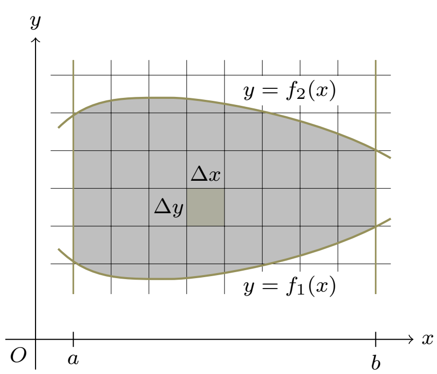
# Chapter 16

# Multiple Integrals

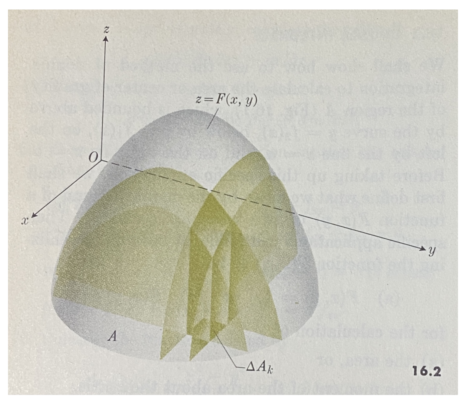
## 16.1 Double Integrals

12/19: • **Double integral** (of $F(x, y)$ over the region $A$): The limit as $\Delta A \to 0$ of the sum of every $\Delta A_k$ (composing a region $A$) times $F(x, y)$ for some $(x, y) \in \Delta A_k$. Mathematically,

$$\int_A \int F(x, y)\,\mathrm{d}A = \lim_{\Delta A \to 0} \sum_{k=1}^{n} F(x_k, y_k)\Delta A_k$$




(a) Subdividing the region $A$.




(b) Parts of $F(x, y)$ over a $\Delta A_k$.

Figure 16.1: The double integral.

- To conceptualize the double integral, first imagine that a region $A$ of the plane is bounded above by the curve $y = f_2(x)$, below by the curve $y = f_1(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ (see Figure 16.1a).
- Now imagine that $A$ is subdivided by a grid into $n$ pieces $\Delta A = \Delta x \Delta y = \Delta y \Delta x$. We disregard the pieces that lie partially or entirely outside of the bounds.
- As discussed in Chapter 15, $F(x, y)$ can be thought of as a surface in three-space. For the sake of simplicity, we will imagine for right now that $F(x, y)$ is positive for all $(x, y) \in A$, i.e., that it lies above $A$ (see Figure 16.1b).
- With this picture, we can imagine summing the partial volumes $F(x, y) \cdot \Delta A_k$ for each $\Delta A_k$ where $(x, y)$ is some point in $\Delta A_k$ to approximate the total volume under the surface (analogous to the area under the curve).

- All that the double integral does at this point is find the exact volume under the surface by taking the limit of this summation as we consider increasingly more increasingly small slivers of volume.

- We can evaluate the double integral of $F(x,y)$ over $A$ if $F(x,y)$ is continuous throughout $A$, if the boundary curves are continuous and have finite total length, and if we let $\Delta y = 2\Delta x$ (or some other similar function) and let $\Delta x \to 0$.
- To evaluate double integrals, we calculate one or the other of the **iterated** integrals[1]

$$\int_A\int F(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad\qquad \int_A\int F(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

- To evaluate the latter integral above, for example, we integrate "$\int F(x,y)\,\mathrm{d}y$ with respect to $y$ (with $x$ held fixed) and [evaluate] the resulting integral between the limits $y=f_1(x)$ and $y=f_2(x)$, and then [integrate] the result... with respect to $x$ between the limits $x=a$ and $x=b$" (Thomas, 1972, p. 549). Mathematically,

$$\int_A\int F(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b\left(\int_{f_1(x)}^{f_2(x)} F(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

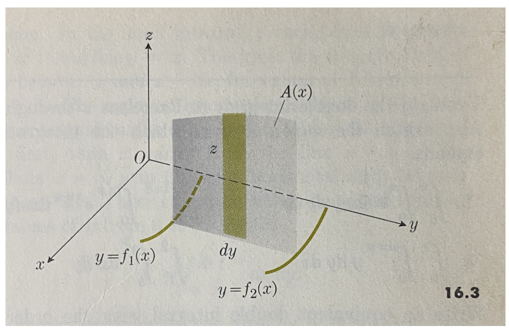



Figure 16.2: Visualizing iterated integration.

- To visualize iterated integration, imagine determining the volume under the surface by summing infinitely many infinitely thin cross sections of the solid parallel to the $y$-axis.
- The cross section at $x=x_0$ would have area $A(x_0)=\int_{f_1(x_0)}^{f_2(x_0)} F(x_0,y)\,\mathrm{d}y$.
- The sum of all such cross sections' contributions to the volume would be the integral $\int_a^b A(x)\,\mathrm{d}x = \int_a^b\left(\int_{f_1(x)}^{f_2(x)} F(x,y)\,\mathrm{d}y\right)\mathrm{d}x$.

## 16.2 Area by Double Integration

- The area of the region of the $xy$-plane is given by either of the following integrals (with proper limits of integration).

$$A = \iint \mathrm{d}x\,\mathrm{d}y = \iint \mathrm{d}y\,\mathrm{d}x$$

- In cases such as that of Figure 16.1a, it makes sense to integrate with respect to $y$ first and $x$ second. However, if we have a region bounded by $y=c$, $y=d$, $x=g_1(y)$, and $x=g_2(y)$, then it would make more sense to do the opposite.
- Conceptualize the iterated integration here as summing the infinitesimal areas of strips parallel to the $x$- or $y$-axis, the lengths of which are given by $f_2(x)-f_1(x)=\int_{f_1(x)}^{f_2(x)}\mathrm{d}y$ or $g_2(y)-g_1(y)=\int_{g_1(y)}^{g_2(y)}\mathrm{d}x$.

[1]Proving that evaluating these integrals is equivalent to evaluating the double integral is a more advanced theorem of analysis.