Chapter 1

Starting at the Beginning: The Natural Numbers

6/15:

- This text will begin by reviewing high school level material, but as rigorously as possible.
  - It will teach the skill of proving complicated properties from simpler ones, allowing you to understand why an "obvious" statement really is obvious.
  - One particularly important skill is the use of **mathematical induction**.
  - We will strive to eliminate **circularity**.

  Circularity: "Using an advanced fact to prove a more elementary fact, and then later using the elementary fact to prove the advanced fact" (14).
- The number systems used in real analysis, listed in order of increasing sophistication, are the **naturals** $\mathbb{N}$[1], the **integers** $\mathbb{Z}$, the **rationals** $\mathbb{Q}$, and the **reals** $\mathbb{R}$.
  - **Complex numbers** $\mathbb{C}$ will only be used much later.
- This chapter will answer the question, "How does one actually *define* the natural numbers?"

1.1 The Peano Axioms

- **Peano Axioms**: First laid out by Guiseppe Peano, these are a standard way to define the natural numbers.
- How do you define operations on the naturals?
  - Complicated operations are defined in terms of simpler ones: Exponentiation is repeated multiplication, multiplication is repeated addition, and addition is repeated **incrementing**.
- **Incrementing**: The most fundamental operation — best thought of as counting forward by one number.
  - Incrementing is one of the fundamental concepts that allows us to define the natural numbers.
  - Let[2] $n{+}{+}$ denote the increment, or **successor**, of n.
    - For example, $3{+}{+} = 4$ and $(3{+}{+}){+}{+} = 5$.
- Let $x := y$ denote the statement, "x is defined to equal y."

[1] Note that in this text, the natural numbers will include 0. The natural numbers without 0 will be called the **positive integers** $\mathbb{Z}^+$.

[2] This notation is pulled from some computer languages such as C.

- At this point, we can begin defining the natural numbers.

  Axiom 1.1. *0 is a natural number.*

  Axiom 1.2. *If n is a natural number, then $n{+}{+}$ is also a natural number.*

- To avoid having to use incrementation notation for every number, we adopt a convention.

  Definition 1.1. We define 1 to be the number $0{+}{+}$, 2 to be the number $(0{+}{+}){+}{+}$, 3 to be the number $((0{+}{+}){+}{+}){+}{+}$, etc.

- From these axioms, we can already prove things.

  Proposition 1.1. *3 is a natural number.*

  Proof. By Axiom 1.1, 0 is a natural number. By Axiom 1.2, $0{+}{+} = 1$ is a natural number. By Axiom 1.2 again, $1{+}{+} = 2$ is a natural number. By Axiom 1.2 again, $2{+}{+} = 3$ is a natural number. □

- It seems like Axioms 1.1 and 1.2 have us pretty well covered. However, what if the number system wraps around (e.g., if $3{+}{+} = 0$)? We can fix this with the following.

  Axiom 1.3. *0 is not the successor of any natural number; i.e., we have $n{+}{+} \neq 0$ for every natural number n.*

- We can now prove that $4 \neq 0$ (because $4 = 3{+}{+}$, $3 \in \mathbb{N}$, and $n{+}{+} \neq 0$).

- However, there are still issues — what if the number system hits a ceiling at 4, e.g., $4{+}{+} = 4$?

- A good way to prevent this kind of behavior is via the following.

  Axiom 1.4. *Different natural numbers must have different successors, i.e., if $n, m \in \mathbb{N}$ and $n \neq m$, then $n{+}{+} \neq m{+}{+}$. Equivalently[3], if $n{+}{+} = m{+}{+}$, then $n = m$.*

- We can now prove propositions like the following, extending our anti-wrap around proving ability.

  Proposition 1.2. *6 is not equal to 2.*

  Proof. Suppose $6 = 2$. Then $5{+}{+} = 1{+}{+}$, so by Axiom 1.4, $5 = 1$. Then $4{+}{+} = 0{+}{+}$, so by Axiom 1.4, $4 = 0$, which contradicts our proof that $4 \neq 0$. □

[3]This is an example of reformulating an implication using its **contrapositive**. In the converse direction, it is the **axiom of substitution**.