

Chapter 3

Set Theory

- 7/4:
- Set theory will be frequently used in the subsequent chapters; it is part of the foundation of almost every other branch of mathematics.
 - Note that Euclidean geometry will not be defined — we will use the Cartesian coordinate system's parallel with the real numbers instead.
 - This chapter covers the elementary aspects, Chapter ?? covers more advanced topics, and the finer subtleties are well beyond the scope of this text.

3.1 Fundamentals

- We define sets axiomatically, as we did with the natural numbers^[1].

Axiom 3.1 (Sets are objects). *If A is a set, then A is also an object. In particular, given two sets A and B , it is meaningful to ask whether A is also an element of B .*

- Note that while all sets are objects, not all objects are sets.
 - For example, 1 is not a set while $\{1\}$ is.
 - Note, though, that **pure set theory** considers all objects to be sets. However, impure set theory (where some objects are not sets) is conceptually easier to deal with.
 - Since both types are equal for the purposes of mathematics, we will take a middle-ground approach.
- If x, y are objects and A a set, then the statement $x \in A$ is either true or false. Note that $x \in y$ is neither true nor false, simply meaningless.
- We now define equality for sets.

Definition 3.1 (Equality of sets). Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B , and every element y of B belongs also to A .

- Note that this implies that repetition of elements does not effect equality ($\{3, 3\} = \{3\}$, for example).
- It can be proven that this notion of equality is reflexive, symmetric, and transitive (see Exercise 3.1.1).

- 7/14:
- Since $x \in A$ and $A = B$ implies $x \in B$, the \in relation obeys the axiom of substitution as well.

¹Note that the following list of axioms will be somewhat overcomplete, as some axioms may be derived from others. However, this is helpful for pedagogical reasons, and there is no real harm being done.

- Thus, any operation defined in terms of the \in relation obeys the axiom of substitution.
- We define sets in an analogous way to how we defined natural numbers from 0, onward.

Axiom 3.2 (Empty set). *There exists a set \emptyset (also denoted $\{\}$), known as the empty set, which contains no elements, i.e., for every object x , we have $x \notin \emptyset$.*

- Note that there is only one empty set — if \emptyset and \emptyset' were supposedly distinct empty sets, then Definition 3.1 would prove that $\emptyset = \emptyset'$.

- **Non-empty set:** A set that “is not equal to the empty set” (Tao, 2016, p. 36).
- Non-empty sets must contain at least one object.

Lemma 3.1 (Single choice). *Let A be a non-empty set. Then there exists an object x such that $x \in A$.*

Proof. Suppose for the sake of contradiction that no object x exists such that $x \in A$, i.e., for all objects x , we have $x \notin A$. By Axiom 3.2, we have $x \notin \emptyset$ either. Thus, $x \in A \iff x \in \emptyset$ (denoting logical equivalence; both statements are equally false), so, by Definition 3.1, $A = \emptyset$, a contradiction. \square

- There exist more sets than just the empty set.

Axiom 3.3 (Singleton sets and pair sets). *If a is an object, then there exists a set $\{a\}$ whose only element is a , i.e., for every object y , we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the **singleton set** whose element is a . Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose elements are a and b ; i.e., for every object y , we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the **pair set** formed by a and b .*

- By Definition 3.1, there exists only one singleton set for each object a and only one pair set for any two objects a, b .
- Note that the singleton set axiom follows from the pair set axiom, and the pair set axiom follows from the singleton set axiom and the pairwise union axiom, below.
- As alluded to, the pairwise union axiom allows us to build sets with more than two elements.

Axiom 3.4 (Pairwise union). *Given any two sets A, B , there exists a set $A \cup B$ called the **union** $A \cup B$ of A and B , whose elements consist of all the elements which belong to A or B or both. In other words, for any object x ,*

$$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$

- The \cup operation obeys the axiom of substitution (if $A = A'$, then $A \cup B = A' \cup B$).
- We now prove some basic properties of unions (one below and three in Exercise 3.1.3).

Lemma 3.2. *If A, B, C are sets, then the union operation is associative, i.e., $(A \cup B) \cup C = A \cup (B \cup C)$.*

Proof. By Definition 3.1, showing that every element x of $(A \cup B) \cup C$ is an element of $A \cup (B \cup C)$ and vice versa will suffice to prove this lemma. Suppose first that $x \in (A \cup B) \cup C$. By Axiom 3.4, this means that at least one of $x \in A \cup B$ or $x \in C$ is true. We now divide into two cases. If $x \in C$, then by Axiom 3.4, $x \in B \cup C$, and, so, by Axiom 3.4 again, we have $x \in A \cup (B \cup C)$. Now suppose instead that $x \in A \cup B$. Then by Axiom 3.4, $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cup C)$ by Axiom 3.4, while if $x \in B$, then by consecutive applications of Axiom 3.4, we have $x \in B \cup C$ and, hence, $x \in A \cup (B \cup C)$. A similar argument shows that every element of $A \cup (B \cup C)$ lies in $(A \cup B) \cup C$, and so $(A \cup B) \cup C = A \cup (B \cup C)$, as desired. \square

- As a consequence of the above, we are free to write $A \cup B \cup C \cup \dots$ to denote repeated unions without having to use parentheses.

- We can also now define triplet sets ($\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\}$), quadruplet sets, and so forth.
 - However, we cannot yet define a set of n objects or an infinite set.
- Note that addition and union are analogous, but importantly *not* identical.
- Some sets are “larger” than others; hence, subsets.

Definition 3.2 (Subsets). Let A, B be sets. We say that A is a **subset** of B , denoted $A \subseteq B$, iff every element of A is also an element of B , i.e., for any object x , $x \in A \implies x \in B$. We say that A is a **proper subset** of B , denoted $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$.

- The \subseteq and \subsetneq operations obey the axiom of substitution (since both $=$ and \in , the two component operations of \subseteq and \subsetneq , obey it).
- Note that $A \subseteq A$ and $\emptyset \subseteq A$ for any set A .
- Note that less than or equal to and subset are analogous, but not identical, either (see below for one related property and Exercise 3.1.4 for two more).

Proposition 3.1 (Sets are partially ordered by set inclusion 1). *Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. Suppose that $A \subseteq B$ and $B \subseteq C$. To prove that $A \subseteq C$, we have to prove that every element of A is an element of C . Let $x \in A$. Then $x \in B$ (since $A \subseteq B$), implying that $x \in C$ (since $B \subseteq C$). \square

- There exist relations between subsets and unions (see Exercise 3.1.7).
- Note this difference between \subsetneq and $<$: Given any two distinct natural numbers n, m , one is smaller than the other (Proposition 2.7). However, given any two distinct sets, it is not in general true that one is a subset of the other. This is why we say that sets are **partially ordered** while the natural numbers (for example) are **totally ordered** (see Definitions ?? and ??, respectively).
- Note that \in and \subseteq are distinct ($2 \in \{1, 2, 3\}$, but $2 \not\subseteq \{1, 2, 3\}$; similarly, $\{2\} \subseteq \{1, 2, 3\}$, but $\{2\} \notin \{1, 2, 3\}$).
- It is important to distinguish sets from their elements, for they can have different properties (\mathbb{N} is an infinite set of finite elements, and $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ is a finite set of infinite objects).
- We now formally state that it is acceptable to create subsets.

Axiom 3.5 (Axiom of specification). *Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$ for short), whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y ,*

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true})$$

- Also known as **axiom of separation**.
- Specification obeys the axiom of substitution (since \in obeys it).
- Sometimes $\{x \in A : P(x)\}$ is denoted by $\{x \in A \mid P(x)\}$ (useful when we need the colon for something else, e.g., $f : X \rightarrow Y$).
- We use Axiom 3.5 to define intersections.

Definition 3.3 (Intersections). The **intersection** $S_1 \cap S_2$ of two sets is defined to be the set

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}$$

In other words, $S_1 \cap S_2$ consists of all the elements which belong to both S_1 and S_2 . Thus, for all objects x ,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2$$

- The \cap operation obeys the axiom of substitution (since \in obeys it).
 - Note that since \cap is defined in terms of more primitive operations, it is well-defined.
- Problems with the English word, “and.”
 - It can mean union or intersection depending on the context.
 - If X, Y are sets, “the set of elements of X and elements of Y ” refers to $X \cup Y$, e.g., “the set of singles and males.”
 - If X, Y are sets, “the set of objects that are elements of X and elements of Y ” refers to $X \cap Y$, e.g., “the set of people who are single and male.”
 - It can also denote addition.
 - “2 and 3 is 5” means $2 + 3 = 5$.
 - “One reason we resort to mathematical symbols instead of English words such as ‘and’ is that mathematical symbols always have a precise and unambiguous meaning, whereas one must often look very carefully at the context in order to work out what an English word means” (Tao, 2016, p. 42).
- **Disjoint** (sets): Two sets A, B such that $A \cap B = \emptyset$.
- **Distinct** (sets): Two sets A, B such that $A \neq B$.
 - Note that \emptyset and \emptyset are disjoint but not distinct.
- We also use Axiom 3.5 to define difference sets.

Definition 3.4 (Difference sets). Given two sets A and B , we define the set $A - B$ or $A \setminus B$ to be the set A with any elements of B removed, i.e.,

$$A \setminus B := \{x \in A : x \notin B\}$$

- For example, $\{1, 2, 3, 4\} \setminus \{2, 4, 6\} = \{1, 3\}$ — in many cases, $B \subseteq A$, but not necessarily.
- 7/15: • See Exercise 3.1.6 for some basic properties of unions, intersections, and difference sets.
 - The de Morgan laws are named after the logician Augustus De Morgan (1806-1871).
 - The **laws of Boolean algebra** (the contents of Proposition 3.4) are named after the mathematician George Boole (1815-1864).
 - They are applicable to a number of objects other than sets (e.g., laws of propositional logic).
 - Note the **duality** in Proposition 3.4, manifesting itself in the certain symmetry between \cup and \cap , and X and \emptyset .
- **Duality**: “Two distinct properties or objects being dual to each other” (Tao, 2016, p. 43).
- We can do a lot, but we still wish to do more. For example, we’d like to transform sets (say, take $\{3, 5, 9\}$ and increment each object to yield $\{4, 6, 10\}$).

Axiom 3.6 (Replacement). *Let A be a set. For any object $x \in A$, and any object y , suppose we have a statement $P(x, y)$ pertaining to x and y such that for each $x \in A$, there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z ,*

$$z \in \{y : P(x, y) \text{ is true for some } x \in A\} \iff P(x, z) \text{ is true for some } x \in A$$

- For example, let $A := \{3, 5, 9\}$ and let $P(x, y)$ be the statement $y = x++$. By Axiom 2.4, for every $x \in A$, there is exactly one y for which $P(x, y)$ is true (namely, the successor of x). Thus, by Axiom 3.6, the set $\{y : y = x++ \text{ for some } x \in \{3, 5, 9\}\}$ exists. It is clearly the same set as $\{4, 6, 10\}$.
- The set obtained can be smaller than the original set, e.g., $\{y : y = 1 \text{ for some } x \in \{3, 5, 9\}\} = \{1\}$.
- We often abbreviate the set specified in Axiom 3.6 to one of the following.

$$\begin{aligned} &\{y : y = f(x) \text{ for some } x \in A\} \\ &\{f(x) : x \in A\} \\ &\{f(x) \mid x \in A\} \end{aligned}$$

- We can combine Axioms 3.6 and 3.5, i.e., $\{f(x) : x \in A; P(x) \text{ is true}\}$, e.g., $\{n++ : n \in \{3, 5, 9\}; n < 6\} = \{4, 6\}$.
- Although we have assumed that natural numbers are objects in several examples up to this point, we must formalize this notion.

Axiom 3.7 (Infinity). *There exists a set \mathbb{N} , whose elements are called the natural numbers, as well as an object 0 in \mathbb{N} , and an object $n++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.*

- Axiom 3.7 is called the **axiom of infinity** because “it introduces the most basic example of an infinite set, namely the set of natural numbers \mathbb{N} ” (Tao, 2016, p. 44).

Exercises

- 7/4: 1. Show that the definition of equality in Definition 3.1 is reflexive, symmetric, and transitive^[2].

Proof. Given a set A , suppose $A \neq A$. Then, by Definition 3.1, every element of A is not an element of A , a contradiction. Thus, $A = A$.

Let sets $A = B$. Then, by Definition 3.1, every element x of A belongs also to B , and every element y of B belongs also to A . Identically, every element y of B belongs also to A , and every element x of A belongs also to B . Thus, $B = A$.

Let sets $A = B$ and $B = C$. Then, by Definition 3.1, every element x of A belongs also to B , and every element y of B belongs also to A . Similarly, every element y of B belongs also to C , and every element z of C belongs also to B . Since $x \in A \Rightarrow x \in B \Rightarrow x \in C$, and $y \in C \Rightarrow y \in B \Rightarrow y \in A$, $A = C$. \square

- 7/14: 2. Using only Definition 3.1, Axiom 3.1, Axiom 3.2, and Axiom 3.3, prove that the sets \emptyset , $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct (i.e., no two of them are equal to each other).

Proof. First, we show that all sets are distinct from the empty set. Axiom 3.1 asserts that \emptyset and $\{\emptyset\}$ are objects. By Axiom 3.3, we have $\emptyset \in \{\emptyset\}$, $\emptyset \in \{\emptyset, \{\emptyset\}\}$, and $\{\emptyset\} \in \{\{\emptyset\}\}$. Since $x \notin \emptyset$ for all objects x (Axiom 3.2), $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ all contain objects that \emptyset does not (namely, \emptyset , $\{\emptyset\}$, and \emptyset , respectively). Thus, by Definition 3.1, $\emptyset \neq \{\emptyset\}$, $\emptyset \neq \{\{\emptyset\}\}$, and $\emptyset \neq \{\emptyset, \{\emptyset\}\}$.

²Note that since Definition 3.1 should be an axiom (should axiomatize equality and all that that entails for sets), this exercise is silly (see Tao, 2020).

Next, we show that $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$. By Axiom 3.3, $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ and $y \in \{\emptyset\}$ iff $y = \emptyset$. Since $\{\emptyset\} \neq \emptyset$ (see above), $\{\emptyset\} \notin \{\emptyset\}$. Thus, $\{\emptyset, \{\emptyset\}\}$ contains an object that $\{\emptyset\}$ does not, implying by Definition 3.1 that $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$.

Lastly, we show that $\{\emptyset\} \neq \{\{\emptyset\}\}$ and that $\{\emptyset, \{\emptyset\}\} \neq \{\{\emptyset\}\}$. We proceed in a similar manner to the above. By Axiom 3.3, $\emptyset \in \{\emptyset\}$, $\emptyset \in \{\emptyset, \{\emptyset\}\}$, and $y \in \{\{\emptyset\}\}$ iff $y = \{\emptyset\}$. Since $\emptyset \neq \{\emptyset\}$ (see above), $\emptyset \notin \{\{\emptyset\}\}$. Thus, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ both contain an object that $\{\{\emptyset\}\}$ does not, implying by Definition 3.1 that $\{\emptyset\} \neq \{\{\emptyset\}\}$ and that $\{\emptyset, \{\emptyset\}\} \neq \{\{\emptyset\}\}$. □

3. Prove the following lemmas.

Lemma 3.3. *If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$.*

Proof. By Definition 3.1, it will suffice to show that every element x of $\{a, b\}$ is an element of $\{a\} \cup \{b\}$ and vice versa. By Axiom 3.3, if $x \in \{a, b\}$, then $x = a$ or $x = b$. By Axiom 3.4, if $x \in \{a\} \cup \{b\}$, then $x \in \{a\}$ or $x \in \{b\}$, implying by Axiom 3.3 that $x = a$ or $x = b$. Thus, the elements of $\{a, b\}$ and of $\{a\} \cup \{b\}$ are both a, b , so by Definition 3.1, the sets are equal. □

Lemma 3.4. *If A, B, C are sets, then the union operation is commutative, i.e., $A \cup B = B \cup A$.*

Proof. By Definition 3.1, it will suffice to show that every element x of $A \cup B$ is an element of $B \cup A$ and vice versa. Let $x \in A \cup B$. By Axiom 3.4, $x \in A$ or $x \in B$. If, on the one hand, $x \in A$, then $x \in B \cup A$ (by Axiom 3.4). If, on the other hand, $x \in B$, then $x \in B \cup A$ (by Axiom 3.4). A similar argument holds if we choose an element $y \in B \cup A$ first. □

Lemma 3.5. *If A is a set, then $A = A \cup \emptyset = \emptyset \cup A = A \cup A$.*

Proof. First, we show that $A \cup \emptyset = \emptyset \cup A$. This is a direct consequence of Lemma 3.3.

Next, we show that $A = A \cup \emptyset$. By Definition 3.1, it will suffice to show that every element x of A is an element of $A \cup \emptyset$ and vice versa. By Axiom 3.4, every element x of A is an element of $A \cup \emptyset$. Now let $x \in A \cup \emptyset$. Then by Axiom 3.4, $x \in A$ or $x \in \emptyset$. By Axiom 3.2, $x \notin \emptyset$, so $x \in A$. Thus, every element of $A \cup \emptyset$ is an element of A . We now have by the transitive property that $A = A \cup \emptyset = \emptyset \cup A$.

Lastly, we show that $A = A \cup A$. By Definition 3.1, it will suffice to show that every element x of A is an element of $A \cup A$ and vice versa. By Axiom 3.4, every element x of A is an element of $A \cup A$. Now let $x \in A \cup A$. Then by Axiom 3.4, $x \in A$ or $x \in A$, implying $x \in A$. We have, at last, by the transitive property that $A = A \cup \emptyset = \emptyset \cup A = A \cup A$. □

4. Prove the following propositions.

Proposition 3.2 (Sets are partially ordered by set inclusion 2). *Let A, B be sets. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.*

Proof. Suppose that $A \subseteq B$ and $B \subseteq A$. By Definition 3.2, $A \subseteq B$ implies that every element of A is also an element of B and $B \subseteq A$ implies that every element of B is also an element of A . Thus, by Definition 3.1, $A = B$. □

Proposition 3.3 (Sets are partially ordered by set inclusion 3). *Let A, B, C be sets. If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$.*

Proof. Suppose that $A \subsetneq B$ and $B \subsetneq C$. Then, by Definition 3.2, $A \subseteq B$ and $B \subseteq C$, implying $A \subseteq C$ by the first claim proved. Since $A \subsetneq B$, $A \neq B$ (implying, by Definition 3.1, that every element of A is not an element of B or every element of B is not an element of A) and every element of A is an element of B ; hence, every element of B is not an element of A . Therefore, B must contain some element that A does not. Similarly, $B \subsetneq C$ implies that C must contain some element that B does not. Hence, C contains at least two elements that A does not, proving that $A \neq C$, too. □

5. Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$, and $A \cap B = A$ are logically equivalent (any one of them implies the other two).

Proof. First, we show that $A \subseteq B \implies (A \cup B = B \text{ and } A \cap B = A)$. Next, we show that $A \cup B = B \implies A \subseteq B$ (which, in turn, implies $A \cap B = A$). Lastly, we show that $A \cap B = A \implies A \subseteq B$ (which, in turn, implies $A \cup B = B$). Let's begin.

Suppose that $A \subseteq B$. To prove $A \cup B = B$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cup B$ is an element of B and vice versa. By Axiom 3.4, $x \in B \implies x \in A \cup B$. By Axiom 3.4, $x \in A \cup B \implies (x \in A \text{ or } x \in B)$. By Definition 3.2, $A \subseteq B$ means that $x \in A \implies x \in B$. Thus, $x \in A \cup B \implies (x \in A \text{ or } x \in B) \implies x \in B$. Therefore, if $A \subseteq B$, then $A \cup B = B$. To prove that $A \cap B = A$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap B$ is an element of A and vice versa. By Definition 3.3, $x \in A \cap B \implies x \in A$ (and $x \in B$). Since $x \in A \implies x \in B$ (see above), $x \in A \implies (x \in A \text{ and } x \in B) \implies x \in A \cap B$ (Definition 3.3). Therefore, if $A \subseteq B$, then $A \cap B = A$.

Suppose that $A \cup B = B$. To prove $A \subseteq B$, Definition 3.2 tells us that it will suffice to show that every element of A is also an element of B . By Axiom 3.4, $x \in A \implies x \in A \cup B$. By Definition 3.1, $y \in A \cup B \implies y \in B$. Thus, $x \in A \implies x \in A \cup B \implies x \in B$. Therefore, if $A \cup B = B$, $A \subseteq B$ (and $A \cap B = A$).

Suppose that $A \cap B = A$. To prove that $A \subseteq B$, Definition 3.2 tells us that it will suffice to show that every element of A is also an element of B . By Definition 3.1, $x \in A \implies x \in A \cap B$. By Definition 3.3, $y \in A \cap B \implies y \in B$. Thus, $x \in A \implies x \in A \cap B \implies x \in B$. Therefore, if $A \cap B = A$, $A \subseteq B$ (and $A \cup B = B$). \square

- 7/15: 6. Prove the following proposition. (Hint: one can use some of these claims to prove others. Some of the claims have also appeared previously in Lemma 3.2 and Exercise 3.1.3.)

Proposition 3.4 (Sets form a boolean algebra). *Let A, B, C be sets, and let X be a set containing A, B, C as subsets.*

- (a) (Minimal element) *We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*

Proof. See Exercise 3.1.3 for the first claim.

To prove $A \cap \emptyset = \emptyset$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap \emptyset$ is an element of \emptyset and vice versa. First off, “every element $x \in \emptyset$ is an element of $A \cap \emptyset$ ” is vacuously true (by Axiom 3.2, there exists no $x \in \emptyset$). In the other direction, suppose for the sake of contradiction that $x \in A \cap \emptyset$ for some object x . By Definition 3.3, $x \in A$ and $x \in \emptyset$. But $x \in \emptyset$ contradicts Axiom 3.2. Therefore, $x \notin A \cap \emptyset$ for all objects x . Thus, the statement “every element $x \in A \cap \emptyset$ is an element of \emptyset ” is vacuously true (there exists no $x \in A \cap \emptyset$). \square

- (b) (Maximal element) *We have $A \cup X = X$ and $A \cap X = A$.*

Proof. See Exercise 3.1.5. \square

- (c) (Identity) *We have $A \cap A = A$ and $A \cup A = A$.*

Proof. To prove that $A \cap A = A$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap A$ is an element of A and vice versa. By Definition 3.3, $x \in A \cap A \implies (x \in A \text{ and } x \in A) \implies x \in A$. On the other hand, $x \in A \implies (x \in A \text{ and } x \in A)$ (idempotent law for conjunction), $(x \in A \text{ and } x \in A) \implies x \in \{x \in A : x \in A\}$ (Axiom 3.5), and $x \in \{x \in A : x \in A\} \implies x \in A \cap A$ (Definition 3.3).

See Exercise 3.1.3 for the second claim. \square

- (d) (Commutativity) *We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*

Proof. See Exercise 3.1.3 for the first claim.

To prove $A \cap B = B \cap A$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap B$ is an element of $B \cap A$ and vice versa. By two applications of Definition 3.3 with the commutative law for conjunction in between, $x \in A \cap B \implies (x \in A \text{ and } x \in B) \implies (x \in B \text{ and } x \in A) \implies x \in B \cap A$. A similar argument works in the opposite direction. \square

- (e) (*Associativity*) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

Proof. See Lemma 3.2 for the first claim.

By Definition 3.1, showing that every element x of $(A \cap B) \cap C$ is an element of $A \cap (B \cap C)$ and vice versa will suffice to prove this lemma. Suppose first that $x \in (A \cap B) \cap C$. By Definition 3.3, $x \in A \cap B$ and $x \in C$, which implies by a second application of Definition 3.3 that $x \in A$ and $x \in B$ and $x \in C$. It follows by consecutive applications of Definition 3.3 that $x \in A$ and $x \in B \cap C$, and that $x \in A \cap (B \cap C)$. A similar argument shows that every element of $A \cap (B \cap C)$ lies in $(A \cap B) \cap C$. \square

- (f) (*Distributivity*) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$ and vice versa. By Definition 3.3, $x \in A \cap (B \cup C) \implies (x \in A \text{ and } x \in B \cup C)$. By Axiom 3.4, $x \in B \cup C \implies x \in B$ or $x \in C$. Thus, we know that $x \in A$, and we know that $x \in B$ or $x \in C$ (or both). We divide into two cases. Suppose first that $x \in B$. Since $x \in A$ as well, $x \in A \cap B$ (Definition 3.3). This implies by Axiom 3.4 that $x \in (A \cap B) \cup (A \cap C)$. Now suppose that $x \in C$. Since $x \in A$ as well, $x \in A \cap C$ (Definition 3.3). This implies by Axiom 3.4 that $x \in (A \cap B) \cup (A \cap C)$. A similar argument shows that every element of $(A \cap B) \cup (A \cap C)$ lies in $A \cap (B \cup C)$.

The second proof is similar to the first (and similar to all the rest of the proofs written for this exercise thus far). Thus, for the sake of variety, we will do this one entirely symbolically, using the laws of propositional logic from Section A.4.

$$\begin{aligned}
 x \in A \cup (B \cap C) &\implies (x \in A) \vee (x \in B \cap C) && \text{Axiom 3.4} \\
 &\implies (x \in A) \vee ((x \in B) \wedge (x \in C)) && \text{Definition 3.3} \\
 &\implies ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) && \text{Distributive law for disjunction} \\
 &\implies (x \in A \cap B) \vee (x \in A \cap C) && \text{Definition 3.3} \\
 &\implies x \in (A \cap B) \cup (A \cap C) && \text{Axiom 3.4}
 \end{aligned}$$

A similar argument works in reverse. \square

- (g) (*Partition*) We have $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

Proof. To prove $A \cup (X \setminus A) = X$, Definition 3.1 tells us that it will suffice to show that every element x of $A \cup (X \setminus A)$ is an element of X and vice versa. Suppose $x \in A \cup (X \setminus A)$. Then by Axiom 3.4, $x \in A$ or $x \in X \setminus A$. We divide into two cases. Suppose first that $x \in A$. Since $A \subseteq X$, Definition 3.2 asserts that $x \in X$. Now suppose that $x \in X \setminus A$. This implies by Definition 3.4 that $x \in \{y \in X : y \notin A\}$, which means by Axiom 3.5 that $x \in X$ (and $x \notin A$, but that's not relevant). On the other hand, suppose $x \in X$. Naturally, either $x \in A$ or $x \notin A$ ($x \in A$ is false). If $x \in A$, then $x \in A \cup (X \setminus A)$ (Axiom 3.4). If $x \notin A$, since $x \in X$ as well, Axiom 3.5 asserts that $x \in \{x \in X : x \notin A\}$, which, by Definition 3.4, implies that $x \in X \setminus A$. This, in turn, implies that $x \in A \cup (X \setminus A)$ (Axiom 3.4).

To prove $A \cap (X \setminus A) = \emptyset$, it suffices to prove that for every object x , we have $x \notin A \cap (X \setminus A)$ (because of the uniqueness of the empty set). Suppose for the sake of contradiction that $x \in A \cap (X \setminus A)$ for some object x . By Definition 3.3, $x \in A$ and $x \in X \setminus A$. By Definition 3.4, $x \in \{y \in X : y \notin A\}$. By Axiom 3.5, $x \in X$ and $x \notin A$. But this contradicts the previously derived fact that $x \in A$. Therefore, $x \notin A \cap (X \setminus A)$ for all objects x . \square

- (h) (*De Morgan laws*) We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Proof. To prove $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$, Definition 3.1 tells us that it will suffice to show that every element x of $X \setminus (A \cup B)$ is an element of $(X \setminus A) \cap (X \setminus B)$ and vice versa. Suppose that $x \in X \setminus (A \cup B)$. Then by Definition 3.4, $x \in \{y \in X : y \notin A \cup B\}$. By Axiom 3.5, $x \in X$ and $x \notin A \cup B$. By the inverse of Axiom 3.4 (which is a valid assertion since Axiom 3.4 asserts the logical equivalence of “ $x \in A \cup B$ ” and “ $x \in A$ and $x \in B$ ”), $x \notin A \cup B \implies (x \notin A \text{ and } x \notin B)$. By Axiom 3.5 and Definition 3.4, $(x \in X \text{ and } x \notin A) \implies x \in \{y \in X : y \notin A\} \implies x \in X \setminus A$. Similarly, $(x \in X \text{ and } x \notin B) \implies x \in \{y \in X : y \notin B\} \implies x \in X \setminus B$. Thus, by Definition 3.3, $x \in (X \setminus A) \cap (X \setminus B)$. A similar, reversed argument will work in the other direction.

To prove $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$, Definition 3.1 tells us that it will suffice to show that every element x of $X \setminus (A \cap B)$ is an element of $(X \setminus A) \cup (X \setminus B)$ and vice versa. By Definition 3.4 and Axiom 3.5, $x \in X \setminus (A \cap B) \implies x \in \{y \in X : y \notin A \cap B\} \implies (x \in X \text{ and } x \notin A \cap B)$. By the inverse of Definition 3.3 (which, again, is a valid assertion since $x \in A \cap B \iff ((x \in A) \wedge (x \in B))$), $x \notin A \cap B \implies (x \notin A \text{ or } x \notin B)$. We divide into two cases. Suppose $x \notin A$. Then by Axiom 3.5 and Definition 3.4, $(x \in X \text{ and } x \notin A) \implies x \in \{y \in X : y \notin A\} \implies x \in X \setminus A$. Thus, by Axiom 3.4, $x \in (X \setminus A) \cup (X \setminus B)$. Now suppose $x \notin B$. Then by Axiom 3.5 and Definition 3.4, $(x \in X \text{ and } x \notin B) \implies x \in \{y \in X : y \notin B\} \implies x \in X \setminus B$. Thus, by Axiom 3.4, $x \in (X \setminus A) \cup (X \setminus B)$. \square

- 7/17: 7. Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ iff $C \subseteq A \cap B$. In a similar spirit, show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and furthermore that $A \subseteq C$ and $B \subseteq C$ iff $A \cup B \subseteq C$.

Proof. To prove that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, Definition 3.2 tells us that it will suffice to show that every element of $A \cap B$ is an element of A and B . By Definition 3.3, $x \in A \cap B \implies (x \in A \text{ and } x \in B)$.

- 7/19: To prove that $C \subseteq A$ and $C \subseteq B$ iff $C \subseteq A \cap B$, it will suffice to show that $C \subseteq A$ and $C \subseteq B$ imply $C \subseteq A \cap B$ and vice versa. Suppose first that $C \subseteq A$ and $C \subseteq B$. Then by two applications of Definition 3.2, $x \in C \implies x \in A$ and $x \in C \implies x \in B$. By Definition 3.3, $(x \in A \text{ and } x \in B) \implies x \in A \cap B$. Thus, every element x of C is an element of $A \cap B$, so by Definition 3.2, $C \subseteq A \cap B$. Now suppose that $C \subseteq A \cap B$. Then by Definition 3.2, $x \in C \implies x \in A \cap B$. By Definition 3.3, $x \in A \cap B \implies (x \in A \text{ and } x \in B)$. Thus, every element x of C is an element of A and B , so by two applications of Definition 3.2, $C \subseteq A$ and $C \subseteq B$.

To prove that $A \subseteq A \cup B$ and that $B \subseteq A \cup B$, Definition 3.2 tells us that it will suffice to show that every element x of A is an element of $A \cup B$ and that every element y of B is an element of $A \cup B$, respectively. By two applications of Axiom 3.4, $x \in A \implies x \in A \cup B$, and $y \in B \implies y \in A \cup B$.

To prove that $A \subseteq C$ and $B \subseteq C$ iff $A \cup B \subseteq C$, it will suffice to show that $A \subseteq C$ and $B \subseteq C$ imply $A \cup B \subseteq C$ and vice versa. Suppose first that $A \subseteq C$ and $B \subseteq C$. By Axiom 3.4, $x \in A \cup B \implies (x \in A \text{ or } x \in B)$. We divide into two cases. If $x \in A$, then Definition 3.2 ensures that $x \in C$. If $x \in B$, then Definition 3.2 similarly ensures that $x \in C$. Thus, either way, $x \in A \cup B \implies x \in C$, so by Definition 3.2, $A \cup B \subseteq C$. Now suppose that $A \cup B \subseteq C$. By Axiom 3.4 followed by Definition 3.2, $x \in A \implies x \in A \cup B \implies x \in C$. A similar argument shows that $x \in B \implies x \in C$. Thus, $A \subseteq C$ and $B \subseteq C$. \square

8. Let A, B be sets. Prove the **absorption laws** $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$.

Proof. By Exercise 3.1.7, $A \subseteq A \cup B$. By Exercise 3.1.6b (with $A = A$ and $X = A \cup B$), $A \cap (A \cup B) = A$. By Exercise 3.1.7, $A \cap B \subseteq A$. By Exercise 3.1.6b (with $A = A \cap B$ and $X = A$), $A \cup (A \cap B) = A$. \square

9. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Show that $A = X \setminus B$ and $B = X \setminus A$.

Proof. To prove $A = X \setminus B$, Definition 3.1 tells us that it will suffice to show that every element x of A is an element of $X \setminus B$ and vice versa. Suppose first that $x \in A$. Then by Axiom 3.4, $x \in A \cup B$. Since $A \cup B = X$, we have by Definition 3.1 that $x \in X$. Now suppose for the sake of contradiction

that $x \in B$. Since $x \in A$ as well, by Definition 3.3, $x \in A \cap B$. But $A \cap B = \emptyset$, so by Definition 3.1, $x \in \emptyset$, which contradicts Axiom 3.2. Therefore, $x \notin B$. Since $x \in X$ and $x \notin B$, by Definition 3.4, $x \in X \setminus B$. Now suppose that $x \in X \setminus B$. Then by Definition 3.4, $x \in X$ and $x \notin B$. By Definition 3.1, $x \in A \cup B$. Thus, by Axiom 3.4, $x \in A$ or $x \in B$. Since $x \notin B$, x must be an element of A . A similar argument shows that $B = X \setminus A$. \square

- 7/22: 10. Let A and B be sets. Show that the three sets $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint, and that their union is $A \cup B$.

Proof. To show that $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint, we must show that

$$(A \setminus B) \cap (A \cap B) = (A \setminus B) \cap (B \setminus A) = (A \cap B) \cap (B \setminus A) = \emptyset$$

We may do this by showing that no object x is an element of any of the left three sets above (because of the uniqueness of the empty set). We do this via three contradiction proofs, as follows.

Suppose for the sake of contradiction that $x \in (A \setminus B) \cap (A \cap B)$. Then by Definition 3.3, $x \in A \setminus B$ and $x \in A \cap B$. Since $x \in A \setminus B$, Definition 3.4 tells us that $x \notin B$ (and $x \in A$). But since $x \in A \cap B$, Definition 3.3 tells us that $x \in B$ (and $x \in A$), a contradiction.

A similar argument to the above can handle $(A \cap B) \cap (B \setminus A)$.

Suppose for the sake of contradiction that $x \in (A \setminus B) \cap (B \setminus A)$. Then by Definition 3.3, $x \in A \setminus B$ and $x \in B \setminus A$. By the first statement, $x \in A$ and $x \notin B$, while by the second statement, $x \in B$ and $x \notin A$, two contradictions.

We now turn our attention to proving the following.

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

We can actually prove this solely on the basis of prior results (and one additional lemma).

Lemma 3.6. *Let A and B be sets. Show that $(A \cup B) \setminus A = B \setminus A$.*

Proof. To prove $(A \cup B) \setminus A = B \setminus A$, Definition 3.1 tells us that it will suffice to show that every element x of $(A \cup B) \setminus A$ is an element of $B \setminus A$ and vice versa. Suppose first that $x \in (A \cup B) \setminus A$. Then by Definition 3.4, $x \in A \cup B$ and $x \notin A$. Since $x \in A \cup B$, by Axiom 3.4, $x \in A$ or $x \in B$. But $x \notin A$, so x must be an element of B . Having established that $x \in B$ and $x \notin A$, Definition 3.4 tells us that $x \in B \setminus A$. Now suppose that $x \in B \setminus A$. Then by Definition 3.4, $x \in B$ and $x \notin A$. Since $x \in B$, by Axiom 3.4, $x \in A \cup B$. Consequently, by Definition 3.4, $x \in (A \cup B) \setminus A$. \square

Now we can begin. By Exercise 3.1.7, $A \cap B \subseteq A$ and $A \subseteq A \cup B$. This implies by Proposition 3.1 that $A \cap B \subseteq A \cup B$. Thus,

$$\begin{aligned} A \cup B &= (A \cap B) \cup ((A \cup B) \setminus (A \cap B)) && \text{Exercise 3.1.6g} \\ &= (A \cap B) \cup ((A \cup B) \setminus A) \cup ((A \cup B) \setminus B) && \text{Exercise 3.1.6h} \\ &= (A \cap B) \cup (B \setminus A) \cup (A \setminus B) && \text{Lemma 3.6} \end{aligned}$$

\square

11. Show that the axiom of replacement implies the axiom of specification.

Proof. Let $P(x, y)$ be the statement “ $y = x$ and $P(y)$ is true.” Then by Axiom 3.6, there exists a set

$$\begin{aligned} \{y : P(x, y) \text{ is true for some } x \in A\} &= \{y : y = x \text{ and } P(y) \text{ is true for some } x \in A\} \\ &= \{y : y \in A \text{ and } P(y) \text{ is true}\} \\ &= \{y \in A : P(y) \text{ is true}\} \end{aligned}$$

Moreover,

$$\begin{aligned}
 z \in \{y \in A : P(y) \text{ is true}\} &\implies z \in \{y : P(x, y) \text{ is true for some } x \in A\} \\
 &\implies P(x, z) \text{ is true for some } x \in A \\
 &\implies z = x \text{ and } P(z) \text{ is true for some } x \in A \\
 &\implies z \in A \text{ and } P(z) \text{ is true}
 \end{aligned}$$

The above logic also works in reverse. Thus, all the tenets of Axiom 3.5 have been shown to follow from Axiom 3.6 (proof modified from Jiang, 2020). \square

3.2 Russell's Paradox

- Suppose that we could unify the multitude of axioms in Section 3.1 into a single axiom. The following would be a good candidate (in fact, it implies the majority of the Section 3.1 axioms — see Exercise 3.2.1).

Axiom 3.8 (Universal specification). (*Dangerous!*) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x , $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x : P(x) \text{ is true}\}$ such that for every object y ,

$$y \in \{x : P(x) \text{ is true}\} \iff P(y) \text{ is true}$$

- Also known as **axiom of comprehension**.
- Basically, Axiom 3.8 asserts that “every property corresponds to a set” (Tao, 2016, p. 46).
- Unfortunately, Axiom 3.8 cannot be introduced into set theory because it creates a logical contradiction known as **Russell's paradox**.

– Discovered by philosopher and logician Bertrand Russell (1872-1970) in 1901.

- **Russell's paradox**: “Let $P(x)$ be the statement... “ x is a set, and $x \notin x$ ”; i.e., $P(x)$ is true only when x is a set which does not contain itself. For instance, $P(\{2, 3, 4\})$ is true, since the set $\{2, 3, 4\}$ is not one of the three elements 2, 3, 4 of $\{2, 3, 4\}$. On the other hand, if we let S be the set of all sets (which we would know to exist from the axiom of universal specification), then since S is itself a set, it is an element of S , and so $P(S)$ is false. Now use the axiom of universal specification to create the set

$$\Omega := \{x : P(x) \text{ is true}\} = \{x : x \text{ is a set and } x \notin x\}$$

i.e., the set of all sets which do not contain themselves. Now ask the question: does Ω contain itself, i.e. is $\Omega \in \Omega$? If Ω did contain itself, then by definition this means that $P(\Omega)$ is true, i.e., Ω is a set and $\Omega \notin \Omega$. On the other hand, if Ω did not contain itself, then $P(\Omega)$ would be true, and hence $\Omega \in \Omega$. Thus in either case we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$, which is absurd” (Tao, 2016, pp. 46–47).

- To clarify the last point: Is $\Omega \in \Omega$? Suppose $\Omega \in \Omega$. Then since Ω contains only sets for which $P(x)$ is true, $P(\Omega)$ must be true. But this implies, by the definition of $P(x)$, that $\Omega \notin \Omega$. Similarly, suppose $\Omega \notin \Omega$. Then since “ Ω is a set and $\Omega \notin \Omega$ ” is a true statement, $P(\Omega)$ must be true. But this implies, since Ω contains all sets for which $P(x)$ is true, that $\Omega \in \Omega$. In either case, we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$ (contradictions).

7/23:

- The main problem highlighted by Russell's paradox is that Axiom 3.8 allows for the creation of sets that are too “large,” i.e., sets that contain themselves, which is somewhat silly.
 - This problem can be informally resolved by creating a hierarchy: primitive objects are below primitive sets (which only contain primitive objects), are below second-level sets (which only contain primitive objects and primitive sets), and so on and so forth. Formalizing this notion is complicated and will not be explored further here.

- Note that in pure set theory, there are no primitive objects — only one primitive set (the empty set).
- To avoid the complications of Russell’s paradox, we create a new axiom.
Axiom 3.9 (Regularity). *If A is a non-empty set, then there is at least one element x of A which is either not a set or is disjoint from A .*
- Also known as **axiom of foundation**.
- Axiom 3.9 asserts that “at least one of the elements of A is so low on the hierarchy of objects that it does not contain any of the other elements of A ” (Tao, 2016, p. 48). It also asserts that sets may not contain themselves (see Exercise 3.2.2).
- As a less intuitive axiom, one might question whether or not Axiom 3.9 is needed. In fact, it is not necessary for the purposes of doing analysis, as all sets considered in analysis are very low on the hierarchy. However, it is necessary to perform more advanced set theory, so Tao, 2016 included it for the sake of completeness.

Exercises

- 7/22: 1. Show that the universal specification axiom, Axiom 3.8, if assumed to be true, would imply Axioms 3.2, 3.3, 3.4, 3.5, and 3.6. (If we assume that all natural numbers are objects, we also obtain Axiom 3.7.) Thus, this axiom, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as “naive set theory”). Unfortunately, as we have seen, Axiom 3.8 is “too good to be true!”

Proof. Axiom 3.2: Let $P(x)$ be a false statement for all objects x . By Axiom 3.8, there exists a set $\{x : P(x) \text{ is true}\}$, which contains no elements. (Suppose for the sake of contradiction that $y \in \{x : P(x) \text{ is true}\}$ for some object y . Then $P(y)$ is true. But $P(y)$ is false by definition, a contradiction. Therefore, $y \notin \{x : P(x) \text{ is true}\}$ for all objects y .) Incidentally, that contradiction proof solidifies the symbolic statement of Axiom 3.2. Lastly, this set may be denoted \emptyset or $\{\}$.

Axiom 3.3: Let $P(x)$ be the statement $x = a$. By Axiom 3.8, there exists a set $\{x : P(x) \text{ is true}\} = \{x : x = a\} = \{a\}$ whose element is a . For every object y , we have $y \in \{a\}$ iff $P(y)$ is true, i.e., iff $y = a$. This set may be called the **singleton set** whose element is a . Now let $P(x)$ be the statement “ $x = a$ or $x = b$.” By Axiom 3.8, there exists a set $\{x : P(x) \text{ is true}\} = \{x : x = a \text{ or } x = b\} = \{a, b\}$ whose elements are a and b . For every object y , we have $y \in \{a, b\}$ iff $P(y)$ is true, i.e., iff $y = a$ or $y = b$. This set may be called the **pair set** formed by a and b .

Axiom 3.4: Let A, B be sets. Let $P(x)$ be the statement “ $x \in A$ or $x \in B$.” By Axiom 3.8, there exists a set $\{x : P(x) \text{ is true}\} = \{x : x \in A \text{ or } x \in B\}$. This set may be called the **union** $A \cup B$ of A and B . By Axiom 3.8, $y \in A \cup B$ iff $P(y)$ is true, i.e., iff $y \in A$ or $y \in B$. Thus, $A \cup B$ is clearly a set whose elements consist of all the elements which belong to A or B or both.

Axiom 3.6: Let A be a set. Let $P(y)$ be the statement “ $P(x, y)$ is true for some $x \in A$,” where $P(x, y)$ is a statement pertaining to x and y such that for each $x \in A$, there is at most one y for which $P(x, y)$ is true. By Axiom 3.8, there exists a set $\{y : P(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some } x \in A\}$. By Axiom 3.8, $z \in \{y : P(x, y) \text{ is true for some } x \in A\}$ iff $P(z)$ is true, i.e., iff $P(x, z)$ is true for some $x \in A$.

Axiom 3.5: Implied by the axiom of replacement (see Exercise 3.1.11). □

- 7/23: 2. Use the axiom of regularity (and the singleton set axiom) to show that if A is a set, then $A \notin A$. Furthermore, show that if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both).

Proof. Suppose for the sake of contradiction that A is a set and $A \in A$. By Axioms 3.1 and 3.3, $A \in \{A\}$. Since $A \in A$ and $A \in \{A\}$, Definition 3.3 tells us that $A \in A \cap \{A\}$. Since there exists an

object x (namely A) such that $x \in A \cap \{A\}$, by Axiom 3.2 and Definition 3.1, $A \cap \{A\} \neq \emptyset$. Thus, we have A is a set and $A \cap \{A\} \neq \emptyset$. But by Axiom 3.9, as the only element of $\{A\}$, A must either not be a set or satisfy $A \cap \{A\} = \emptyset$, a contradiction. Therefore, A is not a set or $A \notin A$. Thus, if A is a set, then $A \notin A$.

Suppose for the sake of contradiction that for two sets A, B , $A \in B$ and $B \in A$. By Axioms 3.1 and 3.3, there exists a set $\{A, B\}$ whose only elements are A and B . Since $A \in B$ and $A \in \{A, B\}$, Definition 3.3, tells us that $A \in B \cap \{A, B\}$. Since there exists an object x (namely A) such that $x \in B \cap \{A, B\}$, by Axiom 3.2 and Definition 3.1, $B \cap \{A, B\} \neq \emptyset$. By a similar argument, $A \cap \{A, B\} \neq \emptyset$. Thus, we have A, B are sets, $B \cap \{A, B\} \neq \emptyset$, and $A \cap \{A, B\} \neq \emptyset$. But by Axiom 3.9, an element x of $\{A, B\}$ (namely A or B) must either not be a set or satisfy $x \cap \{A, B\} = \emptyset$, a contradiction. Therefore, $A \notin B$ or $B \notin A$. \square

3. Show (assuming the other axioms of set theory) that the universal specification axiom, Axiom 3.8, is equivalent to an axiom postulating the existence of a “universal set” Ω consisting of all objects (i.e., for all objects x , we have $x \in \Omega$). In other words, if Axiom 3.8 is true, then a universal set exists, and conversely, if a universal set exists, then Axiom 3.8 is true. (This may explain why Axiom 3.8 is called the axiom of *universal* specification). Note that if a universal set Ω existed, then we would have $\Omega \in \Omega$ by Axiom 3.1, contradicting Exercise 3.2.2. Thus, the axiom of foundation specifically rules out the axiom of universal specification.

Proof. Suppose Axiom 3.8 is true. Let $P(x)$ be a true statement for all objects x . Then there exists a set $\Omega := \{x : P(x) \text{ is true}\}$, and we have $y \in \Omega$ iff $P(y)$ is true, i.e., iff y is an object, i.e., for all objects y . Therefore, a universal set exists. Now suppose that a universal set Ω exists. By Axiom 3.5, there exists a set $\{x \in \Omega : P(x) \text{ is true}\}$ for some property $P(x)$ pertaining to x (note that this implies that $P(x)$ pertains to all x) and $y \in \{x \in \Omega : P(x) \text{ is true}\}$ iff $y \in \Omega$ and $P(y)$ is true. Since $x \in \Omega$ and $y \in \Omega$ are, by the definition of Ω , always true, we have $y \in \{x : P(x) \text{ is true}\}$ iff $P(y)$ is true. Therefore, Axiom 3.8 is true. \square

3.3 Functions

- For analysis, we need not just the notion of a set but the notion of a function from one set to another.

Definition 3.5 (Functions). Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the **vertical line test**). Then we define the **function** $f : X \rightarrow Y$ defined by P on the **domain** X and **range** Y to be the object which, given any **input** $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true}$$

- Also known as **maps**, **transformations**, and **morphisms**.
 - Note, however, that a morphism “refers to a more general class of object, which may or may not correspond to actual functions, depending on the context” (Tao, 2016, p. 49).
- Functions obey the axiom of substitution.
 - Note that equal inputs imply equal outputs, but unequal inputs do not necessarily ensure unequal outputs.
- It can be proven that this notion of equality is reflexive, symmetric, and transitive (see Exercise 3.3.1).
- We can now formally define the increment function: Let $X = \mathbb{N}$, $Y = \mathbb{N}$, and $P(x, y)$ be the property that $y = x++$. By Axiom 2.4, for each $x \in \mathbb{N}$, there is exactly one y for which $P(x, y)$ is true. Thus, we can define the increment function $f : \mathbb{N} \rightarrow \mathbb{N}$ so that $f(x) = x++$ for all x . While we cannot define a *decrement* function $g : \mathbb{N} \rightarrow \mathbb{N}$ ($0 \neq n++$ for any $n \in \mathbb{N}$ by Axiom 2.3), we can define a decrement function $g : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$ (by Lemma 2.4).

- Informally: Note that while we cannot define a square root function $\sqrt{\cdot} : \mathbb{R} \rightarrow \mathbb{R}$, we can define the square root function $\sqrt{\cdot} : [0, +\infty) \rightarrow [0, +\infty)$.
- Functions can be defined **explicitly** or **implicitly**.
 - Explicit definitions: Specify the domain, range, and how one generates the output $f(x)$ from each input.
 - For example, the increment function f could be defined explicitly by saying that the domain and range of f are equal to \mathbb{N} , and $f(x) := x++$ for all $x \in \mathbb{N}$.
 - Implicit definitions: Specify what property $P(x, y)$ links the input x with the output $f(x)$.
 - For example, the square root function $\sqrt{\cdot}$ was defined implicitly by the relation $(\sqrt{x})^2 = x$.
 - Note that implicit definitions are only valid if we know that for every input, there is only one output that obeys the implicit relation.
- Often the domain and range are not specified for the sake of brevity.
 - For example, we could refer to the increment function f as “the function $f(x) := x++$,” “the function $x \mapsto x++$,” “the function $x++$,” or the extremely abbreviated “++.”
 - Note, however, that too much abbreviation can be dangerous, omitting valuable or even necessary information.
- Note that while we now use parentheses to clarify the order of operations and enclose the arguments of functions and properties, the usages should be unambiguous from context.
 - For example, if a is a number, then $a(b+c)$ denotes $a \times (b+c)$, but if a is a function, then $a(b+c)$ denotes the output of a when the input is $b+c$.
 - Note that argument are sometimes subscripted — “a sequence of natural numbers $a_0, a_1, a_2, a_3, \dots$ is, strictly speaking, a function from \mathbb{N} to \mathbb{N} , but is denoted by $n \mapsto a_n$ rather than $n \mapsto a(n)$ ” (Tao, 2016, p. 51).
- Note that functions are not sets and sets are not functions (no $x \in f$ and $A : X \rightarrow Y$), but we can start with a function $f : X \rightarrow Y$ and construct its **graph** $\{(x, f(x)) : x \in X\}$, which describes the function completely (see Section 3.5 for more).
- We now define equality for functions.

Definition 3.6 (Equality of functions). Two functions $f : X \rightarrow Y$, $g : X \rightarrow Y$ with the same domain and range are said to be equal, $f = g$, if and only if $f(x) = g(x)$ for all $x \in X$. (If $f(x)$ and $g(x)$ agree for some values of x , but not others, then we do not consider f and g to be equal.)
- Note that functions can be equal over only a certain domain.
 - For example, $x \mapsto x$ and $x \mapsto |x|$ are equal if defined only on the positive real axis, and are not equal if defined on \mathbb{R} .
- **Empty function:** The function $f : \emptyset \rightarrow X$.
 - We need not specify what f does to any input (since there are none), and Definition 3.6 asserts that for each set X , there is only one function from \emptyset to X .
- A fundamental operation of functions is composition.

Definition 3.7 (Composition). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions, such that the range of f is the same set as the domain of g . We then define the **composition** $g \circ f : X \rightarrow Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) := g(f(x))$$

- It can be proven that composition obeys the axiom of substitution (see Exercise 3.3.1).
- Composition is not commutative, but it is associative.

Lemma 3.7 (Composition is associative). *Let $f : Z \rightarrow W$, $g : Y \rightarrow Z$, and $h : X \rightarrow Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Proof. Since $g \circ h$ is a function from X to Z , $f \circ (g \circ h)$ is a function from X to W . Similarly, $f \circ g$ is a function from $Y \rightarrow W$, and hence $(f \circ g) \circ h$ is a function from $X \rightarrow W$. Thus, $f \circ (g \circ h)$ and $(f \circ g) \circ h$ have the same domain and range. In order to check that they are equal, we see from Definition 3.6 that we have to verify that $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$ for all $x \in X$. But by Definition 3.7, we have

$$\begin{aligned} (f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\ &= f(g(h(x))) \\ &= (f \circ g)(h(x)) \\ &= ((f \circ g) \circ h)(x) \end{aligned}$$

as desired. □

- 7/24: • We now define several special types of functions, beginning with the following.

Definition 3.8 (One-to-one functions). A function f is **one-to-one** if different elements map to different elements:

$$x \neq x' \implies f(x) \neq f(x')$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \implies x = x'$$

- Also known as **injective** (function).
- Informally: Note that while the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(n) := n^2$ is not one-to-one, the function $g : \mathbb{N} \rightarrow \mathbb{Z}$ defined by $g(n) := n^2$ is one-to-one. Thus, being one-to-one can depend not just on the relation, but on the domain.
- **Two-to-one** (function): A function $f : X \rightarrow Y$ such that one can find distinct x and x' in the domain X such that $f(x) = f(x')$.
- Another special type is given by the following.

Definition 3.9 (Onto functions). A function f is **onto** if $f(X) = Y$, i.e., every element in Y comes from applying f to some element in X :

$$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y$$

- Also known as **surjective** (function).
- Informally: Note that while the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(n) := n^2$ is not onto, if we define the set $A := \{n^2 : n \in \mathbb{Z}\}$, then the function $g : \mathbb{Z} \rightarrow A$ defined by $f(n) = n^2$ is onto. Thus, being onto can depend not just on the relation, but on the range.
- Injectivity and surjectivity are rather dual to each other (see Exercises 3.3.2, 3.3.4, and 3.3.5).

- 7/28: • A third special type is given by the following.

Definition 3.10 (Bijective functions). A function $f : X \rightarrow Y$ is **bijective** if it is both one-to-one and onto:

$$\text{For every } y \in Y, \text{ there is exactly one } x \text{ such that } f(x) = y.$$

- Also known as **invertible** (function), **perfect matching**, **one-to-one correspondence**^[3].
 - Instead of being denoted $x \mapsto f(x)$, we sometimes use $x \leftrightarrow f(x)$.
- Bijectivity of functions:
 - $f : \{0, 1, 2\} \rightarrow \{3, 4\}$ defined by $f(0) := 3$, $f(1) := 3$, and $f(2) := 4$ is not bijective (fails injectivity).
 - $f : \{0, 1\} \rightarrow \{2, 3, 4\}$ defined by $f(0) := 2$ and $f(1) := 3$ is not bijective (fails surjectivity).
 - $f : \{0, 1, 2\} \rightarrow \{3, 4, 5\}$ defined by $f(0) := 3$, $f(1) := 4$, and $f(2) := 5$ is bijective.
 - $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) := n++$ is not bijective (fails surjectivity).
 - $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ defined by $f(n) := n++$ is bijective.
 - Bijectivity depends on the domain and range, not just the relation.
- **Inverse** (of a function $f : X \rightarrow Y$): The function $f^{-1} : Y \rightarrow X$ associated with the property $P(y, x)$ defined by $f(x) = y$, i.e., we let $f^{-1}(y)$ be the entity x satisfying $f(x) = y$.

Exercises

- 7/23: 1. Show that the definition of equality in Definition 3.6 is reflexive, symmetric, and transitive^[4]. Also verify the substitution property: if $f, \tilde{f} : X \rightarrow Y$ and $g, \tilde{g} : Y \rightarrow Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $g \circ f = \tilde{g} \circ \tilde{f}$.

Proof. Let $f : X \rightarrow Y$ be a function. For any object $f(x) \in Y$, we have $f(x) = f(x)$ by the reflexive axiom of equality (see Section A.7). Thus, we have $f(x) = f(x)$ for all $f(x) \in Y$, i.e., for all $x \in X$. Therefore, by Definition 3.6, we have $f = f$.

Let $f : X \rightarrow Y$, $g : X \rightarrow Y$ be functions, and let $f = g$. By Definition 3.6, $f(x) = g(x)$ for all $x \in X$. Since equal objects are of the same type (i.e., follow the reflexive axiom of equality), we have $g(x) = f(x)$ for all $x \in X$. Therefore, by Definition 3.6, we have $g = f$.

Let $f : X \rightarrow Y$, $g : X \rightarrow Y$, $h : X \rightarrow Y$ be functions, $f = g$, and $g = h$. By Definition 3.6, $f(x) = g(x)$ for all $x \in X$ and $g(x) = h(x)$ for all $x \in X$. Since equal objects are of the same type (i.e., follow the transitive axiom of equality), we have $f(x) = h(x)$ for all $x \in X$. Therefore, by Definition 3.6, we have $f = h$.

First, we see that both $g \circ f$ and $\tilde{g} \circ \tilde{f}$ are functions from X to Z , i.e., have the same domain and range. To show that $g \circ f = \tilde{g} \circ \tilde{f}$, Definition 3.6 tells us that it will suffice to verify that $(g \circ f)(x) = (\tilde{g} \circ \tilde{f})(x)$ for all $x \in X$. To begin, we see from Definition 3.6 that $f(x) = \tilde{f}(x)$ for all $x \in X$, and that $g(y) = \tilde{g}(y)$ for all $y \in Y$. Since $f(x) \in Y$ for all $f(x)$, we have by Definition 3.7 that

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= \tilde{g}(f(x)) \\ &= \tilde{g}(\tilde{f}(x)) \\ &= (\tilde{g} \circ \tilde{f})(x) \end{aligned}$$

as desired. □

- 7/24: 2. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

³Not to be confused with the notion of a one-to-one function.

⁴Note that since Definition 3.6 should be an axiom (should axiomatize equality and all that that entails for functions), this part of the exercise is silly (see Tao, 2020).

Proof. To prove that $g \circ f$ is injective given the injectivity of f, g , Definition 3.8 tells us that we have to verify that $x \neq x' \implies (g \circ f)(x) \neq (g \circ f)(x')$ for any distinct elements x, x' of X . By Definition 3.8, we have that if x and x' are two distinct elements of X (such that $x \neq x'$), then $f(x) \neq f(x')$. By Definition 3.5, we know that $f(x)$ and $f(x')$ are both elements of Y . Thus, we have by Definition 3.8 that $g(f(x)) \neq g(f(x'))$. Therefore, we have by Definition 3.7 that

$$x \neq x' \implies f(x) \neq f(x') \implies g(f(x)) \neq g(f(x')) \implies (g \circ f)(x) \neq (g \circ f)(x')$$

as desired.

By Definition 3.7, we have $g \circ f : X \rightarrow Z$. Thus, to prove that $g \circ f$ is surjective given the surjectivity of f, g , Definition 3.9 tells us that we have to verify that for every $z \in Z$, there exists $x \in X$ such that $(g \circ f)(x) = z$. Let z be any element of Z . Then by Definition 3.9 and the surjectivity of g , we have $g(y) = z$ for some $y \in Y$. Similarly, we have $f(x) = y$ for that y and for some $x \in X$. Substituting, we have $g(f(x)) = z$ for some $x \in X$. Since $g(f(x)) = (g \circ f)(x)$ by Definition 3.7, we have that for any $z \in Z$, there exists $x \in X$ such that $(g \circ f)(x) = z$. \square

- 7/28: 3. When is the empty function injective? Surjective? Bijective?

Proof. The empty function is always injective: The statement “all distinct inputs map to distinct outputs” is vacuously true, since there exist no distinct inputs, or any inputs to speak of. The empty function is surjective iff its range is the empty set: In this case, the statement “for every $y \in \emptyset$, there exists $x \in \emptyset$ such that $f(x) = y$ ” is similarly vacuously true. Since a function is bijective iff it is both injective and surjective, we must take both take a sort of union of the two constraints above: We can conclude that the empty function is bijective iff “always and its range is the empty set” is true, i.e., iff its range is the empty set. \square

- 7/24: 4. In this section, we give some cancellation laws for composition. Let $f : X \rightarrow Y$, $\tilde{f} : X \rightarrow Y$, $g : Y \rightarrow Z$, and $\tilde{g} : Y \rightarrow Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$. Is the same statement true if f is not surjective?

Proof. To prove that $f = \tilde{f}$, Definition 3.6 tells us that it will suffice to show that $f(x) = \tilde{f}(x)$ for all $x \in X$. By Definition 3.6, $g \circ f = g \circ \tilde{f}$ implies $(g \circ f)(x) = (g \circ \tilde{f})(x)$ for all $x \in X$. Then by Definition 3.7, $g(f(x)) = g(\tilde{f}(x))$ for all $x \in X$. Now by Definition 3.8 and the fact that g is injective, we know that $g(y) = g(y') \implies y = y'$. Therefore, since $g(f(x)) = g(\tilde{f}(x))$ for all $x \in X$, we have $f(x) = \tilde{f}(x)$ for all $x \in X$. Note that f is not necessarily equal to \tilde{f} if we drop the condition that g is injective (if g is not one-to-one, then we have $g(y) = g(y')$ for some elements y and y' of Y such that $y \neq y'$. Thus, if $f(x) \neq \tilde{f}(x)$, we may still have $g(f(x)) = g(\tilde{f}(x))$).

Suppose for the sake of contradiction that $g \neq \tilde{g}$. Then by Definition 3.6, either g and \tilde{g} have a different domain or range, or $g(y) \neq \tilde{g}(y)$ for some $y \in Y$. Since g and \tilde{g} have the same domain and range, we must have $g(y) \neq \tilde{g}(y)$ for some $y \in Y$. Now since f is surjective, by Definition 3.9, for every $y \in Y$, there is some $x \in X$ such that $y = f(x)$. Thus, $g(f(x)) \neq \tilde{g}(f(x))$ for some $x \in X$. Consequently, by Definition 3.7, $(g \circ f)(x) \neq (\tilde{g} \circ f)(x)$ for some $x \in X$. But this contradicts Definition 3.6, which, since $g \circ f = \tilde{g} \circ f$, implies that $(g \circ f)(x) = (\tilde{g} \circ f)(x)$ for all $x \in X$. Note that g is not necessarily equal to \tilde{g} if we drop the condition that f is surjective (if f is not surjective, then, as we’ve only proven the assertion for all $y = f(x)$, we could have some $y \in Y$ not associated with an $f(x)$ for which $g(y) \neq \tilde{g}(y)$). \square

- 7/28: 5. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must also be surjective?

Proof. By Definition 3.8, to prove that f is injective, it will suffice to show that for all $x \neq x'$, $f(x) \neq f(x')$. Suppose for the sake of contradiction that for some $x \neq x'$, $f(x) = f(x')$. Then by Definition 3.5, which guarantees that identical inputs map to a single output, we have $g(f(x)) = g(f(x'))$. Thus, by Definition 3.7, we have $(g \circ f)(x) = (g \circ f)(x')$. But by Definition 3.8, this means that $g \circ f$ is not injective, a contradiction. Therefore, for all $x \neq x'$, $f(x) \neq f(x')$. Note that g need not be injective to prove the above, as this condition was not needed for the above proof to be valid.

By Definition 3.9, to prove that g is surjective, we must verify that for every $z \in Z$, there exists $y \in Y$ such that $g(y) = z$. Let z be any element of Z . Since we know that $g \circ f$ is surjective, Definition 3.9 tells us that there exists an element x of X such that $(g \circ f)(x) = z$. By Definition 3.7, we have $g(f(x)) = z$. Now by Definition 3.5, we know that $f(x) = y$ for some element $y \in Y$. Thus, $g(y) = z$ for some $y \in Y$. Therefore, we have proven that for any (i.e., for every) element $z \in Z$, there exists $y \in Y$ such that $g(y) = z$. Note that f need not be surjective to prove the above, as this condition was not needed for the above proof to be valid. \square

6. Let $f : X \rightarrow Y$ be a bijective function, and let $f^{-1} : Y \rightarrow X$ be its inverse. Verify the cancellation laws $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$. Conclude that f^{-1} is also invertible, and has f as its inverse (thus $(f^{-1})^{-1} = f$).

Proof. Let y be any element of Y . We know that f^{-1} maps y to the element $x \in X$ satisfying $f(x) = y$ (the uniqueness of x being guaranteed by Definition 3.10). Thus, since $y = f(x)$, f^{-1} maps $f(x)$ to x , i.e., $f^{-1}(f(x)) = x$. A similar argument can treat the other cancellation law.

To prove that f^{-1} is invertible, we must verify that for every $x \in X$, there is exactly one y such that $f^{-1}(y) = x$. Since we know that for every $x \in X$, $f^{-1}(f(x)) = x$, and $f(x) = y$ for some $y \in Y$ (Definition 3.5), we know that for every $x \in X$, $f^{-1}(y) = x$ for some $y \in Y$. Now suppose for the sake of contradiction that there exist some $y \neq y'$ such that $f^{-1}(y) = x = f^{-1}(y')$. Then $f(x) = y$ and $f(x) = y'$, which contradicts Definition 3.5. Therefore, for every $x \in X$, $f^{-1}(y) = x$ for some and for not more than one $y \in Y$, i.e., for exactly one $y \in Y$.

The inverse of f^{-1} is, by definition, the function $(f^{-1})^{-1} : X \rightarrow Y$ associated with the property $P(x, y)$ defined by $f^{-1}(y) = x$. Clearly $(f^{-1})^{-1}$ has the same domain and range as f . All that's left is to show that $f(x) = (f^{-1})^{-1}(x)$. $f(x)$ is the element $y \in Y$ such that $f(x) = y$, and $(f^{-1})^{-1}(x)$ is the element $y \in Y$ such that $f^{-1}(y) = x$, i.e., $f(x) = y$. Since the functions are associated with the same property, they must be equal. \square

7. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. By Definition 3.10, f and g are both both injective and surjective. Thus, by consecutive applications of Exercise 3.3.2, $g \circ f$ is both injective and surjective. Thus, by Definition 3.10 again, $g \circ f$ is bijective.

To prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, Definition 3.6 tells us that it will suffice to show that $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have the same domain and range, and $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$ for all $z \in Z$. Since $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, by Definition 3.7, we have $g \circ f : X \rightarrow Z$. Thus, the inverse $(g \circ f)^{-1} : Z \rightarrow X$. Similarly, since $g^{-1} : Z \rightarrow Y$ and $f^{-1} : Y \rightarrow X$, by Definition 3.7, $f^{-1} \circ g^{-1} : Z \rightarrow X$. Thus, $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have the same domain and range. As to the other part of the question, let $(f^{-1} \circ g^{-1})(z) = x$. Then

$$\begin{aligned}
 (f^{-1} \circ g^{-1})(z) = x &\implies f^{-1}(g^{-1}(z)) = x && \text{Definition 3.7} \\
 &\implies g^{-1}(z) = f(x) && \text{Definition of inverse} \\
 &\implies z = g(f(x)) && \text{Definition of inverse} \\
 &\implies z = (g \circ f)(x) && \text{Definition 3.7} \\
 &\implies (g \circ f)^{-1}(z) = x && \text{Definition of inverse}
 \end{aligned}$$

so by transitivity we have $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$, as desired. \square

8. If X is a subset of Y , let $\iota_{X \rightarrow Y} : X \rightarrow Y$ be the **inclusion map** (from X to Y), defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \rightarrow Y}(x) := x$ for all $x \in X$. The map $\iota_{X \rightarrow X}$ is in particular called the **identity map** (on X).

- (a) Show that if $X \subseteq Y \subseteq Z$, then $\iota_{Y \rightarrow Z} \circ \iota_{X \rightarrow Y} = \iota_{X \rightarrow Z}$.

Proof. By Definition 3.7, $\iota_{Y \rightarrow Z} \circ \iota_{X \rightarrow Y} : X \rightarrow Z$, so $\iota_{Y \rightarrow Z} \circ \iota_{X \rightarrow Y}$ and $\iota_{X \rightarrow Z}$ have the same domain and range. Now we must check that $\iota_{Y \rightarrow Z} \circ \iota_{X \rightarrow Y}$ and $\iota_{X \rightarrow Z}$ match the same inputs to the same outputs. Since $X \subseteq Y \subseteq Z$, $X \subseteq Z$ (Proposition 3.1), so $\iota_{X \rightarrow Z}(x) = x$ for all $x \in X$. On the other hand, $X \subseteq Y$ guarantees that $\iota_{X \rightarrow Y}(x) = x$ for all $x \in X$ and $Y \subseteq Z$ guarantees that $\iota_{Y \rightarrow Z}(y) = y$ for all $y \in Y$. Since $x \in Y$ for all $x \in X$ (Definition 3.2),

$$\begin{aligned} \iota_{Y \rightarrow Z} \circ \iota_{X \rightarrow Y}(x) &= \iota_{Y \rightarrow Z}(\iota_{X \rightarrow Y}(x)) && \text{Definition 3.7} \\ &= \iota_{Y \rightarrow Z}(x) \\ &= x \end{aligned}$$

for all $x \in X$. Therefore, $\iota_{Y \rightarrow Z} \circ \iota_{X \rightarrow Y} = \iota_{X \rightarrow Z}$. \square

- (b) Show that if $f : A \rightarrow B$ is any function, then $f = f \circ \iota_{A \rightarrow A} = \iota_{B \rightarrow B} \circ f$.

Proof. We first prove that $f = f \circ \iota_{A \rightarrow A}$, and next prove that $f = \iota_{B \rightarrow B} \circ f$. Transitivity will guarantee that $f = f \circ \iota_{A \rightarrow A} = \iota_{B \rightarrow B} \circ f$.

To prove that $f = f \circ \iota_{A \rightarrow A}$, Definition 3.6 tells us that it will suffice to show that f and $f \circ \iota_{A \rightarrow A}$ have the same domain and range, and that $f(a) = (f \circ \iota_{A \rightarrow A})(a)$ for all $a \in A$. By Definition 3.7, $f \circ \iota_{A \rightarrow A} : A \rightarrow B$, so f and $f \circ \iota_{A \rightarrow A}$ have the same domain and range. Now since $A \subseteq A$, $\iota_{A \rightarrow A}(a) = a$ for all $a \in A$. Thus, by Definition 3.7, $(f \circ \iota_{A \rightarrow A})(a) = f(\iota_{A \rightarrow A}(a)) = f(a)$.

To prove that $f = \iota_{B \rightarrow B} \circ f$, Definition 3.6 tells us that it will suffice to show that f and $\iota_{B \rightarrow B} \circ f$ have the same domain and range, and that $f(a) = (\iota_{B \rightarrow B} \circ f)(a)$ for all $a \in A$. By Definition 3.7, $\iota_{B \rightarrow B} \circ f : A \rightarrow B$, so f and $\iota_{B \rightarrow B} \circ f$ have the same domain and range. Now since $B \subseteq B$, $\iota_{B \rightarrow B}(b) = b$ for all $b \in B$. Since every $f(a) \in B$, by Definition 3.7, $(\iota_{B \rightarrow B} \circ f)(a) = \iota_{B \rightarrow B}(f(a)) = f(a)$. \square

- (c) Show that if $f : A \rightarrow B$ is a bijective function, then $f \circ f^{-1} = \iota_{B \rightarrow B}$, and $f^{-1} \circ f = \iota_{A \rightarrow A}$.

Proof. By Exercise 3.3.6 (and Definition 3.7), $(f \circ f^{-1})(b) = b$ for all $b \in B$. By definition, $\iota_{B \rightarrow B}(b) = b$ for all $b \in B$. Therefore, $f \circ f^{-1} = \iota_{B \rightarrow B}$. A similar argument holds in the other case. \square

- (d) Show that if X and Y are disjoint sets, and $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are functions, then there is a unique function $h : X \cup Y \rightarrow Z$ such that $h \circ \iota_{X \rightarrow X \cup Y} = f$ and $h \circ \iota_{Y \rightarrow X \cup Y} = g$.

Proof. Let $h : X \cup Y \rightarrow Z$ be defined by the following (note that it is the fact that X and Y are disjoint that allows the following definition to make sense, i.e., not be contradictory for some inputs).

$$h(a) := \begin{cases} f(a) & a \in X \\ g(a) & a \in Y \end{cases}$$

To prove that $h \circ \iota_{X \rightarrow X \cup Y} = f$, Definition 3.6 tells us that it will suffice to show that $h \circ \iota_{X \rightarrow X \cup Y}$ and f have the same domain and range, and that $(h \circ \iota_{X \rightarrow X \cup Y})(a) = f(a)$ for all $a \in X$. By Definition 3.7, $h \circ \iota_{X \rightarrow X \cup Y} : X \rightarrow Z$, so $h \circ \iota_{X \rightarrow X \cup Y}$ and f have the same domain and range. Now since $X \subseteq X \cup Y$ (Exercise 3.1.7), $\iota_{X \rightarrow X \cup Y}(a) = a$ for all $a \in X$. Thus, by Definition 3.7, for any $a \in X$, $(h \circ \iota_{X \rightarrow X \cup Y})(a) = h(\iota_{X \rightarrow X \cup Y}(a)) = h(a) = f(a)$. A similar argument holds for the other case^[5]. \square

⁵How do I verify uniqueness, or do I not need to?

3.4 Images and Inverse Images

7/29:

- We now discuss what happens when we apply a function to a set, as opposed to individual, non-set objects.

Definition 3.11 (Images of sets). If $f : X \rightarrow Y$ is a function from X to Y , and S is a set in X (or a subset of X), we define $f(S)$ to be the set

$$f(S) := \{f(x) : x \in S\}$$

This set is a subset of Y , and is sometimes called the **image** (of S under the map f). We sometimes call $f(S)$ the **forward image** (of S) to distinguish it from the concept of the **inverse image** $f^{-1}(S)$ (of S), which is defined below.

Note that, symbolically,

$$x \in S \implies f(x) \in f(S)$$

and

$$y \in f(S) \iff y = f(x) \text{ for some } x \in S$$

- Note that $f(S)$ is well-defined, as Definition 3.11 is based in Axiom 3.6. Also note that Definition 3.11 could be defined in terms of Axiom 3.5.
- As alluded to above, we now define inverse images.

Definition 3.12 (Inverse images). If U is a subset of Y , we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U :

$$f(x) \in U \iff x \in f^{-1}(U)$$

We call $f^{-1}(U)$ the **inverse image** (of U).

- Note that if f is bijective, then we have defined f^{-1} in two different ways. However, the definitions are equivalent (see Exercise 3.4.1).
- Since we wish to treat functions as objects so that we can create sets of functions, we axiomatize this notion.

Axiom 3.10 (Power set axiom). *Let X and Y be sets. Then there exists a set, denoted Y^X , which consists of all the functions from X to Y , thus*

$$f \in Y^X \iff f \text{ is a function with domain } X \text{ and range } Y$$

- “Let $X = \{4, 7\}$ and $Y = \{0, 1\}$. Then the set Y^X consists of four functions: the function that maps $4 \mapsto 0$ and $7 \mapsto 0$; the function that maps $4 \mapsto 0$ and $7 \mapsto 1$; the function that maps $4 \mapsto 1$ and $7 \mapsto 0$; and the function that maps $4 \mapsto 1$ and $7 \mapsto 1$ ” (Tao, 2016, p. 58).
- Note that we use the notation Y^X because if Y has n elements and X has m elements, then Y^X has n^m elements (see Exercise ??).
- See Exercise 3.4.6 for a consequence of Axiom 3.10.
- **Power set** (of X): The set $2^X = \{Y : Y \text{ is a subset of } X\}$.
- Note that we use the notation 2^X because if X has m elements, then 2^X has 2^m elements (see Chapter ??).
- For the sake of completeness, we enhance Axiom 3.4 with the following.

Axiom 3.11 (Union). *Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A . Thus, for all objects x ,*

$$x \in \bigcup A \iff x \in S \text{ for some } S \in A$$

- Note that Axioms 3.11 and 3.3 imply Axiom 3.4 (see Exercise 3.4.8).
- An important consequence of Axiom 3.11 is that “if one has some set I , and for every element $\alpha \in I$ we have some set A_α , then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ ^[6] by defining

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\} \quad (3.1)$$

which is a set thanks to the axiom of replacement and the axiom of union” (Tao, 2016, p. 59).

- Note that for any object y ,

$$y \in \bigcup_{\alpha \in I} A_\alpha \iff y \in A_\alpha \text{ for some } \alpha \in I \quad (3.2)$$

- To explain the above, consider the following example.
 - If $I = \{1, 2, 3\}$, $A_1 := \{2, 3\}$, $A_2 := \{3, 4\}$, and $A_3 := \{4, 5\}$, then $\bigcup_{\alpha \in \{1, 2, 3\}} A_\alpha = \{2, 3, 4, 5\}$.
- There is some notation associated with this paradigm, defined as follows.
- **Index set:** The set I .
- **Labels:** The elements α of the index set I .
- **Family of sets:** The group of sets A_α for all $\alpha \in I$.
 - Note that the family of sets is **indexed** by the labels $\alpha \in I$.
- Note that if I were empty, then $\bigcup_{\alpha \in I} A_\alpha$ would also be empty.
- “Given any non-empty set I , and given an assignment of a set A_α to each $\alpha \in I$, we can define the intersection $\bigcap_{\alpha \in I} A_\alpha$ by first choosing some element β of I (which we can do since I is nonempty), and setting

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} \quad (3.3)$$

which is a set by the axiom of specification” (Tao, 2016, p. 60).

- Note that for any object y ,
- $$y \in \bigcap_{\alpha \in I} A_\alpha \iff y \in A_\alpha \text{ for all } \alpha \in I \quad (3.4)$$
- Note that this definition does not depend on the choice of β (see Exercise 3.4.9).
 - Note that Axioms 3.1-3.11 (excluding Axiom 3.8) are known as the **Zermelo-Fraenkel axioms of set theory**, after Ernest Zermelo (1871-1953) and Abraham Fraenkel (1891-1965).
 - Note that “these axioms are formulated slightly differently in other texts, but all the formulations can be shown to be equivalent to each other” (Tao, 2016, p. 60).
 - Also note that there is one additional axiom (the **axiom of choice** — see Section ??), giving rise to the **Zermelo-Fraenkel-Choice (ZFC) axioms of set theory**, but we do not need this axiom at this moment.

⁶“the union of the family of sets A_α indexed by the labels α in the set I ”

Exercises

1. Let $f : X \rightarrow Y$ be a bijective function, and let $f^{-1} : Y \rightarrow X$ be its inverse. Let V be any subset of Y . Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f ; thus, the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. By Definition 3.11, the forward image of V under f^{-1} is the set $\{f^{-1}(x) : x \in V\}$. By Definition 3.12, the inverse image of V under f is the set $\{x \in X : f(x) \in V\}$. Now to prove that these sets are equal, by Definition 3.1, we must verify that every element y of $\{f^{-1}(x) : x \in V\}$ is an element of $\{x \in X : f(x) \in V\}$ and vice versa. Suppose first that y is an arbitrary element of $\{f^{-1}(x) : x \in V\}$. By Definition 3.11, this implies that $y = f^{-1}(x)$ for some $x \in V$. Thus, by the definition of the inverse (and the fact that f is bijective), $f(y) = x$ for some $x \in V$, or, more simply, $f(y) \in V$. But by Definition 3.12, $f(y) \in V \implies y \in \{x \in X : f(x) \in V\}$. Using the above implications in reverse will suffice to prove that $y \in \{x \in X : f(x) \in V\} \implies y \in \{f^{-1}(x) : x \in V\}$. \square

- 7/30: 2. Let $f : X \rightarrow Y$ be a function from one set X to another set Y , let S be a subset of X , and let U be a subset of Y . What, in general, can one say about $f^{-1}(f(S))$ and S ? What about $f(f^{-1}(U))$ and U ?

Proof. It is hard to generalize much from the rigid definitions, but we will give those and then prove one property. So first off,

$$\begin{aligned} f^{-1}(f(S)) &= \{x \in X : f(x) \in \{f(x') : x' \in S\}\} \\ f(f^{-1}(U)) &= \{f(x) : x \in \{x' \in X : f(x') \in U\}\} \end{aligned}$$

Now, we can actually prove that $S \subseteq f^{-1}(f(S))$ and $f(f^{-1}(U)) \subseteq U$, which we will do as follows.

By Definition 3.2, to prove that $S \subseteq f^{-1}(f(S))$, we must verify that every element $x \in S$ is an element of $f^{-1}(f(S))$. Suppose x is any element of S . To prove that $x \in \{x' \in X : f(x') \in \{f(x'') : x'' \in S\}\}$, Axiom 3.5 tells us that it will suffice to show that $x \in X$ and “ $f(x) \in \{f(x') : x' \in S\}$ ” is a true statement. Since $S \subseteq X$ and $x \in S$, we have by Definition 3.2 that $x \in X$. On the other hand, let $y = f(x)$. Then since $x \in S$, by Axiom 3.6, $y \in \{y' : y' = f(x') \text{ for some } x' \in S\} \implies f(x) \in \{f(x') : x' \in S\}$.

By Definition 3.2, to prove that $f(f^{-1}(U)) \subseteq U$, we must verify that every element $y \in f(f^{-1}(U))$ is an element of U . Suppose y is any element of $f(f^{-1}(U))$. Then by the rigid definition of $f(f^{-1}(U))$, above, and Definition 3.1, we have $y \in \{f(x) : x \in \{x' \in X : f(x') \in U\}\}$. By Axiom 3.6, this means that $y = f(x)$ for some $x \in \{x' \in X : f(x') \in U\}$. But by Axiom 3.5, $x \in \{x' \in X : f(x') \in U\} \implies (x \in X \text{ and } f(x) \in U)$. Thus, we know that $y = f(x)$ for some $f(x) \in U$, so we have $y \in U$. \square

3. Let A, B be two subsets of a set X , and let $f : X \rightarrow Y$ be a function. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$, that $f(A) \setminus f(B) \subseteq f(A \setminus B)$, and that $f(A \cup B) = f(A) \cup f(B)$. For the first two statements, is it true that the \subseteq relation can be improved to $=$?

Proof. Because of the number of statements to prove and the logical simplicity of the implications involved, we will shorthand these proofs.

We begin by proving that $f(A \cap B) \subseteq f(A) \cap f(B)$, for which it will suffice (Definition 3.2) to show that every element y of $f(A \cap B)$ is an element of $f(A) \cap f(B)$. To this end, let y be any element of $f(A \cap B)$. Since $A \cap B \subseteq A$ (Exercise 3.1.7) and $A \subseteq X$, $A \cap B \subseteq X$ (Proposition 3.1). Thus,

$$\begin{aligned} y \in f(A \cap B) &\implies y = f(x) \text{ for some } x \in A \cap B && \text{Definition 3.11} \\ &\implies (y = f(x) \text{ for some } x \in A \text{ and } y = f(x) \text{ for some } x \in B) && \text{Definition 3.3} \\ &\implies (y \in f(A) \text{ and } y \in f(B)) && \text{Definition 3.11} \\ &\implies y \in f(A) \cap f(B) && \text{Definition 3.3} \end{aligned}$$

To prove that $f(A) \setminus f(B) \subseteq f(A \setminus B)$, we must verify (Definition 3.2) that every element y of $f(A) \setminus f(B)$ is an element of $f(A \setminus B)$, which can be done as follows. Note that $A \setminus B \subseteq X$ since $A \setminus B \subseteq A$ (Definition 3.4 and Axiom 3.5) and $A \subseteq X$ (Proposition 3.1).

$$\begin{aligned}
 y \in f(A) \setminus f(B) &\implies y \in \{z \in f(A) : z \notin f(B)\} && \text{Definition 3.4} \\
 &\implies (y \in f(A) \text{ and } y \notin f(B)) && \text{Axiom 3.5} \\
 &\implies (y = f(x) \text{ for some } x \in A \text{ and } y = f(x) \text{ for no } x \in B) && \text{Definition 3.11} \\
 &\implies y = f(x) \text{ for some } x \in A \setminus B && \text{Definition 3.4} \\
 &\implies y \in f(A \setminus B) && \text{Definition 3.11}
 \end{aligned}$$

To prove that $f(A \cup B) = f(A) \cup f(B)$, Definition 3.1 tells us that it will suffice to show that $y \in f(A \cup B) \iff y \in f(A) \cup f(B)$, which can be done as follows.

$$\begin{aligned}
 y \in f(A \cup B) &\iff y = f(x) \text{ for some } x \in A \cup B && \text{Definition 3.11} \\
 &\iff (y = f(x) \text{ for some } x \in A \text{ or } y = f(x) \text{ for some } x \in B) && \text{Axiom 3.4} \\
 &\iff (y \in f(A) \text{ or } y \in f(B)) && \text{Definition 3.11} \\
 &\iff y \in f(A) \cup f(B) && \text{Axiom 3.4}
 \end{aligned}$$

Because of the transitivity of logically equivalent statements (we may induct the result of Exercise A.1.5), we have shown what was desired.

As to the other part of the question, it is *not* true that the \subseteq relation can be improved to $=$ for the first or the second statement, which we may verify thorough two counterexamples, as follows. For the first statement, let $X = \{1, 2\}$, $Y = \{3, 4\}$, $A = \{1\}$, $B = \{2\}$, and $f : \{1, 2\} \rightarrow \{3, 4\}$ be defined by $f(x) := 3$. Then we have $f(A) \cap f(B) = \{3\}$, but $f(A \cap B) = \{\}$. Thus, $3 \in f(A) \cap f(B)$, but $3 \notin f(A \cap B)$. Note that the breakdown in the would-be reverse proof of $f(A \cap B) \subseteq f(A) \cap f(B)$ comes from the fact that although $y \in f(A) \cap f(B) \implies y = f(x)$ for some $x \in A$ and $y = f(x')$ for some $x' \in B$, we cannot guarantee that $x = x'$, i.e., that $x \in A \cap B$. As to the second statement, we consider the same function, except that we let $A = \{1, 2\}$. Thus, we have $f(A \setminus B) = \{3\}$, but $f(A) \setminus f(B) = \{\}$, implying that $3 \in f(A \setminus B)$ but $3 \notin f(A) \setminus f(B)$. Note that the breakdown in the would-be reverse proof of $f(A) \setminus f(B) \subseteq f(A \setminus B)$ comes from the fact that although $y \in f(A \setminus B) \implies y = f(x)$ for some $x \in A$, we cannot guarantee that there exists no $x' \in B$ such that $y = f(x')$. \square

4. Let $f : X \rightarrow Y$ be a function from one set X to another set Y , and let U, V be subsets of Y . Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, and that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$.

Proof. For the same reasons as in Exercise 3.4.3, we shorthand these proofs.

To prove that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, Definition 3.1 tells us that it will suffice to show that $x \in f^{-1}(U \cup V) \iff x \in f^{-1}(U) \cup f^{-1}(V)$, which can be done as follows.

$$\begin{aligned}
 x \in f^{-1}(U \cup V) &\iff f(x) \in U \cup V && \text{Definition 3.12} \\
 &\iff (f(x) \in U \text{ or } f(x) \in V) && \text{Axiom 3.4} \\
 &\iff (x \in f^{-1}(U) \text{ or } x \in f^{-1}(V)) && \text{Definition 3.12} \\
 &\iff x \in f^{-1}(U) \cup f^{-1}(V) && \text{Axiom 3.4}
 \end{aligned}$$

To prove that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, Definition 3.1 tells us that it will suffice to show that $x \in f^{-1}(U \cap V) \iff x \in f^{-1}(U) \cap f^{-1}(V)$, which can be done as follows.

$$\begin{aligned}
 x \in f^{-1}(U \cap V) &\iff f(x) \in U \cap V && \text{Definition 3.12} \\
 &\iff (f(x) \in U \text{ and } f(x) \in V) && \text{Definition 3.3} \\
 &\iff (x \in f^{-1}(U) \text{ and } x \in f^{-1}(V)) && \text{Definition 3.12} \\
 &\iff x \in f^{-1}(U) \cap f^{-1}(V) && \text{Definition 3.3}
 \end{aligned}$$

To prove that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$, Definition 3.1 tells us that it will suffice to show that $x \in f^{-1}(U \setminus V) \iff x \in f^{-1}(U) \setminus f^{-1}(V)$, which can be done as follows.

$$\begin{aligned}
 x \in f^{-1}(U \setminus V) &\iff f(x) \in U \setminus V && \text{Definition 3.12} \\
 &\iff f(x) \in \{y \in U : y \notin V\} && \text{Definition 3.4} \\
 &\iff (f(x) \in U \text{ and } f(x) \notin V) && \text{Axiom 3.5} \\
 &\iff (x \in f^{-1}(U) \text{ and } x \notin f^{-1}(V)) && \text{Definition 3.12} \\
 &\iff x \in \{f^{-1}(U) : x \notin f^{-1}(V)\} && \text{Axiom 3.5} \\
 &\iff x \in f^{-1}(U) \setminus f^{-1}(V) && \text{Definition 3.12}
 \end{aligned}$$

□

5. Let $f : X \rightarrow Y$ be a function from one set X to another set Y . Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Proof. To show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective, we must verify that the implication “if $f(f^{-1}(S)) = S$ for every $S \subseteq Y$, then f is surjective” and its converse are both true.

Suppose first that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$. To show that f is surjective, Definition 3.9 tells us that it will suffice to show that for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Let y be any element of Y . Then by Axiom 3.3, there exists a set $\{y\}$. Since $y \in Y$ and y is the only element of $\{y\}$, by Definition 3.2, $\{y\} \subseteq Y$. Thus, $f(f^{-1}(\{y\})) = \{y\}$. Since $\{y\}$ is a nonempty set, we know that $f^{-1}(\{y\})$ is a nonempty set (if $f^{-1}(\{y\})$ were empty, then $f(f^{-1}(\{y\})) = \{y\}$ would be empty, a contradiction). Thus, there exists some $x \in f^{-1}(\{y\})$. Since Definition 3.12 implies that $x \in \{x' \in X : f(x') \in \{y\}\}$, meaning by Axiom 3.5 that $x \in X$ and $f(x) \in \{y\}$, and, thus, by Axiom 3.3 that $f(x) = y$, we have that there exists $x \in X$ such that $f(x) = y$, as desired.

Now suppose that f is surjective. To show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$, we must verify that for any subset S of Y , every element y of $f(f^{-1}(S))$ is an element of S and vice versa (Definition 3.1). Let S be an arbitrary subset of Y . By Exercise 3.4.2, $f(f^{-1}(S)) \subseteq S$. On the other hand, suppose that y is an arbitrary element of S . Since f is surjective, we have by Definition 3.9 that $f(x) = y$ for some $x \in X$. Now consider this x : since $y \in S$, we know that $f(x) \in S$. Thus, by Definition 3.12, $x \in f^{-1}(S)$. Therefore, we have $y = f(x)$ for some $x \in f^{-1}(S)$, which by Definition 3.11 and the fact that $f^{-1}(S) \subseteq X$ (as implied by Definition 3.12 and Axiom 3.5) implies that $y \in f(f^{-1}(S))$.

To show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective, we must verify that the implication “if $f^{-1}(f(S)) = S$ for every $S \subseteq X$, then f is injective” and its converse are both true.

Suppose first that $f^{-1}(f(S)) = S$ for every $S \subseteq X$. To show that f is injective, Definition 3.8 tells us that it will suffice to show that if $x \neq x'$ for two elements x, x' of X , then $f(x) \neq f(x')$. Let x, x' be any two elements of X such that $x \neq x'$. By Axiom 3.3, the sets $\{x\}$ and $\{x'\}$ exist. Note that since $x \in X$ and x is the only element of $\{x\}$, by Definition 3.2, $\{x\} \subseteq X$, and, similarly, $\{x'\} \subseteq X$. Since there exists an element of $\{x\}$ (namely x) that is not an element of $\{x'\}$, by Definition 3.1, $\{x\} \neq \{x'\}$. Thus, since $f^{-1}(f(\{x\})) = \{x\}$ and $f^{-1}(f(\{x'\})) = \{x'\}$, we have $f^{-1}(f(\{x\})) \neq f^{-1}(f(\{x'\}))$. This implies that $f(\{x\}) \neq f(\{x'\})$ (if $f(\{x\})$ were equal to $f(\{x'\})$, then $f^{-1}(f(\{x\}))$ would be equal to $f^{-1}(f(\{x'\}))$, a contradiction). Since $f(\{x\}) = \{f(x'') : x'' \in \{x\}\} = \{f(x)\}$, and, similarly, $f(\{x'\}) = \{f(x')\}$, we have $\{f(x)\} \neq \{f(x')\}$. Therefore, $f(x) \neq f(x')$ (if $f(x) = f(x')$, then, as singleton sets (Axiom 3.3), every element of $\{f(x)\}$ would be an element of $\{f(x')\}$, and vice versa, proving their equality by Definition 3.1, a contradiction).

Now suppose that f is injective. To show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$, we must verify that for any subset S of X , every element x of $f^{-1}(f(S))$ is an element of S and vice versa (Definition 3.1). Let S be an arbitrary subset of X , and suppose that x is an arbitrary element of $f^{-1}(f(S))$. First off, $f(S) \subseteq Y$. (We must show that every $y \in f(S)$ is an element of Y (Definition 3.2). If $y \in f(S)$,

then we know that $y = f(x)$ for some $x \in S$ by Definition 3.11. Since $S \subseteq X$, we have that $y = f(x)$ for some $x \in X$, which, since $f(x) \in Y$, implies that $y \in Y$. Thus, by Definition 3.12, $f(x) \in f(S)$. By Definition 3.11 and the fact that f is injective, this implies that $x \in S$ (if f were not injective, then it would be possible that $x \notin S$ but $f(x) \in f(S)$, since there could exist some $x' \in S$ such that $f(x') = f(x)$). On the other hand, by Exercise 3.4.2, $S \subseteq f^{-1}(f(S))$. \square

- 7/29: 6. Prove the following lemma. (Hint: start with the set $\{0,1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^{-1}(\{1\})$.) See also Exercise 3.5.11.

Lemma 3.8. *Let X be a set. Then the set*

$$\{Y : Y \text{ is a subset of } X\}$$

is a set.

Proof. We first define a set composed entirely of subsets of X . We then guarantee that this set includes all subsets of X .

Consider the set $\{0,1\}^X$. For any $f \in \{0,1\}^X$, since $\{1\} \subseteq \{0,1\}$ (i.e., is a subset of the range of f), Definition 3.12 allows us to define the (unique) set $f^{-1}(\{1\})$. Now let $P(f, A)$ be the property $A = f^{-1}(\{1\})$. Then for every element $f \in \{0,1\}^X$, there is exactly, i.e., at most one A for which $P(f, A)$ is true. Consequently, by Axiom 3.6, there exists a set $\{A : P(f, A) \text{ is true for some } f \in \{0,1\}^X\}$, or, more briefly, $\{f^{-1}(\{1\}) : f \in \{0,1\}^X\}$. Now by Definition 3.12, each $f^{-1}(\{1\})$ represents the corresponding set $\{x \in X : f(x) \in \{1\}\}$. This implies by Axiom 3.5 that every $f^{-1}(\{1\})$ is a subset of X . Thus, $\{f^{-1}(\{1\}) : f \in \{0,1\}^X\}$ is a set composed entirely of subsets of X .

At this point, all that's left is to guarantee that $\{f^{-1}(\{1\}) : f \in \{0,1\}^X\}$ includes *all* subsets of X , which may be done as follows. Let Z be any subset of X . Then we may define the function $f : X \rightarrow \{0,1\}$ by the following.

$$f(x) := \begin{cases} 1 & x \in Z \\ 0 & x \notin Z \end{cases}$$

By Axiom 3.10, $f \in \{0,1\}^X$. Thus, by the first part of this proof, we know that $f^{-1}(\{1\})$ exists and is defined to be equal to $\{x \in X : f(x) \in \{1\}\}$. But Axiom 3.3, $f(x) \in \{1\}$ iff $f(x) = 1$, and by the definition of f , $f(x) = 1$ iff $x \in Z$. Thus, $f^{-1}(\{1\}) = \{x \in X : x \in Z\} = Z$. Since $f^{-1}(\{1\}) \in \{f^{-1}(\{1\}) : f \in \{0,1\}^X\}$, we have $Z \in \{f^{-1}(\{1\}) : f \in \{0,1\}^X\}$. Therefore, there exists a set that contains exactly the subsets of X , which we may denote by $\{Y : Y \text{ is a subset of } X\}$. \square

- 7/31: 7. Let X, Y be sets. Define a **partial function** from X to Y to be any function $f : X' \rightarrow Y'$ whose domain X' is a subset of X , and whose range Y' is a subset of Y . Show that the collection of all partial functions from X to Y is itself a set. (Hint: use Exercise 3.4.6, the power set axiom, the replacement axiom, and the union axiom.)

Proof. To create the set of all partial functions from X to Y , we follow a four step plan. First, we define the set 2^X of all subsets of X , and the set 2^Y of all subsets of Y . Second, we define the set $A_{X'}$ of all sets of all partial functions from X to Y with domain X' and a certain range (namely, a range that is any particular element of 2^Y). Third, we create the set $\bigcup_{X' \in 2^X} A_{X'}$ of all sets of all partial functions from X to Y with a certain domain (namely, any particular element of 2^X) and a certain range (namely, any particular element of 2^Y). Fourth, we create the set $\bigcup (\bigcup_{X' \in 2^X} A_{X'})$, which is the set of all partial functions from X to Y , as desired⁷. Let's begin.

- (1) By Exercise 3.4.6, there exists a set $\{X' : X' \text{ is a subset of } X\}$ (which we may denote by 2^X) and a set $\{Y' : Y' \text{ is a subset of } Y\}$ (which we may denote by 2^Y). In sum, we have defined the sets

$$2^X := \{X' : X' \text{ is a subset of } X\} \qquad 2^Y := \{Y' : Y' \text{ is a subset of } Y\}$$

⁷Do I need to verify that every element of this set is a partial function from X to Y and that every partial function from X to Y is an element of this set?

- (2) Let X' be an element of 2^X , let Y' be any element of 2^Y , and let $P(Y', Z)$ be the statement $Z = Y'^{X'}$. (Note that $Y'^{X'}$ is the set of all functions with domain X' and range Y' , i.e., the set of all partial functions from X to Y with domain X' and range Y' , as defined in Axiom 3.10.) Since Axiom 3.10 guarantees the uniqueness of B^A for any sets A, B , for every $Y' \in 2^Y$, there is exactly one Z for which $P(Y', Z)$ is true. Thus, by Axiom 3.6, there exists a set $\{Z : P(Y', Z) \text{ is true for some } Y' \in 2^Y\}$ (which we may denote by $A_{X'}$). In sum, we have defined for every $X' \in 2^X$ the set

$$A_{X'} := \{Y'^{X'} : Y' \in 2^Y\}$$

- (3) Let 2^X be an index set. Then by (2), for every $X' \in 2^X$, we have some set $A_{X'}$. Thus, by Equation 3.1, we can form the union set $\bigcup_{X' \in 2^X} A_{X'}$ by defining $\bigcup_{X' \in 2^X} A_{X'} := \bigcup \{A_{X'} : X' \in 2^X\}$. If we inspect this definition, we see that this set is the set of all elements in every possible $A_{X'}$ (i.e., every $A_{X'}$ such that $X' \in 2^X$). Since every element of any $A_{X'}$ is a set of the type $Y'^{X'}$, the above union is really just the set of all $Y'^{X'}$ for any $X' \in 2^X$ and any $Y' \in 2^Y$. In sum, we have defined the set

$$\bigcup_{X' \in 2^X} A_{X'} := \{Y'^{X'} : X' \in 2^X; Y' \in 2^Y\}$$

- (4) By Axiom 3.10, every $Y'^{X'}$ is a set. Thus, the set defined in (3) is a set whose elements are all themselves sets. Therefore, Axiom 3.11 applies, and we can create the set $\bigcup (\bigcup_{X' \in 2^X} A_{X'})$, whose elements are precisely those objects which are elements of the elements of $\bigcup_{X' \in 2^X} A_{X'}$; in other words, whose elements are every partial function in every $Y'^{X'}$, i.e., all partial functions from X to Y .

□

7/29: 8. Show that Axiom 3.4 can be deduced from Axiom 3.1, Axiom 3.3, and Axiom 3.11.

Proof. To demonstrate Axiom 3.4, we must show that for any two sets A, B , there exists a set $A \cup B$ satisfying $x \in A \cup B$ iff $x \in A$ or $x \in B$.

Let A, B be sets. Then by Axiom 3.1, A and B are objects. Thus, by Axiom 3.3, we can create the pair set $\{A, B\}$. Since $\{A, B\}$ has only sets for elements, Axiom 3.11 applies, and implies the existence of $\bigcup \{A, B\}$, which we may denote by $A \cup B$. Axiom 3.11 also asserts that

$$x \in A \cup B \iff x \in S \text{ for some } S \in \{A, B\}$$

But by Axiom 3.3, $S \in A \cup B$ iff $S = A$ or $S = B$. Therefore, we have

$$x \in A \cup B \iff x \in S, S \text{ being a set that satisfies } S = A \text{ or } S = B$$

or

$$x \in A \cup B \iff x \in A \text{ or } x \in B$$

□

9. Show that if β and β' are two elements of a set I , and to each $\alpha \in I$ we assign a set A_α , then

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}$$

and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in Equation 3.3 does not depend on β . Also explain why Equation 3.4 is true.

Proof. To prove $\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}$, Definition 3.1 tells us that it will suffice to show that

$$y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} \iff y \in \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}$$

By Equations 3.3 and 3.4, we have

$$y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} \iff y \in A_\alpha \text{ for all } \alpha \in I$$

Similarly, we have

$$y \in \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\} \iff y \in A_\alpha \text{ for all } \alpha \in I$$

Therefore, by Exercise A.1.5, we have

$$y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} \iff y \in \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}$$

as desired.

As to the other question, by Definition 3.1, $y \in \bigcap_{\alpha \in I} A_\alpha \implies y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$. Thus, by Axiom 3.5, $y \in A_\beta$ and $y \in A_\alpha$ for all $\alpha \in I$. But $A_\beta = A_\alpha$ for some $\alpha \in I$, so the condition that $y \in A_\beta$ is a tautology. Thus, $y \in A_\alpha$ for all $\alpha \in I$. A similar argument works in the other direction. \square

- 7/31: 10. Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that $(\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in I \cup J} A_\alpha$. If I and J are non-empty, show that $(\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in I \cup J} A_\alpha$.

Proof. To prove that $(\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in I \cup J} A_\alpha$, Definition 3.1 tells us that it will suffice to show that every element x of $(\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$ is an element of $\bigcup_{\alpha \in I \cup J} A_\alpha$ and vice versa. Suppose first that $x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$. Then by Axiom 3.4, $x \in \bigcup_{\alpha \in I} A_\alpha$ or $x \in \bigcup_{\alpha \in J} A_\alpha$. We now divide into two cases. If $x \in \bigcup_{\alpha \in I} A_\alpha$, then by Equation 3.2, $x \in A_\alpha$ for some $\alpha \in I$. By Axiom 3.4, this implies that $x \in A_\alpha$ for some $\alpha \in I \cup J$. Therefore, by Equation 3.2, $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$. On the other hand, if $x \in \bigcup_{\alpha \in J} A_\alpha$, then by an analogous argument to the other case, we have $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$. Now suppose that $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$. Then by Equation 3.2, $x \in A_\alpha$ for some $\alpha \in I \cup J$. Thus, by Axiom 3.4, $x \in A_\alpha$ for some $\alpha \in I$, or $x \in A_\alpha$ for some $\alpha \in J$. We now divide into two cases. If $x \in A_\alpha$ for some $\alpha \in I$, then by Equation 3.2, we have $x \in \bigcup_{\alpha \in I} A_\alpha$. Therefore, by Axiom 3.4, we have $x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$. On the other hand, if $x \in A_\alpha$ for some $\alpha \in J$, then by an analogous argument to the other case, we have $x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$.

To prove that $(\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in I \cup J} A_\alpha$, Definition 3.1 tells us that it will suffice to show that every element x of $(\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha)$ is an element of $\bigcap_{\alpha \in I \cup J} A_\alpha$ and vice versa. Suppose first that $x \in (\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha)$. Then by Definition 3.3, $x \in \bigcap_{\alpha \in I} A_\alpha$ and $x \in \bigcap_{\alpha \in J} A_\alpha$. By Equation 3.4, this implies that $x \in A_\alpha$ for all $\alpha \in I$ and $x \in A_\alpha$ for all $\alpha \in J$. Thus, by Axiom 3.4, $x \in A_\alpha$ for all $\alpha \in I \cup J$. Therefore, by Equation 3.4, $x \in \bigcap_{\alpha \in I \cup J} A_\alpha$. Now suppose that $x \in \bigcap_{\alpha \in I \cup J} A_\alpha$. Then by Equation 3.4, $x \in A_\alpha$ for all $\alpha \in I \cup J$. By Axiom 3.4, this implies that $x \in A_\alpha$ for all $\alpha \in I$ and $x \in A_\alpha$ for all $\alpha \in J$. Thus, by Equation 3.4, we have $x \in \bigcap_{\alpha \in I} A_\alpha$ and $x \in \bigcap_{\alpha \in J} A_\alpha$. Therefore, by Definition 3.3, $x \in (\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha)$. \square

11. Let X be a set, let I be a non-empty set, and for all $\alpha \in I$ let A_α be a subset of X . Show that $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$. This should be compared with de Morgan's laws in Proposition 3.4 (although one cannot derive the above identities directly from de Morgan's laws, as I could be infinite).

Proof. To prove that $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$, Definition 3.1 tells us that it will suffice to show that every element x of $X \setminus \bigcup_{\alpha \in I} A_\alpha$ is an element of $\bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and vice versa. Suppose first that $x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$. Then by Definition 3.4 and Axiom 3.5, $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_\alpha$. By Equation 3.2, this implies that $x \notin A_\alpha$ for any $\alpha \in I$. Thus, by Axiom 3.5 and Definition 3.4, $x \in X \setminus A_\alpha$ for all $\alpha \in I$. Therefore, by Equation 3.4, $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$. Now suppose that $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$. Then by Equation 3.4, $x \in X \setminus A_\alpha$ for all $\alpha \in I$. By Definition 3.4 and Axiom 3.5, this implies that $x \in X$ and $x \notin A_\alpha$ for any $\alpha \in I$. Thus, by Equation 3.2, $x \notin \bigcup_{\alpha \in I} A_\alpha$. Therefore, by Axiom 3.5 and Definition 3.4, $x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$.

A similar argument can treat the other case. \square

3.5 Cartesian Products

- We begin by defining the ordered pair.

Definition 3.13 (Ordered pair). If x and y are any objects (possibly equal), we define the **ordered pair** (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal if and only if both their components match, i.e.,

$$(x, y) = (x', y') \iff (x = x' \text{ and } y = y')$$

- It can be proven that this notion of equality obeys the usual axioms of equality (see Exercise 3.5.3).

8/1:

- Note that while order doesn't matter in sets, it does matter in ordered pairs.

– For example, $\{3, 5\} = \{5, 3\}$, but $(3, 5) \neq (5, 3)$.

- Note that Definition 3.13 is stated as an axiom (since we have simply postulated that for any two objects x, y , an object of the form (x, y) exists), but it *is* possible to define ordered pairs solely in terms of the axioms of set theory (see Exercise 3.5.1).

8/4:

- Note that we now use parentheses to enclose ordered pairs, in addition to clarifying the order of operations and enclosing the arguments of functions and properties. However, the usages should still be unambiguous from context.

- We now define the Cartesian product.

Definition 3.14 (Cartesian product). If X and Y are sets, then we define the **Cartesian product** $X \times Y$ to be the collection of ordered pairs whose first component lies in X and second component lies in Y ; thus,

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

or equivalently

$$a \in (X \times Y) \iff a = (x, y) \text{ for some } x \in X \text{ and } y \in Y$$

- Although we simply state it here, it can be proven that the Cartesian product is a set (see Exercise 3.5.1).

- Although $X \times Y$ and $Y \times X$ are often different sets, they are very similar.

– For example, they have the same number of elements (see Exercise ??).

- Cartesian products allow us to consider functions of two variables.

– “Let $f : X \times Y \rightarrow Z$ be a function whose domain $X \times Y$ is a Cartesian product of two other sets X and Y . Then f can either be thought of as a function of one variable, mapping the single input of an ordered pair (x, y) in $X \times Y$ to an output $f(x, y)$ in Z , or as a function of two variables, mapping an input $x \in X$ and another input $y \in Y$ to a single output $f(x, y)$ in Z ” (Tao, 2016, p. 63).

– While the notions are technically different, Tao, 2016 will not distinguish between the two (will hold them both equally true).

– Note that this notion allows us to reinterpret previously defined operations. For example, addition can be defined as the function $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, defined by $(x, y) \mapsto x + y$.

- We now generalize the ordered pair and Cartesian product.

Definition 3.15 (Ordered n -tuple and n -fold Cartesian product). Let n be a natural number. An **ordered n -tuple** $(x_i)_{1 \leq i \leq n}$ (also denoted (x_1, \dots, x_n)) is a collection of objects x_i , one for every natural number i between 1 and n ; we refer to x_i as the i^{th} **component** of the ordered n -tuple. Two ordered n -tuples $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$. If $(X_i)_{1 \leq i \leq n}$ is an ordered n -tuple of sets, we define their **Cartesian product** $\prod_{1 \leq i \leq n} X_i$ (also denoted $\prod_{i=1}^n X_i$ or $X_1 \times \dots \times X_n$) by

$$\prod_{1 \leq i \leq n} X_i := \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

- Also known as $[n\text{-tuple}]$ **ordered sequence** (of n elements), **finite sequence**.
 - There also exist **infinite sequences** (see Chapter ??).
- Note, again, that while Definition 3.15 simply postulates that an ordered n -tuple and a Cartesian product always exist when needed, it is possible to construct these objects solely in terms of the axioms of set theory (see Exercise 3.5.2).
- The construction of n -fold Cartesian products in Definition 3.15 can be generalized to infinite Cartesian products (see Definition ??).
- Although $X_1 \times X_2 \times X_3$ and $(X_1 \times X_2) \times X_3$ (or $X_1 \times (X_2 \times X_3)$) are often different sets, there exist obvious **bijections** (bijective functions) between them, and it is common to treat them as equal.
- n -fold Cartesian products allow us to consider functions of three or more variables.
 - Again, we can make trouble by using parentheses as above, and by considering different notions of what qualifies as an input, but Tao, 2016 will not distinguish.
- “If x is an object, then (x) is a 1-tuple, which we shall identify with x itself (even though the two are, strictly speaking, not the same object)” (Tao, 2016, p. 64).
 - It follows that if X_1 is any set, then the Cartesian product $\prod_{1 \leq i \leq 1} X_i$ is just X_1 (since the set of all 1-tuples of the elements of X_1 is equal to the set of all elements of X_1 , i.e., X_1).
- **0-tuple**: The ordered n -tuple $()$ containing no objects. Also known as **empty tuple**.
- **Empty Cartesian product**: The Cartesian product $\prod_{1 \leq i \leq 0} X_i$, which is equal to the singleton set $\{()\}$ whose only element is the 0-tuple.
- “If n is a natural number, we often write X^n as shorthand for the n -fold Cartesian product $X^n := \prod_{1 \leq i \leq n} X$ ” (Tao, 2016, p. 65).
 - For example, $X^2 = X \times X$, $X^1 = X$ (assuming $(x) = x$), and $X^0 = \{()\}$.
- We now generalize Lemma 3.1.

Lemma 3.9 (Finite choice). *Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a nonempty set. Then there exists an n -tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. In other words, if each X_i is nonempty, then the set $\prod_{1 \leq i \leq n} X_i$ is also non-empty.*

Proof. We induct on n (starting with the base case $n = 1$; the claim is also vacuously true with $n = 0$ but is not particularly interesting in that case). When $n = 1$, the claim follows from Lemma 3.1. Now suppose inductively that the claim has already been proven for some n ; we will now prove it for $n++$. Let X_1, \dots, X_{n++} be a collection of non-empty sets. By induction hypothesis, we can find an n -tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. Also, since X_{n++} is non-empty, by Lemma 3.1 we may find an object a such that $a \in X_{n++}$. If we thus define the $n++$ -tuple $(y_i)_{1 \leq i \leq n++}$ by setting $y_i := x_i$ when $1 \leq i \leq n$ and $y_i := a$ when $i = n++$ it is clear that $y_i \in X_i$ for all $1 \leq i \leq n++$, thus closing the induction. \square

- Note that Lemma 3.9 can be extended to allow for an infinite number of choices, but doing so requires the **axiom of choice** (see Section ??).

Exercises

- 8/1: 1. Suppose we define the ordered pair (x, y) for any objects x and y by the formula $(x, y) := \{\{x\}, \{x, y\}\}$ (thus using several applications of Axiom 3.3). Thus for instance $(1, 2)$ is the set $\{\{1\}, \{1, 2\}\}$, $(2, 1)$ is the set $\{\{2\}, \{2, 1\}\}$, and $(1, 1)$ is the set $\{\{1\}\}$. Show that such a definition indeed obeys the definition of equality in Definition 3.13, and also whenever X and Y are sets, the Cartesian product $X \times Y$ is also a set. Thus this definition can be validly used as a definition of an ordered pair. For an additional challenge, show that the alternate definition $(x, y) := \{x, \{x, y\}\}$ also verifies the definition of equality in Definition 3.13 and is thus also an acceptable definition of an ordered pair. (For this latter task, one needs the axiom of regularity, and in particular Exercise 3.2.2.)

Proof. To answer the first part of this question (to prove that the given definition of ordered pairs in terms of set theory obeys the appropriate definition of equality), a lemma will be helpful.

Lemma 3.10. *Let a, b, c be objects. Then the singleton sets $\{a\}$ and $\{b\}$ are equal if and only if $a = b$. Additionally, if $a = b$, then the pair sets $\{a, c\}$ and $\{b, c\}$ are equal (the inverse of this implication is also true). To summarize, we have the following three implications:*

$$\begin{aligned} a = b &\iff \{a\} = \{b\} \\ a = b &\implies \{a, c\} = \{b, c\} \\ a \neq b &\implies \{a, c\} \neq \{b, c\} \end{aligned}$$

Proof. To verify that the singleton sets $\{a\}$ and $\{b\}$ are equal if and only if $a = b$, we must prove that the implication “if $\{a\} = \{b\}$, then $a = b$ ” and its converse are both true. Suppose first that $\{a\} = \{b\}$. By Axiom 3.3, $a \in \{a\}$. By Definition 3.1 and the fact that $\{a\} = \{b\}$, this implies that $a \in \{b\}$. But by Axiom 3.3, $a \in \{b\} \implies a = b$. Now suppose that $a = b$. To prove that $\{a\} = \{b\}$, Definition 3.1 tells us that it will suffice to show that every element of $\{a\}$ is an element of $\{b\}$ and vice versa. By Axiom 3.3, a is the only element of $\{a\}$, meaning that to show that every element of $\{a\}$ is an element of $\{b\}$, we need only verify that $a \in \{b\}$. But since $a = b$, Axiom 3.3 asserts that $a \in \{b\}$. A similar argument works in the other direction.

Suppose that $a = b$. To prove that $\{a, c\} = \{b, c\}$, Definition 3.1 tells us that it will suffice to show that every element of $\{a, c\}$ is an element of $\{b, c\}$ and vice versa. By Axiom 3.3, the only elements of $\{a, c\}$ are a and c , so to show that every element of $\{a, c\}$ is an element of $\{b, c\}$, we need only verify that both a and c are elements of $\{b, c\}$. But since $a = b$, Axiom 3.3 implies that $a \in \{b, c\}$, and since $c = c$, Axiom 3.3 similarly implies that $c \in \{b, c\}$. A similar argument works in the other direction.

Suppose that $a \neq b$. To prove that $\{a, c\} \neq \{b, c\}$, Definition 3.1 tells us that it will suffice to find a single element x of $\{a, c\}$ that is not an element of $\{b, c\}$, or a single element y of $\{b, c\}$ that is not an element of $\{a, c\}$. Now for this part of the problem, we must consider separately three cases of possible relations between the three objects a, b, c : $a = c$ (and, hence, $b \neq c$), $b = c$ (and, hence, $a \neq c$), and $a \neq c$ and $b \neq c$. Suppose first that $a = c$. By Axiom 3.3, $b \in \{b, c\}$. Since $b \neq a$ and $b \neq c$, Axiom 3.3 implies that $b \notin \{a, c\}$. Therefore, there exists an element y of $\{b, c\}$ (namely b) such that $y \in \{b, c\}$ and $y \notin \{a, c\}$. Suppose second that $b = c$. By Axiom 3.3, $a \in \{a, c\}$. Since $a \neq b$ and $a \neq c$, Axiom 3.3 implies that $a \notin \{b, c\}$. Therefore, there exists an element x of $\{a, c\}$ (namely a) such that $x \in \{a, c\}$ and $x \notin \{b, c\}$. Suppose third that $a \neq c$ and $b \neq c$. By Axiom 3.3, $a \in \{a, c\}$. Since $a \neq b$ and $a \neq c$, Axiom 3.3 implies that $a \notin \{b, c\}$. Therefore, there exists an element x of $\{a, c\}$ (namely a) such that $x \in \{a, c\}$ and $x \notin \{b, c\}$. \square

Now for the (first part of the) primary proof: To verify that the definition $(x, y) := \{\{x\}, \{x, y\}\}$ of ordered pairs obeys the definition of equality in Definition 3.13, we must prove that the sets $\{\{x\}, \{x, y\}\}$ and $\{\{x'\}, \{x', y'\}\}$ are equal if and only if $x = x'$ and $y = y'$. To prove this, we must show that the implication “if $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, then $x = x'$ and $y = y'$ ” and its converse are both true. Suppose first that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, and suppose for the sake of contradiction that $x \neq x'$ or $y \neq y'$. We divide into two cases.

- (1) ($x \neq x'$): By Axiom 3.3, $\{x\} \in \{\{x\}, \{x, y\}\}$. Since $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, this implies by Definition 3.1 that $\{x\} \in \{\{x'\}, \{x', y'\}\}$. Thus, by Axiom 3.3, $\{x\} = \{x'\}$ or $\{x\} = \{x', y'\}$. Since $x \neq x'$ by hypothesis, Lemma 3.10 implies that $\{x\} \neq \{x'\}$. Thus, we have that $\{x\} = \{x', y'\}$. Since $x' \in \{x', y'\}$ (Axiom 3.3), Definition 3.1 implies that $x' \in \{x\}$. But by Axiom 3.3, this implies that $x' = x$, a contradiction.
- (2) ($y \neq y'$): By (1), $x = x'$, implying by Lemma 3.10 that $\{x\} = \{x'\}$. Additionally, by consecutive applications^[8] of Lemma 3.10, the facts that $x = x'$ and $y \neq y'$ imply that $\{x, y\} \neq \{x', y'\}$. Since $\{x\} = \{x'\}$ and $\{x, y\} \neq \{x', y'\}$, consecutive applications of Lemma 3.10 again imply that $\{\{x\}, \{x, y\}\} \neq \{\{x'\}, \{x', y'\}\}$, a contradiction.

Therefore, $x = x'$ and $y = y'$. Now suppose that $x = x'$ and $y = y'$. To prove that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, Definition 3.1 tells us that it will suffice to show that every element of $\{\{x\}, \{x, y\}\}$ is an element of $\{\{x'\}, \{x', y'\}\}$ and vice versa. By Axiom 3.3, the only elements of $\{\{x\}, \{x, y\}\}$ are $\{x\}$ and $\{x, y\}$, so to show that every element of $\{\{x\}, \{x, y\}\}$ is an element of $\{\{x'\}, \{x', y'\}\}$, we need only verify that both $\{x\}$ and $\{x, y\}$ are elements of $\{\{x'\}, \{x', y'\}\}$. Since $x = x'$, Lemma 3.10 guarantees that $\{x\} = \{x'\}$. But by Axiom 3.3, this implies that $\{x\} \in \{\{x'\}, \{x', y'\}\}$. On the other hand, since $x = x'$ and $y = y'$, we have by consecutive applications of Lemma 3.10 that $\{x, y\} = \{x', y'\}$. But by Axiom 3.3, this implies that $\{x, y\} \in \{\{x'\}, \{x', y'\}\}$. A similar argument can show that every element of $\{\{x'\}, \{x', y'\}\}$ is an element of $\{\{x\}, \{x, y\}\}$. Therefore, $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$.

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Having shown that ordered pairs are indeed well-defined, we now turn our attention to proving that if X and Y are sets, then the Cartesian product $X \times Y$ is also a set. For this purpose, we adapt the procedure from Exercise 3.4.7. First, for each $x \in X$, we define the set A_x of all ordered pairs whose first component is x and whose second component is some element of Y . Second, we create the set $\bigcup_{x \in X} A_x$, which is the set of all ordered pairs whose first component is some element of X and whose second component is some element of Y , as desired. Let's begin.

- (1) Let x be an element of X , let y be any element of Y , and let $P(y, z)$ be the statement $z = (x, y)$. (Note that (x, y) is the ordered pair with first component x and second component y , as defined in Definition 3.13). Since Definition 3.13 guarantees the uniqueness of the ordered pair (a, b) for any objects a, b , for every $y \in Y$, there is exactly one z for which $P(y, z)$ is true. Thus, by Axiom 3.6, there exists a set $\{z : P(y, z) \text{ is true for some } y \in Y\}$ (which we may denote by A_x). In sum, we have defined for every $x \in X$ the set

$$A_x := \{(x, y) : y \in Y\}$$

- (2) Let X be an index set. Then by (1), for every $x \in X$, we have some set A_x . Thus, by Equation 3.1, we can form the union set $\bigcup_{x \in X} A_x$ by defining $\bigcup_{x \in X} A_x := \bigcup \{A_x : x \in X\}$. If we inspect this definition, we can see that this set is the set of all elements in every possible A_x (i.e., every A_x such that $x \in X$). Since every element of any A_x is an ordered pair with first component x and second component in Y , the set $\bigcup_{x \in X} A_x$ is the set whose elements are every ordered pair whose first component is some element of X and whose second component is some element of Y . In sum, we have by Definition 3.14

$$\bigcup_{x \in X} A_x = \{(x, y) : x \in X, y \in Y\} = X \times Y$$

Lastly, we treat the additional challenge. To verify that the definition $(x, y) := \{x, \{x, y\}\}$ of ordered pairs obeys the definition of equality in Definition 3.13, we must prove that the sets $\{x, \{x, y\}\}$ and $\{x', \{x', y'\}\}$ are equal if and only if $x = x'$ and $y = y'$. To prove this, we must show that the implication “if $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$, then $x = x'$ and $y = y'$ ” and its converse are both true. Suppose first that $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$, and suppose for the sake of contradiction that $x \neq x'$ or $y \neq y'$. We divide into two cases.

⁸By Lemma 3.10, $x = x' \implies \{x, y\} = \{x', y\}$, and by Lemma 3.10 again, $y \neq y' \implies \{x', y\} \neq \{x', y'\}$. Thus, transitivity implies that $\{x, y\} \neq \{x', y'\}$.

- (1) ($x \neq x'$): By Axiom 3.3, $x \in \{x, \{x, y\}\}$. Since $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$, this implies by Definition 3.1 that $x \in \{x', \{x', y'\}\}$. Thus, by Axiom 3.3, $x = x'$ or $x = \{x', y'\}$. Since $x \neq x'$ by hypothesis, we have $x = \{x', y'\}$. A similar argument can show that $x' = \{x, y\}$. Substituting, we have that $x = \{\{x, y\}, y'\}$. Since $\{x, y\} \in \{\{x, y\}, y'\}$ (Axiom 3.3), Definition 3.1 implies that $\{x, y\} \in x$. But by Axiom 3.3, we also have $x \in \{x, y\}$, which contradicts Exercise 3.2.2 (which states that we cannot have $A \in B$ and $B \in A$ for two sets A, B).
- (2) ($y \neq y'$): By (1), $x = x'$. Additionally, by consecutive applications of Lemma 3.10, the facts that $x = x'$ and $y \neq y'$ imply that $\{x, y\} \neq \{x', y'\}$. Since $x = x'$ and $\{x, y\} \neq \{x', y'\}$, consecutive applications of Lemma 3.10 again imply that $\{x, \{x, y\}\} \neq \{x', \{x', y'\}\}$, a contradiction.

Therefore, $x = x'$ and $y = y'$. Now suppose that $x = x'$ and $y = y'$. To prove that $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$, Definition 3.1 tells us that it will suffice to show that every element of $\{x, \{x, y\}\}$ is an element of $\{x', \{x', y'\}\}$ and vice versa. By Axiom 3.3, the only elements of $\{x, \{x, y\}\}$ are x and $\{x, y\}$, so to show that every element of $\{x, \{x, y\}\}$ is an element of $\{x', \{x', y'\}\}$, we need only verify that both x and $\{x, y\}$ are elements of $\{x', \{x', y'\}\}$. Since $x = x'$, Axiom 3.3 implies that $x \in \{x', \{x', y'\}\}$. On the other hand, since $x = x'$ and $y = y'$, we have by consecutive applications of Lemma 3.10 that $\{x, y\} = \{x', y'\}$. But by Axiom 3.3, this implies that $\{x, y\} \in \{x', \{x', y'\}\}$. A similar argument can show that every element of $\{x', \{x', y'\}\}$ is an element of $\{x, \{x, y\}\}$. Therefore, $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$. \square

2. Suppose we define an ordered n -tuple to be a surjective function $x : \{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow X$ whose range is some arbitrary set X (so different ordered n -tuples are allowed to have different ranges); we then write x_i for $x(i)$, and also write x as $(x_i)_{1 \leq i \leq n}$. Using this definition, verify that we have $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ if and only if $x_i = y_i$ for all $1 \leq i \leq n$. Also, show that if $(X_i)_{1 \leq i \leq n}$ are an ordered n -tuple of sets, then the Cartesian product, as defined in Definition 3.15, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

Proof. To verify that the definition $x : \{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow X$ of ordered n -tuples makes the statement “ $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ if and only if $x_i = y_i$ for all $1 \leq i \leq n$ ” true, we must show that the implication “if $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$, then $x_i = y_i$ for all $1 \leq i \leq n$ ” and its converse are both true under this definition. Suppose first that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$. Since $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are both functions, Definition 3.6 implies that $((x_i)_{1 \leq i \leq n})(i) = ((y_i)_{1 \leq i \leq n})(i)$ for all $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$. But notationally, this is the same as $x_i = y_i$ for all $1 \leq i \leq n$. Now suppose that $x_i = y_i$ for all $1 \leq i \leq n$. Let $X := \{x_i : 1 \leq i \leq n\}$ and $Y := \{y_i : 1 \leq i \leq n\}$. It can be shown that X and Y are equal by Definition 3.1. Similarly, Exercise 3.1.1 implies that $\{i \in \mathbb{N} : 1 \leq i \leq n\} = \{i \in \mathbb{N} : 1 \leq i \leq n\}$ (note that we are allowed to create this set because Axiom 3.7 implies that the set \mathbb{N} exists, and Axiom 3.5 implies that we are allowed to cherry-pick elements from it). Thus, since the domain, range, and mappings are equal, we have by Definition 3.6 that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$.

As to the second part of the question, begin by using Equation 3.1 to define the set $\bigcup_{i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}} X_i$ containing every element in every set in the ordered n -tuple $(X_i)_{1 \leq i \leq n}$. Now we define the function $(x_i)_{1 \leq i \leq n} : \{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow \bigcup_{i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}} X_i$ (the mapping of this specific function will be irrelevant to our discussion, so we forego defining one). By Exercise 3.4.7, there exists a set $\{(x_i)_{1 \leq i \leq n} : I \rightarrow X \mid I \subseteq \{i \in \mathbb{N} : 1 \leq i \leq n\}, X \subseteq \bigcup_{i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}} X_i\}$ of all partial functions $(x_i)_{1 \leq i \leq n}$ from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to $\bigcup_{i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}} X_i$. Let $P((x_i)_{1 \leq i \leq n} : I \rightarrow X)$ be the statement “ $(I = \{i \in \mathbb{N} : 1 \leq i \leq n\})$ and (for any x, x' in X , $x \in X_i$ and $x' \in X_i$ implies that $x = x'$) and (for all $x \in X$, $((x_i)_{1 \leq i \leq n})(i) = x$ if and only if $x \in X_i$).” As defined, $P((x_i)_{1 \leq i \leq n} : I \rightarrow X)$ is a property pertaining to each $(x_i)_{1 \leq i \leq n} : I \rightarrow X$ in $\{(x_i)_{1 \leq i \leq n} : I \rightarrow X \mid I \subseteq \{i \in \mathbb{N} : 1 \leq i \leq n\}, X \subseteq \bigcup_{i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}} X_i\}$ that is either true or false. Thus, we can use Axiom 3.5 to create the set

$$\{(x_i)_{1 \leq i \leq n} : I \rightarrow X \mid I \subseteq \{i \in \mathbb{N} : 1 \leq i \leq n\}, X \subseteq \bigcup_{i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}} X_i \mid P((x_i)_{1 \leq i \leq n} : I \rightarrow X) \text{ is true}\}$$

which, in layman's terms, is the set of all partial functions that map every number i in $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to an element x_i of X_i , and that are surjective (we know [Definition 3.9] that every $x \in X$ maps to an $i \in I$ since every x is an element of X_i for some i [Equation 3.2] and there exists only one element from each X_i in X [if there were two, $P((x_i)_{1 \leq i \leq n} : I \rightarrow X)$ guarantees that they are equal]). Thus, by the function definition of ordered n -tuples, the above set could be expressed as follows

$$\{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

making it, by Definition 3.15, the set of all ordered n -tuples that should be in the n -fold Cartesian product of $(X_i)_{1 \leq i \leq n}$. \square

- 7/31: 3. Show that the definitions of equality for ordered pair and ordered n -tuple obey the reflexivity, symmetry, and transitivity axioms.

Proof. We first treat ordered pairs. To prove that ordered pairs obey the reflexivity axiom, we must show that for any ordered pair (x, y) , we have $(x, y) = (x, y)$. By Definition 3.13, to prove that $(x, y) = (x, y)$, we must verify that $x = x$ and $y = y$. But since x and y are clearly objects of the same type(s) as themselves, respectively, the reflexive axiom of equality (see Section A.7) implies that $x = x$ and $y = y$. To prove that ordered pairs obey the symmetry axiom, we must show that for any two ordered pairs $(x, y), (x', y')$, if $(x, y) = (x', y')$, then $(x', y') = (x, y)$. Since (x, y) and (x', y') are both objects of the same type (i.e., both ordered pairs), we know by the symmetry axiom of equality that if $(x, y) = (x', y')$, then $(x', y') = (x, y)$. To prove that ordered pairs obey the transitivity axiom, we must show that for any three ordered pairs $(x, y), (x', y'), (x'', y'')$, if $(x, y) = (x', y')$ and $(x', y') = (x'', y'')$, then $(x, y) = (x'', y'')$. By Definition 3.13, $(x, y) = (x', y') \implies (x = x' \text{ and } y = y')$ and $(x', y') = (x'', y'') \implies (x' = x'' \text{ and } y' = y'')$. Since $x = x'$ and $x' = x''$, we have by the transitive axiom of equality that $x = x''$. Similarly, we have $y = y''$. Thus, by Definition 3.13, we have $(x, y) = (x'', y'')$.

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We now turn our attention to ordered n -tuples. To prove that ordered n -tuples obey the reflexivity axiom, we must show that for any ordered n -tuple $(x_i)_{1 \leq i \leq n}$, we have $(x_i)_{1 \leq i \leq n} = (x_i)_{1 \leq i \leq n}$. By Definition 3.15, to prove that $(x_i)_{1 \leq i \leq n} = (x_i)_{1 \leq i \leq n}$, we must verify that $x_i = x_i$ for all $1 \leq i \leq n$. But since each x_i is clearly an object of the same type as itself, the reflexive axiom of equality implies that $x_i = x_i$ for all $1 \leq i \leq n$. To prove that ordered n -tuples obey the symmetry axiom, we must show that for any two ordered n -tuples $(x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n}$, if $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$, then $(y_i)_{1 \leq i \leq n} = (x_i)_{1 \leq i \leq n}$. Since $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are both objects of the same type (i.e., both ordered n -tuples), we know by the symmetry axiom of equality that if $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$, then $(y_i)_{1 \leq i \leq n} = (x_i)_{1 \leq i \leq n}$. To prove that ordered pairs obey the transitivity axiom, we must show that for any three ordered pairs $(x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n}, (z_i)_{1 \leq i \leq n}$, if $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n} = (z_i)_{1 \leq i \leq n}$, then $(x_i)_{1 \leq i \leq n} = (z_i)_{1 \leq i \leq n}$. By Definition 3.15, $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n} \implies x_i = y_i$ for all $1 \leq i \leq n$ and $(y_i)_{1 \leq i \leq n} = (z_i)_{1 \leq i \leq n} \implies y_i = z_i$ for all $1 \leq i \leq n$. Since $x_i = y_i$ and $y_i = z_i$ for all $1 \leq i \leq n$, we have by the transitive axiom of equality that $x_i = z_i$ for all $1 \leq i \leq n$. Thus, by Definition 3.15, we have $(x_i)_{1 \leq i \leq n} = (z_i)_{1 \leq i \leq n}$. \square

4. Let A, B, C be sets. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$, that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, and that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$. (One can of course prove similar identities in which the roles of the left and right factors of the Cartesian product are reversed.)

Proof. Because of the number of statements to prove and the logical simplicity of the implications involved, we will shorthand these proofs.

We begin by proving that $A \times (B \cup C) = (A \times B) \cup (A \times C)$, for which it will suffice (Definition 3.1)

to show that $x \in A \times (B \cup C) \iff x \in (A \times B) \cup (A \times C)$, which can be done as follows.

$$\begin{aligned}
 x \in (A \times (B \cup C)) &\iff x = (a, b) \text{ for some } a \in A \text{ and } b \in B \cup C && \text{Definition 3.14} \\
 &\iff (x = (a, b) \text{ for some } a \in A \text{ and } b \in B \text{ or } x = (a, b) \text{ for some } a \in A \text{ and } b \in C) && \text{Axiom 3.4} \\
 &\iff (x \in (A \times B) \text{ or } x \in (A \times C)) && \text{Definition 3.14} \\
 &\iff x \in (A \times B) \cup (A \times C) && \text{Axiom 3.4}
 \end{aligned}$$

To prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, we must verify (Definition 3.1) that $x \in A \times (B \cap C) \iff x \in (A \times B) \cap (A \times C)$, which can be done as follows.

$$\begin{aligned}
 x \in A \times (B \cap C) &\iff x = (a, b) \text{ for some } a \in A \text{ and } b \in B \cap C && \text{Definition 3.14} \\
 &\iff (x = (a, b) \text{ for some } a \in A \text{ and } b \in B \text{ and } x = (a, b) \text{ for some } a \in A \text{ and } b \in C) && \text{Definition 3.3} \\
 &\iff (x \in A \times B \text{ and } x \in A \times C) && \text{Definition 3.14} \\
 &\iff x \in (A \times B) \cap (A \times C) && \text{Definition 3.3}
 \end{aligned}$$

To prove that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$, Definition 3.1 tells us that it will suffice to show that $x \in A \times (B \setminus C) \iff x \in (A \times B) \setminus (A \times C)$, which can be done as follows.

$$\begin{aligned}
 x \in A \times (B \setminus C) &\iff x = (a, b) \text{ for some } a \in A \text{ and } b \in B \setminus C && \text{Definition 3.14} \\
 &\iff x = (a, b) \text{ for some } a \in A \text{ and } b \in \{y \in B : y \notin C\} && \text{Definition 3.4} \\
 &\iff (x = (a, b) \text{ for some } a \in A \text{ and } b \in B \text{ and } x \neq (a, b) \text{ for any } a \in A \text{ and } b \in C) && \text{Axiom 3.5} \\
 &\iff (x \in A \times B \text{ and } x \notin A \times C) && \text{Definition 3.14} \\
 &\iff x \in \{y \in A \times B : y \notin A \times C\} && \text{Axiom 3.5} \\
 &\iff x \in (A \times B) \setminus (A \times C) && \text{Definition 3.4}
 \end{aligned}$$

□

5. Let A, B, C, D be sets. Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Is it true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$? Is it true that $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$?

Proof. To prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$, Definition 3.1 tells us that it will suffice to show that every element x of $(A \times B) \cap (C \times D)$ is an element of $(A \cap C) \times (B \cap D)$ and vice versa. Suppose first that $x \in (A \times B) \cap (C \times D)$. Then by Definition 3.3, $x \in (A \times B)$ and $x \in (C \times D)$. By Definition 3.14, this implies that $x = (a, b)$ for some $a \in A$ and $b \in B$, and $x = (c, d)$ for some $c \in C$ and $d \in D$. By transitivity, we have that $(a, b) = (c, d)$, implying by Definition 3.13 that $a = c$ and $b = d$. Thus, we know that $x = (a, b)$ for some element a of A that is also an element of C and for some element b of B that is also an element of D . In fact, this implies by Definition 3.3 that $x = (a, b)$ for some $a \in A \cap C$ and $b \in B \cap D$. Therefore, by Definition 3.14, we have $x \in (A \cap C) \times (B \cap D)$. A similar argument works in the reverse direction.

It is *not* necessarily true^[9] that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$; we may verify this through the following counterexample. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then $A \times B = \{(1, 2)\}$ and $C \times D = \{(3, 4)\}$, so $(A \times B) \cup (C \times D) = \{(1, 2), (3, 4)\}$. However, $A \cup C = \{1, 3\}$ and $B \cup D = \{2, 4\}$,

⁹Note, however, that it *is* true that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$: to verify as much will require (Definition 3.2) showing that every element x of $(A \times B) \cup (C \times D)$ is an element of $(A \cup C) \times (B \cup D)$. Suppose that $x \in (A \times B) \cup (C \times D)$. Then by Axiom 3.4, $x \in (A \times B)$ or $x \in (C \times D)$. We now divide into two cases. If $x \in (A \times B)$, then $x = (a, b)$ for some $a \in A$ and $b \in B$ (Definition 3.14). But by Axiom 3.4, $a \in A \implies a \in A \cup C$ and $b \in B \implies b \in B \cup D$. Thus, we have $x = (a, b)$ for some $a \in A \cup C$ and $b \in B \cup D$, implying by Definition 3.14 that $x \in (A \cup C) \times (B \cup D)$. A similar argument can treat the other case.

so $(A \cup C) \times (B \cup D) = \{(1, 2), (1, 4), (3, 2), (3, 4)\}$. Thus, for example, $(1, 4) \in (A \cup C) \times (B \cup D)$, but $(1, 4) \notin (A \times B) \cup (C \times D)$.

Similarly, it is *not* necessarily true^[10] that $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$; we may verify this through the following counterexample. Let $A = \{1\}$, $B = \{2\}$, $C = \{1\}$, and $D = \{4\}$. Then $A \times B = \{(1, 2)\}$ and $C \times D = \{(1, 4)\}$, so $(A \times B) \setminus (C \times D) = \{(1, 2)\}$. However, $A \setminus C = \emptyset$, so $(A \setminus C) \times (B \setminus D) = \emptyset$ as well. Thus, for example, $(1, 2) \in (A \times B) \setminus (C \times D)$, but $(1, 2) \notin (A \setminus C) \times (B \setminus D)$. \square

6. Let A, B, C, D be non-empty sets. Show that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$, and that $A \times B = C \times D$ if and only if $A = C$ and $B = D$. What happens if the hypotheses that the A, B, C, D are all non-empty are removed?

Proof. To prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$, we must verify that the implication “if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$ ” and its converse are both true. Suppose first that $A \times B \subseteq C \times D$. To prove that $A \subseteq C$ and $B \subseteq D$, Definition 3.2 tells us that it will suffice to show that every element a of A is an element of C , and that every element b of B is an element of D , respectively. Let a be an arbitrary element of A , and let b be an arbitrary element of B . Consider (a, b) , the ordered pair whose first component is a and whose second component is b . By Definition 3.14, $(a, b) \in (A \times B)$, implying by Definition 3.2 that $(a, b) \in (C \times D)$. Therefore, by Definition 3.14, $a \in C$ and $b \in D$. Now suppose that $A \subseteq C$ and $B \subseteq D$. To prove that $A \times B \subseteq C \times D$, Definition 3.2 tells us that it will suffice to show that every element x of $A \times B$ is an element of $C \times D$. Let x be an arbitrary element of $A \times B$. By Definition 3.14, $x = (a, b)$ for some $a \in A$ and $b \in B$. It follows by Definition 3.2 that $a \in C$ and $b \in D$. Therefore, $x = (a, b)$ for some $a \in C$ and $b \in D$, so by Definition 3.14, $x \in C \times D$.

To prove that $A \times B = C \times D$ if and only if $A = C$ and $B = D$, we must verify that the implication “if $A \times B = C \times D$, then $A = C$ and $B = D$ ” and its converse are both true. Suppose first that $A \times B = C \times D$. To prove that $A = C$ and $B = D$, Definition 3.1 tells us that it will suffice to show that every element a of A is an element of C and vice versa, and that every element b of B is an element of D and vice versa, respectively. Let a be an arbitrary element of A , and let b be an arbitrary element of B . Consider (a, b) , the ordered pair whose first component is a and whose second component is b . By Definition 3.14, $(a, b) \in (A \times B)$, implying by Definition 3.1 that $(a, b) \in (C \times D)$. Therefore, by Definition 3.14, $a \in C$ and $b \in D$. A similar argument works in the other direction. Now suppose that $A = C$ and $B = D$. To prove that $A \times B = C \times D$, Definition 3.1 tells us that it will suffice to show that every element x of $A \times B$ is an element of $C \times D$ and vice versa. Let x be an arbitrary element of $A \times B$. By Definition 3.14, $x = (a, b)$ for some $a \in A$ and $b \in B$. It follows by Definition 3.1 that $a \in C$ and $b \in D$. Therefore, $x = (a, b)$ for some $a \in C$ and $b \in D$, so by Definition 3.14, $x \in C \times D$. A similar argument works in the other direction.

If the hypotheses that A, B, C, D are all non-empty are removed, not much happens. If C or D is empty, though, then $A \subseteq C$ and $B \subseteq D$ becomes a strong enough condition to prove $A \times B = C \times D$ (since the only subset of the empty set is the empty set, $A \subseteq C = \emptyset \implies A = \emptyset$, and similarly for B and D). \square

7. Let X, Y be sets, and let $\pi_{X \times Y \rightarrow X} : X \times Y \rightarrow X$ and $\pi_{X \times Y \rightarrow Y} : X \times Y \rightarrow Y$ be the maps $\pi_{X \times Y \rightarrow X}(x, y) := x$ and $\pi_{X \times Y \rightarrow Y}(x, y) := y$; these maps are known as the **co-ordinate functions** on $X \times Y$. Show that for any functions $f : Z \rightarrow X$ and $g : Z \rightarrow Y$, there exists a unique function $h : Z \rightarrow X \times Y$ such that $\pi_{X \times Y \rightarrow X} \circ h = f$ and $\pi_{X \times Y \rightarrow Y} \circ h = g$. (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function h is known as the **direct sum** of f and g and is denoted $h = f \oplus g$.

¹⁰Note, however, that it *is* true that $(A \setminus C) \times (B \setminus D) \subseteq (A \times B) \setminus (C \times D)$: to verify as much will require (Definition 3.2) showing that every element x of $(A \setminus C) \times (B \setminus D)$ is an element of $(A \times B) \setminus (C \times D)$. Suppose that $x \in (A \setminus C) \times (B \setminus D)$. Then by Definition 3.14, $x = (a, b)$ for some $a \in A \setminus C$ and $b \in B \setminus D$. By Definition 3.4 and, subsequently, Axiom 3.5, $a \in A \setminus C \implies (a \in A \text{ and } a \notin C)$. Similarly, $b \in B \setminus D \implies (b \in B \text{ and } b \notin D)$. Thus, we know that $x = (a, b)$ for some $a \in A$ and $b \in B$, and $x \neq (c, d)$ for any combination of elements $c \in C$ and $d \in D$. Therefore, by Definition 3.14, $x \in (A \times B)$ and $x \notin (C \times D)$, which implies by Axiom 3.5 and, subsequently, Definition 3.4 that $x \in (A \times B) \setminus (C \times D)$.

Proof. Let $h : Z \rightarrow X \times Y$ be defined by the following.

$$h(z) := (f(z), g(z))$$

To prove that $\pi_{X \times Y \rightarrow X} \circ h = f$, Definition 3.6 tells us that it will suffice to show that $\pi_{X \times Y \rightarrow X} \circ h$ and f have the same domain and range, and that $(\pi_{X \times Y \rightarrow X} \circ h)(z) = f(z)$ for all $z \in Z$. By Definition 3.7, $\pi_{X \times Y \rightarrow X} \circ h : Z \rightarrow X$, so $\pi_{X \times Y \rightarrow X} \circ h$ and f have the same domain and range. Moving on to the next test, we can see by Definition 3.5 that $f(z) \in X$ and $g(z) \in Y$; thus, $(f(z), g(z)) \in X \times Y$ by Definition 3.14. Therefore, by Definition 3.7, for any $z \in Z$, $(\pi_{X \times Y \rightarrow X} \circ h)(z) = \pi_{X \times Y \rightarrow X}(h(z)) = \pi_{X \times Y \rightarrow X}(f(z), g(z)) = f(z)$. A similar argument holds for the other case. \square

8. Let X_1, \dots, X_n be sets. Show that the Cartesian product $\prod_{i=1}^n X_i$ is empty if and only if at least one of the X_i is empty.

Proof. To prove that the Cartesian product $\prod_{i=1}^n X_i$ is empty if and only if at least one of the X_i is empty, we must verify that the implication “if the Cartesian product $\prod_{i=1}^n X_i$ is empty, then at least one of the X_i is empty” and its converse are both true. Suppose first that the Cartesian product $\prod_{i=1}^n X_i$ is empty. Then by the contrapositive of Lemma 3.9, at least one of the X_i is empty. Now suppose that at least one of the X_i is empty, say X_j for some $j \in \{1 \leq i \leq n\}$. Thus, there exists no ordered n -tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$ (because there will always be no $x_j \in X_j$, by Axiom 3.2). Consequently, there are no elements in the set $\{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}$ since the property “ $P((x_i)_{1 \leq i \leq n}) = x_i \in X_i \text{ for all } 1 \leq i \leq n$ ” is false for all $(x_i)_{1 \leq i \leq n}$, as previously established (also see Axiom 3.5). Since $\prod_{i=1}^n X_i = \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}$, the uniqueness of the empty set guarantees that $\prod_{i=1}^n X_i$ is empty. \square

- 8/5: 9. Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that $(\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta) = \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$.

Proof. To prove that $(\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta) = \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$, Definition 3.1 tells us that it will suffice to show that every element x of $(\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta)$ is an element of $\bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$ and vice versa. Suppose first that x is any element of $(\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta)$. Then by Definition 3.3, $x \in \bigcup_{\alpha \in I} A_\alpha$ and $x \in \bigcup_{\beta \in J} B_\beta$. By Equation 3.2, this implies that $x \in A_\alpha$ for some $\alpha \in I$ and $x \in B_\beta$ for some $\beta \in J$. Since $\alpha \in I$ and $\beta \in J$, Definition 3.14 implies that the ordered pair (α, β) is an element of $I \times J$. Additionally, since x is an element of some A_α and some B_β , Definition 3.3 implies that $x \in A_\alpha \cap B_\beta$ for some $(\alpha, \beta) \in I \times J$. Although we officially take an agnostic position as to whether or not the family of sets $A_\alpha \cap B_\beta$ is indexed by one or two labels, if we consider (α, β) to be a single label and $I \times J$ to be a single index set, then we have by Equation 3.2 that $x \in \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$. A similar argument works in the reverse direction. \square

10. If $f : X \rightarrow Y$ is a function, define the **graph** of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$. Show that two functions $f : X \rightarrow Y$, $\tilde{f} : X \rightarrow Y$ are equal if and only if they have the same graph. Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the **vertical line test**), show that there is exactly one function $f : X \rightarrow Y$ whose graph is equal to G .

Proof. To prove that two functions $f : X \rightarrow Y$, $\tilde{f} : X \rightarrow Y$ are equal if and only if they have the same graph, we must verify that the implication “if $f = \tilde{f}$, then they have the same graph” and its converse are both true. Suppose first that $f = \tilde{f}$. By definition, the graph of f is the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$, and the graph of \tilde{f} is the subset of $X \times Y$ defined by $\{(x, \tilde{f}(x)) : x \in X\}$. Thus, the question becomes the problem of demonstrating that $\{(x, f(x)) : x \in X\} = \{(x, \tilde{f}(x)) : x \in X\}$. By Definition 3.1, this will require showing that every element of $\{(x, f(x)) : x \in X\}$ is an element of $\{(x, \tilde{f}(x)) : x \in X\}$ and vice versa. Let $(x, f(x))$ be an arbitrary element of $\{(x, f(x)) : x \in X\}$. Since $x = x$ reflexively and $f(x) = \tilde{f}(x)$ by Definition 3.6, Definition 3.13 implies that $(x, f(x)) = (x, \tilde{f}(x))$ for some $(x, \tilde{f}(x))$ (such that $x \in X$). Therefore, $(x, f(x)) \in \{(x, \tilde{f}(x)) : x \in X\}$. A similar

argument works in the reverse direction. Now suppose that f and \tilde{f} have the same graph, i.e., that $\{(x, f(x)) : x \in X\} = \{(x, \tilde{f}(x)) : x \in X\}$. To prove that $f = \tilde{f}$, already knowing that they have the same domain and range, Definition 3.6 tells us that we need only show in addition that $f(x) = \tilde{f}(x)$ for all $x \in X$. Let x be an arbitrary element of X . To x corresponds the unique ordered pair $(x, f(x))$, an object which satisfies all requirements necessary to be an element of $\{(x, f(x)) : x \in X\}$. By Definition 3.1, this implies that $(x, f(x)) \in \{(x, \tilde{f}(x)) : x \in X\}$, i.e., that $(x, f(x)) = (x, \tilde{f}(x))$ for some $(x, \tilde{f}(x)) \in \{(x, \tilde{f}(x)) : x \in X\}$. But by Definition 3.13, this implies that $f(x) = \tilde{f}(x)$.

As to the other part of the question, we must first guarantee that for every suitable G , there exists a function $f : X \rightarrow Y$ whose graph is equal to G . We must then guarantee the uniqueness of this f . Let's begin. First off, we are given that X and Y are sets. Now for any $x \in X$, let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$. More specifically, let $P(x, y)$ be the property $y \in \{y' \in Y : (x, y') \in G\}$. By the definition of G , we know that for each $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (namely the element $y \in Y$ such that $(x, y) \in G$ — see Axiom 3.5). Therefore, by Definition 3.5, we can define a function $f : X \rightarrow Y$ defined by P on the domain X and range Y . To show that f has G as its graph, we must show that $G = \{(x', f(x')) : x' \in X\}$; showing as much will require (Definition 3.1) verifying that every element of G is an element of $\{(x', f(x')) : x' \in X\}$ and vice versa. Suppose first that $(x, f(x))$ is an arbitrary element of G . Since $x \in X$, Axiom 3.5 implies that $(x, f(x)) \in \{(x', f(x')) : x' \in X\}$. On the other hand, let $(x, f(x))$ be an arbitrary element of $\{(x', f(x')) : x' \in X\}$. By the definition of f (specifically, the property P), $f(x) \in \{y' \in Y : (x, y') \in G\}$, implying by Axiom 3.5 that $(x, f(x)) \in G$. Now suppose for the sake of contradiction that $f : X \rightarrow Y$ and $\tilde{f} : X \rightarrow Y$ are two different functions whose graphs are both equal to G , as defined. Since the functions are distinct but have the same domain and range, Definition 3.6 implies the existence of at least one element $x \in X$ such that $f(x) \neq \tilde{f}(x)$. Now by the definition of a graph, we have $(x, f(x)) \in G$ and $(x, \tilde{f}(x)) \in G$. But since $f(x) \neq \tilde{f}(x)$ (and $f(x) \in Y$ and $\tilde{f}(x) \in Y$), we thus have two distinct elements in the set $\{y \in Y : (x, y) \in G\}$ pertaining to the x in question, a contradiction. Therefore, either f and \tilde{f} have distinct graphs, or they are not different functions, implying that if the graph of both f and \tilde{f} is G , then $f = \tilde{f}$. All of this together implies that there is exactly one function $f : X \rightarrow Y$ corresponding to any suitable G . \square

11. Show that Axiom 3.10 can in fact be deduced from Exercise 3.4.6^[11] and the other axioms of set theory, and thus Exercise 3.4.6 can be used as an alternate formulation of the power set axiom. (Hint: for any two sets X and Y , use Exercise 3.4.6 and the axiom of specification to construct the set of all subsets of $X \times Y$ which obey the vertical line test. Then use Exercise 3.5.10 and the axiom of replacement.)

Proof. To verify Axiom 3.10, we must show that the collection of all the functions from X to Y is in fact a set. To begin, let X and Y be sets. By Definition 3.14 and Exercise 3.5.1, the Cartesian product $X \times Y$ is a set. Thus, by Exercise 3.4.6, there exists a set $2^{X \times Y} = \{Z : Z \text{ is a subset of } X \times Y\}$ of all subsets of $X \times Y$ (note that any subset of $X \times Y$ is a set of ordered pairs whose first component is an element $x \in X$ and whose second component is an element $y \in Y$). Now for every $Z \in 2^{X \times Y}$, let $P(Z)$ be a property pertaining to Z . More specifically, let $P(Z)$ be the statement “for every $x \in X$, exactly one ordered pair (x, y) is an element of Z , where y is some element of Y .” Thus, by Axiom 3.5, we can create the set $\{Z \in 2^{X \times Y} : P(Z) \text{ is true}\}$, which by definition is the set of all subsets of $X \times Y$ which obey the vertical line test. But by Exercise 3.5.10, for every set that obeys the vertical line test, there is exactly one function $f : X \rightarrow Y$ whose graph is equal to that set. Thus, if we let $P(Z, f)$ be the statement “ Z is the graph of f ,” then we know that for each $Z \in \{Z \in 2^{X \times Y} : P(Z) \text{ is true}\}$, there is exactly one, i.e., at most one f for which $P(Z, f)$ is true. Therefore, Axiom 3.6 implies the existence of the set $Y^X = \{f : P(Z, f) \text{ is true for some } Z \in \{Z \in 2^{X \times Y} : P(Z) \text{ is true}\}\}$, which is the set of all functions from X to Y . \square

¹¹Isn't this circular since Exercise 3.4.6 was deduced from Axiom 3.10? I suppose it doesn't matter — it just means that we must postulate either Axiom 3.10 or Exercise 3.4.6 (we cannot forego axiomatizing both in the same way that we could forego axiomatizing specification [Exercise 3.1.11]).

12. This exercise will establish a rigorous version of Proposition 2.3. Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a function, and let c be a natural number. Show that there exists a function $a : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$a(0) = c$$

and

$$a(n++) = f(n, a(n)) \text{ for all } n \in \mathbb{N}$$

and furthermore that this function is unique. (Hint: first show inductively, by a modification of the proof of Lemma 3.9, that for every natural number $N \in \mathbb{N}$, there exists a unique function $a_N : \{n \in \mathbb{N} : n \leq N\} \rightarrow \mathbb{N}$ such that $a_N(0) = c$ and $a_N(n++) = f(n, a_N(n))$ for all $n \in \mathbb{N}$ such that $n < N$.) For an additional challenge, prove this result without using any properties of the natural numbers other than the Peano axioms directly (in particular, without using the ordering of the natural numbers, and without appealing to Proposition 2.3). (Hint: first show inductively, using only the Peano axioms and basic set theory, that for every natural number $N \in \mathbb{N}$, there exists a unique pair A_N, B_N of subsets of \mathbb{N} which obeys the following properties: (a) $A_N \cap B_N = \emptyset$, (b) $A_N \cup B_N = \mathbb{N}$, (c) $0 \in A_N$, (d) $N++ \in B_N$, (e) Whenever $n \in B_N$, we have $n++ \in B_N$, and (f) Whenever $n \in A_N$ and $n \neq N$, we have $n++ \in A_N$. Once one obtains these sets, use A_N as a substitute for $\{n \in \mathbb{N} : n \leq N\}$ in the previous argument.)

Proof. We first prove that for every natural number $N \in \mathbb{N}$, there exists a unique function $a_N : \{n \in \mathbb{N} : n \leq N\} \rightarrow \mathbb{N}$ such that $a_N(0) = c$ and $a_N(n++) = f(n, a_N(n))$ for all $n \in \mathbb{N}$ such that $n < N$. We induct on N (starting with the base case $N = 1$; the claim is also vacuously true with $N = 0$ but is not particularly interesting in that case). Let's begin.

When $N = 1$, we seek to prove that there exists a unique function $a_1 : \{n \in \mathbb{N} : n \leq 1\} \rightarrow \mathbb{N}$ such that $a_1(0) = c$ and $a_1(n++) = f(n, a_1(n))$ for all $n \in \mathbb{N}$ such that $n < 1$. Since 0 is the only natural number less than 1, we need only show that there exists a unique function $a_1 : \{n \in \mathbb{N} : n \leq 1\} \rightarrow \mathbb{N}$ such that $a_1(0) = c$ and $a_1(1) = f(0, a_1(0))$. We begin by showing that such a function exists. Let the function $a_1 : \{n \in \mathbb{N} : n \leq 1\} \rightarrow \mathbb{N}$ be defined by $a_1(0) := c$ and $a_1(1) := f(0, a_1(0))$ (note that 0 and 1 are the only elements in the domain of a_1 , so defining mappings for only them suffices to define a_1). Since f is defined for all ordered pairs whose components are natural numbers, and 0 and $a_1(0) = c$ are natural numbers, $f(0, a_1(0))$ is defined (and is a natural number, i.e., is an element of the defined range of a_1). Thus, for the base case $N = 1$, there exists a function $a_1 : \{n \in \mathbb{N} : n \leq 1\} \rightarrow \mathbb{N}$ such that $a_1(0) = c$ and $a_1(n++) = f(n, a_1(n))$ for all $n \in \mathbb{N}$ such that $n < 1$. Now we must show that a_1 is unique. Suppose for the sake of contradiction that $\tilde{a}_1 : \{n \in \mathbb{N} : n \leq 1\} \rightarrow \mathbb{N}$ is another function such that $\tilde{a}_1(0) = c$ and $\tilde{a}_1(n++) = f(n, \tilde{a}_1(n))$ for all $n \in \mathbb{N}$ such that $n < 1$ and $a_1 \neq \tilde{a}_1$. Since the functions are distinct but have the same domain and range, Definition 3.6 implies the existence of at least one element $n \in \{n \in \mathbb{N} : n \leq 1\}$ such that $a_1(n) \neq \tilde{a}_1(n)$. As before, \tilde{a}_1 is only defined for 0 and 1, so either $a_1(0) \neq \tilde{a}_1(0)$ or $a_1(1) \neq \tilde{a}_1(1)$. Since both $a_1(0)$ and $\tilde{a}_1(0)$ are equal to c by definition, transitivity implies that $a_1(0) = \tilde{a}_1(0)$, so we must have $a_1(1) \neq \tilde{a}_1(1)$. But $a_1(1) = f(0, a_1(0)) = f(0, \tilde{a}_1(0)) = \tilde{a}_1(1)$, a contradiction. Therefore, if we assert that \tilde{a}_1 is defined like a_1 , then $a_1 = \tilde{a}_1$, proving the uniqueness of a_1 and, hence, the claim for the base case $N = 1$.

Now suppose inductively that we have proven the claim for N , i.e., we know that there exists a unique function $a_N : \{n \in \mathbb{N} : n \leq N\} \rightarrow \mathbb{N}$ such that $a_N(0) = c$ and $a_N(n++) = f(n, a_N(n))$ for all $n \in \mathbb{N}$ such that $n < N$. We seek to prove that there exists a unique function $a_{N++} : \{n \in \mathbb{N} : n \leq N++\} \rightarrow \mathbb{N}$ such that $a_{N++}(0) = c$ and $a_{N++}(n++) = f(n, a_{N++}(n))$ for all $n \in \mathbb{N}$ such that $n < N++$. We begin by showing that such a function exists. If we define a_{N++} by setting $a_{N++}(n) := a_N(n)$ when $n \in \{n \in \mathbb{N} : n \leq N\}$ and $a_{N++}(N++) := f(N, a_{N++}(N))$ (which we know exists for the same reasons detailed in the proof of the base case), it is clear that $a_{N++}(0) = c$ and $a_{N++}(n++) = f(n, a_{N++}(n))$ for all $n \in \mathbb{N}$ such that $n < N++$, thus closing the induction. The uniqueness of a_{N++} can be verified in much the same way that the uniqueness of a_1 was demonstrated for the base case.

Now we tie it all together. Let $a : \mathbb{N} \rightarrow \mathbb{N}$ be a function defined by $a(n) := a_n(n)$, where a_n is the appropriate function defined above. Thus, we have

$$a(0) = a_0(0) = c$$

and

$$a(n++) = a_{n++}(n++) = f(n, a_{n++}(n)) = f(n, a_n(n)) = f(n, a(n)) \text{ for all } n \in \mathbb{N}$$

as desired. Now we must verify the uniqueness of a . Suppose for the sake of contradiction that $\tilde{a} : \mathbb{N} \rightarrow \mathbb{N}$ is another function such that $\tilde{a}(0) = c$ and $\tilde{a}(n++) = f(n, \tilde{a}(n))$ for all $n \in \mathbb{N}$ and $a \neq \tilde{a}$. Since the functions are distinct but have the same domain and range, Definition 3.6 implies the existence of at least one element $n \in \mathbb{N}$ such that $a(n) \neq \tilde{a}(n)$. Thus, to find a contradiction, we must show that $a(n) = \tilde{a}(n)$ for all $n \in \mathbb{N}$, which we may do by induction. For the base case $n = 0$, we have $a(0) = c = \tilde{a}(0)$. Now suppose inductively that $a(n) = \tilde{a}(n)$. Then $a(n++) = f(n, a(n)) = f(n, \tilde{a}(n)) = \tilde{a}(n++)$, which closes the induction. Thus, we have found the desired contradiction. Therefore, if we assert that \tilde{a} is defined like a , then $a = \tilde{a}$, proving the uniqueness of a and, hence, completing the proof.

To treat the additional challenge, a couple of lemmas will be helpful.

Lemma 3.11. *Let A, B be sets. Then we have $A \setminus B \subseteq A$.*

Proof. To prove that $A \setminus B \subseteq A$, Definition 3.2 tells us that it will suffice to show that every element x of $A \setminus B$ is an element of A . Let x be an arbitrary element of $A \setminus B$. By Definition 3.4, this implies that $x \in \{y \in A : y \notin B\}$. Thus, by Axiom 3.5, $x \in A$ (and $x \notin B$, but this is not relevant to our present discussion). \square

8/6: **Lemma 3.12.** *Let A, B, C, D be sets. Then if $A \subsetneq C$, $B \subsetneq D$, and $C \cap D = \emptyset$, we have $A \cup B \subsetneq C \cup D$.*

Proof. Suppose $A \subsetneq C$ and $B \subsetneq D$. To show that $A \cup B \subsetneq C \cup D$, Definition 3.2 tells us that we must show that $A \cup B \subseteq C \cup D$ and $A \cup B \neq C \cup D$. We begin by showing that $A \cup B \subseteq C \cup D$. To do so, Definition 3.2 tells us that we must verify that every element x of $A \cup B$ is an element of $C \cup D$. Let x be an arbitrary element of $A \cup B$. Then by Axiom 3.4, $x \in A$ or $x \in B$. We divide into two cases. If $x \in A$, then Definition 3.2 implies that $x \in C$, and Axiom 3.4 implies that $x \in C \cup D$. Similarly, if $x \in B$, then Definition 3.2 implies that $x \in D$, and Axiom 3.4 implies that $x \in C \cup D$. Therefore, $A \cup B \subseteq C \cup D$. Now to show that $A \cup B \neq C \cup D$, Definition 3.1 tells us that we must find an element of $A \cup B$ that is not an element of $C \cup D$, or an element of $C \cup D$ that is not an element of $A \cup B$. Since every element of $A \cup B$ is an element of $C \cup D$ as previously established, we must find an element of $C \cup D$ that is not an element of $A \cup B$. First off, there exists an element (which we shall call x) of C that is not an element of A ¹² (if such an element did not exist, then every element of C would be an element of A , which, when combined with the fact that every element of A is an element of C , would imply that $A = C$, contradicting the fact that $A \subsetneq C$). We wish to show that this object x is an element of $C \cup D$ but not an element of $A \cup B$. Clearly, since $x \in C$, $x \in C \cup D$ (Axiom 3.4). On the other hand, if we suppose for the sake of contradiction that $x \in B$, we will find that $x \in D$ (Definition 3.2), implying by Definition 3.3 that $x \in C \cap D$ and contradicting the fact that $C \cap D = \emptyset$. Thus, $x \notin B$. Since it is also not an element of A by hypothesis, Axiom 3.4 implies that $x \notin A \cup B$. Therefore, $A \cup B \neq C \cup D$ and, hence, $A \cup B \subsetneq C \cup D$. \square

8/5: Now for the primary proof (of the additional challenge). We first prove that for every natural number $N \in \mathbb{N}$, there exists a unique pair A_N, B_N of subsets of \mathbb{N} which obeys the following properties: (a) $A_N \cap B_N = \emptyset$, (b) $A_N \cup B_N = \mathbb{N}$, (c) $0 \in A_N$, (d) $N++ \in B_N$, (e) Whenever $n \in B_N$, we have $n++ \in B_N$, and (f) Whenever $n \in A_N$ and $n \neq N$, we have $n++ \in A_N$. We induct on N . Let's begin.

For the base case $N = 0$, let $A_0 = \{0\}$ and let $B_0 = \mathbb{N} \setminus A_0$. First, we show that A_0, B_0 are in fact subsets of \mathbb{N} . By Axiom 3.3, 0 is the only element of A_0 . Thus, by Definition 3.2, to show that $A_0 \subseteq \mathbb{N}$, we need only show that $0 \in \mathbb{N}$. But by Axiom 3.7, $0 \in \mathbb{N}$. On the other hand, Lemma 3.11 implies that $B_0 = \mathbb{N} \setminus A_0 \subseteq \mathbb{N}$. Next, we check that A_0, B_0 satisfy the properties (a)-(f).

(a) By Exercise 3.1.6g, we have $A_0 \cap B_0 = A_0 \cap (\mathbb{N} \setminus A_0) = \emptyset$.

¹²Note that we could also choose to study an element of D that is not an element of B ; the choice to study an element of C that is not an element of A is wholly arbitrary.

- (b) By Exercise 3.1.6g, we also have $A_0 \cup B_0 = A_0 \cup (\mathbb{N} \setminus A_0) = \mathbb{N}$.
- (c) By Axiom 3.3, $0 \in A_0$.
- (d) By Axiom 2.2, $0++$ is a natural number, so Axiom 3.7 implies that $0++ \in \mathbb{N}$. Since Axiom 2.3 asserts that $0++ \neq 0$, $0++ \notin A_0$ by Axiom 3.3. Thus, by Axiom 3.5, $0++ \in \{n \in \mathbb{N} : n \notin A_0\}$. Consequently, by Definition 3.4, $0++ \in \mathbb{N} \setminus A_0$, implying that $0++ \in B_0$.
- (e) For any $n \in B_0$, $n \in \mathbb{N}$ (Definition 3.2). Thus, by Axiom 3.7, $n++ \in \mathbb{N}$. Since $n++ \neq 0$ (Axiom 2.3), $n++ \notin A_0$ by Axiom 3.3. Thus, by Axiom 3.5 and subsequently Definition 3.4, $n++ \in \mathbb{N} \setminus A_0$, which implies that $n++ \in B_0$.
- (f) By Axiom 3.3, 0 is the only element of A_0 , so there is no $n \in A_0$ such that $n \neq 0$. Thus, the statement “whenever $n \in A_0$ and $n \neq 0$, we have $n++ \in A_0$ ” is vacuously true.

Last, we prove the uniqueness of A_0 and B_0 . Suppose for the sake of contradiction that there exist a pair of subsets A_0' and B_0' of \mathbb{N} that satisfy all the same properties as A_0 and B_0 and such that $A_0' \neq A_0$ and $B_0' \neq B_0$. We begin by examining the construction of B_0' . By property (d), $0++ \in B_0'$, and by property (e), whenever $n \in B_0'$, we have $n++ \in B_0'$; thus, B_0' contains the successors of every natural number. Now consider A_0' . By property (c), $0 \in A_0'$. Additionally, since $A_0 \neq A_0'$ by hypothesis, Definition 3.1 implies that there is some element of A_0 that is not in A_0' or there is some element of A_0' that is not in A_0 . However, since 0 is the only element of A_0 (Axiom 3.3) and $0 \in A_0'$ as previously established, there is no element of A_0 that is not in A_0' . Thus, there must be some element $n \neq 0$ of A_0' that is not in A_0 . Let's take a closer look at this n : since A_0' is a subset of \mathbb{N} , n is a natural number. Since $n \neq 0$, n is the successor of some natural number (Axiom 2.3); this implies that n is an element of B_0' . Thus, $n \in A_0' \cap B_0'$ (Definition 3.3), contradicting property (a). Therefore, if we assert that A_0' and B_0' satisfy all the same properties as A_0 and B_0 , then $A_0' = A_0$ and $B_0' = B_0$, proving the uniqueness of A_0 and B_0 .

Now suppose inductively that we have proven that for some natural number N , there exists a unique pair A_N, B_N of subsets of \mathbb{N} which obeys properties (a)-(f) above. We seek to prove the same for $N++$. Let $A_{N++} = A_N \cup \{N++\}$ and let $B_{N++} = \mathbb{N} \setminus A_{N++}$. First, we show that A_{N++}, B_{N++} are in fact subsets of \mathbb{N} . Since N is a natural number, by Axiom 2.2, $N++$ is a natural number, i.e., is an element of \mathbb{N} (Axiom 3.7). Thus, since the singleton set $\{N++\}$ contains only $N++$ (Axiom 3.3), $\{N++\} \subseteq \mathbb{N}$ (Definition 3.2). This, combined with the fact that $A_N \subseteq \mathbb{N}$ by hypothesis, implies that $A_{N++} = A_N \cup \{N++\} \subseteq \mathbb{N}$ (Exercise 3.1.7). On the other hand, Lemma 3.11 implies that $B_{N++} = \mathbb{N} \setminus A_{N++} \subseteq \mathbb{N}$. Next, we check that A_N, B_N satisfy the properties (a)-(f).

- (a) By Exercise 3.1.6g, we have $A_{N++} \cap B_{N++} = A_{N++} \cap (\mathbb{N} \setminus A_{N++}) = \emptyset$.
- (b) By Exercise 3.1.6g, we also have $A_{N++} \cup B_{N++} = A_{N++} \cup (\mathbb{N} \setminus A_{N++}) = \mathbb{N}$.
- (c) Since $0 \in A_N$, Axiom 3.4 implies that $0 \in A_N \cup \{N++\} = A_{N++}$.
- (d) As previously established, $N++$ is a natural number. Thus, by Axiom 2.2, $(N++)++$ is a natural number. Consequently, Axiom 3.7 implies that $(N++)++ \in \mathbb{N}$. Now suppose for the sake of contradiction that $(N++)++ \in A_{N++}$. Then $(N++)++ \in A_N \cup \{N++\}$, so by Axiom 3.4, $(N++)++ \in A_N$ or $(N++)++ \in \{N++\}$. Since $(N++)++ \neq N++$ (Axiom 2.4), $(N++)++ \notin \{N++\}$ (Axiom 3.3); thus, we must have $(N++)++ \in A_N$. But by property (d), $N++ \in B_N$, implying by property (e) that $(N++)++ \in B_N$. Thus, $(N++)++ \in A_N \cap B_N$ by Definition 3.3, implying that $A_N \cap B_N \neq \emptyset$, which contradicts property (a). Thus, $(N++)++ \notin A_{N++}$. Therefore, by Axiom 3.5 and Definition 3.4, $(N++)++ \in \mathbb{N} \setminus A_{N++} = B_{N++}$.
- (e) We want to show that for any $n \in B_{N++}$, we have $n++ \in B_{N++}$. As such, let n be an arbitrary element of B_{N++} . By Definition 3.4 and Axiom 3.5, to show that $n++ \in B_{N++}$, it will suffice to show that $n++ \in \mathbb{N}$ and $n++ \notin A_{N++}$. We first show that $n++ \in \mathbb{N}$. Since $n \in B_{N++}$, Definition 3.4 and Axiom 3.5 imply that $n \in \mathbb{N}$. Thus, by Axiom 3.7, $n++ \in \mathbb{N}$. We now turn our attention to demonstrating that $n++ \notin A_{N++}$, which we may do by the following contradiction argument. Suppose for the sake of contradiction that $n++ \in A_{N++}$. Then by Axiom 3.4, $n++ \in A_N$ or $n++ \in \{N++\}$. We now divide into two cases. Suppose first that $n++ \in A_N$. Since $n \in B_{N++} = \mathbb{N} \setminus A_{N++} = \mathbb{N} \setminus (A_N \cup \{N++\})$ by hypothesis, $n \in (\mathbb{N} \setminus A_N) \cap (\mathbb{N} \setminus \{N++\}) = B_N \cap (\mathbb{N} \setminus \{N++\})$

by Exercise 3.1.6h. Thus, by Definition 3.3, $n \in B_N$, so by property (e), $n++ \in B_N$. Since $n++$ is also an element of A_{N++} by hypothesis, this implies by Definition 3.3 that $n \in A_{N++} \cap B_{N++}$, contradicting the fact that $A_{N++} \cap B_{N++} = \emptyset$ by property (a). Now suppose that $n++ \in \{N++\}$. Then Axiom 3.3 implies that $n++ = N++$. Thus, Axiom 2.4 implies that $n = N$. Since $N \in A_N$ (if N were an element of B_N then either $N = N++$ or N equals some iterated successor of $N++$; both eventualities are forebidden by Axioms 2.3 and 2.4), we have that $n \in A_N$. Thus, by Axiom 3.4, $n \in A_{N++}$. Since n is also an element of B_{N++} by hypothesis, this implies by Definition 3.3 that $n \in A_{N++} \cap B_{N++}$, contradicting the fact that $A_{N++} \cap B_{N++} = \emptyset$ by property (a). Therefore, $n++ \notin A_{N++}$.

- (f) For any $n \in A_{N++}$, Axiom 3.4 implies that $n \in A_N$ or $n \in \{N++\}$. We divide into two cases. Suppose first that $n \in A_N$. Then either $n = N$ or $n \neq N$. We divide into two cases again. If $n = N$, then $n++ = N++ \in \{N++\}$ as previously established, implying that $n++ \in A_N \cup \{N++\} = A_{N++}$ by Axiom 3.4. On the other hand, if $n \neq N$, then $n++ \in A_N$ since A_N obeys property (f), implying by Axiom 3.4 again that $n++ \in A_{N++}$. Now suppose that $n \in \{N++\}$. Then by Axiom 3.3, $n = N++$, and property (f) does not apply. Therefore, whenever $n \in A_{N++}$ and $n \neq N++$, we have $n++ \in A_{N++}$.

8/6:

Last, we prove the uniqueness of A_{N++} and B_{N++} . Suppose for the sake of contradiction that there exist a pair of subsets A_{N++}' and B_{N++}' of \mathbb{N} that satisfy all the same properties as A_{N++} and B_{N++} and such that $A_{N++}' \neq A_{N++}$ and $B_{N++}' \neq B_{N++}$. By property (c), $0 \in A_{N++}'$, and by property (f), whenever $n \in A_{N++}'$ and $n \neq N++$, $n++ \in A_{N++}'$. Since this also defines exactly the elements of A_{N++} , we know that at least every element of A_{N++} is an element of A_{N++}' . This, combined with the fact that $A_{N++}' \neq A_{N++}$, implies that $A_{N++} \subsetneq A_{N++}'$ (Definition 3.2). Similarly, we can use properties (d) and (e) to show that $B_{N++} \subsetneq B_{N++}'$. These two proper subset relations combined with the fact that $A_{N++}' \cap B_{N++}' = \emptyset$ (property (a)) imply by Lemma 3.12 that $A_{N++} \cup B_{N++} \subsetneq A_{N++}' \cup B_{N++}'$. But by property (b), this implies that $\mathbb{N} \subsetneq \mathbb{N}$, i.e., that $\mathbb{N} \neq \mathbb{N}$, a contradiction. Therefore, if we assert that A_{N++}' and B_{N++}' have all the same properties as A_{N++} and B_{N++} , then $A_{N++}' = A_{N++}$ and $B_{N++}' = B_{N++}$, proving the uniqueness of A_{N++} and B_{N++} .

We now prove that for every natural number $N \in \mathbb{N}$, there exists a unique function $a_N : A_N \rightarrow \mathbb{N}$ such that $a_N(0) = c$ and $a_N(n++) = f(n, a_N(n))$ for all $n \in A_N$ such that $n \neq N$. We induct on N (starting with the base case $N = 1$; the claim is also vacuously true with $N = 0$ but is not particularly interesting in that case). Let's begin.

When $N = 1$, we seek to prove that there exists a unique function $a_1 : A_1 \rightarrow \mathbb{N}$ such that $a_1(0) = c$ and $a_1(n++) = f(n, a_1(n))$ for all $n \in A_1$ such that $n \neq 1$. By the preceding proof of the existence and uniqueness of A_N, B_N , we know that $A_1 = A_0 \cup \{1\} = \{0\} \cup \{1\}$. By Exercise 3.1.3, $\{0\} \cup \{1\} = \{0, 1\}$, so A_1 is the pair set formed by 0 and 1. Now since 0 is the only element of A_1 that is not equal to 1 (we know that $0 \neq 1$ by Axiom 2.3), we need only show that there exists a unique function $a_1 : A_1 \rightarrow \mathbb{N}$ such that $a_1(0) = c$ and $a_1(1) = f(0, a_1(0))$. We begin by showing that such a function exists. Let the function $a_1 : A_1 \rightarrow \mathbb{N}$ be defined by $a_1(0) := c$ and $a_1(1) := f(0, a_1(0))$ (since 0 and 1 are the only elements of A_1 by Axiom 3.3, defining mappings for only them suffices to define a_1). Since f is defined for all ordered pairs whose components are natural numbers, and 0 and $a_1(0) = c$ are natural numbers, $f(0, a_1(0))$ is defined (and is a natural number, i.e., is an element of the defined range of a_1). Thus, for the base case $N = 1$, there exists a function $a_1 : A_1 \rightarrow \mathbb{N}$ such that $a_1(0) = c$ and $a_1(n++) = f(n, a_1(n))$ for all $n \in A_1$ such that $n \neq 1$. Now we must show that a_1 is unique. Suppose for the sake of contradiction that $\tilde{a}_1 : A_1 \rightarrow \mathbb{N}$ is another function such that $\tilde{a}_1(0) = c$ and $\tilde{a}_1(n++) = f(n, \tilde{a}_1(n))$ for all $n \in A_1$ such that $n \neq 1$ and $a_1 \neq \tilde{a}_1$. Since the functions are distinct but have the same domain and range, Definition 3.6 implies the existence of at least one element $n \in A_1$ such that $a_1(n) \neq \tilde{a}_1(n)$. As before, \tilde{a}_1 is only defined for 0 and 1, so either $a_1(0) \neq \tilde{a}_1(0)$ or $a_1(1) \neq \tilde{a}_1(1)$. Since both $a_1(0)$ and $\tilde{a}_1(0)$ are equal to c by definition, transitivity implies that $a_1(0) = \tilde{a}_1(0)$, so we must have $a_1(1) \neq \tilde{a}_1(1)$. But $a_1(1) = f(0, a_1(0)) = f(0, \tilde{a}_1(0)) = \tilde{a}_1(1)$, a contradiction. Therefore, if we assert that \tilde{a}_1 is defined like a_1 , then $a_1 = \tilde{a}_1$, proving the uniqueness of a_1 and, hence, the claim for the base case $N = 1$.

Now suppose inductively that we have proven the claim for N , i.e., we know that there exists a unique

function $a_N : A_N \rightarrow \mathbb{N}$ such that $a_N(0) = c$ and $a_N(n++) = f(n, a_N(n))$ for all $n \in A_N$ such that $n \neq N$. We seek to prove that there exists a unique function $a_{N++} : A_{N++} \rightarrow \mathbb{N}$ such that $a_{N++}(0) = c$ and $a_{N++}(n++) = f(n, a_{N++}(n))$ for all $n \in A_{N++}$ such that $n \neq N++$. We begin by showing that such a function exists. If we define a_{N++} by setting $a_{N++}(n) := a_N(n)$ when $n \in A_N$ (recall that since $A_{N++} = A_N \cup \{N++\}$, every element save one of A_{N++} is an element of A_N and, thus, is mapped by a_N by hypothesis) and $a_{N++}(N++) := f(N, a_{N++}(N))$ (which we know exists for the same reasons detailed in the proof of the base case), it is clear that $a_{N++}(0) = c$ and $a_{N++}(n++) = f(n, a_{N++}(n))$ for all $n \in A_{N++}$ such that $n \neq N++$, thus closing the induction. The uniqueness of a_{N++} can be verified in much the same way that the uniqueness of a_1 was demonstrated for the base case.

Now we tie it all together. Let $a : \mathbb{N} \rightarrow \mathbb{N}$ be a function defined by $a(n) := a_n(n)$, where a_n is the appropriate function defined above. Thus, we have

$$a(0) = a_0(0) = c$$

and

$$a(n++) = a_{n++}(n++) = f(n, a_{n++}(n)) = f(n, a_n(n)) = f(n, a(n)) \text{ for all } n \in \mathbb{N}$$

as desired. Now we must verify the uniqueness of a . Suppose for the sake of contradiction that $\tilde{a} : \mathbb{N} \rightarrow \mathbb{N}$ is another function such that $\tilde{a}(0) = c$ and $\tilde{a}(n++) = f(n, \tilde{a}(n))$ for all $n \in \mathbb{N}$ and $a \neq \tilde{a}$. Since the functions are distinct but have the same domain and range, Definition 3.6 implies the existence of at least one element $n \in \mathbb{N}$ such that $a(n) \neq \tilde{a}(n)$. Thus, to find a contradiction, we must show that $a(n) = \tilde{a}(n)$ for all $n \in \mathbb{N}$, which we may do by induction. For the base case $n = 0$, we have $a(0) = c = \tilde{a}(0)$. Now suppose inductively that $a(n) = \tilde{a}(n)$. Then $a(n++) = f(n, a(n)) = f(n, \tilde{a}(n)) = \tilde{a}(n++)$, which closes the induction. Thus, we have found the desired contradiction. Therefore, if we assert that \tilde{a} is defined like a , then $a = \tilde{a}$, proving the uniqueness of a and, hence, completing the additional challenge. \square

13. The purpose of this exercise is to show that there is essentially only one version of the natural number system in set theory (cf. the note in Section 2.1 on isomorphic number systems). Suppose we have a set \mathbb{N}' of “alternative natural numbers,” an “alternative zero” $0'$, and an “alternative increment operation” which takes any alternative natural number $n' \in \mathbb{N}'$ and returns another alternative natural number $n'++' \in \mathbb{N}'$, such that the Peano axioms (Axioms 2.1-2.5) all hold with the natural numbers, zero, and increment replaced by their alternative counterparts. Show that there exists a bijection $f : \mathbb{N} \rightarrow \mathbb{N}'$ from the natural numbers to the alternative natural numbers such that $f(0) = 0'$, and such that for any $n \in \mathbb{N}$ and $n' \in \mathbb{N}'$, we have $f(n) = n'$ if and only if $f(n++) = n'++'$. (Hint: use Exercise 3.5.12.)

Proof. Let $g : \mathbb{N} \times \mathbb{N}' \rightarrow \mathbb{N}'$ be defined by $g(a, b) = b++'$. Then by Exercise 3.5.12, there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}'$ such that $f(0) = c$ and $f(n++) = g(n, f(n))$ for all $n \in \mathbb{N}$. If we let $c = 0'$, then we have $f(0) = 0'$. On the other hand, if $f(n) = n'$, then we have $f(n++) = g(n, f(n)) = g(n, n') = n'++'$ and vice versa. \square

3.6 Cardinality of Sets

8/8:

- In Chapter 2, the natural numbers were defined axiomatically — we assumed an object 0, an increment operation, and five axioms. As previously noted, this is quite different from the intuitive conceptualization of the natural numbers as measuring how many elements are in a set. More formally...
 - Chapter 2 treats natural numbers as **ordinals**: “The ordinals are First, Second, Third, ..., and are used to order a sequence of objects” (Tao, 2016, p. 68).
 - Intuitively, natural numbers are treated as **cardinals**: “The cardinals are One, Two, Three, ..., and are used to count how many things there are in a set” (Tao, 2016, p. 68).
 - There is a subtle difference between the two (especially in the infinite case), but this is beyond the scope of Tao, 2016.

- 8/9:
- This section is concerned with using the natural numbers to count sets.
 - We first define a rigorous basis for two sets having the same number of elements. Tao, 2016 intuitively rationalizes Cantor's one-to-one matching principle by saying "Indeed, this is how we first learn to count a set: we correspond the set we are trying to count with another set, such as a set of fingers on your hand" (Tao, 2016, p. 68).

Definition 3.16 (Equal cardinality). We say that two sets X and Y have **equal cardinality** iff there exists a bijection $f : X \rightarrow Y$ from X to Y .

- Definition 3.16 allows us to easily prove that sets have equal cardinality, but does not make it immediately easy to prove that sets have unequal cardinality.
 - For example, an obvious bijection exists between $\{0, 1, 2\}$ and $\{3, 4, 5\}$, but to prove that $\{0, 1, 2\}$ and $\{3, 4\}$ have unequal cardinality, we would have to verify that all functions $f : \{0, 1, 2\} \rightarrow \{3, 4\}$ are not bijective, or make use of Proposition 3.5.
- Note that Definition 3.16 treats both the finite and infinite case (though remember that we have not yet rigorously defined this distinction).
- Note that "the fact that two sets have equal cardinality does not preclude one of the sets from containing the other" (Tao, 2016, p. 69).
 - For example, "if X is the set of natural numbers and Y is the set of even natural numbers, then the map $f : X \rightarrow Y$ defined by $f(n) := 2n$ is a bijection from X to Y ... and so X and Y have equal cardinality, despite Y being a subset of X and seeming intuitively as if it should only have 'half' of the elements of X " (Tao, 2016, p. 69).

- The notion of having equal cardinality is an equivalence relation (see Exercise 3.6.1).
- We now define precisely what it means for a set to have n elements.

Definition 3.17. Let n be a natural number. A set X is said to have **cardinality n** iff it has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. We also say that **X has n elements** iff it has cardinality n .

- Note that we could compare cardinality with the set $\{i \in \mathbb{N} : i < n\}$ as opposed to the set $\{i \in \mathbb{N} : 1 \leq i \leq n\}$.
- It would make sense that each set should have exactly one cardinality (i.e., not multiple). Let's make sure that Definition 3.17 implies this.

Proposition 3.5 (Uniqueness of cardinality). *Let X be a set with some cardinality n . Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.*

We first need a lemma.

Lemma 3.13. *Suppose that $n \geq 1$, and X has cardinality n . Then X is non-empty, and if x is any element of X , then the set $X - \{x\}$ (i.e., X with the element x removed) has cardinality $n - 1$ ^[13].*

Proof. If X is empty, then it clearly cannot have the same cardinality as the nonempty set $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, as there is no bijection from the empty set to a non-empty set^[14]. Now let x be an element of X . Since X has the same cardinality as $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, we thus have a bijection f from X to $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. In particular, $f(x)$ is a natural number between 1 and n . Now define the

¹³"Strictly speaking, $n - 1$ has not yet been defined in this text. For the purposes of this lemma, we define $n - 1$ to be the unique natural number m such that $m++ = n$; this m is given by [Exercise 2.2.2]" (Tao, 2016, p. 70).

¹⁴Any potential bijection would be the empty function $f : \emptyset \rightarrow \{i \in \mathbb{N} : 1 \leq i \leq n\}$. To be bijective, we would have to have exactly one $x \in \emptyset$ such that $f(x) = i$ for all $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$. But even if there were only one i in $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, there would be no $x \in \emptyset$ (because there are no $x \in \emptyset$) such that $f(x) = i$.

function $g : X - \{x\} \rightarrow \{i \in \mathbb{N} : 1 \leq i \leq n - 1\}$ by the following rule: for any $y \in X - \{x\}$, we define $g(y) := f(y)$ if $f(y) < f(x)$, and define $g(y) := f(y) - 1$ if $f(y) > f(x)$. (Note that $f(y)$ cannot equal $f(x)$ since $y \neq x$ and f is a bijection.) It is easy to check that this map is also a bijection^[15], and so $X - \{x\}$ has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. In particular, $X - \{x\}$ has cardinality $n - 1$, as desired. \square

Now we prove the proposition.

Proof of Proposition 3.5. We induct on n . First suppose that $n = 0$. Then X must be empty, and so X cannot have any non-zero cardinality. Now suppose that the proposition is already proven for some n ; we now prove it for $n++$. Let X have cardinality $n++$, and suppose that X also has some other cardinality $m \neq n++$. By Lemma 3.13, X is non-empty, and if x is any element of X , then $X - \{x\}$ has cardinality n and also has cardinality $m - 1$. By induction hypothesis, this means that $n = m - 1$, which implies that $m = n++$, a contradiction. This closes the induction. \square

8/10:

- As previously alluded to, we can now show that $\{0, 1, 2\}$ and $\{3, 4\}$ do not have equal cardinality since $\{0, 1, 2\}$ has cardinality 3 and $\{3, 4\}$ has cardinality 2.
- We now rigorously define the distinction between finite and infinite sets.

Definition 3.18 (Finite sets). A set is **finite** iff it has cardinality n for some natural number n ; otherwise, the set is called **infinite**. If X is a finite set, we use $\#(X)$ to denote the cardinality of X .

- Here's an example of an infinite set.

Theorem 3.1. *The set of natural number \mathbb{N} is infinite.*

Proof. Suppose for the sake of contradiction that the set of natural numbers \mathbb{N} was finite, so it has some cardinality $\#(\mathbb{N}) = n$. Then there is a bijection f from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to \mathbb{N} . One can show that the sequence $f(1), f(2), \dots, f(n)$ is bounded, or more precisely that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$ (see Exercise 3.6.3). But then the natural number $M + 1$ is not equal to any of the $f(i)$, contradicting the hypothesis that f is a bijection. \square

- Note that a similar argument can show that other unbounded sets are infinite.
- While the naturals and the even naturals are of equal size, note that it is possible for some sets to be “more” infinite than others (see Section ??).
- See Exercise 3.6.4 for some relations between cardinality and the arithmetic of natural numbers.

Exercises

8/9:

1. Prove the following proposition.

Proposition 3.6. *Let X, Y, Z be sets. Then X has equal cardinality with X . If X has equal cardinality with Y , then Y has equal cardinality with X . If X has equal cardinality with Y and Y has equal cardinality with Z , then X has equal cardinality with Z .*

¹⁵We know that for every $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$, there is exactly one $y \in X$ such that $f(y) = i$. We want to show that for every $i \in \{i \in \mathbb{N} : 1 \leq i \leq n - 1\}$, there is exactly one $y \in X - \{x\}$ such that $g(y) = i$. Consider any $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$. If $i < f(x)$, choose the unique element $y \in X - \{x\}$ such that $f(y) = i$. By definition, $g(y) = f(y) = i$, and $g(y') \neq g(y)$ unless $y' = y$. If $i \geq f(x)$, choose the unique element $y \in X - \{x\}$ such that $f(y) = i + 1$. By definition, $g(y) = f(y) - 1 = i$, and $g(y') \neq g(y)$ unless $y' = y$.

Proof. To prove that X has equal cardinality with X , Definition 3.16 tells us that it will suffice to show that there exists a bijection $f : X \rightarrow X$ from X to X . Let f be the identity map on X , i.e., $f(x) := x$ for all $x \in X$. To show that f is bijective, we must verify that the statement “for every $x \in X$, there is exactly one x such that $f(x) = x$ ” is true (Definition 3.10). But since f is a function defined as above, Definition 3.5 implies that this statement is true.

To prove that Y has equal cardinality with X , Definition 3.16 tells us that it will suffice to show that there exists a bijection $f : Y \rightarrow X$ from Y to X . To begin, since X has equal cardinality with Y , there exists a bijection $f^{-1} : X \rightarrow Y$ from X to Y (Definition 3.16). Let $f = (f^{-1})^{-1}$. Thus, f maps Y to X by definition, and f is bijective by Exercise 3.3.6.

To prove that X has equal cardinality with Z , Definition 3.16 tells us that it will suffice to show that there exists a bijection $f : X \rightarrow Z$ from X to Z . To begin, since X has equal cardinality with Y and Y has equal cardinality with Z , there exists a bijection $g : X \rightarrow Y$ from X to Y and there exists a bijection $h : Y \rightarrow Z$ from Y to Z , respectively (Definition 3.16). Let $f = h \circ g$. Thus, f maps X to Z by Definition 3.7, and f is bijective by Exercise 3.3.7. \square

- 8/12: 2. Show that a set X has cardinality 0 if and only if X is the empty set.

Proof. We first need a lemma.

Lemma 3.14. *There exists no natural number i satisfying $1 \leq i \leq 0$.*

Proof. Suppose for the sake of contradiction that there exists some natural number i satisfying $1 \leq i \leq 0$. Since $i \leq 0$ by hypothesis but $i \geq 0$ by Exercise 2.2.4a, Exercise 2.2.3c implies that $i = 0$. However, we also have $i \geq 1$, which implies that $i > 0$ by Exercise 2.2.3e, and that $i \neq 0$ by Definition 2.3, a contradiction. Therefore, no natural number i satisfies $1 \leq i \leq 0$. \square

Now for the primary proof. Suppose first that X has cardinality 0. Then by Definition 3.17, there exists a bijection $f : \{i \in \mathbb{N} : 1 \leq i \leq 0\} \rightarrow X$. Since there is no natural number i (i.e., no element of \mathbb{N}) satisfying $1 \leq i \leq 0$ (Lemma 3.14), $\{i \in \mathbb{N} : 1 \leq i \leq 0\} = \emptyset$. Thus, the bijection f maps \emptyset to X . But since f is an empty function and a bijection, Exercise 3.3.3 asserts that the range of f is the empty set, i.e., $X = \emptyset$. Now suppose that X is the empty set. Then for $f : X \rightarrow \{i \in \mathbb{N} : 1 \leq i \leq n\}$ to be a bijection, we must have $\{i \in \mathbb{N} : 1 \leq i \leq n\} = \emptyset$ (Exercise 3.3.3 again). As previously established, this happens when $n = 0$ (note that we need not rule out this happening for all other n since we have Proposition 3.5). Thus, by Definition 3.17, $\#(X) = 0$. \square

- 8/10: 3. Let n be a natural number, and let $f : \{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow \mathbb{N}$ be a function. Show that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$. (Hint: induct on n . You may also want to peek at Lemma ??.) Thus, finite subsets of the natural numbers are bounded.

Proof. We induct on n (starting with the case where $n = 1$; the claim is also vacuously true when $n = 0$, but is not particularly interesting in that case). When $n = 1$, there is only one object (namely, the natural number 1) in the domain of f . Thus, there is only one $f(i)$ (namely, $f(1)$). Therefore, if we choose $M := f(1)$, then it is true that $f(i) \leq M$ for all $1 \leq i \leq n$. Now suppose inductively that we have proven the claim for some n ; we wish to prove it for $n++$. By induction hypothesis, $f(i) \leq M$ for all $1 \leq i \leq n$. Since $M \leq M + f(n++)$ and $f(n++) \leq M + f(n++)$, we have $f(i) \leq M + f(n++)$ for all $1 \leq i \leq n$ and $f(n++) \leq M + f(n++)$. Thus, if we choose $M' := M + f(n++)$, then $f(i) \leq M'$ for all $1 \leq i \leq n++$, thus closing the induction. \square

4. Prove the following proposition.

Proposition 3.7 (Cardinal arithmetic).

- (a) *Let X be a finite set, and let x be an object which is not an element of X . Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.*

Proof. Consider the set X . Since X is finite, Definition 3.18 implies that $\#(X) = n$ for some natural number n . Thus, by Definition 3.17, X has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$; it follows by Definition 3.16 that there exists a bijection $f : X \rightarrow \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Now consider the set $\{x\}$. Let $g : \{x\} \rightarrow \{n+1\}$ be the function defined by $g(x) := n+1$. Then because the single element $n+1$ of $\{n+1\}$ comes from exactly one element of $\{x\}$, g is bijective. At this point, define the function $h : X \cup \{x\} \rightarrow \{i \in \mathbb{N} : 1 \leq i \leq n+1\}$ by the following rule: for any $y \in X$, we define $h(y) := f(y)$, and define $h(x) := g(x)$. (Note that $f(y)$ cannot equal $g(x)$ since $g(x)$ is not in the range of f , i.e., $n+1 \notin \{i \in \mathbb{N} : 1 \leq i \leq n\}$.) As defined, h is bijective (let i be an arbitrary element of $\{i \in \mathbb{N} : 1 \leq i \leq n+1\}$. If $1 \leq i \leq n$, then the bijectivity of f (and the previously established fact that $f(y) \neq g(x)$ for any $y \in X$) implies that there is exactly one element z of $X \cup \{x\}$ such that $h(z) = i$. If it is not true that $1 \leq i \leq n$, then $i = n+1$; thus, the bijectivity of g (and the previously established fact that $f(y) \neq g(x)$ for any $y \in X$) implies that there is exactly one element z of $X \cup \{x\}$ such that $h(z) = i$). Therefore, $X \cup \{x\}$ has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n+1\}$ (Definition 3.16), implying by Definition 3.17 that $\#(X \cup \{x\}) = n+1 = \#(X) + 1$. Also, since $n+1$ is a natural number, Definition 3.18 implies that $X \cup \{x\}$ is finite. \square

8/12:

- (b) Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint (i.e., $X \cap Y = \emptyset$), then $\#(X \cup Y) = \#(X) + \#(Y)$.

Proof. We first treat the case where X and Y are not necessarily disjoint. To begin, we need several lemmas.

Lemma 3.15. Let n, m be a natural numbers such that $n = m++$. Then $n \geq 1$.

Proof. By Axiom 2.3, the fact that n is the successor of some natural number proves that $n \neq 0$. Additionally, by Exercise 2.2.4, we have $n \geq 0$. But since $n \neq 0$, Definition 2.3 implies that $n > 0$. Consequently, by Exercise 2.2.3e, $n \geq 0++$, i.e., $n \geq 1$. \square

Lemma 3.16. Let X be a finite set, and let x be an object which is an element of X . Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X)$.

Proof. By Axiom 3.3, x is the only element of $\{x\}$. Thus, by Definition 3.2, to show that $\{x\} \subseteq X$, it will suffice to show that $x \in X$. But x is an element of X by hypothesis. It follows by Exercise 3.1.5 that $X \cup \{x\} = X$. Since $X \cup \{x\}$ is equal to a finite set, $X \cup \{x\}$ is finite, itself. Consequently, Definition 3.18 implies that $\#(X \cup \{x\})$ exists, and, substituting, $\#(X \cup \{x\}) = \#(X)$. \square

Lemma 3.17. Let X, Y be sets. If y is any element of Y , then $X \cup Y = (X \cup (Y \setminus \{y\})) \cup \{y\}$.

Proof. If $y \in Y$, then for the same reasons outlined in the proof of Lemma 3.16, $\{y\} \subseteq Y$. By Exercise 3.1.6g, this implies that $(Y \setminus \{y\}) \cup \{y\} = Y$. Thus, Lemma 3.2 and the fact that the \cup operation obeys the axiom of substitution show that $X \cup Y = (X \cup (Y \setminus \{y\})) \cup \{y\}$, as desired. \square

Lemma 3.18. Let n be a natural number, and let $n-1$ be defined as in Lemma 3.13. Then $n-1+1 = n$.

Proof. By Lemma 3.13, $n-1$ equals the unique element m satisfying $m++ = n$. Substituting, Lemma 2.2 and Definition 2.2 give us $n-1+1 = m+1 = (m+0)++ = m++ = n$. \square

Lemma 3.19. Let X, Y be disjoint sets (i.e., we have $X \cap Y = \emptyset$), and let y be any element of Y . Then we have $y \notin X \cup (Y \setminus \{y\})$.

Proof. Suppose for the sake of contradiction that $y \in X \cup (Y \setminus \{y\})$. Then by Axiom 3.4, $y \in X$ or $y \in Y \setminus \{y\}$. If $y \in Y \setminus \{y\}$, then by Definition 3.4 and subsequently Axiom 3.5, $y \notin \{y\}$ (and $y \in Y$, but we already know this). However, $y \notin \{y\}$ contradicts Axiom 3.3, so $y \notin Y \setminus \{y\}$. Thus, we must have $y \in X$. But then since $y \in X$ and $y \in Y$, Definition 3.3 implies that $y \in X \cap Y$. Consequently, Definition 3.1 implies that $y \in \emptyset$, contradicting Axiom 3.2. Therefore, $y \notin X \cup (Y \setminus \{y\})$. \square

Lemma 3.20. Let A, X, Y be sets, and let $X \cap Y = \emptyset$. Then $X \cap (Y \setminus A) = \emptyset$.

Proof. Suppose for the sake of contradiction that for some object x , $x \in X \cap (Y \setminus A)$. By Definition 3.3, this implies that $x \in X$ and $x \in Y \setminus A$. By Definition 3.4 followed by Axiom 3.5, $x \in X$ and $x \in Y$ (and $x \notin A$, but this is not relevant to our present discussion). Since $x \in X$ and $x \in Y$, Definition 3.3 implies that $x \in X \cap Y$. But by Definition 3.1, this implies that $x \in \emptyset$, which contradicts Axiom 3.2. Therefore, $x \notin X \cap (Y \setminus A)$ for all objects x , which, by the uniqueness of the empty set, proves that $X \cap (Y \setminus A) = \emptyset$. \square

Now for the primary proof (of the first part of the question). Since X and Y are finite, Definition 3.18 implies that $\#(X) = n$ and $\#(Y) = m$ for some natural numbers n, m . We induct on m (holding n constant).

For the base case $m = 0$, we have by Exercise 3.6.2 that $Y = \emptyset$. Thus, by Exercise 3.1.3, $X \cup Y = X \cup \emptyset = X$. Since $X \cup Y$ is equal to a finite set, $X \cup Y$ is finite. It follows that $\#(X \cup Y) = \#(X) \leq \#(X) + \#(Y)$ (note that we could use $=$ instead of \leq here, but we use \leq because of the nature of the claim). This completes the proof of the base case.

Now suppose inductively that we have proven the claim for some natural number m ; we now seek to prove it for $m++$. Let X and Y be finite sets with $\#(Y) = m++$. As we are taking an induction approach, we want to reduce the case of $\#(Y) = m++$ to a case of $\#(Y') = m$, where Y' is some set. We do this by partitioning Y and then analyzing the partition with previously derived statements. However, any proof relating cardinalities will require the established finitude of a couple of sets, so we begin there.

As mentioned above, we begin by demonstrating that $X \cup Y$ is finite. Since $\#(Y) = m++$ where m is a natural number, we know that $\#(Y) \geq 1$ (Lemma 3.15). Thus, Lemma 3.13 applies, and asserts that $\#(Y \setminus \{y\}) = (m++) - 1 = m$ (note that $(m++) - 1 = m$ since m is clearly the unique element such that $m++ = m++$; refer to the definition of $n - 1$ for natural numbers n in Lemma 3.13). Thus, by induction hypothesis, $X \cup (Y \setminus \{y\})$ is finite (note that the finitude of this set, itself, will be important later). It follows by either Exercise 3.6.4a or Lemma 3.16 (depending on whether or not y is an element of $X \cup (Y \setminus \{y\})$ — remember, while we know that $y \notin Y \setminus \{y\}$ because of Definition 3.4, Axiom 3.5, and Axiom 3.3, it is still possible that $y \in X$) that $(X \cup (Y \setminus \{y\})) \cup \{y\}$ is finite. But by Lemma 3.17, this implies that $X \cup Y$ is finite.

The remainder of the first part of the proof will be an inequality with comments interjected in between steps. By the above, we have that $\#(X \cup Y)$ exists, and by Lemma 3.17, we have that

$$\#(X \cup Y) = \#((X \cup (Y \setminus \{y\})) \cup \{y\})$$

If we treat $X \cup (Y \setminus \{y\})$ as a single finite set (which we are allowed to because of the above), then either Exercise 3.6.4a or Lemma 3.16 applies, once again depending on whether or not y is an element of $X \cup (Y \setminus \{y\})$. If $y \notin X \cup (Y \setminus \{y\})$, then Exercise 3.6.4a tells us that $\#((X \cup (Y \setminus \{y\})) \cup \{y\}) = \#(X \cup (Y \setminus \{y\})) + 1$. On the other hand, if $y \in X \cup (Y \setminus \{y\})$, then Lemma 3.16, then $\#((X \cup (Y \setminus \{y\})) \cup \{y\}) = \#(X \cup (Y \setminus \{y\}))$. Either way, we can conclude that $\#((X \cup (Y \setminus \{y\})) \cup \{y\}) \leq \#(X \cup (Y \setminus \{y\})) + 1$. Continuing from the equation above, we can write

$$\leq \#(X \cup (Y \setminus \{y\})) + 1$$

Since X and $Y \setminus \{y\}$ are finite and $\#(Y \setminus \{y\}) = m$, as established above, the induction hypothesis applies and asserts that $\#(X \cup (Y \setminus \{y\})) \leq \#(X) + \#(Y \setminus \{y\})$. Continuing from the above inequality, this allows us to write

$$\leq \#(X) + \#(Y \setminus \{y\}) + 1$$

By Lemma 3.13, $\#(Y \setminus \{y\}) = \#(Y) - 1$. Substituting, we may write

$$= \#(X) + \#(Y) - 1 + 1$$

Since X and Y are finite, $\#(X)$ and $\#(Y)$ are natural numbers. Thus, by Definition 2.2, $\#(X) + \#(Y)$ is a natural number. Consequently, Lemma 3.18 applies, and we can at last conclude the following.

$$= \#(X) + \#(Y)$$

We now address the second part of the question, which we may do in a very similar manner to the first part. Let's begin.

Since X and Y are finite, Definition 3.18 implies that $\#(X) = n$ and $\#(Y) = m$ for some natural numbers n, m . We induct on m (holding n constant). For the base case $m = 0$, we have by Exercise 3.6.2 that $Y = \emptyset$. Thus, by Exercise 3.1.3, $\#(X \cup Y) = \#(X) = \#(X) + 0 = \#(X) + \#(Y)$. Now suppose inductively that we have proven the claim for some natural number m ; we now seek to prove it for $m++$. Let X and Y be finite sets with $\#(Y) = m++$ and $X \cap Y = \emptyset$. The remainder of the proof will be equations with comments interjected between steps. By the proof of the first part, we have that $\#(X \cup Y)$ exists, and by Lemma 3.17, we have that

$$\#(X \cup Y) = \#((X \cup (Y \setminus \{y\})) \cup \{y\})$$

Since $X \cup (Y \setminus \{y\})$ is a finite set and y is not an element of it (Lemma 3.19), Exercise 3.6.4a applies. As such, we have that $\#((X \cup (Y \setminus \{y\})) \cup \{y\}) = \#(X \cup (Y \setminus \{y\})) + 1$. Thus, continuing from the above equation, we can write

$$= \#(X \cup (Y \setminus \{y\})) + 1$$

Since X and $Y \setminus \{y\}$ are finite by hypothesis, $X \cap (Y \setminus \{y\}) = \emptyset$ by Lemma 3.20, and $\#(Y \setminus \{y\}) = m$ as established above, the induction hypothesis applies and asserts that $\#(X \cup (Y \setminus \{y\})) = \#(X) + \#(Y \setminus \{y\})$. Thus, we can write

$$= \#(X) + \#(Y \setminus \{y\}) + 1$$

By Lemma 3.13, $\#(Y \setminus \{y\}) = \#(Y) - 1$. Substituting, we may write

$$= \#(X) + \#(Y) - 1 + 1$$

Since X and Y are finite, $\#(X)$ and $\#(Y)$ are natural numbers. Thus, by Definition 2.2, $\#(X) + \#(Y)$ is a natural number. Consequently, Lemma 3.18 applies, and we can at last conclude the following.

$$= \#(X) + \#(Y)$$

□