Chapter 15

Partial Differentiation

15.1 Functions of Two or More Variables

- **Function** (from D to E^1): A mapping that assigns a unique number w to each point $(x_1, \ldots, x_n) \in D \subset E^n$.
  - We write $w = f(x_1, \ldots, x_n)$ and say that w is the value of the function f at $(x_1, \ldots, x_n)$.
- **Continuous** (function $f(x, y)$): A function $f(x, y)$ such that $w \to w_0 = f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$.

15.2 The Directional Derivative: Special Cases

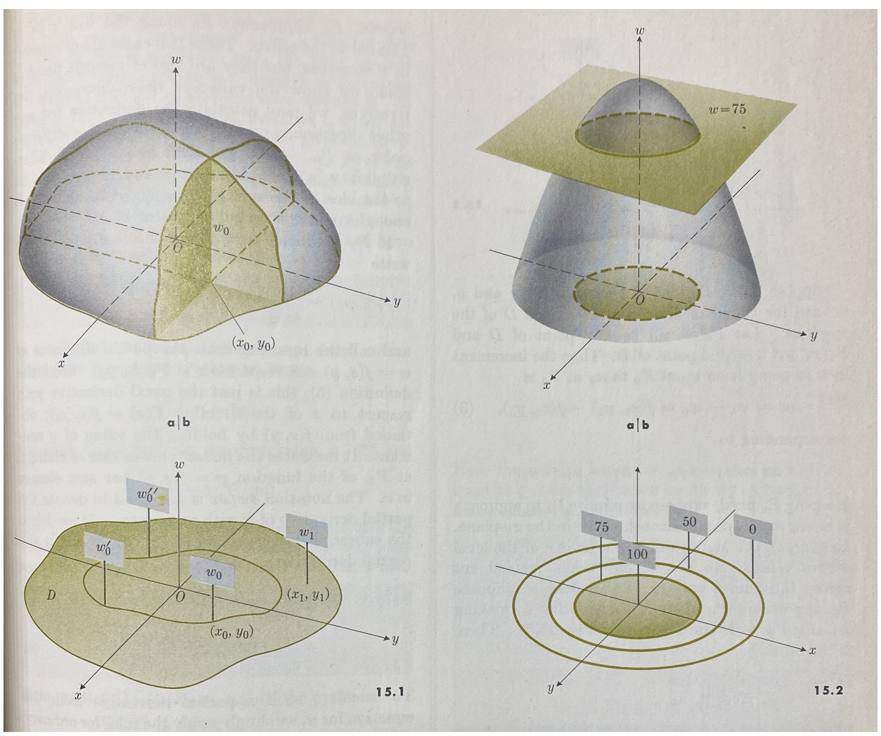

Figure 15.1: Surface plots and contour maps of 2D functions.

- The equation $w = f(x, y)$ can be interpreted as representing a surface in xyw-space, or as a base region D in the xy-plane with a marker bearing a corresponding w-value attached to each point.

- To introduce order into the second interpretation, we can construct a **contour map** with a number of **contour curves**.

- **Contour curve**: A curve consisting of points $(x, y) \in D$ with equal w-values.
  - The formula for such a curve can be derived by setting $w_0 = f(x, y)$, where $w_0 \in R_f$.

- **Directional derivative** (of $f(x, y)$ at (x_0, y_0) in the direction of L): The limit

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s} = \lim_{P_1 \to P_0} \frac{f(x_1, y_1) - f(x_0, y_0)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

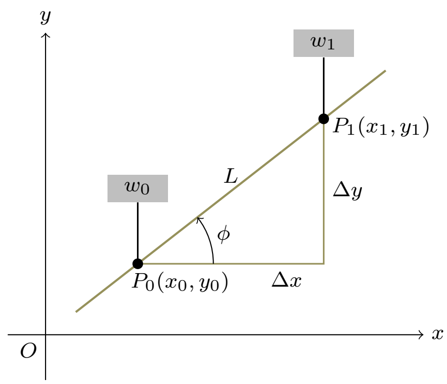

Figure 15.2: The directional derivative.

  - Basically, we let P_1 approach P_0 along a smooth curve (a line for simplicity and to be definite) and watch how $\Delta w = w_1 - w_0 = f(x_1, y_1) - f(x_0, y_0)$, $\Delta x = x_1 - x_0$, and $\Delta y = y_1 - y_0$ change.
  - Note that the directional derivative does depend on the *direction* from which P_1 approaches P_0, not just the absolute distance between P_1 and P_0.

- We now consider two special cases: When "P_1 approaches P_0 along the line $y = y_0$ parallel to the x-axis, [and when] P_1 approaches P_0 along the line $x = x_0$ parallel to the y-axis" (Thomas, 1972, p. 498).
  - These cases are important because if $f(x, y)$ is **differentiable** at P_0, we can calculate the directional derivative in any direction from them.

- **Partial derivative** (of $f(x, y)$ with respect to x at $P_0(x_0, y_0)$): The value

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

  - Essentially, this is the derivative with respect to x of the function $g(x) = f(x, y)$ with y held constant.
  - It measures "the instantaneous rate of change, at P_0, of the function $[f(x, y)]$ per unit change in x" (Thomas, 1972, p. 498).

- **Partial derivative** (of $w = f(x, y)$ with respect to x): The function

$$\frac{\partial w}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

  - To evaluate this, we apply the ordinary rules of differentiation, treating y as a constant.

- In either of the partial derivative definitions, Δx can be positive or negative. However, if we take the directional derivative in the positive x direction (for example), then Δx in the partial derivative definitions can only be positive.
  - Similarly, if f_x exists, it gives the directional derivative in the positive x-direction, whereas $-f_x$ is the directional derivative in the negative x-direction.

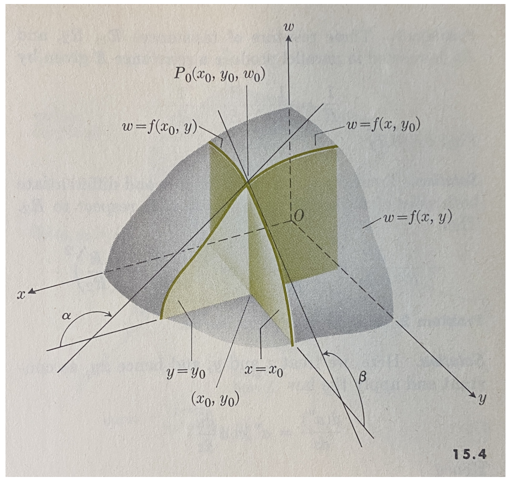

Figure 15.3: Geometric interpretation of the partial derivative.

- As in Figure 15.3, the geometric interpretation of the partial derivative (wrt. x) at a point $P(x_0, y_0, w_0)$ is as the slope of the curve $f(x, y_0)$, and symetrically wrt. y.
- We can define the partial derivative with respect to y similarly to how it is defined for x.

$$\frac{\partial w}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

- With higher order derivatives $\partial w/\partial z$, $\partial w/\partial u$, $\partial w/\partial v$, and more as in $w = f(x, y, z, u, v)$, we evaluate by holding all but the variable of interest constant.
- To denote the partial derivative at a point, we have two notations:

$$\left(\frac{\partial w}{\partial x}\right)_{(x_0, y_0)} \qquad\qquad f_x(x_0, y_0)$$

15.3 Tangent Plane and Normal Line

- **Tangent plane** (to $w = f(x, y)$ at $P_0(x_0, y_0, w_0)$): A plane T such that for any point P on the surface described by $f(x, y)$, as $P \to P_0$, the angle between T and $\overline{PP_0}$ approaches 0.
- **Normal line** (to $w = f(x, y)$ at $P_0(x_0, y_0, w_0)$): The line through P_0 which is normal to the tangent plane to $f(x, y)$ at P_0.
- The tangent plane is determined by the lines L_1 and L_2 tangent to the curves $C_1 : w = f(x_0, y)$ and $C_2 : w = f(x, y_0)$; the slopes of these lines are given by $\partial w/\partial y$ and $\partial w/\partial x$, respectively.
- Formulae for the tangent plane and normal line follow easily after finding a normal vector $\mathbf{N}$ to the plane of L_1 and L_2. To find $\mathbf{N}$, we can use the cross product of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ lying along L_1 and L_2, respectively.

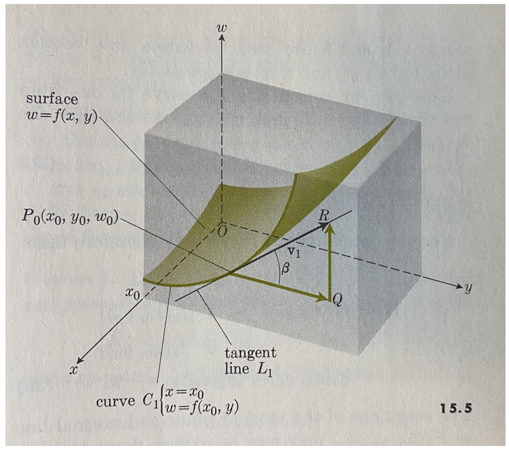

Figure 15.4: Deriving formulae for the tangent plane and normal line.

- From Figure 15.4, we can see that

$$\mathbf{v}_1 = \mathbf{j} + f_y(x_0, y_0)\mathbf{k} \qquad\qquad \mathbf{v}_2 = \mathbf{i} + f_x(x_0, y_0)\mathbf{k}$$

- Thus,

$$\mathbf{N} = \mathbf{i}f_x(x_0, y_0) + \mathbf{j}f_y(x_0, y_0) - \mathbf{k}$$

- Therefore, the formulae for the tangent plane and normal line, respectively, are

$$A(x - x_0) + B(y - y_0) + C(w - w_0) = 0 \qquad\qquad (x, y, w) = (x_0, y_0, w_0) + t(A, B, C)$$

  where $A = f_x(x_0, y_0)$, $B = f_y(x_0, y_0)$, $C = -1$, and $t \in (-\infty, \infty)$.
- In vector form, if $\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ and $\mathbf{R}_0 = \mathbf{i}x_0 + \mathbf{j}y_0 + \mathbf{k}z_0$, then

$$\mathbf{N} = \mathbf{i}f_x(x_0, y_0) + \mathbf{j}f_y(x_0, y_0) - \mathbf{k} \qquad \mathbf{N} \cdot (\mathbf{R} - \mathbf{R}_0) = 0 \qquad \mathbf{R} = \mathbf{R}_0 + t\mathbf{N}$$

15.4 Approximate Value of Δw

- **Linearization** (of f at P_0): The function (based off of the tangent plane)

$$w = f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$$

- Note that

$$\Delta w_{\text{tan}} = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

  meaning that to calculate Δw_{tan}, we need only add the tangential components; no other interaction term is needed.

- Important results:

  Theorem 15.1. Let the function $w = f(x, y)$ be continuous and possess partial derivatives f_x, f_y throughout a region $R : |x - x_0| < h,\ |y - y_0| < k$ of the xy-plane. Let f_x and f_y be continuous at (x_0, y_0). Let $\Delta w = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. Then

$$\Delta w = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

  where $\epsilon_1, \epsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$.

  Corollary 15.1. *Let $w = f(x, y)$ be continuous in a region $R : |x - x_0| < h,\ |y - y_0| < k$. Let f_x and f_y exist in R and be continuous at (x_0, y_0). Then the surface $w = f(x, y)$ has a tangent plane at $P_0(x_0, y_0, w_0)$, where $w_0 = f(x_0, y_0)$.*

  - These results extend into finitely higher dimensions.