Chapter 20

Differential Equations

20.1 Introduction

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- **Differential equation**: An equation that involves one or more derivatives, or differentials.
- **Type** (of a differential equation): A differential equation is either an **ordinary differential equation** or a **partial differential equation**.
- **Order** (of a differential equation): The order of the highest-order derivative that occurs in the equation.
- **Degree** (of a differential equation): The exponent of the highest power of the highest-order derivative, after the equation has been cleared of fractions and radicals in the dependent variable and its derivatives.
- **Ordinary differential equation**: A differential equation where the only derivatives that appear are those of a dependent variable y varying as a function of a single independent variable x.
- **Partial differential equation**: A differential equation where partial derivatives appear.
- "For example,
$$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)^2 + \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^5 + \frac{y}{x^2+1} = \mathrm{e}^x$$
is an ordinary differential equation, of order three and degree two" (Thomas, 1972, p. 691).
- Thomas (1972) does not include a systematic treatment of partial differential equations, so he recommends Chapter 10 of Kaplan (1952).
- If A, B, C are radioactive substances such that A decomposes into B at a rate proportional to the amount of A present (proportionality constant k_1), B decomposes into C at a rate proportional to the amount of A present (proportionality constant k_2), and C decomposes into A at a rate proportional to the amount of C present (proportionality constant k_3), we can write the following system of differential equations.
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k_1x + k_3z \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = k_1x - k_2y \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = k_2y - k_3z$$
  - Since $\mathrm{d}x/\mathrm{d}t + \mathrm{d}y/\mathrm{d}t + \mathrm{d}z/\mathrm{d}t = 0$ in this case, our solution is that $x + y + z = C$, i.e., the statement that if the amounts of substances change in this manner, the total amount of substance present remains constant.

20.2 Solutions

- **Solution** (of a differential equation): A function $y = f(x)$ such that the differential equation is identically satisfied when y and its derivatives are replaced throughout by $f(x)$ and its corresponding derivatives.
- Differential equations often have solutions in which certain arbitrary constants occur.
  - However, these constants can often be resolved into specific values with the help of initial conditions.
  - In fact, "a differential equation of order n will in general possess a solution involving n arbitrary constants" (Thomas, 1972, p. 693).
    - There is a more precise theorem that implies this result, but Thomas (1972) neither states nor proves it.
- **General solution**: A solution that still contains all arbitrary constants arising from the solving process.
- Since finding general solutions requires calculus and finding specific solutions from general solutions and initial conditions only requires algebra, we will focus on finding general solutions.
- Note that this is only an introduction; for a more exhaustive treatment of differential equations, refer to Martin and Reissner (1961).

20.3 First-Order Equations with Variables Separable

- If it is possible to collect all y-terms with $\mathrm{d}y$ and all x-terms with $\mathrm{d}x$, i.e., if it is possible to write the equation in the form
$$f(y)\,\mathrm{d}y + g(x)\,\mathrm{d}x = 0$$
then the general solution is
$$\int f(y)\,\mathrm{d}y + \int g(x)\,\mathrm{d}x = C$$
where C is an arbitrary constant.

20.4 First-Order Homogeneous Equations

- **Homogeneous** (differential equation): A differential equation that can be put into the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(y/x)$$
where $F(y/x)$ is some function of y/x.
- Solving a first-order homogeneous differential equation.
  - We use u-substitution.
  - If we let $u = y/x$, then $\mathrm{d}y/\mathrm{d}x = u + x\ \mathrm{d}u/\mathrm{d}x$ by the product rule.
  - Thus, the differential equation can be rewritten in the form $u + x\ \mathrm{d}u/\mathrm{d}x = F(u)$, which can be solved in terms of u and x via separation of variables as follows.
$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}u}{u - F(u)} = 0$$
  - Once the above is solved, we can return the substitution to obtain our final solution.

- "Show that the equation $(x^2 + y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0$ is homogeneous, and solve it" (Thomas, 1972, p. 694).
  - Via basic algebra, we can rewrite the above in the form

    $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x^2 + y^2}{2xy} \\ &= -\frac{1 + (y/x)^2}{2(y/x)}\end{aligned}$$

    implying that it is homogeneous with $F(u) = -(1 + u^2)/(2u)$.
  - Therefore, the only remaining task is to solve

    $$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}u}{u - [-(1 + u^2)/(2u)]} = 0$$

    via separation of variables integration.
  - After doing so, we obtain

    $$\begin{aligned}\ln|x| + \frac{1}{3}\ln\left(1 + 3u^2\right) &= \frac{1}{3}\ln C \\ x^3(1 + 3u^2) &= \pm C\end{aligned}$$

  - Therefore, returning our substitution, we have that

    $$x(x^2 + 3y^2) = C$$

    is our solution.

20.5 First-Order Linear Equations

- **Degree** (of a term in a differential equation): The sum of the exponents of the dependent variable and any of its derivatives in a given term.
  - For example, $(\mathrm{d}^2y/\mathrm{d}x^2)$ is of degree one but $y(\mathrm{d}y/\mathrm{d}x)$ is of degree two.
- **Linear differential equation**: A differential equation such that every term is of degree zero or degree one.

  $$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

  - A linear differential equation of first order can always be put into the above form, where P, Q are functions of x.
- **Integrating factor**: A function ρ of the independent variable x such that if the differential equation at hand is multiplied by ρ, it will compress into a form that is easier to integrate.
  - For first-order linear differential equations, the function ρ that we seek makes the left-hand side becomes the derivative of the product ρy.
- Deriving ρ in terms of the values given in a general first-order linear differential equation.
  - Multiplying by ρ, we have

    $$\rho\frac{\mathrm{d}y}{\mathrm{d}x} + \rho Py = \rho Q$$

- Thus, since we want
$$\rho\frac{\mathrm{d}y}{\mathrm{d}x}+\rho Py=\frac{\mathrm{d}}{\mathrm{d}x}(\rho y)=\rho\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}\rho}{\mathrm{d}x}y$$
we must have by comparing terms that
$$\frac{\mathrm{d}\rho}{\mathrm{d}x}=\rho P$$
- It follows by separation of variables integration that
$$\begin{aligned}\frac{\mathrm{d}\rho}{\rho}&=P\,\mathrm{d}x\\ \ln|\rho|&=\int P\,\mathrm{d}x+\ln C\\ \rho&=\pm C\mathrm{e}^{\int P\mathrm{d}x}\\ \rho&=\mathrm{e}^{\int P\mathrm{d}x}\end{aligned}$$
  - Note that we can choose $\pm C=1$ since in the equation $\rho\;\mathrm{d}y/\mathrm{d}x+\rho Py=\rho Q$, any C term can be divided out of both sides anyways.

- With the help of the integrating factor, we can now derive the general solution to a first-order linear differential equation as follows.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}+Py&=Q\\ \rho\frac{\mathrm{d}y}{\mathrm{d}x}+\rho Py&=\rho Q\\ \frac{\mathrm{d}}{\mathrm{d}x}(\rho y)&=\rho Q\\ \rho y&=\int\rho Q\,\mathrm{d}x+C\\ y&=\frac{1}{\mathrm{e}^{\int P\mathrm{d}x}}\left(\int\mathrm{e}^{\int P\mathrm{d}x}Q\,\mathrm{d}x+C\right)\end{aligned}$$
- Note that a first-order linear differential equation may also be separable, or homogeneous. In such cases, we have a choice of solution methods.

20.6 First-Order Equations With Exact Differentials

- Refer to Section 15.13 for the method of solving exact differentials.
- Every first-order differential equation $P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y=0$ can be made exact by multiplication by an integrating factor $\rho(x,y)$ having the property that
$$\frac{\partial}{\partial y}[\rho(x,y)P(x,y)]=\frac{\partial}{\partial x}[\rho(x,y)Q(x,y)]$$
  - "It is not easy to determine ρ from this equation. However, one can often recognize certain combinations of differentials that can be made exact by 'ingenious devices'" (Thomas, 1972, p. 696).
- For example, consider $x\,\mathrm{d}y-y\,\mathrm{d}x=xy^2\,\mathrm{d}x$.
  - We may solve this differential equation by recognizing that it can be rewritten as
$$-x\,\mathrm{d}x=\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2}=\mathrm{d}(x/y)$$
  - Alternatively, we can multiply by the integrating factor $1/y^2$.