Analysis I (Tao) Notes

Steven Labalme

June 15, 2020

Contents

Chapter 1

Introduction

1.1 What is analysis?

6/14:

- **Analysis**: "The rigorous study of such objects, with a focus on trying to pin down precisely and accurately the qualitative and quantitative behavior of those objects" (1).
- **Real analysis**: "The analysis of the real numbers, sequences and series of real numbers, and real-valued functions" (1).
- Real analysis is the theoretical foundation for **calculus**.
- **Calculus**: The collection of computational algorithms which one uses to manipulate functions.
- Lists questions that can be answered with real analysis (motivation for studying it).

1.2 Why do analysis?

- Lists examples of contradictions in naïve calculus that must be resolved (and can be resolved with real analysis).

Chapter 2

Starting at the Beginning: The Natural Numbers

6/15:

- This text will begin by reviewing high school level material, but as rigorously as possible.
  - It will teach the skill of proving complicated properties from simpler ones, allowing you to understand why an "obvious" statement really is obvious.
  - One particularly important skill is the use of **mathematical induction**.
  - We will strive to eliminate **circularity**.

  Circularity: "Using an advanced fact to prove a more elementary fact, and then later using the elementary fact to prove the advanced fact" (14).

- The number systems used in real analysis, listed in order of increasing sophistication, are the **naturals** $\mathbb{N}$[1], the **integers** $\mathbb{Z}$, the **rationals** $\mathbb{Q}$, and the **reals** $\mathbb{R}$.
  - **Complex numbers** $\mathbb{C}$ will only be used much later.
- This chapter will answer the question, "How does one actually *define* the natural numbers?"

2.1 The Peano Axioms

- **Peano Axioms**: First laid out by Guiseppe Peano, these are a standard way to define the natural numbers.
- How do you define operations on the naturals?
  - Complicated operations are defined in terms of simpler ones: Exponentiation is repeated multiplication, multiplication is repeated addition, and addition is repeated **incrementing**.
- **Incrementing**: The most fundamental operation — best thought of as counting forward by one number.
  - Incrementing is one of the fundamental concepts that allows us to define the natural numbers.
  - Let[2] $n{+}{+}$ denote the increment, or **successor**, of n.
    - For example, $3{+}{+} = 4$ and $(3{+}{+}){+}{+} = 5$.
- Let $x := y$ denote the statement, "x is defined to equal y."

[1]Note that in this text, the natural numbers will include 0. The natural numbers without 0 will be called the **positive integers** $\mathbb{Z}^+$.

[2]This notation is pulled from some computer languages such as *C*.

- At this point, we can begin defining the natural numbers.

  Axiom 2.1. *0 is a natural number.*

  Axiom 2.2. *If n is a natural number, then n++ is also a natural number.*

- To avoid having to use incrementation notation for every number, we adopt a convention.

  Definition 2.1. We define 1 to be the number 0++, 2 to be the number (0++)++, 3 to be the number ((0++)++)++, etc.

- From these axioms, we can already prove things.

  Proposition 2.1. *3 is a natural number.*

  Proof. By Axiom 2.1, 0 is a natural number. By Axiom 2.2, 0++ = 1 is a natural number. By Axiom 2.2 again, 1++ = 2 is a natural number. By Axiom 2.2 again, 2++ = 3 is a natural number. □

- It seems like Axioms 2.1 and 2.2 have us pretty well covered. However, what if the number system wraps around (e.g., if 3++ = 0)? We can fix this with the following.

  Axiom 2.3. *0 is not the successor of any natural number; i.e., we have n++ $\neq 0$ for every natural number n.*

- We can now prove that $4 \neq 0$ (because 4 = 3++, $3 \in \mathbb{N}$, and n++ $\neq 0$).

- However, there are still issues — what if the number system hits a ceiling at 4, e.g., 4++ = 4?

- A good way to prevent this kind of behavior is via the following.

  Axiom 2.4. *Different natural numbers must have different successors, i.e., if $n, m \in \mathbb{N}$ and $n \neq m$, then n++ $\neq m$++. Equivalently[3], if n++ = m++, then $n = m$.*

- We can now prove propositions like the following, extending our anti-wrap around proving ability.

  Proposition 2.2. *6 is not equal to 2.*

  Proof. Suppose 6 = 2. Then 5++ = 1++, so by Axiom 2.4, 5 = 1. Then 4++ = 0++, so by Axiom 2.4, 4 = 0, which contradicts our proof that $4 \neq 0$. □

[3]This is an example of reformulating an implication using its **contrapositive**. In the converse direction, it is the **axiom of substitution**.