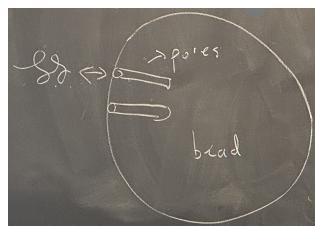


Figure 3.14: Theoretical porous resin microstructure.

- SEM and TEM images reveal that the porous beads used in GPC are highly irregular, but we will assume that these irregularities cancel to something more uniform so that our math is easier.
- Specifically, we will assume that our beads have a bunch of identical, cylindrical pores of diameter D .
- Nobel Prize (1991) to de Gennes for the “blob concept.”



Figure 3.15: Segmenting a polymer into blobs.

- Very interesting guy who used to smoke while giving lectures (despite No Smoking signs), and sit in the front of other lectures and smoke.
- Consider a polymer with $N = 10$ Kuhn steps.
- Now double the MW to $N = 20$ Kuhn steps, still of the same length. This corresponds to the addition of the 10 orange Kuhn steps in Figure 3.15.
- We could say that the orange part has its own mean squared end-to-end distance instead of looking at the whole polymer! In other words, we can split a single polymer into self-similar subsections called **blobs**.
- Stuffing our polymer into a pore.



Figure 3.16: Polymer blobs in a pore.

- If we stuff a small enough part of the total polymer into the pore, that blob doesn't know it's in a pore! This is because there's no conformational restriction to this blob on average.
- Essentially, the polymer chain doesn't know about the pore on length scales $\leq D$.
- Assume a blob has g Kuhn steps. Let the blob size be equal to D . Then $D = g^{1/2}l_k$. It follows that

$$g = \left(\frac{D}{l_k}\right)^2$$

- Two important consequences of this.
 - It follows that the number of blobs is N/g : The total number of Kuhn steps divided by the number of Kuhn steps per blob.
 - The energy penalty per blob is $k_B T$, so the entropy penalty per blob is $k_B T/T = k_B$. This is the amount of freedom we're taking away by fixing the orientation between each blob.
- Thus, the total entropic penalty is $(N/g) \cdot k_B$.
- It follows by using some of the above substitutions that

$$\Delta S_{pp} = -\frac{k_B N l_k^2}{D^2}$$

- But in the unconfined bulk, $\langle R^2 \rangle = N l_k^2$, so the change in entropy *per polymer* is

$$\Delta S_{pp} = -\frac{k_B \langle R^2 \rangle}{D^2}$$

- It follows that the change in entropy *per mole* of polymer is

$$\Delta \hat{S}_{pp} = -\frac{R \langle R^2 \rangle}{D^2}$$

- Be careful here! R denotes both the ideal gas constant and the end-to-end distance; which is which depends on context.

- Thus,

$$K = \exp\left(-\frac{\langle R^2 \rangle}{D^2}\right)$$

- Most of the stuff in the slides reflects what we did above; Doyle does not go through it in any sort of depth.
- Now how do we measure the molecular weight of the peaks? One common way is with standards.

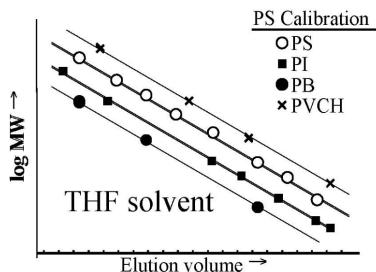


Figure 3.17: Calibrant response curves.

- Use essentially monodisperse GPC calibration standards that generate response curves between which you can interpolate.
- Example: Malvern's standards.
- An alternative method of measuring molecular weights: Universal calibration.

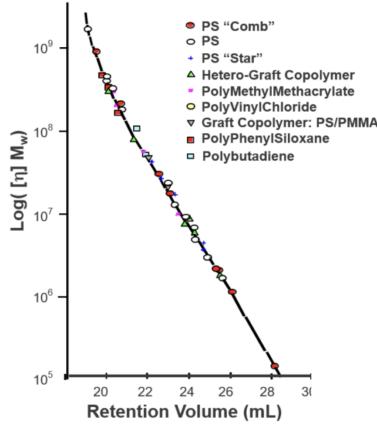


Figure 3.18: Universal calibration curve.

- Recall that

$$[\eta] \propto \frac{R_g^3}{M} \propto \frac{V_H}{M} \propto \frac{V_r}{M}$$
 - Hence, the retention volume V_r is proportional to $M[\eta]$.
 - It follows — assuming that the proportionality between $M[\eta]$ and V_r is independent of molecular structure — that if two species have the same retention volume, they have the same value of $M[\eta]$. This is how we are able to generate Figure 3.18, which maps every polymer to the same $\log(M[\eta])$ value based only on its V_r .
 - We will now use this universal calibration curve to derive an unknown polymer's molar mass in terms of a known polymer's molar mass.
- Recall from the Mark-Houwink model that $[\eta] = kM^a$.
- Let some standard s (with retention volume V_r , as measured by GPC) have Mark-Houwink parameters k_s, a_s . Then for the standard,

$$([\eta]M)_s = k_s M_s^{a_s} \cdot M_s = k_s M_s^{1+a_s}$$
 - Similarly, for an unknown u (with equal retention volume V_r , as measured by GPC), we can calculate

$$([\eta]M)_u = k_u M_u^{1+a_u}$$
 - But if the standard and unknown have the same retention volume, then

$$k_s M_s^{1+a_s} = ([\eta]M)_s = \text{const} \cdot V_r = ([\eta]M)_u = k_u M_u^{1+a_u}$$
and hence

$$M_u = \left[\frac{k_s M_s^{1+a_s}}{k_u} \right]^{\frac{1}{1+a_u}}$$
 - We can look up k_s, k_u, a_s, a_u , and then we can convert (from one measurement of M_s) to M_u without having to create a purpose-built calibration curve with multiple samples of u .
- We now switch gears to osmometry, specifically membrane osmometry.

- Membrane osmometry overview, as in a capillary membrane osmometer.

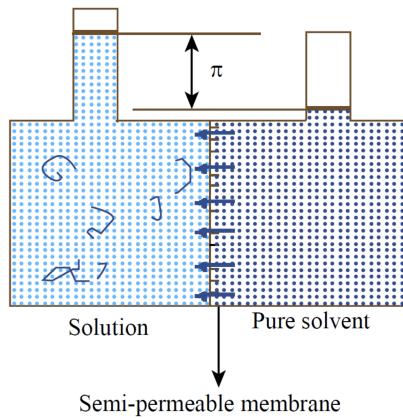


Figure 3.19: Membrane osmometer.

- Pure solvent on one side of a membrane, polymer solution on the other side. The membrane is semipermeable, letting solvent through but not polymer.
- Initially, you have solutions (open to atmosphere, and hence under constant pressure) of equal volume on both side of the membrane.
- Entropy will cause the solvent to want to mix with the polymer, with solvent flowing until the chemical potentials are equal.
- **Osmotic pressure:** A colligative property which depends only on the number of solute molecules in the solution. *Denoted by Π .*
 - Your equilibrium is established when the chemical potential μ_1° of the pure solvent is equal to that of the other one (μ_1) plus a mechanical force.

$$\mu_1^\circ = \mu_1 + \underbrace{\int_{P_0}^{P_0+\Pi} \left(\frac{\partial \mu_1}{\partial P} \right) dP}_{\text{work term}}$$

- We now invoke a Maxwell relation.

$$\left(\frac{\partial \mu}{\partial P} \right) = \frac{\partial}{\partial P} \left(\frac{\partial G}{\partial n_1} \right) = \frac{\partial}{\partial n_1} \left(\frac{\partial G}{\partial P} \right) = \frac{\partial V}{\partial n_1} = \bar{V}_1$$

- This is based on $dG = V dP - T dS + \sum_i \mu_i dn_i$.
- \bar{V}_1 is the partial molar volume of the solvent. It is a material property parameter.
- It follows by evaluating the above integral that

$$\begin{aligned} \mu_1^\circ &= \mu_1 + \Pi \bar{V}_1 \\ \mu_1 - \mu_1^\circ &= -\Pi \bar{V}_1 \end{aligned}$$

- The term on the right above is the **osmotic pressure**.
- The full derivation is on the slides, but Doyle will not go through it.
- We will now skip ahead to getting the Flory χ parameter.
 - Just above, we derived an expression for $\mu_1 - \mu_1^\circ$.

- But recall from Lecture 2.2 the additional expression

$$\mu_1 - \mu_1^\circ = RT \left[\ln \phi_1 + \left(1 - \frac{1}{N_2} \right) \phi_2 + \chi \phi_2^2 \right]$$

- Additionally, recall that with a dilute polymer solution ($\phi_2 < \phi_1$), we may obtain from the Taylor series expansion of $\ln(1 - \phi_2)$ that

$$\mu_1 - \mu_1^\circ = RT \left[-\frac{\phi_2}{N_2} + \left(\chi - \frac{1}{2} \right) \phi_2^2 \right]$$

- Setting equal the two expressions, we obtain

$$RT \left[-\frac{\phi_2}{N_2} + \left(\chi - \frac{1}{2} \right) \phi_2^2 \right] = -\Pi \bar{V}_1$$

- Comments on the osmometry derivation.

- Some rearranging, and then putting it in terms of concentration.
- Then the slope in a Π/c vs. c plot gives us χ , via the following equation.

$$\frac{\Pi}{c} = RT \left[\frac{1}{M_n} + \left(\frac{1}{2} - \chi \right) \frac{V_1 N_2^2}{M_n^2} \cdot c \right]$$

- This is the most important result, and main takeaway from osmometry theory.
- We also get something analogous to the ideal gas law ($PV = nRT$ becomes $\Pi V = n_2 RT$).
- The virial expansion has an analogous osmotic pressure form. A_1 is related to the molecular weight; A_2 is related to χ , the density, and some other things.
- Takeaway: The full osmometry derivation is not testable material, but he would expect us to be able to bring in Flory-Huggins and work with limits of small/big ϕ to do the first couple of steps.
- Last note on our osmometry theory: This theory works best when either (1) $\phi_2 \ll \phi_1$ or (2) Flory-Huggins theory takes over in the melt case, i.e., $c > c^*$.
- So we want to be concentrated, but not too concentrated. “Thermodynamically concentrated but mathematically dilute.”

3.5 Light Scattering

10/16:

- Lecture outline.
 - The theory of light scattering.
 - Context and geometry of the experimental setup.
 - Derivation of the scattering equation.
 - Experimental measurement of light scattering.
 - Zimm plots.
- Readings on light scattering.
 - Chapter 8 in Lodge and Hiemenz (2020) is excellent.
 - Young and Lovell (1991, pp. 178–90) is good.

- Geometry of a light scattering apparatus.

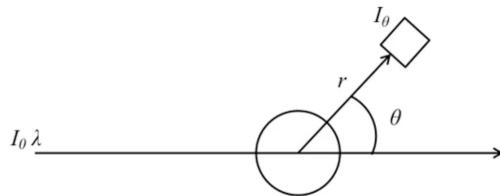


Figure 3.20: Light scattering geometry.

- The circle denotes our system of study.
- Impinging upon this system, we have light of wavelength λ and intensity I_0 .
- We want to understand how much light gets scattered out by the system, which we will measure using the square detector positioned at an angle θ from the incident light and distance r from the system. The detector measures the intensity I_θ of the light scattered in its direction.
- Based on this data, we build a **wavevector**.
- **Wavevector:** A dimensionless scattering vector. *Denoted by q . Given by*

$$q := \frac{4\pi}{\lambda/n_o} \sin\left(\frac{\theta}{2}\right)$$
 - n_o is the index of refraction of the medium. n_o , specifically, denotes that of a vacuum.
- Types of light scattering.
 - We are most interested in **small angle light scattering**.
 - Depending on the wavelength, there is also **small angle X-ray scattering** and **small angle neutron scattering**.
- **Small angle light scattering:** An experiment measuring the scattering of longer wavelength light at small angles θ , which arises from polarizability fluctuations $(\Delta\alpha)^2$. *Also known as SALS.*
- **Small angle X-ray scattering:** An experiment measuring the scattering of X-rays at small angles θ , which arises from electron density variations $(\Delta\rho)^2$. *Also known as SAXS.*
- **Small angle neutron scattering:** An experiment measuring the scattering of neutrons at small angles θ , which arises from neutron scattering length variations $(\Delta b)^2$. *Also known as SANS.*
- These three methods are governed by the **scattering equation**, which we will derive in the theoretical half of class today.
- **Scattering equation:** The relationship between (a) the measured light scattering caused specifically by the polymers in a dilute solution and (b) the fundamental constants describing the experimental setup and the characteristics of said polymers. *Given by*

$$\frac{Kc}{\Delta R} = \frac{1}{P(\theta)} \left[\frac{1}{M_w} + 2A_2c + \dots \right]$$

- Variables to be aware of.
 - $\Delta R(\theta, r) = R_{\text{solution}} - R_{\text{solvent}}$ is the excess scattering intensity measured at angle θ , i.e., the additional scattering caused by the polymers in solution beyond that caused by the solvent.
 - K is a bundle of optical constants that are invariant for a given experimental setup.
 - We will derive the exact form of K shortly.
 - c is the polymer concentration in g/L.

- $P(\theta)$ is the particle scattering factor.
 - It has something to do with molecular shape.
 - M_w is the molecular weight.
 - Throughout much of the following derivation, we will use “ M ” since we will theoretically treat a monodisperse collection of polymers, as we have been in every other derivation so far.
 - A_2 is the 2nd virial coefficient.

$$A_2 = \frac{\frac{1}{2} - \chi}{\rho_2^2 \bar{V}_1}$$
 - χ is the Flory χ parameter.
 - $\rho_2 = M/\bar{V}_2$ is the density of component 2 (the polymer species) in grams per liter.
 - \bar{V}_1 is the (partial) molar volume of component 1 (the solvent).
 - Overall, A_2 gives information on the quality of the solvent.
- Note that this equation relates to many of the properties we've been looking at characterizing!

- How does SALS work?

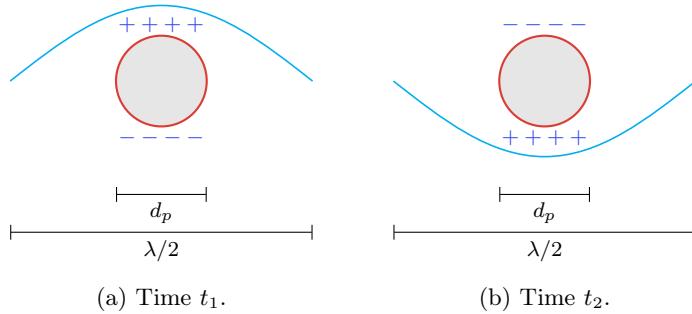


Figure 3.21: Polarizability fluctuations in SALS.

- When light impinges upon a system, it subjects the system to a time-dependent electric field $E(t)$.
 - $E(t)$ then causes polarizability fluctuations.
 - In particular, if we have a very long wavelength relative to the diameter of the particles (i.e., $\lambda \gg d_p$), then when the electrical field points one direction (e.g., at time t_1), we will get one polarity of the substance. When we move to a time $t_2 = t_1 + \tau/2$ half a period τ later, the polarity flips.
- We now begin deriving the scattering equation.
 - Let's start by analyzing the SALS caused by a dilute gas of small particles in a vacuum. The results of this derivation are also applicable to small molecules to low-weight oligomers in solution!
 - Since we're considering SALS, we are in the $\lambda \gg d_p$ regime (as in Figure 3.21).
 - Each particle in our dilute gas acts like a **spherical antenna**, propagating waves out at the frequency at which its dipole is flipping.
 - Each particle's dipole changes as a function of time t via

$$\text{dipole}(t) = \alpha E(t)$$

- This means that the dipole depends on the polarizability α of the particle, and the electric field $E(t)$ induced by the incident light.

- This type of scattering is called **Rayleigh scattering**, and the scattering produced by a single particle is described by the following equation.

$$I_\theta = \frac{I_0 8\pi^4 (1 + \cos^2 \theta)}{\lambda^4 r^2} \alpha^2$$

- All variables here are the same as in Figure 3.20.
- This equation reveals that $I_\theta \propto \lambda^{-4}$, so small wavelengths scatter much more than large wavelengths.
- It is also important that $I_\theta \propto I_0$ and $I_\theta \propto \alpha^2$.
- When we have more than one particle — say N particles in a total volume V — the total scattering intensity I'_θ goes as

$$I'_\theta = \frac{N}{V} I_\theta$$

- Note that this equation relies on the assumption of a *dilute* solution. Otherwise, we would need additional terms in concentration N/V to account for **coherent scattering** and the resultant correlations among the centers of the molecules.
- The Rayleigh scattering equation expresses I_θ in terms a series of observables (I_0, θ, λ, r) and one molecular property (α). As such, we will now seek to express α in terms of observables.
- **Clausius-Mossotti equation:** The general equation relating the index of refraction to the polarizability of the molecules. *Also known as Lorentz-Lorenz equation. Given by*

$$\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} = \frac{4}{3}\pi \left(\frac{N}{V}\right) \alpha$$

- $\varepsilon/\varepsilon_0$ is the dielectric constant in a medium of interest.
- ε_0 is the permittivity of a vacuum (from high school physics!).
- In the dilute limit (i.e., $N/V \rightarrow 0$), we have that $\varepsilon \rightarrow \varepsilon_0$ and hence we may approximate the Clausius-Mossotti equation as follows.

$$\begin{aligned} \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} &= \frac{4}{3}\pi \left(\frac{N}{V}\right) \alpha \\ \frac{\varepsilon/\varepsilon_0 - 1}{3} &\approx \frac{4}{3}\pi \left(\frac{N}{V}\right) \alpha \\ \frac{\varepsilon}{\varepsilon_0} &\approx 1 + 4\pi \left(\frac{N}{V}\right) \alpha \end{aligned}$$

- The idea here is that as $\varepsilon \rightarrow \varepsilon_0$, $\varepsilon/\varepsilon_0 - 1 \rightarrow 0$. Thus, in the dilute limit, small changes in the numerator affect the equality far more than small changes in the denominator (which approaches the fixed value of 3).
- Let's now relate the approximated Clausius-Mossotti equation to the index of refraction n of our dilute gas of small particles in a vacuum.
 - First off, this process will involve a series of approximations, so note that we will use “=” to denote relations that may not strictly be equality.
 - Since $\varepsilon/\varepsilon_0 = n^2$, we have that

$$n = \sqrt{1 + \frac{4\pi\alpha N}{V}}$$

- Additional complication: The dielectric constant $\varepsilon/\varepsilon_0$ depends on the frequency $\nu = c/\lambda$ at which the light-induced electric field is alternating.

- In the dilute limit, we can assume a linear relationship with concentration $c \propto N/V$, and thus

$$n = 1 + c \cdot \frac{dn}{dc}$$

- Thus,

$$n^2 = \left(1 + c \cdot \frac{dn}{dc}\right)^2 = 1 + 2\left(\frac{dn}{dc}\right)c + \dots$$

- It follows by combining this expansion with the original approximation that

$$\begin{aligned} 1 + \frac{4\pi\alpha N}{V} &= 1 + 2c \cdot \frac{dn}{dc} \\ \alpha &= \frac{1}{2\pi} \frac{c \cdot dn/dc}{N/V} \end{aligned}$$

- This equation expresses α solely in terms of observables!

- While we were not interested in the polarizability, we *are* interested in the molecular weight M of the polymers. Thus, let's see if we can relate the light scattering observables to M .

- The scattering equation (dilute gas of small particles in a vacuum) at this point is as follows.

$$\frac{I'_\theta}{I_0} = \left(\frac{N}{V}\right) \frac{8\pi^4(1 + \cos^2\theta)}{\lambda^4 r^2} \left(\frac{c \cdot dn/dc}{2\pi N/V}\right)^2$$

- We introduce M by recalling that

$$\frac{N}{V} = \frac{c}{M} \cdot N_A$$

- Expanding and substituting this in, we obtain

$$\frac{I'_\theta}{I_0} = \frac{2\pi^2(1 + \cos^2\theta)c^2(dn/dc)^2}{\lambda^4 r^2 N/V} = \frac{2\pi^2(1 + \cos^2\theta)(dn/dc)^2 Mc}{\lambda^4 r^2 N_A}$$

- The above equation contains several parameters ($I'_\theta, I_0, \theta, r$) which depend purely on the scattering geometry. We can combine these into the **Rayleigh ratio**, so as to obtain a relation between (a) a measure of the overall scattering obtained by our system and (b) intrinsic parameters describing our system.
- **Rayleigh ratio:** The parameters in the scattering equation which depend on scattering geometry. Denoted by R . Given by

$$R := \frac{I'_\theta r^2}{I_0(1 + \cos^2\theta)}$$

- Plugging the Rayleigh ratio into the scattering equation gives

$$R = \underbrace{\frac{2\pi^2(dn/dc)^2}{\lambda^4 N_A}}_K \cdot Mc$$

- Every term in the optical constant K is constant, provided we fix the wavelength λ .
- This equation — $R = KM_c$ — is the foundational descriptor of scattering in a solution of small molecules to oligomers.
- For reasons we will explain momentarily, in a polymer-solvent solution we have an additional n_0^2 factor, as follows.

$$K = \frac{2\pi^2 n_0^2 (dn/dc)^2}{\lambda^4 N_A}$$

- We now begin analyzing a polymer-solvent solution.
 - To do so, we will go back to the Rayleigh scattering equation (which still holds) and find α in terms of our polymer-solution parameters instead of dilute gas parameters.
- Debye figured out how to treat the polymer-solution case. In particular, he observed four key features of such solutions.

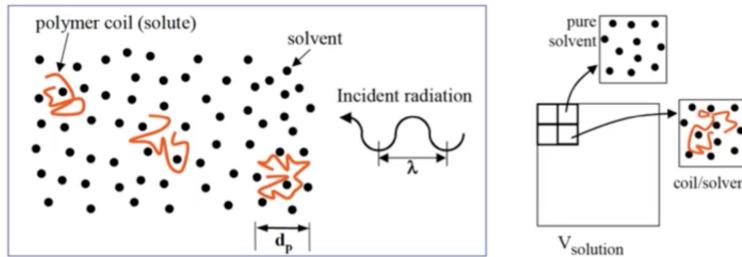


Figure 3.22: Fluctuations in concentration induce scattering.

1. Because we have a solvent, we need to discard the Rayleigh ratio of the pure solvent. Thus, we define the **excess Rayleigh ratio**.
 - How does a solvent induce scattering?
 - If the solvent molecules were distributed equally everywhere, scattered light from all the tightly packed particles would interfere and cancel out (think destructive interference of waves).
 - However, solvent molecules are *not* distributed equally.
 - To quantify these changes in distribution, let's divide the volume of our solution into bins of volume δV as in Figure 3.22. Some of these bins will have pure solvent, and some will have bits of polymer.
 - On average, each bin will have the same number of solvent molecules. However, fluctuations in solvent density give some regions higher or lower numbers. These fluctuations lead to weak scattering.
 - Beyond random fluctuations in solvent concentration, polymer solutes induce *additional* inequalities in solvent distribution!
 - This is an osmotic pressure effect. Indeed, even when the membrane is an imaginary “bin,” solvent molecules will prefer to move into bins with more polymer.
2. Remaining scattering arises from fluctuations in solute concentration. Namely, ΔR depends on $(dn/dc)^2$ and $\langle (\delta c)^2 \rangle$, where the latter quantity is given by the **Einstein-Smoluchowski equation**.
 - Just like osmotic pressure creates inequalities in solvent concentration, it creates inequalities in solute (polymer) concentration. Such fluctuations in polymer concentration give rise to significant scattering.
 - The Einstein-Smoluchowski equation, then, is what we use to quantify the fluctuations δc in solute concentration as a function of osmotic pressure (Π) effects.
3. To relate the quantities in the Einstein-Smoluchowski equation to polymer parameters we care about (e.g., M_w and χ), we can use the main osmometry result from Lecture 3.4 (restated at left below). This equation in turn allows us to calculate the derivative at right below.

$$\frac{\Pi}{RTc} = \frac{1}{M} + \frac{\frac{1}{2} - \chi}{\bar{V}_1 \rho_2^2} c + \dots \quad \frac{\partial \Pi}{\partial c} = RT \left[\frac{1}{M} + 2A_2 c + \dots \right]$$

4. Polymer chains are large and cannot be treated as point scatterers.

- In other words, we are now in the $\lambda \approx d_p$ regime, and we can no longer consider the light-induced electric field to be homogeneous over the entire volume of the polymer.
 - Thus, we need a scalar correction $P(\theta) \neq 1$ to $R = KM_c$.
 - We will explore the form of $P(\theta)$ further below.
- **Excess Rayleigh ratio:** A measure of the additional scattering in a binary solution caused strictly by the polymers, in excess of the scattering caused by the solvent. *Denoted by ΔR . Given by*

$$\Delta R := R_{\text{solution}} - R_{\text{pure solvent}}$$

- **Einstein-Smoluchowski equation.** *Given by*

$$\langle (\delta c)^2 \rangle = \frac{RTc}{\delta V N_A \cdot \partial \Pi / \partial c}$$

- Note that relations of the above type between concentration fluctuations and osmotic pressure are far more general than just the polymer-solvent solution case; the above form is just one special case of it.
- We are now ready to put everything together.

- The equation for Rayleigh scattering still largely holds in this case. We restate it for reference.

$$I_\theta = \frac{I_0 8\pi^4 (1 + \cos^2 \theta)}{\lambda^4 r^2} \alpha^2$$

- We do not have N particles in a volume V this time, but rather N “bins” in a volume V , each of volume δV . Thus, the total scattering intensity is

$$\begin{aligned} I'_\theta &= \frac{N}{V} \cdot I_\theta \\ &= \frac{1}{\delta V} \cdot \frac{I_0 8\pi^4 (1 + \cos^2 \theta)}{\lambda^4 r^2} \alpha^2 \\ \underbrace{\frac{I'_\theta r^2}{I_0 (1 + \cos^2 \theta)}}_R &= \frac{8\pi^4}{\delta V \lambda^4} \alpha^2 \end{aligned}$$

- As discussed, the excess Rayleigh ratio (what we observe) is related to fluctuations in the polarizability. This means that we transform the above equation into

$$\Delta R = \frac{8\pi^4}{\delta V \lambda^4} \langle (\delta \alpha)^2 \rangle$$

- Analogously to our equation for α in a dilute gas (left equation below for reference), we postulate the new expression on the right below for the polymer-solvent case.

$$\alpha = \frac{1}{2\pi} \frac{dn/dc}{N/V} c \quad \langle (\delta \alpha)^2 \rangle = \left(\frac{1}{2\pi} \frac{n_0 \cdot dn/dc}{1/\delta V} \right)^2 \langle (\delta c)^2 \rangle$$

- Thus,

$$\Delta R = \frac{8\pi^4}{\delta V \lambda^4} \cdot \frac{n_0^2 (dn/dc)^2 \delta V^2}{4\pi^2} \langle (\delta c)^2 \rangle = \frac{2\pi^2 n_0^2 (dn/dc)^2}{\lambda^4} \cdot \delta V \langle (\delta c)^2 \rangle$$

- We can now use the Einstein-Smoluchowski equation, and the osmotic pressure derivative.

$$\begin{aligned}
 \Delta R &= \frac{2\pi^2 n_0^2 (\mathrm{d}n/\mathrm{d}c)^2}{\lambda^4} \cdot \frac{RTc}{N_A} \left(\frac{\partial \Pi}{\partial c} \right)^{-1} \\
 &= \frac{2\pi^2 n_0^2 (\mathrm{d}n/\mathrm{d}c)^2}{\lambda^4} \cdot \frac{RTc}{N_A} \left[RT \left(\frac{1}{M} + 2A_2c + \dots \right) \right]^{-1} \\
 &= \frac{2\pi^2 n_0^2 (\mathrm{d}n/\mathrm{d}c)^2 c}{N_A \lambda^4} \left(\frac{1}{M} + 2A_2c + \dots \right)^{-1} \\
 &= Kc \left(\frac{1}{M} + 2A_2c + \dots \right)^{-1}
 \end{aligned}$$

- Note that when $A_2 = 0$, we recover $\Delta R = KMc!$ This is another way of expressing the idea that in the theta condition, the scattering of a polymer-solvent solution is equivalent to that of an ideal gas in a solution.

- Debye's feature 4: How do we account for polymers not being point scatterers?

- Let $P(\theta)$ denote a finite size chain scattering factor. It influences the scattering equation as follows.

$$\Delta R = Kc \left(\frac{1}{M} + 2A_2c + \dots \right)^{-1} P(\theta)$$

- What is the form of $P(\theta)$? Debye calculated in 1939 that for a Gaussian coil (i.e., ideal polymer obeying random walk Gaussian statistics),

$$P(\theta) = \frac{2}{u^2} (u - 1 + e^{-u})$$

where

$$u = \underbrace{\left[\frac{4\pi n_0}{\lambda} \sin\left(\frac{\theta}{2}\right) \right]^2}_{q^2} \langle R_g^2 \rangle$$

- q is the same wavevector from earlier, but with n_0 instead of n_o .

- In the limit of small θ (i.e., $u^{1/2} \ll 1$), a good approximation is

$$P(\theta) \approx 1 - \frac{u}{3} \quad \frac{1}{P(\theta)} \approx 1 + \frac{u}{3}$$

- At this point, we can put together the original equation we wanted: For a SALS experiment in a general, dilute polymer-solvent solution (i.e., $P(\theta) \neq 1$, $A_2 \neq 0 \neq c_2$, and $u^{1/2} \ll 1$), we have

$$\frac{Kc}{\Delta R} = \left(1 + \frac{u}{3} \right) \left(\frac{1}{M} + 2A_2c + \dots \right) = \left[1 + \frac{16\pi^2 n_0^2}{3\lambda^2} \sin^2\left(\frac{\theta}{2}\right) \langle R_g^2 \rangle \right] \left(\frac{1}{M} + 2A_2c + \dots \right)$$

- We now look into how to experimentally extract M, A_2, R_g by plotting scattered intensity vs. c, q^2 .

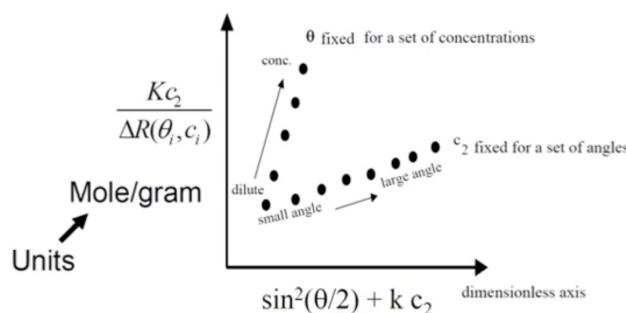


Figure 3.23: Zimm plot features.

- We typically collect scattered intensity at 8 angles: $\theta = 30^\circ, 37.5^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ, 105^\circ, 120^\circ$.
- We collect this data for 5 solute concentrations, c_1 through c_5 .
- Referring to the master scattering equation just above: If $\theta \rightarrow 0^\circ$, the first term goes to 1 and we can get the second term's dependent variables (A_{2c}).
 - Measuring c through orthogonal means then gives us A_2 .
 - Now, we cannot actually measure *scattering* for $\theta = 0^\circ$ because of competition from transmission, so we measure for a series of positive values of θ and *extrapolate* the data to 0° .
- Analogously, extrapolating $c_2 \rightarrow 0$ gives us $\langle R_g^2 \rangle$ (where we know n_0, λ, θ through orthogonal means, or because we're controlling them).
- Note that in the actual plot, our dimensionless x -axis is $\sin^2(\theta/2) + kc$ (where k is an arbitrary constant that neutralizes the units of c).

- Structure of a typical data set.

		θ							
		30°	37.5°	45°	60°	75°	90°	105°	120°
c (g/cm ³)	2×10^{-3}	3.18	3.26	3.25	3.45	3.56	3.72	3.78	4.01
	1.5×10^{-3}	2.73	2.76	2.81	2.94	3.08	3.27	3.40	3.57
	1×10^{-3}	2.29	2.33	2.37	2.53	2.66	2.85	2.96	3.12
	7.5×10^{-4}	2.10	2.14	2.17	2.32	2.47	2.64	2.79	2.93
	5×10^{-4}	1.92	1.95	1.98	2.16	2.33	2.51	2.66	2.79

Table 3.2: Zimm plot sample data set.

- A 2D table of $40 = 8 \times 5$ values of $Kc/\Delta R$, measured at 8 angles θ and 5 concentrations c .
 - You may observe that some of these angles are not exactly “small.” However, this is ok because while the small angle approximation was necessary to do the derivation, it turns out that the result we derived is only at most 10% off from the actual relationship at large angles!
 - In other words, our scattering equation is still empirically useful, *even* at large θ .
 - Orthogonal measurements of n_0 , dn/dc , λ .
 - Table 3.2 shows an example of a typical set.
 - From this data set, we can get extrapolate twice to get two new lines, that collectively will give us all our dependent variables ($A_2, M, \langle R_g^2 \rangle$).
 - Setting $k = 100 \text{ cm}^3/\text{g}$ will normalize the x -axis to a roughly 0-1 scale, so let's do that.
 - Example calculation for a point: For $\theta = 30^\circ$ and $c = 2 \times 10^{-3}$, we have
- $$x = \sin^2\left(\frac{30^\circ}{2}\right) + k \cdot 2 \times 10^{-3} = 0.067 + 100 \cdot 2 \times 10^{-3} = 0.267$$
- The circles in the plots of Figure 3.24 correspond to actual data points in Table 3.2.
 - The lines correspond to best fit lines.
 - In Figure 3.24a, lines correspond to constant θ .
 - In Figure 3.24b, lines correspond to constant c .
 - The squares in the plots correspond to best-fit extrapolations to either $c = 0$ (Figure 3.24a) or $\theta = 0^\circ$ (Figure 3.24b).
 - The slopes of the extrapolated lines are equal to $16\pi^2 n_0^2 \langle R_g^2 \rangle / 3\lambda^2$ (Figure 3.24a) or $2A_2$ (Figure 3.24b).

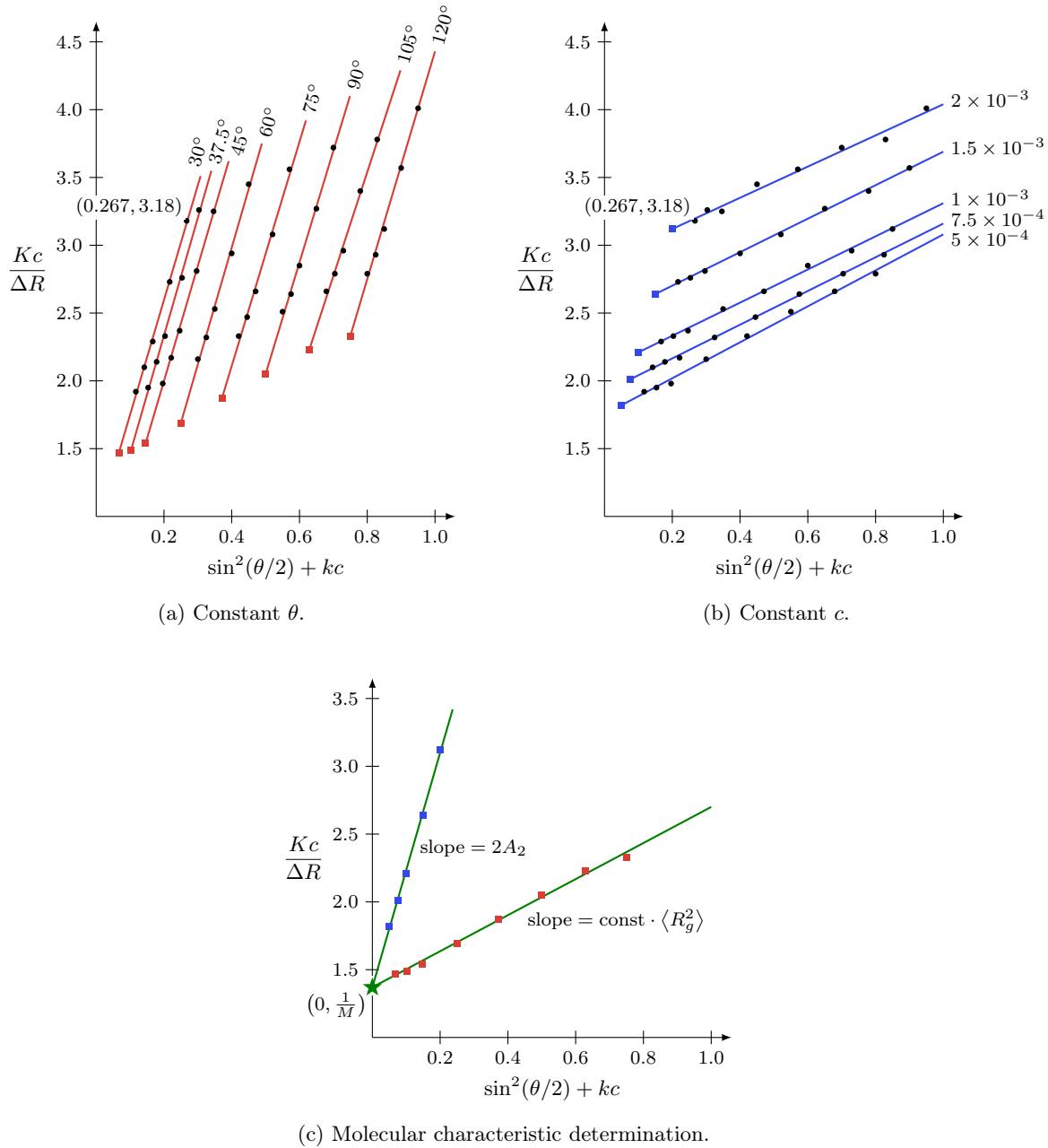


Figure 3.24: Example Zimm plots.

- The extrapolated lines share a y -intercept, which is at $(0, M^{-1})$ per Figure 3.24c.
- Remember that given the information from a Zimm plot, we can then use our previous formulas to obtain even more information!
 - For example, if we know ρ_2 and \bar{V}_1 , then we can use A_2 to calculate χ via the definition of the 2nd virial coefficient.

3.6 Office Hours (Alexander-Katz)

- 10/21:
- PSet 2, Q5.
 - Without excluded volume, we consider just the stretching induced by closeness. This brush could exist in a vacuum or theta solvent.
 - With excluded volume, we consider the volume to be already reduced by the volume of the chain. This brush *must* be in a good or bad solvent; something where excluded volume effects matter.
 - I forgot to address my concerns about the $HD^2 = Na^3$ substitution that renders one of the balancing terms a constant. It seems that Alfredo forgot about this complicating factor and solved the problem without it. I think he may also be solving something of a different problem in general than the one explicitly asked for by the PSet.
 - End of the light scattering derivation.
 - See the additional notes on Canvas.
 - The core idea is that Rayleigh scattering is foundational for all particles, but small molecules and oligomers in solution can be approximated with an α for a dilute gas while polymers need a $\langle(\delta\alpha)^2\rangle$.

3.7 Chapter 9: Dynamics of Dilute Polymer Solutions

From Lodge and Hiemenz (2020).

- 10/9:
- What relates “the viscosity of polymer solutions, the diffusion of polymer molecules, the technique of dynamic light scattering, the phenomenon of hydrodynamic interaction, and the separation and analysis of polymers by size exclusion chromatography” (Lodge & Hiemenz, 2020, p. 377)? They all help determine polymer molecular weight, and they all depend on the spatial extent of polymer coils.
 - We have studied *static* properties of polymer up to this point. Now, we will discuss *time-dependent* dynamics.
 - We begin our study of dynamics with dilute solutions, so as to highlight the properties of individual polymer molecules.
 - **Molecular friction factor:** The proportionality factor between the force \mathbf{F} directionally applied to a polymer in a solvent, and the polymer’s equilibrium velocity \mathbf{v} after accelerating to the point where drag counterbalances \mathbf{F} . *Units kg/s. Denoted by f . Given by*

$$f = \frac{|\mathbf{F}|}{|\mathbf{v}|}$$

- Figure 3.4 may be helpful in visualizing this definition.
- The textbook uses the units of g/s.
- Typical values of f for a polymer in water fall between 10^{-7} - 10^{-6} g/s.
- Two questions we will focus on answering.
 - How can we experimentally measure f ?
 - What can f tell us about polymers?
- First model we will use: The impenetrable sphere.
 - Surprisingly accurate for certain polymers (e.g., floppy random coils, as discussed in class).
 - Allows us to use the computationally simple ideal solution model to underpin our analysis!

- Defining the viscosity of a fluid, and related terms (see Figure 3.1).
 - A *fluid* is preliminarily defined to be “a set of infinitesimally thin layers moving parallel to each other, each with a characteristic velocity” (Lodge & Hiemenz, 2020, p. 377).
 - We postulate that our fluid has no **slip** at the interface between the stationary and flowing phases.
 - This is a good approximation for the systems in which we are interested.
 - This leads our fluid to behave as a **steady flow**.
- **Slip:** Any difference in velocity between those fluid layers that are adjacent to nonflowing surfaces and the nonflowing surfaces themselves.
- **Steady flow:** The time-independent velocity profile developed when the upper plate (Figure 3.1) is moving at a constant velocity.
- We now build up to justifying the notation “ $\dot{\gamma}$ ” for the shear rate.
- **Shear displacement:** The distance Δx that the top layer in the velocity profile moves relative to the bottom layer during a short time interval Δt .
- **Shear strain:** The shear displacement per unit distance H between the two plates. *Denoted by γ . Given by*

$$\gamma := \frac{\Delta x}{H}$$

- Thus, the shear rate (i.e., the rate at which the shear strain develops) is given by the first time derivative of the shear strain $d\gamma/dt$. But in Newton’s notation, first derivatives are represented by putting a single dot over the variable being differentiated. Thus, we represent the shear rate as $\dot{\gamma}$.
- **Shear force:** The force applied to the top plate (Figure 3.1) to develop a velocity profile. *Also known as viscous force. Denoted by \mathbf{F} .*
- **Newton’s law of viscosity:** The statement that the shear stress τ depends bilinearly on the viscosity η and shear rate $\dot{\gamma}$ with no proportionality constant. *Given by*

$$\tau = \eta \dot{\gamma}$$

- **Newtonian** (fluid): A fluid that satisfies Newton’s law of viscosity, i.e., for which the viscosity η is independent of the shear rate $\dot{\gamma}$.
 - Liquids of low molecular weight compounds are generally Newtonian.
 - Newtonian fluids are characterized by a single viscosity.
- Any liquid that undergoes shear thinning is non-Newtonian, as a Newtonian liquid would have a straight line on the graphs in Figure 3.2 (η independent of $\dot{\gamma}$).
 - Shear thinning is often observed for polymer solutions or melts.

- Most viscometers can determine if a fluid is Newtonian by varying the shear rate $\dot{\gamma}$ and measuring whether or not the viscosity stays constant.
- Lodge and Hiemenz (2020, pp. 379–80) covers **viscous heating**, which was not mentioned in class.

10/14:

- **Equation of motion** (of a fluid): The differential equation — obtained by considering all x -, y -, and z -forces of gravitational or mechanical origin acting on a volume element of liquid — whose solution would give the velocity v of the flowing liquid as a function of time and position within the sample.
- Comments on Einstein’s model for the viscosity of a suspension of hard spheres (Lodge & Hiemenz, 2020, p. 384).

- The viscosity does not depend on the radius of the spheres, only their total volume fraction.
- By describing the concentration dependence of viscosity as a power series, Einstein's theory plays a comparable role for viscosity as our “virial expansion” in Flory-Huggins theory.
- The volume fraction emerges from the Einstein derivation as the natural concentration unit to describe viscosity. This parallels the way volume fraction arises as a natural thermodynamic concentration unit in the Flory-Huggins theory.
- Common features and assumptions between Stokes' law and Einstein's model.
 - The liquid medium is a continuum.
 - It follows that the results may be suspect for spheres so small that the molecular nature of the solvent cannot be ignored.
 - However, even in this case, the results often hold up experimentally surprisingly well.
 - Both relationships hold up well to experimental verification, across many systems and spheres of many diameters.
 - For particles that are shaped differently than spheres (e.g., elongated ellipsoids of revolution), there are related derivations.
 - The disturbance of the flow streamlines is assumed to be produced by a single particle, hence the limitation to dilute solutions. In other words, the net effect of an array of spheres is treated as the sum of the individual nonoverlapping disturbances. Moreover, contributions from the walls of the container are assumed to be absent.
- On the hydrodynamic volume.
 - By assuming that hydrodynamic volume is related to the volume of a sphere based on the radius of gyration, we assume that “the volume that matters in the viscosity experiment is not the volume actually occupied by the polymer segments (which would be the degree of polymerization times the volume of the monomer), but the volume pervaded by the entire molecule” (Lodge & Hiemenz, 2020, p. 386).
- **Poiseuille equation:** An expression for the flow rate through a vertical capillary. *Given by*

$$Q = \frac{\pi r^4}{8\eta L} (\rho g L + \Delta P)$$
 - Lodge and Hiemenz (2020, pp. 393–95) fully derives this. The not-too-difficult calculus may be worth going through!
- On co-centric cylinder rheometers (Lodge & Hiemenz, 2020, pp. 397–98).
 - The range of applicability is very wide, extending at least from $\eta \approx 0.01 - 10^{10}$ Pas.
 - The design permits different velocity gradients to be considered, so that non-Newtonian behavior (e.g., shear thinning) can be investigated.
 - Some of the mathematics is derived.
- **Diffusion coefficient:** A measure of how fast molecules move. *Denoted by D .*
- **Tracer (diffusion coefficient):** A diffusion coefficient describing the motion of a single Brownian particle. *Units m^2/s . Denoted by D_t .*
 - The factor of 6 in $\langle \Delta r^2 \rangle = 6D\tau$ is a historical convention.
- **van Hove space-time self-correlation function:** The probability of finding a particle a distance r away from where it was at time $t = 0$ at time $t = \tau$. *Given by*

$$P(r, \tau) = 4\pi r^2 \left(\frac{1}{4\pi D_t \tau} \right)^{3/2} \exp\left(-\frac{r^2}{4D_t \tau}\right)$$

- Notice that the form is mathematically equivalent to the three-dimensional Gaussian distribution for a polymer's end-to-end distance derived in Lecture 1.3, using the substitution $6D_t\tau = \langle \Delta r^2 \rangle = Na^2$.
- **Stokes-Einstein equation:** An expression for the diffusion of a hard-sphere tracer particle of radius r_s in a continuous solvent. *Given by*

$$D_t = \frac{k_B T}{6\pi\eta_s r_s}$$

- **Hydrodynamic radius:** For any polymer or particle, the radius of the hard sphere that would have the same friction factor or diffusivity. *Denoted by R_H . Given by*

$$R_H := \frac{k_B T}{6\pi\eta_s D_t}$$

- Since flexible molecules under hydrodynamic conditions behave like hard spheres with radius proportional to the radius of gyration, they also diffuse this way. This is where $R_H \propto R_g$ comes from!
- Another important set of proportionalities is $D_t \propto R_H^{-1} \propto M^{-\nu}$, for the typical values of ν depending on solvent quality.
- **Shape factor:** The dimensionless proportionality constant between the hydrodynamic radius and radius of gyration. *Denoted by γ . Given by*

$$\gamma := \frac{R_g}{R_H}$$

- As the name suggests, γ provides insight into how spherical particles are: $\gamma \rightarrow 1$ as particles become more spherical.
- **Mutual** (diffusion coefficient): A diffusion coefficient describing how a collection of Brownian particles will distribute themselves in space. *Denoted by D_m .*
 - “Mutual diffusion acts to eliminate any gradients in concentration” (Lodge & Hiemenz, 2020, p. 401).
 - On Fick’s law.
 - This equation quantifies the idea that mass diffusion is analogous to thermal diffusion (in the sense of eliminating concentration gradients), and is mathematically an adaptation of Fourier’s law of heat conduction to the transport of material.
 - J has units of mass per area per time.
 - The statement of Fick’s law given corresponds to one-dimensional diffusion in the x -direction.
 - Lodge and Hiemenz (2020, pp. 401–05) covers Fick’s other laws, and relates diffusion to chemical potential!
 - Lodge and Hiemenz (2020, pp. 406–09) covers dynamic light scattering.