Chapter 3

Set Theory

• Set theory will be frequently used in the subsequent chapters; it is part of the foundation of almost every other branch of mathematics.

- Note that Euclidean geometry will not be defined — we will use the Cartesian coordinate system's parallel with the real numbers instead.

• This chapter covers the elementary aspects, Chapter **??** covers more advanced topics, and the finer subtleties are well beyond the scope of this text.

3.1 Fundamentals

• We define sets axiomatically, as we did with the natural numbers[1].

Axiom 3.1 (Sets are objects). *If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.*

• Note that while all sets are objects, not all objects are sets.

- For example, 1 is not a set while $\{1\}$ is.
- Note, though, that **pure set theory** considers all objects to be sets. However, impure set theory (where some objects are not sets) is conceptually easier to deal with.
  - ■ Since both types are equal for the purposes of mathematics, we will take a middle-ground approach.

• If x, y are objects and A a set, then the statement $x \in A$ is either true or false. Note that $x \in y$ is neither true nor false, simply meaningless.

• We now define equality for sets.

Definition 3.1 (Equality of sets). Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

• Note that this implies that repetition of elements does not effect equality ($\{3,3\} = \{3\}$, for example).

• It can be proven that this notion of equality is reflexive, symmetric, and transitive (see Exercise 3.1.1).

• Since $x \in A$ and $A = B$ implies $x \in B$, the $\in$ relation obeys the axiom of substitution as well.

---

[1]Note that the following list of axioms will be somewhat overcomplete, as some axioms may be derived from others. However, this is helpful for pedagogical reasons, and there is no real harm being done.

– Thus, any operation defined in terms of the $\in$ relation obeys the axiom of substitution.

- We define sets in an analogous way to how we defined natural numbers from 0, onward.

   Axiom 3.2 (Empty set). *There exists a set $\emptyset$ (also denoted $\{\}$), known as the empty set, which contains no elements, i.e., for every object x, we have $x \notin \emptyset$.*

   – Note that there is only one empty set — if $\emptyset$ and $\emptyset'$ were supposedly distinct empty sets, then Definition 3.1 would prove that $\emptyset = \emptyset'$.

- **Non-empty set**: A set that "is not equal to the empty set" (Tao, 2016, p. 36).

- Non-empty sets must contain at least one object.

   Lemma 3.1 (Single choice). *Let A be a non-empty set. Then there exists an object x such that $x \in A$.*

   Proof. Suppose for the sake of contradiction that no object x exists such that $x \in A$, i.e., for all objects x, we have $x \notin A$. By Axiom 3.2, we have $x \notin \emptyset$ either. Thus, $x \in A \iff x \in \emptyset$ (denoting logical equivalence; both statements are equally false), so, by Definition 3.1, $A = \emptyset$, a contradiction. $\square$

- There exist more sets than just the empty set.

   Axiom 3.3 (Singleton sets and pair sets). *If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the **singleton set** whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the **pair set** formed by a and b.*

- By Definition 3.1, there exists only one singleton set for each object a and only one pair set for any two objects a, b.

- Note that the singleton set axiom follows from the pair set axiom, and the pair set axiom follows from the singleton set axiom and the pairwise union axiom, below.

- As alluded to, the pairwise union axiom allows us to build sets with more than two elements.

   Axiom 3.4 (Pairwise union). *Given any two sets A, B, there exists a set $A \cup B$ called the **union** $A \cup B$ of A and B, whose elements consist of all the elements which belong to A or B or both. In other words, for any object x,*
   $$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$

- The $\cup$ operation obeys the axiom of substitution (if $A = A'$, then $A \cup B = A' \cup B$).

- We now prove some basic properties of unions (one below and three in Exercise 3.1.3).

   Lemma 3.2. *If A, B, C are sets, then the union operation is associative, i.e., $(A \cup B) \cup C = A \cup (B \cup C)$.*

   Proof. By Definition 3.1, showing that every element x of $(A \cup B) \cup C$ is an element of $A \cup (B \cup C)$ and vice versa will suffice to prove this lemma. Suppose first that $x \in (A \cup B) \cup C$. By Axiom 3.4, this means that at least one of $x \in A \cup B$ or $x \in C$ is true. We now divide into two cases. If $x \in C$, then by Axiom 3.4, $x \in B \cup C$, and, so, by Axiom 3.4 again, we have $x \in A \cup (B \cup C)$. Now suppose instead that $x \in A \cup B$. Then by Axiom 3.4, $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cup C)$ by Axiom 3.4, while if $x \in B$, then by consecutive applications of Axiom 3.4, we have $x \in B \cup C$ and, hence, $x \in A \cup (B \cup C)$. A similar argument shows that every element of $A \cup (B \cup C)$ lies in $(A \cup B) \cup C$, and so $(A \cup B) \cup C = A \cup (B \cup C)$, as desired. $\square$

- As a consequence of the above, we are free to write $A \cup B \cup C \cup \cdots$ to denote repeated unions without having to use parentheses.

- We can also now define triplet sets ($\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\}$), quadruplet sets, and so forth.

  - However, we cannot yet define a set of n objects or an infinite set.

- Note that addition and union are analogous, but importantly *not* identical.

- Some sets are "larger" than others; hence, subsets.

  Definition 3.2 (Subsets). Let A, B be sets. We say that A is a **subset** of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e., for any object x, $x \in A \Longrightarrow x \in B$. We say that A is a **proper subset** of B, denoted $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$.

- The $\subseteq$ and $\subsetneq$ operations obey the axiom of substitution (since both $=$ and $\in$, the two component operations of $\subseteq$ and $\subsetneq$, obey it).

- Note that $A \subseteq A$ and $\emptyset \subseteq A$ for any set A.

- Note that less than or equal to and subset are analogous, but not identical, either (see below for one related property and Exercise 3.1.4 for two more).

  Proposition 3.1 (Sets are partially ordered by set inclusion 1). *Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

  Proof. Suppose that $A \subseteq B$ and $B \subseteq C$. To prove that $A \subseteq C$, we have to prove that every element of A is an element of C. Let $x \in A$. Then $x \in B$ (since $A \subseteq B$), implying that $x \in C$ (since $B \subseteq C$). $\quad\square$

- There exist relations between subsets and unions (see Exercise 3.1.7).

- Note this difference between $\subsetneq$ and $<$: Given any two distinct natural numbers n, m, one is smaller than the other (Proposition 2.7). However, given any two distinct sets, it is not in general true that one is a subset of the other. This is why we say that sets are **partially ordered** while the natural numbers (for example) are **totally ordered** (see Definitions **??** and **??**, respectively).

- Note that $\in$ and $\subseteq$ are distinct ($2 \in \{1, 2, 3\}$, but $2 \nsubseteq \{1, 2, 3\}$; similarly, $\{2\} \subseteq \{1, 2, 3\}$, but $\{2\} \notin \{1, 2, 3\}$).

- It is important to distinguish sets from their elements, for they can have different properties ($\mathbb{N}$ is an infinite set of finite elements, and $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ is a finite set of infinite objects).

- We now formally state that it is acceptable to create subsets.

  Axiom 3.5 (Axiom of specification). *Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$ for short), whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,*

  $$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true})$$

- *Also known as* **axiom of separation**.

- Specification obeys the axiom of substitution (since $\in$ obeys it).

- Sometimes $\{x \in A : P(x)\}$ is denoted by $\{x \in A \,\big|\, P(x)\}$ (useful when we need the colon for something else, e.g., $f : X \to Y$).

- We use Axiom 3.5 to define intersections.

Definition 3.3 (Intersections). The **intersection** $S_1 \cap S_2$ of two sets is defined to be the set

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}$$

In other words, $S_1 \cap S_2$ consists of all the elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2$$

- The $\cap$ operation obeys the axiom of substitution (since $\in$ obeys it).

    - Note that since $\cap$ is defined in terms of more primitive operations, it is well-defined.

- Problems with the English word, "and."

    - It can mean union or intersection depending on the context.

        ■ If X, Y are sets, "the set of elements of X and elements of Y" refers to $X \cup Y$, e.g., "the set of singles and males."
        ■ If X, Y are sets, "the set of objects that are elements of X and elements of Y" refers to $X \cap Y$, e.g., "the set of people who are single and male."

    - It can also denote addition.

        ■ "2 and 3 is 5" means $2 + 3 = 5$.

    - "One reason we resort to mathematical symbols instead of English words such as 'and' is that mathematical symbols always have a precise and unambiguous meaning, whereas one must often look very carefully at the context in order to work out what an English word means" (Tao, 2016, p. 42).

- **Disjoint** (sets): Two sets A, B such that $A \cap B = \emptyset$.

- **Distinct** (sets): Two sets A, B such that $A \neq B$.

    - Note that $\emptyset$ and $\emptyset$ are disjoint but not distinct.

- We also use Axiom 3.5 to define difference sets.

    Definition 3.4 (Difference sets). Given two sets A and B, we define the set $A - B$ or $A \setminus B$ to be the set A with any elements of B removed, i.e.,

    $$A \setminus B := \{x \in A : x \notin B\}$$

- For example, $\{1, 2, 3, 4\} \setminus \{2, 4, 6\} = \{1, 3\}$ — in many cases, $B \subseteq A$, but not necessarily.

7/15:
- See Exercise 3.1.6 for some basic properties of unions, intersections, and difference sets.

    - The de Morgan laws are named after the logician Augustus De Morgan (1806-1871).
    - The **laws of Boolean algebra** (the contents of Proposition 3.4) are named after the mathematician George Boole (1815-1864).

        ■ They are applicable to a number of objects other than sets (e.g., laws of propositional logic).

    - Note the **duality** in Proposition 3.4, manifesting itself in the certain symmetry between $\cup$ and $\cap$, and X and $\emptyset$.

- **Duality**: "Two distinct properties or objects being dual to each other" (Tao, 2016, p. 43).

- We can do a lot, but we still wish to do more. For example, we'd like to transform sets (say, take $\{3, 5, 9\}$ and increment each object to yield $\{4, 6, 10\}$).

Axiom 3.6 (Replacement). *Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x,y)$ pertaining to x and y such that for each $x \in A$, there is at most one y for which $P(x,y)$ is true. Then there exists a set $\{y : P(x,y)$ is true for some $x \in A\}$, such that for any object z,*

$$z \in \{y : P(x,y) \text{ is true for some } x \in A\} \Longleftrightarrow P(x,z) \text{ is true for some } x \in A$$

- For example, let $A := \{3,5,9\}$ and let $P(x,y)$ be the statement $y = x{+}{+}$. By Axiom 2.4, for every $x \in A$, there is exactly one y for which $P(x,y)$ is true (namely, the successor of x). Thus, by Axiom 3.6, the set $\{y : y = x{+}{+}$ for some $x \in \{3,5,9\}\}$ exists. It is clearly the same set as $\{4,6,10\}$.

- The set obtained can be smaller than the original set, e.g., $\{y : y = 1$ for some $x \in \{3,5,9\}\} = \{1\}$.

- We often abbreviate the set specified in Axiom 3.6 to one of the following.

$$\{y : y = f(x) \text{ for some } x \in A\}$$
$$\{f(x) : x \in A\}$$
$$\{f(x) \big| x \in A\}$$

- We can combine Axioms 3.6 and 3.5, i.e., $\{f(x) : x \in A; P(x)$ is true$\}$, e.g., $\{n{+}{+} : n \in \{3,5,9\}; n < 6\} = \{4,6\}$.

- Although we have assumed that natural numbers are objects in several examples up to this point, we must formalize this notion.

  Axiom 3.7 (Infinity). *There exists a set $\mathbb{N}$, whose elements are called the natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.*

- Axiom 3.7 is called the **axiom of infinity** because "it introduces the most basic example of an infinite set, namely the set of natural numbers $\mathbb{N}$" (Tao, 2016, p. 44).

Exercises

7/4:   1. Show that the definition of equality in Definition 3.1 is reflexive, symmetric, and transitive.

Proof. Given a set A, suppose $A \neq A$. Then, by Definition 3.1, every element of A is not an element of A, a contradiction. Thus, $A = A$.

Let sets $A = B$. Then, by Definition 3.1, every element x of A belongs also to B, and every element y of B belongs also to A. Identically, every element y of B belongs also to A, and every element X of A belongs also to B. Thus, $B = A$.

Let sets $A = B$ and $B = C$. Then, by Definition 3.1, every element x of A belongs also to B, and every element y of B belongs also to A. Similarly, every element y of B belongs also to C, and every element z of C belongs also to B. Since $x \in A \Rightarrow x \in B \Rightarrow x \in C$, and $y \in C \Rightarrow y \in B \Rightarrow y \in A$, $A = C$.   □

7/14:   2. Using only Definition 3.1, Axiom 3.1, Axiom 3.2, and Axiom 3.3, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct (i.e., no two of them are equal to each other).

Proof. First, we show that all sets are distinct from the empty set. Axiom 3.1 asserts that $\emptyset$ and $\{\emptyset\}$ are objects. By Axiom 3.3, we have $\emptyset \in \{\emptyset\}$, $\emptyset \in \{\emptyset, \{\emptyset\}\}$, and $\{\emptyset\} \in \{\{\emptyset\}\}$. Since $x \notin \emptyset$ for all objects x (Axiom 3.2), $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ all contain objects that $\emptyset$ does not (namely, $\emptyset$, $\{\emptyset\}$, and $\emptyset$, respectively). Thus, by Definition 3.1, $\emptyset \neq \{\emptyset\}$, $\emptyset \neq \{\{\emptyset\}\}$, and $\emptyset \neq \{\emptyset, \{\emptyset\}\}$.

Next, we show that $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$. By Axiom 3.3, $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ and $y \in \{\emptyset\}$ iff $y = \emptyset$. Since $\{\emptyset\} \neq \emptyset$ (see above), $\{\emptyset\} \notin \{\emptyset\}$. Thus, $\{\emptyset, \{\emptyset\}\}$ contains an object that $\{\emptyset\}$ does not, implying by Definition 3.1 that $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$.

Lastly, we show that $\{\emptyset\} \neq \{\{\emptyset\}\}$ and that $\{\emptyset, \{\emptyset\}\} \neq \{\{\emptyset\}\}$. We proceed in a similar manner to the above. By Axiom 3.3, $\emptyset \in \{\emptyset\}$, $\emptyset \in \{\emptyset, \{\emptyset\}\}$, and $y \in \{\{\emptyset\}\}$ iff $y = \{\emptyset\}$. Since $\emptyset \neq \{\emptyset\}$ (see above), $\emptyset \notin \{\{\emptyset\}\}$. Thus, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ both contain an object that $\{\{\emptyset\}\}$ does not, implying by Definition 3.1 that $\{\emptyset\} \neq \{\{\emptyset\}\}$ and that $\{\emptyset, \{\emptyset\}\} \neq \{\{\emptyset\}\}$.

$\square$

3. Prove the following lemmas.

   Lemma 3.3. *If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$.*

   Proof. By Definition 3.1, it will suffice to show that every element x of $\{a, b\}$ is an element of $\{a\} \cup \{b\}$ and vice versa. By Axiom 3.3, if $x \in \{a, b\}$, then $x = a$ or $x = b$. By Axiom 3.4, if $x \in \{a\} \cup \{b\}$, then $x \in \{a\}$ or $x \in \{b\}$, implying by Axiom 3.3 that $x = a$ or $x = b$. Thus, the elements of $\{a, b\}$ and of $\{a\} \cup \{b\}$ are both a, b, so by Definition 3.1, the sets are equal. $\square$

   Lemma 3.4. *If A, B, C are sets, then the union operation is commutative, i.e., $A \cup B = B \cup A$.*

   Proof. By Definition 3.1, it will suffice to show that every element x of $A \cup B$ is an element of $B \cup A$ and vice versa. Let $x \in A \cup B$. By Axiom 3.4, $x \in A$ or $x \in B$. If, on the one hand, $x \in A$, then $x \in B \cup A$ (by Axiom 3.4). If, on the other hand, $x \in B$, then $x \in B \cup A$ (by Axiom 3.4). A similar argument holds if we choose an element $y \in B \cup A$ first. $\square$

   Lemma 3.5. *If A is a set, then $A = A \cup \emptyset = \emptyset \cup A = A \cup A$.*

   Proof. First, we show that $A \cup \emptyset = \emptyset \cup A$. This is a direct consequence of Lemma 3.3.

   Next, we show that $A = A \cup \emptyset$. By Definition 3.1, it will suffice to show that every element x of A is an element of $A \cup \emptyset$ and vice versa. By Axiom 3.4, every element x of A is an element of $A \cup \emptyset$. Now let $x \in A \cup \emptyset$. Then by Axiom 3.4, $x \in A$ or $x \in \emptyset$. By Axiom 3.2, $x \notin \emptyset$, so $x \in A$. Thus, every element of $A \cup \emptyset$ is an element of A. We now have by the transitive property that $A = A \cup \emptyset = \emptyset \cup A$.

   Lastly, we show that $A = A \cup A$. By Definition 3.1, it will suffice to show that every element x of A is an element of $A \cup A$ and vice versa. By Axiom 3.4, every element x of A is an element of $A \cup A$. Now let $x \in A \cup A$. Then by Axiom 3.4, $x \in A$ or $x \in A$, implying $x \in A$. We have, at last, by the transitive property that $A = A \cup \emptyset = \emptyset \cup A = A \cup A$. $\square$

4. Prove the following propositions.

   Proposition 3.2 (Sets are partially ordered by set inclusion 2)**.** *Let A, B be sets. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.*

   Proof. Suppose that $A \subseteq B$ and $B \subseteq A$. By Definition 3.2, $A \subseteq B$ implies that every element of A is also an element of B and $B \subseteq A$ implies that every element of B is also an element of A. Thus, by Definition 3.1, $A = B$. $\square$

   Proposition 3.3 (Sets are partially ordered by set inclusion 3)**.** *Let A, B, C be sets. If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$.*

   Proof. Suppose that $A \subsetneq B$ and $B \subsetneq C$. Then, by Definition 3.2, $A \subseteq B$ and $B \subseteq C$, implying $A \subseteq C$ by the first claim proved. Since $A \subsetneq B$, $A \neq B$ (implying, by Definition 3.1, that every element of A is not an element of B or every element of B is not an element of A) and every element of A is an element of B; hence, every element of B is not an element of A. Therefore, B *must* contain some element that A does not. Similarly, $B \subsetneq C$ implies that C must contain some element that B does not. Hence, C contains at least two elements that A does not, proving that $A \neq C$, too. $\square$

5. Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$, and $A \cap B = A$ are logically equivalent (any one of them implies the other two).

Proof. First, we show that $A \subseteq B \implies (A \cup B = B$ and $A \cap B = A)$. Next, we show that $A \cup B = B \implies A \subseteq B$ (which, in turn, implies $A \cap B = A$). Lastly, we show that $A \cap B = A \implies A \subseteq B$ (which, in turn, implies $A \cup B = B$). Let's begin.

Suppose that $A \subseteq B$. To prove $A \cup B = B$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cup B$ is an element of B and vice versa. By Axiom 3.4, $x \in B \implies x \in A \cup B$. By Axiom 3.4, $x \in A \cup B \implies (x \in A$ or $x \in B)$. By Definition 3.2, $A \subseteq B$ means that $x \in A \implies x \in B$. Thus, $x \in A \cup B \implies (x \in A$ or $x \in B) \implies x \in B$. Therefore, if $A \subseteq B$, then $A \cup B = B$. To prove that $A \cap B = A$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap B$ is an element of A and vice versa. By Definition 3.3, $x \in A \cap B \implies x \in A$ (and $x \in B$). Since $x \in A \implies x \in B$ (see above), $x \in A \implies (x \in A$ and $x \in B) \implies x \in A \cap B$ (Definition 3.3). Therefore, if $A \subseteq B$, then $A \cap B = A$.

Suppose that $A \cup B = B$. To prove $A \subseteq B$, Definition 3.2 tells us that it will suffice to show that every element of A is also an element of B. By Axiom 3.4, $x \in A \implies x \in A \cup B$. By Definition 3.1, $y \in A \cup B \implies y \in B$. Thus, $x \in A \implies x \in A \cup B \implies x \in B$. Therefore, if $A \cup B = B$, $A \subseteq B$ (and $A \cap B = A$).

Suppose that $A \cap B = A$. To prove that $A \subseteq B$, Definition 3.2 tells us that it will suffice to show that every element of A is also an element of B. By Definition 3.1, $x \in A \implies x \in A \cap B$. By Definition 3.3, $y \in A \cap B \implies y \in B$. Thus, $x \in A \implies x \in A \cap B \implies x \in B$. Therefore, if $A \cap B = A$, $A \subseteq B$ (and $A \cup B = B$). $\qquad\square$

7/15:   6. Prove the following proposition. (Hint: one can use some of these claims to prove others. Some of the claims have also appeared previously in Lemma 3.2 and Exercise 3.1.3.)

Proposition 3.4 (Sets form a boolean algebra)**.** *Let A, B, C be sets, and let X be a set containing A, B, C as subsets.*

(a) *(Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*

Proof. See Exercise 3.1.3 for the first claim.
To prove $A \cap \emptyset = \emptyset$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap \emptyset$ is an element of $\emptyset$ and vice versa. First off, "every element $x \in \emptyset$ is an element of $A \cap \emptyset$" is vacuously true (by Axiom 3.2, there exists no $x \in \emptyset$). In the other direction, suppose for the sake of contradiction that $x \in A \cap \emptyset$ for some object x. By Definition 3.3, $x \in A$ and $x \in \emptyset$. But $x \in \emptyset$ contradicts Axiom 3.2. Therefore, $x \notin A \cap \emptyset$ for all objects x. Thus, the statement "every element $x \in A \cap \emptyset$ is an element of $\emptyset$" is vacuously true (there exists no $x \in A \cap \emptyset$). $\qquad\square$

(b) *(Maximal element) We have $A \cup X = X$ and $A \cap X = A$.*

Proof. See Exercise 3.1.5. $\qquad\square$

(c) *(Identity) We have $A \cap A = A$ and $A \cup A = A$.*

Proof. To prove that $A \cap A = A$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap A$ is an element of A and vice versa. By Definition 3.3, $x \in A \cap A \implies (x \in A$ and $x \in A) \implies x \in A$. On the other hand, $x \in A \implies (x \in A$ and $x \in A)$ (idempotent law for conjunction), $(x \in A$ and $x \in A) \implies x \in \{x \in A : x \in A\}$ (Axiom 3.5), and $x \in \{x \in A : x \in A\} \implies x \in A \cap A$ (Definition 3.3).
See Exercise 3.1.3 for the second claim. $\qquad\square$

(d) *(Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*

Proof. See Exercise 3.1.3 for the first claim.
To prove $A \cap B = B \cap A$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap B$ is an element of $B \cap A$ and vice versa. By two applications of Definition 3.3 with the commutative law for conjunction in between, $x \in A \cap B \implies (x \in A$ and $x \in B) \implies (x \in B$ and $x \in A) \implies x \in B \cap A$. A similar argument works in the opposite direction. $\qquad\square$

(e) *(Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*

Proof. See Lemma 3.2 for the first claim.

By Definition 3.1, showing that every element x of $(A \cap B) \cap C$ is an element of $A \cap (B \cap C)$ and vice versa will suffice to prove this lemma. Suppose first that $x \in (A \cap B) \cap C$. By Definition 3.3, $x \in A \cap B$ and $x \in C$, which implies by a second application of Definition 3.3 that $x \in A$ and $x \in B$ and $x \in C$. It follows by consecutive applications of Definition 3.3 that $x \in A$ and $x \in B \cap C$, and that $x \in A \cap (B \cap C)$. A similar argument shows that every element of $A \cap (B \cap C)$ lies in $(A \cap B) \cap C$. □

(f) *(Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Proof. To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$ and vice versa. By Definition 3.3, $x \in A \cap (B \cup C) \implies (x \in A$ and $x \in B \cup C)$. By Axiom 3.4, $x \in B \cup C \implies x \in B$ or $x \in C$. Thus, we know that $x \in A$, and we know that $x \in B$ or $x \in C$ (or both). We divide into two cases. Suppose first that $x \in B$. Since $x \in A$ as well, $x \in A \cap B$ (Definition 3.3). This implies by Axiom 3.4 that $x \in (A \cap B) \cup (A \cap C)$. Now suppose that $x \in C$. Since $x \in A$ as well, $x \in A \cap C$ (Definition 3.3). This implies by Axiom 3.4 that $x \in (A \cap B) \cup (A \cap C)$. A similar argument shows that every element of $(A \cap B) \cup (A \cap C)$ lies in $A \cap (B \cup C)$.

The second proof is similar to the first (and similar to all the rest of the proofs written for this exercise thus far). Thus, for the sake of variety, we will do this one entirely symbolically, using the laws of propositional logic from Section A.4.

$$
\begin{aligned}
x \in A \cup (B \cap C) &\implies (x \in A) \vee (x \in B \cap C) & \text{Axiom 3.4} \\
&\implies (x \in A) \vee ((x \in B) \wedge (x \in C)) & \text{Definition 3.3} \\
&\implies ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) & \text{Distributive law for disjunction} \\
&\implies (x \in A \cap B) \vee (x \in A \cap C) & \text{Definition 3.3} \\
&\implies x \in (A \cap B) \cup (A \cap C) & \text{Axiom 3.4}
\end{aligned}
$$

A similar argument works in reverse. □

(g) *(Partition) We have $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.*

Proof. To prove $A \cup (X \setminus A) = X$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cup (X \setminus A)$ is an element of X and vice versa. Suppose $x \in A \cup (X \setminus A)$. Then by Axiom 3.4, $x \in A$ or $x \in X \setminus A$. We divide into two cases. Suppose first that $x \in A$. Since $A \subseteq X$, Definition 3.2 asserts that $x \in X$. Now suppose that $x \in X \setminus A$. This implies by Definition 3.4 that $x \in \{y \in X : y \notin A\}$, which means by Axiom 3.5 that $x \in X$ (and $x \notin A$, but that's not relevant). On the other hand, suppose $x \in X$. Naturally, either $x \in A$ or $x \notin A$ ($x \in A$ is false). If $x \in A$, then $x \in A \cup (X \setminus A)$ (Axiom 3.4). If $x \notin A$, since $x \in X$ as well, Axiom 3.5 asserts that $x \in \{x \in X : x \notin A\}$, which, by Definition 3.4, implies that $x \in X \setminus A$. This, in turn, implies that $x \in A \cup (X \setminus A)$ (Axiom 3.4).

To prove $A \cap (X \setminus A) = \emptyset$, it suffices to prove that for every object x, we have $x \notin A \cap (X \setminus A)$ (because of the uniqueness of the empty set). Suppose for the sake of contradiction that $x \in A \cap (X \setminus A)$ for some object x. By Definition 3.3, $x \in A$ and $x \in X \setminus A$. By Definition 3.4, $x \in \{y \in X : y \notin A\}$. By Axiom 3.5, $x \in X$ and $x \notin A$. But this contradicts the previously derived fact that $x \in A$. Therefore, $x \notin A \cap (X \setminus A)$ for all objects x. □

(h) *(De Morgan laws) We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.*

Proof. To prove $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$, Definition 3.1 tells us that it will suffice to show that every element x of $X \setminus (A \cup B)$ is an element of $(X \setminus A) \cap (X \setminus B)$ and vice versa. Suppose that $x \in X \setminus (A \cup B)$. Then by Definition 3.4, $x \in \{y \in X : y \notin A \cup B\}$. By Axiom 3.5, $x \in X$ and $x \notin A \cup B$. By the inverse of Axiom 3.4 (which is a valid assertion since Axiom 3.4 asserts the logical equivalence of "$x \in A \cup B$" and "$x \in A$ and $x \in B$"), $x \notin A \cup B \implies (x \notin A$ and $x \notin B)$.

By Axiom 3.5 and Definition 3.4, $(x \in X$ and $x \notin A) \implies x \in \{y \in X : y \notin A\} \implies x \in X \setminus A$. Similarly, $(x \in X$ and $x \notin B) \implies x \in \{y \in X : y \notin B\} \implies x \in X \setminus B$. Thus, by Definition 3.3, $x \in (X \setminus A) \cap (X \setminus B)$. A similar, reversed argument will work in the other direction.

To prove $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$, Definition 3.1 tells us that it will suffice to show that every element x of $X \setminus (A \cap B)$ is an element of $(X \setminus A) \cup (X \setminus B)$ and vice versa. By Definition 3.4 and Axiom 3.5, $x \in X \setminus (A \cap B) \implies x \in \{y \in X : y \notin A \cap B\} \implies (x \in X$ and $x \notin A \cap B)$. By the inverse of Definition 3.3 (which, again, is a valid assertion since $x \in A \cap B \leftrightarrow ((x \in A) \wedge (x \in B)))$, $x \notin A \cap B \implies (x \notin A$ or $x \notin B)$. We divide into two cases. Suppose $x \notin A$. Then by Axiom 3.5 and Definition 3.4, $(x \in X$ and $x \notin A) \implies x \in \{y \in X : y \notin A\} \implies x \in X \setminus A$. Thus, by Axiom 3.4, $x \in (X \setminus A) \cup (X \setminus B)$. Now suppose $x \notin B$. Then by Axiom 3.5 and Definition 3.4, $(x \in X$ and $x \notin B) \implies x \in \{y \in X : y \notin B\} \implies x \in X \setminus B$. Thus, by Axiom 3.4, $x \in (X \setminus A) \cup (X \setminus B)$. $\qquad\square$

7/17:   7. Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ iff $C \subseteq A \cap B$. In a similar spirit, show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and furthermore that $A \subseteq C$ and $B \subseteq C$ iff $A \cup B \subseteq C$.

Proof. To prove that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, Definition 3.2 tells us that it will suffice to show that every element of $A \cap B$ is an element of A and B. By Definition 3.3, $x \in A \cap B \implies (x \in A$ and $x \in B)$.

7/19:   To prove that $C \subseteq A$ and $C \subseteq B$ iff $C \subseteq A \cap B$, it will suffice to show that $C \subseteq A$ and $C \subseteq B$ imply $C \subseteq A \cap B$ and vice versa. Suppose first that $C \subseteq A$ and $C \subseteq B$. Then by two applications of Definition 3.2, $x \in C \implies x \in A$ and $x \in C \implies x \in B$. By Definition 3.3, $(x \in A$ and $x \in B) \implies x \in A \cap B$. Thus, every element x of C is an element of $A \cap B$, so by Definition 3.2, $C \subseteq A \cap B$. Now suppose that $C \subseteq A \cap B$. Then by Definition 3.2, $x \in C \implies x \in A \cap B$. By Definition 3.3, $x \in A \cap B \implies (x \in A$ and $x \in B)$. Thus, every element x of C is an element of A and B, so by two applications of Definition 3.2, $C \subseteq A$ and $C \subseteq B$.

To prove that $A \subseteq A \cup B$ and that $B \subseteq A \cup B$, Definition 3.2 tells us that it will suffice to show that every element x of A is an element of $A \cup B$ and that every element y of B is an element of $A \cup B$, respectively. By two applications of Axiom 3.4, $x \in A \implies x \in A \cup B$, and $y \in B \implies y \in A \cup B$.

To prove that $A \subseteq C$ and $B \subseteq C$ iff $A \cup B \subseteq C$, it will suffice to show that $A \subseteq C$ and $B \subseteq C$ imply $A \cup B \subseteq C$ and vice versa. Suppose first that $A \subseteq C$ and $B \subseteq C$. By Axiom 3.4, $x \in A \cup B \implies (x \in A$ or $x \in B)$. We divide into two cases. If $x \in A$, then Definition 3.2 ensures that $x \in C$. If $x \in B$, then Definition 3.2 similarly ensures that $x \in C$. Thus, either way, $x \in A \cup B \implies x \in C$, so by Definition 3.2, $A \cup B \subseteq C$. Now suppose that $A \cup B \subseteq C$. By Axiom 3.4 followed by Definition 3.2, $x \in A \implies x \in A \cup B \implies x \in C$. A similar argument shows that $x \in B \implies x \in C$. Thus, $A \subseteq C$ and $B \subseteq C$. $\qquad\square$

8. Let A, B be sets. Prove the **absorption laws** $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$.

Proof. By Exercise 3.1.7, $A \subseteq A \cup B$. By Exercise 3.1.6b (with $A = A$ and $X = A \cup B$), $A \cap (A \cup B) = A$. By Exercise 3.1.7, $A \cap B \subseteq A$. By Exercise 3.1.6b (with $A = A \cap B$ and $X = A$), $A \cup (A \cap B) = A$. $\quad\square$

9. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Show that $A = X \setminus B$ and $B = X \setminus A$.

Proof. To prove $A = X \setminus B$, Definition 3.1 tells us that it will suffice to show that every element x of A is an element of $X \setminus B$ and vice versa. Suppose first that $x \in A$. Then by Axiom 3.4, $x \in A \cup B$. Since $A \cup B = X$, we have by Definition 3.1 that $x \in X$. Now suppose for the sake of contradiction that $x \in B$. Since $x \in A$ as well, by Definition 3.3, $x \in A \cap B$. But $A \cap B = \emptyset$, so by Definition 3.1, $x \in \emptyset$, which contradicts Axiom 3.2. Therefore, $x \notin B$. Since $x \in X$ and $x \notin B$, by Definition 3.4, $x \in X \setminus B$. Now suppose that $x \in X \setminus B$. Then by Definition 3.4, $x \in X$ and $x \notin B$. By Definition 3.1, $x \in A \cup B$. Thus, by Axiom 3.4, $x \in A$ or $x \in B$. Since $x \notin B$, x must be an element of A. A similar argument shows that $B = X \setminus A$. $\qquad\square$

7/22:   10. Let A and B be sets. Show that the three sets $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint, and that their union is $A \cup B$.

Proof. To show that $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint, we must show that

$$(A \setminus B) \cap (A \cap B) = (A \setminus B) \cap (B \setminus A) = (A \cap B) \cap (B \setminus A) = \emptyset$$

We may do this by showing that no object x is an element of any of the left three sets above (because of the uniqueness of the empty set). We do this via three contradiction proofs, as follows.

Suppose for the sake of contradiction that $x \in (A \setminus B) \cap (A \cap B)$. Then by Definition 3.3, $x \in A \setminus B$ and $x \in A \cap B$. Since $x \in A \setminus B$, Definition 3.4 tells us that $x \notin B$ (and $x \in A$). But since $x \in A \cap B$, Definition 3.3 tells us that $x \in B$ (and $x \in A$), a contradiction.

A similar argument to the above can handle $(A \cap B) \cap (B \setminus A)$.

Suppose for the sake of contradiction that $x \in (A \setminus B) \cap (B \setminus A)$. Then by Definition 3.3, $x \in A \setminus B$ and $x \in B \setminus A$. By the first statement, $x \in A$ and $x \notin B$, while by the second statement, $x \in B$ and $x \notin A$, two contradictions.

We now turn our attention to proving the following.

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

We can actually prove this solely on the basis of prior results (and one additional lemma).

Lemma 3.6. *Let A and B be sets. Show that $(A \cup B) \setminus A = B \setminus A$.*

Proof. To prove $(A \cup B) \setminus A = B \setminus A$, Definition 3.1 tells us that it will suffice to show that every element x of $(A \cup B) \setminus A$ is an element of $B \setminus A$ and vice versa. Suppose first that $x \in (A \cup B) \setminus A$. Then by Definition 3.4, $x \in A \cup B$ and $x \notin A$. Since $x \in A \cup B$, by Axiom 3.4, $x \in A$ or $x \in B$. But $x \notin A$, so x must be an element of B. Having established that $x \in B$ and $x \notin A$, Definition 3.4 tells us that $x \in B \setminus A$. Now suppose that $x \in B \setminus A$. Then by Definition 3.4, $x \in B$ and $x \notin A$. Since $x \in B$, by Axiom 3.4, $x \in A \cup B$. Consequently, by Definition 3.4, $x \in (A \cup B) \setminus A$. $\square$

Now we can begin. By Exercise 3.1.7, $A \cap B \subseteq A$ and $A \subseteq A \cup B$. This implies by Proposition 3.1 that $A \cap B \subseteq A \cup B$. Thus,

$$\begin{aligned} A \cup B &= (A \cap B) \cup ((A \cup B) \setminus (A \cap B)) && \text{Exercise 3.1.6g} \\ &= (A \cap B) \cup ((A \cup B) \setminus A) \cup ((A \cup B) \setminus B) && \text{Exercise 3.1.6h} \\ &= (A \cap B) \cup (B \setminus A) \cup (A \setminus B) && \text{Lemma 3.6} \end{aligned}$$

$\square$

11. Show that the axiom of replacement implies the axiom of specification.

Proof. Let $P(x, y)$ be the statement "$y = x$ and $P(y)$ is true." Then by Axiom 3.6, there exists a set

$$\begin{aligned} \{y : P(x, y) \text{ is true for some } x \in A\} &= \{y : y = x \text{ and } P(y) \text{ is true for some } x \in A\} \\ &= \{y : y \in A \text{ and } P(y) \text{ is true}\} \\ &= \{y \in A : P(y) \text{ is true}\} \end{aligned}$$

Moreover,

$$\begin{aligned} z \in \{y \in A : P(y) \text{ is true}\} &\implies z \in \{y : P(x, y) \text{ is true for some } x \in A\} \\ &\implies P(x, z) \text{ is true for some } x \in A \\ &\implies z = x \text{ and } P(z) \text{ is true for some } x \in A \\ &\implies z \in A \text{ and } P(z) \text{ is true} \end{aligned}$$

The above logic also works in reverse. Thus, all the tenets of Axiom 3.5 have been shown to follow from Axiom 3.6 (proof modified from Jiang, 2020). $\square$

3.2   Russell's Paradox

- Suppose that we could unify the multitude of axioms in Section 3.1 into a single axiom. The following would be a good candidate (in fact, it implies the majority of the Section 3.1 axioms — see Exercise 3.2.1).

  Axiom 3.8 (Universal specification). *(Dangerous!) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x : P(x)$ is true$\}$ such that for every object y,*

  $$y \in \{x : P(x) \text{ is true}\} \iff P(y) \text{ is true}$$

- *Also known as* **axiom of comprehension**.

- Basically, Axiom 3.8 asserts that "every property corresponds to a set" (Tao, 2016, p. 46).

- Unfortunately, Axiom 3.8 cannot be introduced into set theory because it creates a logical contradiction known as **Russell's paradox**.

  - Discovered by philosopher and logician Bertrand Russell (1872-1970) in 1901.

- **Russell's paradox**: "Let $P(x)$ be the statement... "x is a set, and $x \notin x$"; i.e., $P(x)$ is true only when x is a set which does not contain itself. For instance, $P(\{2, 3, 4\})$ is true, since the set $\{2, 3, 4\}$ is not one of the three elements 2, 3, 4 of $\{2, 3, 4\}$. On the other hand, if we let S be the set of all sets (which we would know to exist from the axiom of universal specification), then since S is itself a set, it is an element of S, and so $P(S)$ is false. Now use the axiom of universal specification to create the set

  $$\Omega := \{x : P(x) \text{ is true}\} = \{x : x \text{ is a set and } x \notin x\}$$

  i.e., the set of all sets which do not contain themselves. Now ask the question: does Ω contain itself, i.e. is $\Omega \in \Omega$? If Ω did contain itself, then by definition this means that $P(\Omega)$ is true, i.e., Ω is a set and $\Omega \notin \Omega$. On the other hand, if Ω did not contain itself, then $P(\Omega)$ would be true, and hence $\Omega \in \Omega$. Thus in either case we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$, which is absurd" (Tao, 2016, pp. 46–47).

  - To clarify the last point: Is $\Omega \in \Omega$? Suppose $\Omega \in \Omega$. Then since Ω contains only sets for which $P(x)$ is true, $P(\Omega)$ must be true. But this implies, by the definition of $P(x)$, that $\Omega \notin \Omega$. Similarly, suppose $\Omega \notin \Omega$. Then since "Ω is a set and $\Omega \notin \Omega$" is a true statement, $P(\Omega)$ must be true. But this implies, since Ω contains all sets for which $P(x)$ is true, that $\Omega \in \Omega$. In either case, we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$ (contradictions).

Exercises

1. Show that the universal specification axiom, Axiom 3.8, if assumed to be true, would imply Axioms 3.2, 3.3, 3.4, 3.5, and 3.6. (If we assume that all natural numbers are objects, we also obtain Axiom 3.7.) Thus, this axiom, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as "naive set theory"). Unfortunately, as we have seen, Axiom 3.8 is "too good to be true!"

   Proof. Axiom 3.2: Let $P(x)$ be a false statement for all objects x. By Axiom 3.8, there exists a set $\{x : P(x)$ is true$\}$, which contains no elements. (Suppose for the sake of contradiction that $y \in \{x : P(x)$ is true$\}$ for some object y. Then $P(y)$ is true. But $P(y)$ is false by definition, a contradiction. Therefore, $y \notin \{x : P(x)$ is true$\}$ for all objects y.) Incidentally, that contradiction proof solidifies the symbolic statement of Axiom 3.2. Lastly, this set may be denoted $\emptyset$ or $\{\}$.

   Axiom 3.3: Let $P(x)$ be the statement $x = a$. By Axiom 3.8, there exists a set $\{x : P(x)$ is true$\} = \{x : x = a\} = \{a\}$ whose element is a. For every object y, we have $y \in \{a\}$ iff $P(y)$ is true, i.e., iff $y = a$. This set may be called the **singleton set** whose element is a. Now let $P(x)$ be the statement "$x = a$ or $x = b$." By Axiom 3.8, there exists a set $\{x : P(x)$ is true$\} = \{x : x = a$ or $x = b\} = \{a, b\}$

whose elements are a and b. For every object y, we have $y \in \{a, b\}$ iff $P(y)$ is true, i.e., iff $y = a$ or $y = b$. This set may be called the **pair set** formed by a and b.

Axiom 3.4: Let A, B be sets. Let $P(x)$ be the statement "$x \in A$ or $x \in B$." By Axiom 3.8, there exists a set $\{x : P(x) \text{ is true}\} = \{x : x \in A \text{ or } x \in B\}$. This set may be called the **union** $A \cup B$ of A and B. By Axiom 3.8, $y \in A \cup B$ iff $P(y)$ is true, i.e., iff $y \in A$ or $y \in B$. Thus, $A \cup B$ is clearly a set whose elements consist of all the elements which belong to A or B or both.

Axiom 3.6: Let A be a set. Let $P(y)$ be the statement "$P(x, y)$ is true for some $x \in A$," where $P(x, y)$ is a statement pertaining to x and y such that for each $x \in A$, there is at most one y for which $P(x, y)$ is true. By Axiom 3.8, there exists a set $\{y : P(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some } x \in A\}$. By Axiom 3.8, $z \in \{y : P(x, y) \text{ is true for some } x \in A\}$ iff $P(z)$ is true, i.e., iff $P(x, z)$ is true for some $x \in A$.

Axiom 3.5: Implied by the axiom of replacement (see Exercise 3.1.11). $\qquad\qquad\qquad\square$