Chapter 3

Set Theory

7/4: • Set theory will be frequently used in the subsequent chapters; it is part of the foundation of almost every other branch of mathematics.

  - Note that Euclidean geometry will not be defined — we will use the Cartesian coordinate system's parallel with the real numbers instead.

- This chapter covers the elementary aspects, chapter **??** covers more advanced topics, and the finer subtleties are well beyond the scope of this text.

3.1 Fundamentals

- We define sets axiomatically, as we did with the natural numbers[1].

  Axiom 3.1 (Sets are objects)**.** *If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.*

- Note that while all sets are objects, not all objects are sets.
  - For example, 1 is not a set while $\{1\}$ is.
  - Note, though, that **pure set theory** considers all objects to be sets. However, impure set theory (where some objects are not sets) is conceptually easier to deal with.
    - Since both types are equal for the purposes of mathematics, we will take a middle-ground approach.
- If x, y are objects and A a set, then the statement $x \in A$ is either true or false. Note that $x \in y$ is neither true nor false, simply meaningless.
- We now define equality for sets.

  Definition 3.1 (Equality of sets)**.** Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

- Note that this implies that repetition of elements does not effect equality ($\{3, 3\} = \{3\}$, for example).
- It can be proven that this notion of equality is reflexive, symmetric, and transitive (see Exercise 3.1.1).

[1] Note that the following list of axioms will be somewhat overcomplete, as some axioms may be derived from others. However, this is helpful for pedagogical reasons, and there is no real harm being done.

Exercises

1. Show that the definition of equality in Definition 3.1 is reflexive, symmetric, and transitive.

   Proof. Given a set A, suppose $A \neq A$. Then, by Definition 3.1, every element of A is not an element of A, a contradiction. Thus, $A = A$.

   Let sets $A = B$. Then, by Definition 3.1, every element x of A belongs also to B, and every element y of B belongs also to A. Identically, every element y of B belongs also to A, and every element X of A belongs also to B. Thus, $B = A$.

   Let sets $A = B$ and $B = C$. Then, by Definition 3.1, every element x of A belongs also to B, and every element y of B belongs also to A. Similarly, every element y of B belongs also to C, and every element z of C belongs also to B. Since $x \in A \Rightarrow x \in B \Rightarrow x \in C$, and $y \in C \Rightarrow y \in B \Rightarrow y \in A$, $A = C$. $\square$