Calculus and Analytic Geometry (Thomas) Notes

Steven Labalme

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Chapter 1

The Rate of Change of a Function

1.1 Introduction

7/3: • Discusses the importance of calculus, when it should be used, and why one should study it.

- **Analytic geometry**: "Uses algebraic methods and equations to study geometric problems. Conversely, it permits us to visualize algebraic equations in terms of geometric curves" (Thomas, 1972, p. 2).

1.2 Coordinates

- "The basic idea in analytic geometry is the establishment of a one-to-one correspondence between the points of a plane on the one hand and pairs of numbers (x, y) on the other hand" (Thomas, 1972, p. 2).
- Such a correspondence is most commonly established as follows.

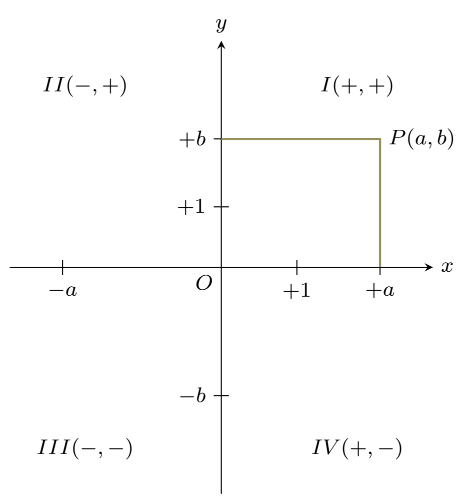


Figure 1.1: Cartesian coordinates.

  - "A horizontal line in the plane, extending indefinitely to the left and to the right, is chosen as the x-axis or axis of **abscissas**. A reference point O on this line and a unit of length are then chosen. The axis is scaled off in terms of this unit of length in such a way that the number zero is attached to O, the number $+a$ is attached to the point which is a units to the right of O, and $-a$ is attached to the symmetrically located point to the left of O. In this way, a one-to-one correspondence is established between points of the x-axis and the set of all **real numbers**" (Thomas, 1972, pp. 2–3).

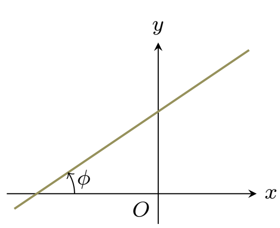
- "Now through O take a second, vertical line in the plane, extending indefinitely up and down. This line becomes the y-axis, or axis of **ordinates**. The unit of length used to represent $+1$ on the y-axis need not be the same as the unit of length used to represent $+1$ on the x-axis. The y-axis is scaled off in terms of the unit of length adopted for it, with the positive number $+b$ attached to the point b units above O and negative number $-b$ attached to the symmetrically located point b units below O" (Thomas, 1972, p. 3).
- "If a line parallel to the y-axis is drawn through the point marked a on the x-axis, and a line parallel to the x-axis is drawn through the point marked b on the y-axis, their point of intersection P is to be labeled $P(a, b)$. Thus, given the pair of real numbers a and b, we find one and only one point with abscissa a and ordinate b, and this point we denote by $P(a, b)$" (Thomas, 1972, p. 3).
- "Conversely, if we start with any point P in the plane, we may draw lines through it parallel to the coordinate axes. If these lines intersect the x-axis at a and the y-axis at b, we then regard the pair of numbers (a, b) as corresponding to the point P. We say that the coordinates of P are (a, b)" (Thomas, 1972, p. 3).
- "The two axes divide the plane into four quadrants, called the first quadrant, second quadrant, and so on, and labeled I, II, III, IV in [Figure 1.1]. Points in the first quadrant have both coordinates positive, and in the second quadrant the x-coordinate (abscissa) is negative and the y-coordinate (ordinate) is positive. The notations $(-, -)$ and $(+, -)$ in quadrants III and IV of [Figure 1.1] represent the signs of the coordinates of points in these quadrants" (Thomas, 1972, p. 3).

1.3 Increments

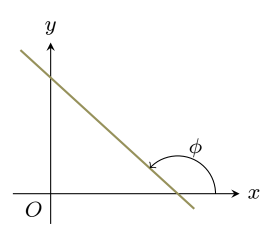
- **Increments**: The values $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ concerning a particle, the initial position of which is $P_1(x_1, y_1)$ and the terminal position of which is $P_2(x_2, y_2)$.
- If the unit of measurement for both axes is the same, then we may express distances in the plane in terms of this unit using the Pythagorean theorem.

1.4 Slope of a Straight Line

- Let L be a straight line not parallel to the y-axis intersecting distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Then L has a **rise**, **run**, and **slope**.
- **Rise**: The increment Δy.
- **Run**: The increment Δx.
- **Slope**: The rate of rise per run $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. *Also known as* **inclination**.



(a) Negative x-intercept.



(b) Positive x-intercept.

Figure 1.2: The slope and the angle of inclination.

  - If we chose different distinct points, the slope would be same because the triangles in the Cartesian plane would be similar.
  - Δy is proportional to Δx with m as the proportionality factor.
  - On interpolation: If we're given the values of a function at (x_1, y_1) and (x_2, y_2), then we may approximate the function by a straight line L passing through those two points and approximate the value $f(x)$ for any $x_1 \leq x \leq x_2$.
  - If the scales on both axes are equal, then the slope of L is equal to the tangent of the **angle of inclination** that L makes with the positive x-axis. That is, $m = \tan\phi$ (see Figure 1.2).
- **Parallel** (lines): Two lines with equal inclinations ($m_1 = m_2$).
- **Perpendicular** (lines): Two lines with inclinations that differ by 90° ($m_1 = -\frac{1}{m_2}$).
  - Note that we can prove the relation between the slopes using the angles of inclination as follows.

$$\begin{aligned}
m_1 &= \tan\phi_1 \\
&= \tan(\phi_2 + 90^\circ) \\
&= -\cot\phi_2 \\
&= -\frac{1}{\tan\phi_2} \\
&= -\frac{1}{m_2}
\end{aligned}$$

1.5 Equations of a Straight Line

7/5:
- How do you know if $P(x, y)$ is a point on the line P_1P_2 through distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$?
  - If $x_1 = x_2$, then P_1P_2 is vertical and P lies on P_1P_2 iff $x = x_1$.
  - If $x_1 \neq x_2$, then the slope of P_1P_2 $m_{P_1P_2} = \frac{y_2 - y_1}{x_2 - x_1}$. Thus, P lies on P_1P_2 iff $P = P_1$ or, for the line PP_1 through P and P_1, $m_{P_1P_2} = m_{PP_1} = \frac{y - y_1}{x - x_1}$. In other words, the coordinates x, y of P must satisfy $y - y_1 = m_{P_1P_2}(x - x_1)$.
- Thomas, 1972 calls the above equation the **point-slope form**.
- **Variable**: "A symbol, such as x, which may take on any value in some set of numbers" (Thomas, 1972, p. 10).
- **Slope-intercept form**: $y = mx + b$.
- **General form**: $Ax + By + C = 0$.
  - Such an equation (one that contains only first powers of x and y and constants) is said to be **linear in *x* and *y***.
  - "Every straight line in the plane is represented by a linear equation and, conversely, every linear equation represents a straight line" (Thomas, 1972, p. 10).
- ***y*-intercept**: The constant b in the above equation.
- Let L be a line with the equation $Ax + By + C = 0$. The shortest distance d from a point $P_1(x_1, y_1)$ not on L to L is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

  - Derive by finding a line perpendicular to L through P_1.

1.6 Functions and Graphs

- **Domain** (of a variable x): "The set of numbers over which x may vary" (Thomas, 1972, p. 12).
- Defines **open intervals**, **half-open intervals**, and **closed intervals**.

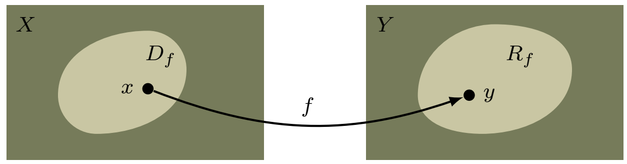


Figure 1.3: A function f maps the domain D_f onto the range R_f. The image of x is $y = f(x)$.

- **Function**: For two nonempty sets X, Y, the collection f of ordered pairs (x, y) with $x \in X$ and $y \in Y$ that assigns to every $x \in X$ a unique $y \in Y$. *Also known as* **mapping** (from X to Y), $\boldsymbol{y = f(x)}$, $\boldsymbol{f : x \to y}$[1].
  - When using the latter notation, it is understood that the domain is $\mathbb{R}$ unless this is impossible (e.g., $f : x \to \frac{1}{x}$ must exclude 0 from the domain).
- **Domain** (of a function f): "The collection of all first elements x of the pairs (x, y) in f" (Thomas, 1972, p. 13). *Also known as* $\boldsymbol{D_f}$.
- **Range** (of a function f): "The set of all second elements y of the pairs (x, y) in f" (Thomas, 1972, p. 13). *Also known as* $\boldsymbol{R_f}$.
- **Image** (of x): The value y to which a function maps x.
- Thomas, 1972 considers functions from the reals to the reals, but also more abstract functions.
  - For example, it considers the function from all triangles (a set of decidedly nonnumerical objects) to their enclosed areas (the set of positive real numbers).

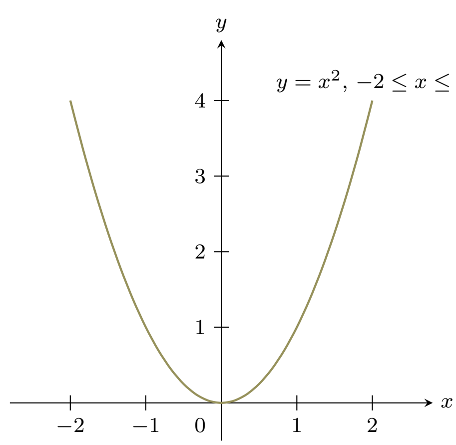


Figure 1.4: Graph of a function.

- **Graph** (of a function): "The set of points which correspond to members of the function" (Thomas, 1972, p. 14).

[1] "eff sends ex into wy"

  - For example, let X be the closed interval $[-2,2]$. To each $x \in X$, assign the number x^2. This describes the function
$$f = \left\{(x,y) : -2 \leq x \leq 2,\ y = x^2\right\}$$
The graph of f can be seen in Figure 1.4.

- **Independent variable**: The first variable x in the ordered pair (x,y). *Also known as* **argument**.
- **Dependent variable**: The second variable y in the ordered pair (x,y).
- **Real-valued function of a real variable**: "A function f whose domain and range are sets of real numbers" (Thomas, 1972, p. 14).
  - As a general rule, *function* indicates a real-valued function of a real variable for the first seven chapters of Thomas, 1972.
- f can be represented by...
  - A table of corresponding values (this will be incomplete, though).
  - Corresponding numerical scales, as on a slide rule (this will be incomplete, though).
  - A simple formula, such as $f(x) = x^2$ (this may be less exact than ordered pairs, but it is more easily understood/applicable/complete).
  - A graph (for any value x in the domain, begin x units from the origin along the x-axis, move vertically until intersecting the curve, and then move horizontally until intersecting the image y on the y-axis).
- Some mappings cannot be expressed in terms of algebraic operations on x.
  - For example, the **greatest-integer function** "maps any real number x onto that unique integer which is the largest among all integers that are less than or equal to x" (Thomas, 1972, p. 15).
    - The image of x is represented by $[x]$, and the function by $f : x \to [x]$.
    - An example of a **step function**.
    - It exhibits points of **discontinuity**.
- Note: The fact that a one-to-one mapping exists between the points in the interval $(0,1]$ and $[1,\infty)$ (namely, $f : x \to \frac{1}{x}$) proves that there are equally many points in both intervals.
- The absolute value function can be geometrically interpreted in the context of distance from a point. As such, it is useful in describing **neighborhoods**.

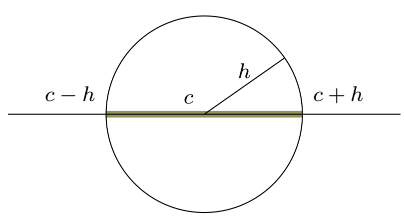


Figure 1.5: The symmetric neighborhood $N_h(c)$, centered at c, with radius h.

- **Symmetric neighborhood** (of a point c): "The open interval $(c-h, c+h)$, where h may be any positive number" (Thomas, 1972, p. 17). *Also known as* $\boldsymbol{N_h(c)}$.
- **Radius** (of a symmetric neighborhood): The value h (see Figure 1.5).
- **Neighborhood** (of a point c): The open interval $(c-h, c+k)$, where h, k may be any positive numbers.
  - Like requiring that $|x-c|$ is small.

- **Deleted neighborhood** (of a point c): "A neighborhood of c from which c itself has been removed" (Thomas, 1972, p. 17).
  - Like requiring that $|x-c|>0$.
- **Intersection** (of the neighborhoods $(c-h_1, c+k_1)$ and $(c-h_2, c+k_2)$): The neighborhood $(c-h, c+k)$, where $h=\min(h_1,h_2)$ and $k=\min(k_1,k_2)$.
  - "The intersection of two neighborhoods of c is a neighborhood of c, and the intersection of two deleted neighborhoods of c is a deleted neighborhood of c" (Thomas, 1972, p. 18).
- Let A be a neighborhood of c. Then denote the deleted neighborhood equivalent to A with c removed by A^-.

7/6:

- "A function is determined by the domain and by any rule that tells what image in the range is to be associated with each element of the domain" (Thomas, 1972, p. 18).
  - Thus, we can think of a "function machine" that takes in elements of the domain and computes the image based on the rule.
  - Thomas, 1972 visualizes function machines as flow charts.
  - In theory, a function machine could store every pair (x,y) in its memory to be recalled later. Since machines have limited memory, a fractional set of pairs could also be stored and the values in between calculated by interpolation.
  - In practice, though, calculating as we go is usually best.
- Two restrictions should be inferred to apply to the domain of a function, even if they are unstated: First, never divide by 0. Second, do not consider complex outputs (yet).
- Sometimes we have functions of more than one independent variable.
  - For example, the volume $v=\frac{1}{3}\pi r^2h$ of a right circular cone is uniquely determined only when r,h are given definite, positive, nonzero values.
  - "Its domain is the set of all pairs (r,h) with $r>0$, $h>0$. Its range is the set of positve numbers $v>0$" (Thomas, 1972, p. 20).
  - r,h are independent variables. v is a dependent variable.
- "More generally, suppose that some quantity y is uniquely determined by n other quantities x_1,x_2, $x_3,\dots,x_n$. The set of all ordered $(n+1)$-tuples $(x_1,x_2,x_3,\dots,x_n,y)$ that can be obtained by substituting permissible values of the variables $x_1,x_2,\dots,x_n$ and the corresponding values of y is a function whose domain is the set of all allowable n-tuples $(x_1,x_2,x_3,\dots,x_n)$ and whose range is the set of all possible values of y corresponding to this domain. If values can be assigned independently to each of the x's, we call them independent variables and say that y is a function of the x's. We also write $y=f(x_1,x_2,x_3,\dots,x_n)$ to indicate that y is a function of the n x's, just as we write $y=f(x)$ to indicate that y is a function of one independent variable x" (Thomas, 1972, p. 20).
  - Note, though, that functions of a single variable will be the primary concern of this book.
- **Signum function**: The function

$$\operatorname{sgn} x=\begin{cases}-1 & x<0\\ 0 & x=0\\ 1 & x>0\end{cases}$$

1.7 Ways of Combining Functions

7/7:

- The domains of the sum, product, difference, and quotient of two functions f and g are the intersections of D_f and D_g.
  - Note that for the quotient, we must also exclude points where $g(x) = 0$.
- There is a distinction between $x \cdot \frac{1}{x}$ and 1 (namely, the fact that the latter includes 0 in its domain while the former excludes it).
- Translation.

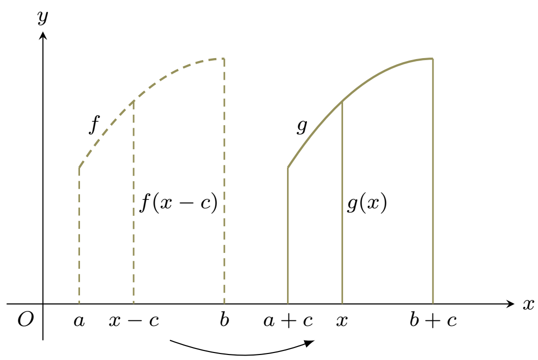
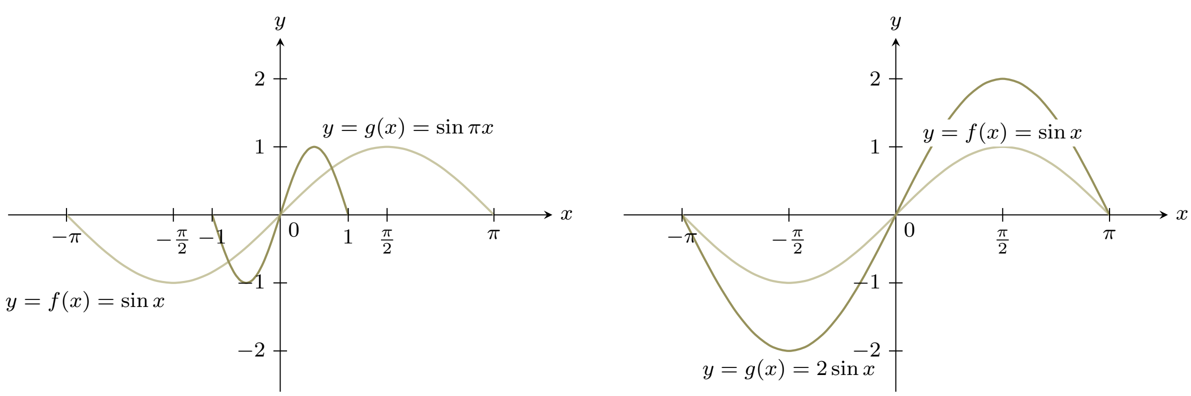


Figure 1.6: Translation.

  - "Take the ordinate of f at $x - c$ and shift it to the right c units to get the ordinate of g at x" (Thomas, 1972, p. 23).
  - $g(x) = f(x - c)$ implies that the graph of g is that of f translated c units to the right.
  - If $D_f = [a, b]$ ($f(x)$ is only defined when $a \leq x \leq b$), then $f(x-c)$ is only defined when $a \leq x-c \leq b$. This implies that $g(x)$ is only defined when $a + c \leq x \leq b + c$; hence, $D_g = [a + c, b + c]$.
- Change of scale.



(a) Horizontal scaling. (b) Vertical scaling.

Figure 1.7: Change of scale.

  - Suppose $g(x) = f(kx)$. Then to transform the graph of f into that of g, "compress or stretch the x-axis by shrinking (if $k > 1$) or stretching (if $k < 1$) every interval of length k on the x-axis into an interval of length 1" (Thomas, 1972, p. 24).

- Let $g(x) = f(kx)$. If $D_f = [a, b]$ ($f(x)$ is only defined when $a \leq x \leq b$), then $f(kx)$ is only defined when $a \leq kx \leq b$. This implies that $g(x)$ is only defined when $a/k \leq x \leq b/k$; hence, $D_g = [a/k, b/k]$.
- Be wary when $k = 0$.

– Suppose $g(x) = k \cdot f(x)$. Then to transform the graph of f into that of g, "stretch the f curve vertically (if $k > 1$), or compress it (if $k < 1$) [or] change the scale on the y-axis so that points labeled $1, 2, 3, \ldots$ for the graph of f are relabled $k, 2k, 3k, \ldots$ for the graph of [g]" (Thomas, 1972, p. 25).

- This kind of stretching may cause R_g to differ from R_f by some factor k, but it will not affect D_f and D_g.

1.8 Behavior of Functions

- Linear functions (refer to Figure 1.8 throughout the following).

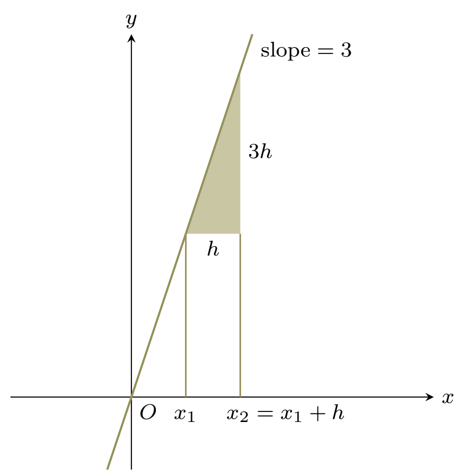


Figure 1.8: Behavior of linear functions.

– For example, let $f(x) = 3x$. The graph of f is a straight line through the origin with slope $+3$.

– Let x_1 be an initial value of x, and let $x_2 = x_1 + h$ be a new value of x obtained by increasing x_1 by h units[2]. The corresponding increase in $f(x)$ is given by

$$f(x_1 + h) - f(x_1) = 3(x_1 + h) - 3x_1 = 3h$$

Thus, we see that f everywhere changes three times as fast as x.

- Quadratic functions (refer to Figure 1.9 throughout the following).

– Unlike linear functions, quadratic functions do not everywhere have a constant rate of change.

– Let $f(x) = x^2$. In an analogous manner to the previous example, we find that the change between x_1 and $x_2 = x_1 + h$ is

$$f(x_1 + h) - f(x_1) = (x_1 + h)^2 - x_1^2 = 2x_1 h + h^2$$

Thus, we see that the rate of change of f is dependent on both the initial value of x and the amount of increase in x.

[2]Note that h is an alternative notation for Δx.

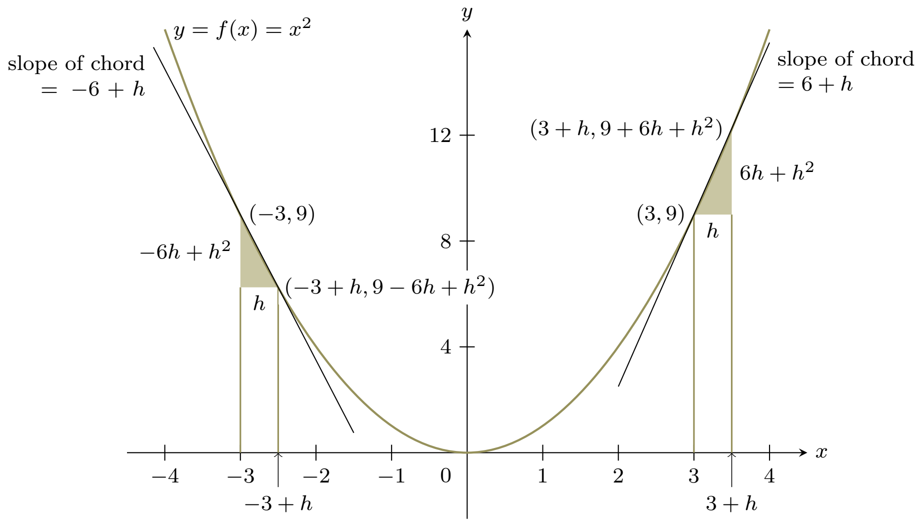


Figure 1.9: Behavior of quadratic functions.

- While a linear function increases at a rate directly proportional to the increase in x, the above demonstrates that "the increase in x^2, as x increases from x_1 to $x_1 + h$, is $2x_1 + h$ times the increase in x" (Thomas, 1972, p. 27).
- We can now define the **average rate of increase**.
- Thus, the average rate of increase of x^2 is $2x_1 + h$.
- Now let's see what happens as h shrinks.

		x_1			
		2	3	-2	-3
h	1	5	7	-3	-5
	0.5	4.5	6.5	-3.5	-5.5
	0.25	4.25	6.25	-3.75	-5.75
	0.1	4.1	6.1	-3.9	-5.9
	0.01	4.01	6.01	-3.99	-5.99
	0.001	4.001	6.001	-3.999	-5.999

Table 1.1: Average rate of change of x^2 versus h.

- From Table 1.1, we see that smaller values of h cause the average rate of change to tend toward $2x_1$. This is the beginning of **differential calculus**.

- **Average rate of increase** (of $f(x)$, per unit of increase in x, from x_1 to $x_1 + h$): The ratio,

$$\frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} = \frac{\text{change in } f(x)}{\text{change in } x}$$

- **Differential calculus**: The branch of calculus concerned with the *instantaneous* rate of increase of a function, as opposed to the *average* rate of increase.
- So constant and linear functions are easy to analyze. But for more complicated functions, we need more advanced tools.
- Let's begin exploring the instantaneous rate of change, continuing with the parabola example.

- For x^2, the average rate of change is given by $\frac{f(x_1+h)-f(x_1)}{h} = 2x_1 + h,\ h \neq 0$.
- The $h \neq 0$ is critical — we wish to consider the case where $h = 0$, but we cannot. However, we can consider values of the slope function $m(h) = 2x_1 + h$ in a deleted neighborhood of $h = 0$. By decreasing the radius of the neighborhood, we can get progressively closer to analytically approximating $m(0)$.

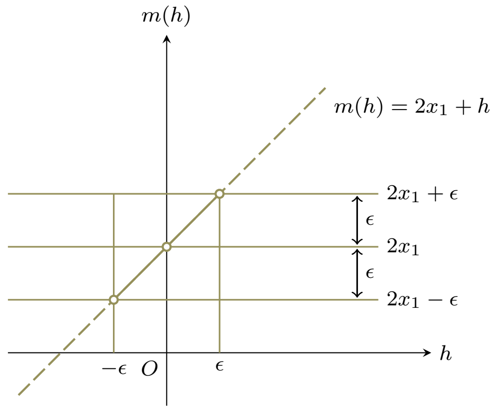


Figure 1.10: Deleted neighborhood of a slope function.

- From Figure 1.10, we can see that $m(h)$ is bounded between $2x_1 + \epsilon$ and $2x_1 - \epsilon$ when $|h|$ is less than ϵ (or any positive number smaller than ϵ)[3].
- In fact, $0 < |h| < \epsilon$, or $-\epsilon < h < \epsilon$, directly implies $2x_1 - \epsilon < 2x_1 + h < 2x_1 + \epsilon$.
- We can now formally define **approximation**, such as what was just described.

- **Approximation** (of $f(x)$ by L to within ϵ on (a, b)): The value L approximates $f(x)$ to within ϵ on the interval (a, b) if $L - \epsilon < f(x) < L + \epsilon$ when $a < x < b$.

1.9 Slope of a Curve

- **Slope of the curve** (at P): The limiting value of the slope of the secant between distinct points P, Q on the curve $y = f(x)$ as Q moves along the curve progressively closer to P. *Also known as* **slope of the tangent to the curve** (at P).
  - A purely geometric definition also exists: "Let C be a curve and P a point on C. If there exists a line L through P such that the measure of one of the angles between L and the secand line PQ approaches zero as Q approaches P along C, then L is said to be tangent to C at P" (Thomas, 1972, p. 30).
  - An advantage of the geometric definition is that it does not depend on the coordinate axes and allows vertical lines.
  - However, in most cases, we will stick with the algebraic definition.
- Thomas, 1972 considers the average rate of increase equation for a cubic function, informally allowing Δx to tend towards 0 to derive a slope function.

1.10 Derivative of a Function

- We now formalize our notion of a slope function.

[3]Note that ϵ is used to denote an arbitrary (often arbitrarily small) positive number.

- We know that the slope m_{sec} of the secant from $P(x,y)$ to a point on the curve $y=f(x)$ at $(x+\Delta x, f(x+\Delta x))$ is given by

$$m_{\text{sec}} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- Now as Δx tends toward 0, m_{sec} tends toward the slope $m_{\tan}$ of the tangent at P. The mathematical symbols which summarize this discussion are

$$m_{\tan} = \lim_{Q\to P} m_{\text{sec}} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- The number given by the last operation in the above equation is clearly related to f. Thus, to indicate relation, we define

$$\boldsymbol{f'(x)} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Also known as $\boldsymbol{y'}$, $\boldsymbol{\frac{dy}{dx}}$, $\boldsymbol{D_x y}$.

- This limit may sometimes fail to exist. However, at each point where it does exist, f is said to have a **derivative**, or to be **differentiable**. Similarly, $f'(x)$ is said to be the **derivative** (of f at x).
- Differential calculus is concerned with two problems.
  1. "Given a function f, determine those values of x (in the domain of f) at which the function possesses a derivative" (Thomas, 1972, p. 32).
  2. "Given a function f and an x at which the derivative exists, find $f'(x)$" (Thomas, 1972, p. 32).
- **Derived function**: "The set of all pairs of numbers $(x, f'(x))$ that can be formed by this process" (Thomas, 1972, p. 33). *Also known as* **derivative** (of f).
  - "The domain of f' is a subset of the domain of f" (Thomas, 1972, p. 33).
  - Symbolically, $D_{f'} \subset D_f$. However, for most functions considered in this book, $D_{f'} = D_f$ with maybe a few exceptions.
- On computing $f'(x)$ by eliminating the division by 0 and then substituting: "We may say that after the division by Δx has been carried out and the expression has been reduced to a form... which 'makes sense' (that is, does not involve division by zero) when Δx is taken equal to zero, then the limit as Δx approaches zero does exist and may be found by simply replacing Δx by zero in this reduced form" (Thomas, 1972, p. 34).
- Essentially, what we are doing when we eliminate the division by zero is we are expanding the domain of the function, the limit of which we are taking, to include a point of interest (Thomas, 1972 elaborates quite a bit on this point).

1.11 Velocity and Rates

- Mainly just applies average and instantaneous rates of change to the physical problem of distance and velocity. However...
- "Derivatives are important in economic theory, where they are usually indicated by the adjective **marginal**" (Thomas, 1972, p. 37).
  - "Suppose that in order to produce $x+\Delta x$ tons of steel weekly, it would cost $y+\Delta y$ dollars. The increase in cost per unit increase in output would be $\Delta y/\Delta x$. The limit of this ratio, as Δx tends to zero, is called the **marginal cost**" (Thomas, 1972, p. 37).
  - There also exists **marginal revenue** $\mathrm{d}P/\mathrm{d}x$ and **marginal profit** $\mathrm{d}T/\mathrm{d}x$.

- Note: "The *average* rate of change of y per unit change in x, $\Delta y/\Delta x$, when multiplied by the number of units change in x, Δx, gives the actual change in y:

  $$\Delta y = \frac{\Delta y}{\Delta x}\,\Delta x$$

  The *instantaneous* rate of change of y per unit change in x, $f'(x)$, multiplied by the number of units change in x, Δx, gives the change that would be produced in y if the point (x, y) were to move along the tangent line instead of moving along the curve; that is

  $$\Delta y_{\text{tan}} = f'(x)\,\Delta x$$

  One reason calculus is important is that it enables us to find quantitatively how a change in one of two related variables affects the second variable" (Thomas, 1972, p. 38).

Chapter 4

Applications

4.1 Increasing or Decreasing Functions: The Sign of $\mathrm{d}y/\mathrm{d}x$

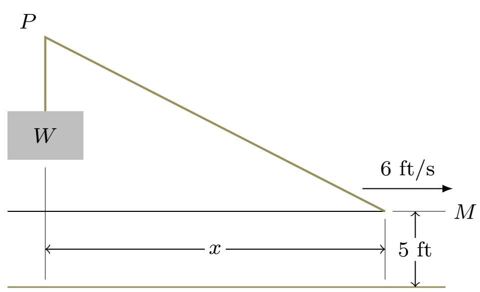
7/8:

- **Increasing** (function f on $[a,b]$): A function f such that $f(x_1) > f(x_2)$ when $x_1 > x_2$ for all x_1, x_2 in the interval $[a,b]$. *Also known as* **rising**.
- **Decreasing** (function f on $[a,b]$): A function f such that $f(x_1) < f(x_2)$ for $a \leq x_2 < x_1 \leq b$. *Also known as* **falling**.
  - Sometimes, we consider functions that increase or decrease on open or half-open intervals.
- **Increasing** (function f at a point c): A function f such that in some neighborhood N of c, $x > c \Rightarrow f(x) > f(x)$ and $x < c \Rightarrow f(x) < f(x)$ for all $x \in N$.
- **Decreasing** (function f at a point c): A function f such that in some neighborhood N of c, $x > c \Rightarrow f(x) < f(x)$ and $x < c \Rightarrow f(x) > f(x)$ for all $x \in N$.
  - As an odd example, $\operatorname{sgn} x$ is increasing at $x = 0$.
- A function may oscillate sufficiently fast at a point to be neither increasing nor decreasing.
  - For example, for
    $$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
    "no matter how small a neighborhood of zero N may be, there are x's in N for which $f(x)$ is positive and those for which it is negative. This function oscillates infinitely often between positive and negative values in ever neighborhood of $x = 0$" (Thomas, 1972, p. 107).
- When $\mathrm{d}y/\mathrm{d}x > 0$, y is increasing. When $\mathrm{d}y/\mathrm{d}x < 0$, y is decreasing. When $\mathrm{d}y/\mathrm{d}x = 0$, y may be increasing (consider $y = x^3$), decreasing (consider $y = -x^3$), or neither (consider $y = x^2$).
- There is a relation between increasing and decreasing points, and positive and negative slopes, respectively, of the tangent lines to those points.
- Knowing where a function is increasing or decreasing can help in sketching the curve.

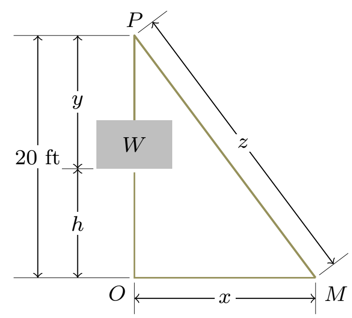
4.2 Related Rates

- In certain physical settings, we must consider not only quantities but the rates at which those quantities are changing to answer questions.

- For a **problem in related rates**, it is typical that "(a) certain variables are related in a definite way for all value of t under consideration, (b) the values of some or all of these variables and the rates of change of some of them are given at some particular instant, and (c) it is required to find the rate of change of one or more of them at this instant" (Thomas, 1972, p. 110).
  - "The variables may then all be considered to be functions of time, and if the equations which relate them for all values of t are differentiated with respect to t, the new equations so obtained will tell how their rates of change are related" (Thomas, 1972, p. 110).
- We explore three examples to illustrate the most common techniques used.
- Suppose (see Figure 4.1a) there is a "rope running through a pulley at P, bearing a weight W at one end. The other end is held in a man's hand M at a distance of 5 feet above the ground as he walks in a straight line at the rate of 6 [ft/s]" (Thomas, 1972, p. 108). Additionally (see Figure 4.1b), "suppose that the pulley is 25 ft above the ground, the rope is 45 ft long, and at a given instant the distance x is 15 ft and the man is walking away from the pulley. How fast is the weight being raised at this particular instant?" (Thomas, 1972, p. 109).



(a) The man and the pulley.



(b) Construction of the pulley.

Figure 4.1: Related rates: The pulley.

  - We begin by assessing what is given and what we want to find.
    We are given...
    (a) Relationships between the variables which are to hold for all instants of time:
    $$y + z = 45 \qquad h + y = 20 \qquad 20^2 + x^2 = z^2$$
    (b) Quantities at a given instant in time, which we may take to be $t = 0$:
    $$x = 15 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 6$$
    We want to find...
    $$\frac{\mathrm{d}h}{\mathrm{d}t}$$
    at the instant $t = 0$.
  - We obtain a relationship between x (whose rate is given) and h (whose rate we want).
    $$y = 20 - h$$
    $$z = 45 - (20 - h) = 25 + h$$
    $$20^2 + x^2 = (25 + h)^2$$

– We now implicitly differentiate the above equation with respect to t and solve for $\mathrm{d}h/\mathrm{d}t$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(20^2+x^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(25+h)^2\\ 0+2x\frac{\mathrm{d}x}{\mathrm{d}t} &= 2(25+h)\frac{\mathrm{d}h}{\mathrm{d}t}\\ \frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{x}{25+h}\frac{\mathrm{d}x}{\mathrm{d}t}\end{aligned}$$

– We see that we will need the value of h at $t=0$. This can be found via the equation $20^2+x^2 = (25+h)^2$ since we know the value of x at $t=0$.

$$\begin{aligned}(25+h)^2 &= 20^2+(15)^2\\ h &= 0\end{aligned}$$

– Since we now have every value that we have set equal to $\mathrm{d}h/\mathrm{d}t$, all that is left is to plug and chug.

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{x}{25+h}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{15}{25+0}\cdot 6\\ &= \frac{18}{5}\text{ ft/s}\end{aligned}$$

- Suppose (see Figure 4.2) there is a "ladder 26 ft long which leans against a vertical wall. At a particular instant, the foot of the laddeer is 10 ft out from the base of the wall and is being drawn away from the wall at the rate of 4 [ft/s]. How fast is the top of the ladder moving down the wall at this instant?" (Thomas, 1972, p. 110).

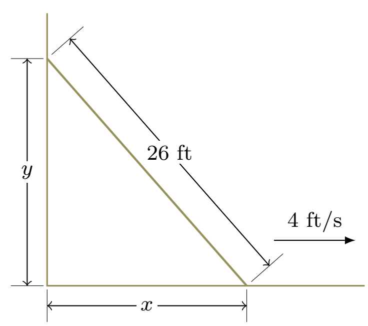


Figure 4.2: Related rates: The ladder.

– Symbolically, the problem is asking this: given

$$x^2+y^2=26^2 \qquad x=10 \qquad \frac{\mathrm{d}x}{\mathrm{d}t}=4$$

find

$$\frac{\mathrm{d}y}{\mathrm{d}t}$$

– As before, differentiate and solve for $\mathrm{d}y/\mathrm{d}t$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2+y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(26^2\right)\\ 2x\frac{\mathrm{d}x}{\mathrm{d}t}+2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}\end{aligned}$$

– Now find y and substitute.

$$\begin{aligned}10^2 + y^2 &= 26^2 \\ y &= 24\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -\frac{10}{24} \cdot 4 \\ &= -\frac{5}{3} \text{ ft/s}\end{aligned}$$

– Note that the negative sign indicates that y is decreasing; that the top of the ladder is moving *down* at 5/3 ft/s (or up at $-5/3$ ft/s).

- Suppose there is an inverted right "conical reservoir [of height 10 ft and base radius 5 ft] into which water runs at the constant rate of 2 ft^3 per minute. How fast is the water level rising when it is 6 ft deep?" (Thomas, 1972, p. 111).

  – Let h be the height (in ft) of the reservoir, r be the base radius (in ft) of the reservoir, x be the radius (in ft) of the section of the cone at the water line at time t (in min), y be the depth (in ft) of water in the tank at time t (in min), and v be the volume (in ft^3) of water in the tank at time t (in min).

  – Thus, the problem is asking this: given

$$v = \frac{1}{3}\pi x^2 y \qquad \frac{x}{y} = \frac{r}{h}$$

$$h = 10 \qquad r = 5 \qquad y = 6 \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = 2$$

find

$$\frac{\mathrm{d}y}{\mathrm{d}t}$$

  – Like with the pulley, we need to find an equation relating just v and y. Use a substitution based on similar triangles.

$$\begin{aligned}v &= \frac{1}{3}\pi x^2 y \\ &= \frac{1}{3}\pi\left(\frac{ry}{h}\right)^2 y \\ &= \frac{\pi r^2}{3h^2} y^3\end{aligned}$$

  – Differentiate, solve, and substitute.

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\pi r^2}{h^2} y^2 \frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{h^2}{\pi r^2 y^2}\frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{10^2}{\pi 5^2 6^2} \cdot 2 \\ &= \frac{2}{9\pi} \approx 0.071 \text{ ft/min}\end{aligned}$$

4.3 Significance of the Sign of the Second Derivative

- Note: if $\mathrm{d}y/\mathrm{d}x$ fails to exist at some point P, *but* $\mathrm{d}x/\mathrm{d}y = 0$, the tangent to P is vertical.
  - On obtaining $\mathrm{d}x/\mathrm{d}y$[1]:
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$$
- "The sign of the second derivative tells whether the graph of $y = f(x)$ is concave upward (y'' positive) or downward (y'' negative)" (Thomas, 1972, p. 113).
- **Point of inflection**: "A point where the curve changes the direction of its concavity from downward to upward or vice versa [that is not a **cusp**]" (Thomas, 1972, p. 114). *Also known as* **inflection point**.
  - Inflection points occur where y'' changes sign. This can happen when $y'' = 0$ or when y'' fails to exist.
- **Cusp**: A sharp corner on a graph (a place where y'' fails to exist).

4.4 Curve Plotting

- When sketching curves given the equation, use the following procedure.
  A. "Calculate $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}^2y/\mathrm{d}x^2$" (Thomas, 1972, p. 115).
  B. "Find the values of x for which $\mathrm{d}y/\mathrm{d}x$ is positive and for which it is negative. Calculate y and $\mathrm{d}^2y/\mathrm{d}x^2$ at the points of transition between positive and negative values of $\mathrm{d}y/\mathrm{d}x$. These may give maximum or minimum points on the curve" (Thomas, 1972, p. 115).
  C. "Find the values of x for which $\mathrm{d}^2y/\mathrm{d}x^2$ is positive and for which it is negative. Calculate y and $\mathrm{d}y/\mathrm{d}x$ at the points of transition between positive and negative values of $\mathrm{d}^2y/\mathrm{d}x^2$. These may give points of inflection of the curve" (Thomas, 1972, p. 115).
  D. "Plot a few additional points. In particular, points which lie between the transition points already determined or points which lie to the left and to the right of all of them will ordinarily be useful. The nature of the curve for large values of $|x|$ should also be indicated" (Thomas, 1972, p. 115).
  E. "Sketch a smooth curve through the points found above, unless there are discontinuitites in the curve or its slope. Have the curve pass through its points rising or falling as indicated by the sign of $\mathrm{d}y/\mathrm{d}x$, and concave upward or downward as indicated by the sign of $\mathrm{d}^2y/\mathrm{d}x^2$" (Thomas, 1972, p. 115).
- As you plot points, consider sketching their tangents, too.
- Consider making a table with columns of significant x values, their assigned y, y', and y'' values, and any important remarks before starting to draw.
- If $f(x) = \frac{P(x)}{Q(x)}$, solve $Q(x) = 0$ to find vertical asymptotes.

4.5 Maxima and Minima: Theory

- **Relative maximum** (of f): A point $(a, f(a))$ of a function f such that $f(a) \geq f(a+h)$ for all positive and negative values of h sufficiently near zero. *Also known as* **local maximum**.
- **Relative minimum** (of f): A point $(b, f(b))$ of a function f such that $f(b) \leq f(x)$ for all x in some neighborhood of a. *Also known as* **local minimum**.

[1]This is another place where Leibniz's notation is particularly useful.

- **Absolute maximum** (of f): A point $(a, f(a))$ of a function f such that $f(a) \geq f(x)$ for all $x \in D_f$.
- **Absolute minimum** (of f): A point $(b, f(b))$ of a function f such that $f(b) \geq f(x)$ for all $x \in D_f$.
- We now prove a relationship between f' and the maxima and minima of f.

  Theorem 4.1. Let the function f be defined for $a \leq x \leq b$ and have a relative maximum or minimum at $x = c$, where $a < c < b$. If thee derivative $f'(x)$ exists as a finite number at $x = c$, then $f'(c) = 0$.

  Proof. If $f'(c)$ were positive, then f would be increasing. But f is neither increasing nor decreasing at c because f has a local maximum or minimum at c. Hence, $f'(c)$ cannot be positive. Likewise, $f'(c)$ cannot be negative. Therefore, $f'(c) = 0$. □

  - Note that the theorem does not pertain to cases where $f'(c)$ does not exist, nor does it pertain to cases where c is at one of the endpoints of the interval $[a, b]$.
  - Also note that the converse of the theorem does not hold.
- The inverse of an increasing function is increasing. This also holds for decreasing functions.
- If f is only defined on $[a, b]$, then $f'(a)$ and $f'(b)$ do not exist (because the limit is different on both sides). However,

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \qquad f'(b^-) = \lim_{h]to0^-} \frac{f(x+h) - f(x)}{h}$$

  may exist. These are called the **right-hand derivative** and **left-hand derivative** respectively.

  - It's imprecise to say that f is differentiable at these endpoints, but many mathematicians will allow it[2] as they expect us to just know that what is really meant is it is differentiable on (a, b) and one-side differentiable at the endpoints.
- "If the domain of f is the bounded, closed interval $[a, b]$ and if $f'(a^+)$ and $f'(b^-)$ exist, then it is easy to verify that f has a local [maximum or minimum] at a if $[f'(a^+) < 0$ or $f'(a^+) > 0$, respectively] and f has a local [minimum or maximum] at b if $[f'(b^-) < 0$ or $f'(b^-) > 0]$" (Thomas, 1972, p. 121).
- Maxima and minima are more generally referred to as **critical points** or **extrema**.
- Candidates for extrema exist at points where (1) the derivative is zero, (2) the derivative fails to exist, and (3) the domain of the function has an end.

[2] Really?

Chapter 8

Hyperbolic Functions

8.1 Introduction

6/24:
- **Hyperbolic functions**: Certain combinations of e^x and e^{-x} that are used to solve certain engineering problems (the hanging cable) and are useful in connection with differential equations.

8.2 Definitions and Identities

- Let

$$\cosh u = \frac{1}{2}(e^u + e^{-u}) \qquad \sinh u = \frac{1}{2}(e^u - e^{-u})$$

  - These combinations of exponentials occur sufficiently frequently that we give a special name to them.
  - Although the names may seem random, $\sinh u$ and $\cosh u$ do share many analogous properties with $\sin u$ and $\cos u$.
  - Pronounced to rhyme with "gosh you" and as "cinch you," respectively.
- Like $x = \cos u$ and $y = \sin u$ are associated with the point (x, y) on the unit circle $x^2 + y^2 = 1$, $x = \cosh u$ and $y = \sinh u$ are associated with the point (x, y) on the unit hyperbola $x^2 - y^2 = 1$.
  - Note that $x = \cosh u$ and $y = \sinh u$ are associated with the *right-hand* branch of the unit hyperbola.
  - Also note that sine and cosine are sometimes referred to as the **circular functions**.
- Analogous to sine and cosine, we have the identity

$$\cosh^2 u - \sinh^2 u = 1$$

- We define the remaining hyperbolic trig functions as would be expected.

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} \qquad \operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}}$$
$$\coth u = \frac{\cosh u}{\sinh u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} \qquad \operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}}$$

- Since $\cosh u + \sinh u = e^u$, we can replace any combination of exponentials with hyperbolic sines and cosines and vice versa.
- Note that the hyperbolic functions are *not* periodic.
  - This does mean, though, that they have more easily defined properties at infinity.
- "Practically all the circular trigonometric identities have hyperbolic analogies" (Thomas, 1972, p. 267).

8.3 Derivatives and Integrals

6/25:

- Derivatives of the hyperbolic functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = \cosh u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh u) = \sinh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh u) = \operatorname{sech}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth u) = -\operatorname{csch}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{\mathrm{d}u}{\mathrm{d}x}$$

  - Note that these are almost exact analogs of the formulas for the corresponding circular functions, the exception being that the negative signs are not associated with the cofunctions but with the latter three.

- We now introduce the hanging cable problem, deriving the differential equation that represents the condition for equilibrium of forces acting on a section AP of a hanging cable (Figure 8.1).

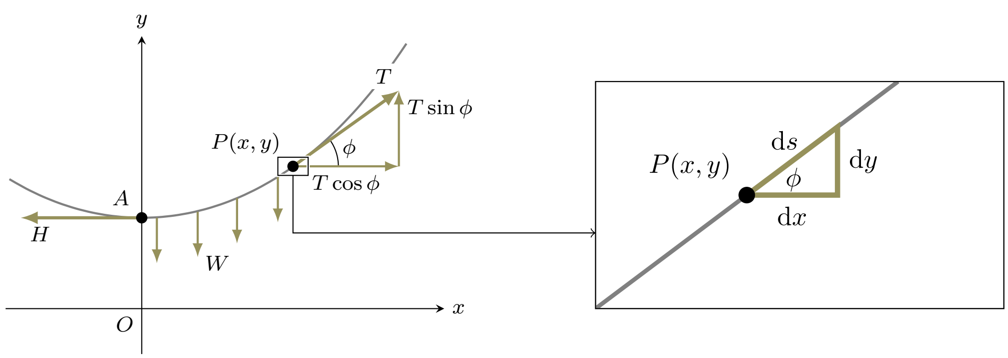


Figure 8.1: A section AP of a hanging cable.

- Let point A to be the lowest point in the arc of the hanging cable, and let it be at $(0, y_0)$ in the Cartesian plane.
- Continue along the right arc of the cable until arriving at some point $P(x, y)$.
- We wish to consider only segment AP, so we need to anchor points A and P as if the rest of the cable were still there. Now every infinitesimal sliver of the cable is being pulled (downward) slightly by gravity, but significantly (tangentially) by the rest of the cable. Thus, we can compensate at point A by pulling it tangentially left with some force H, and at point P by pulling it tangentially up and to the right with some force T.
- Since the cable is at equilibrium, the three forces acting on the cable as a whole (T, H, and W) are balanced. Thus,

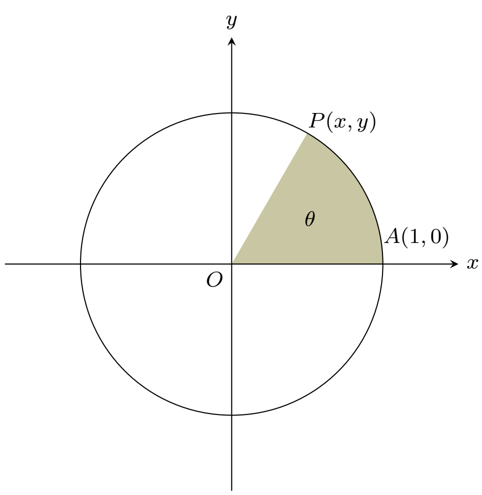
$$T \sin\phi = W$$
$$T \cos\phi = H$$

- Combining these two equations gives an important result:

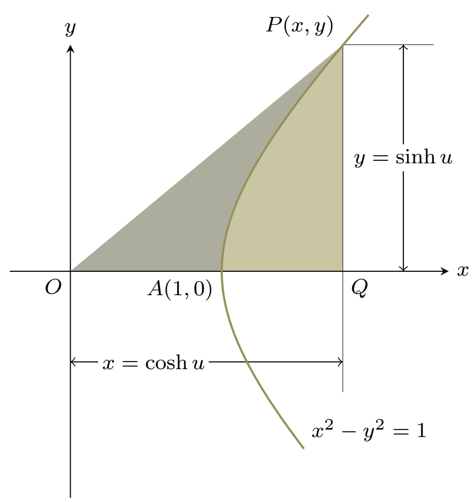
$$\frac{T \sin\phi}{T \cos\phi} = \frac{W}{H}$$
$$\tan\phi = \frac{W}{H}$$

- Now $\tan\phi$ is a particularly important piece of the puzzle, because it actually equals $\frac{\mathrm{d}y}{\mathrm{d}x}$ (see the zoomed-in section of Figure 8.1). Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{W}{H}$$
- Contrary to how it may look, W is actually not a constant — the weight of section AP is dependent on P (i.e., is dependent on how long the section is). If we assume that the cable has a uniform weight per length ratio w and that section AP is s units long, then we have $W = ws$. Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ws}{H}$$
- s is just arc length. Thus, $s = \int_A^P \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$. However, because we cannot have an integral in a differential equation, we differentiate to find $\mathrm{d}s = \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$.
  - Note that this expression for $\mathrm{d}s$ makes sense because, by the zoomed-in section of Figure 8.1, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{(\mathrm{d}x^2/\mathrm{d}x^2 + \mathrm{d}y^2/\mathrm{d}x^2)\mathrm{d}x^2} = \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$.
- If we differentiate $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ws}{H}$, we can get $\mathrm{d}s$ into the equation and substitute, as follows, to yield the final differential equation.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\frac{\mathrm{d}s}{\mathrm{d}x}$$
$$\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}$$

8.4 Geometric Meaning of the Hyperbolic Radian



(a) Circular radians.



(b) Hyperbolic radians.

Figure 8.2: Geometric meaning of radians.

- For circular sine and cosine, the "meaning of the variable θ in the equations $x = \cos\theta$, $y = \sin\theta$ as they relatee to the point $P(x, y)$ on the unit circle $x^2 + y^2 = 1$ [is] the radian measure of the angle AOP in [Figure 8.2a], that is $\theta = \frac{\text{arc } AP}{\text{radius } OA}$" (Thomas, 1972, p. 271).
  - However, since $A = \frac{1}{2}r^2\theta = \frac{\theta}{2}$ for $r = 1$, θ also equals twice the area of the sector AOP.

- To understand the meaning of the variable u, calculate the area of the sector AOP in Figure 8.2b as an analog to circular area.

$$\begin{aligned} A_{AOP} &= A_{OQP} - A_{AQP} \\ &= \frac{1}{2}bh - \int_A^P y\,\mathrm{d}x \end{aligned}$$

$$y = \sinh u, \quad x = \cosh u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \sinh u \Rightarrow \mathrm{d}x = \sinh u\,\mathrm{d}u$$

$$\begin{aligned} &= \frac{1}{2}xy - \int_A^P \sinh^2 u\,\mathrm{d}u \\ &= \frac{1}{2}\cosh u \sinh u - \frac{1}{2}\int_A^P (\cosh 2u - 1)\,\mathrm{d}u \\ &= \frac{1}{2}\sinh u \cosh u - \frac{1}{2}\left[\frac{1}{2}\sinh 2u - u\right]_{A(u=0)}^{P(u=u)} \\ &= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{4}\sinh 2u - \frac{1}{2}u\right) \\ &= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{2}\sinh u \cosh u - \frac{1}{2}u\right) \\ &= \frac{1}{2}u \end{aligned}$$

  - This implies that u also equals twice the area of the sector AOP (the hyperbolic sector, that is).
  - This means, for example, that "$\cosh 2$ and $\sinh 2$ may be interpreted as the coordinates of P when the area of the sector AOP is just equal to the area of a square having OA as side" (Thomas, 1972, p. 272).

8.5 The Inverse Hyperbolic Functions

6/26:

- The inverse of $x = \sinh y$ is $y = \sinh^{-1} x$.
  - Since there is a one-to-one correspondence between x and y values for the inverse hyperbolic sine function, there is no need to define a principal branch (as there was with circular sine).
- The inverse of $x = \cosh y$ is $y = \cosh^{-1} x$, where $y \geq 0$ and $x \geq 1$.
  - Since there is a two-to-one correspondence between x and y values this time, we choose the positive values to be the principal branch and let the negative values be defined by the function $y = -\cosh^{-1} x$.
- Note that the only other inverse hyperbolic trig function that needs a principal branch is (rather appropriately) $\operatorname{sech} x$. Likewise, the positive values make up the principal branch.
- Like the hyperbolic trig functions have exponential forms, the inverse hyperbolic trig functions have logarithmic forms.

– For example,

$$\begin{aligned}
y &= \sinh^{-1} x \\
\sinh y &= x \\
\frac{1}{2}(\mathrm{e}^y - \mathrm{e}^{-y}) &= x \\
\mathrm{e}^y - \frac{1}{\mathrm{e}^y} &= 2x \\
\mathrm{e}^{2y} - 1 &= 2x\mathrm{e}^y \\
0 &= \mathrm{e}^{2y} - 2x\mathrm{e}^y - 1 \\
\mathrm{e}^y &= x \pm \sqrt{x^2+1} \\
y &= \ln\left(x + \sqrt{x^2+1}\right)
\end{aligned}$$

- Like the inverse circular trig functions, the inverse hyperbolic functions are quite useful as the end results of integration of radicals. First, however, we must derive their derivatives.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} u\right) = \frac{1}{\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} && \frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} u\right) = \frac{1}{\sqrt{u^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} u\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| < 1 && \frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{sech}^{-1} u\right) = \frac{-1}{u\sqrt{1-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\coth^{-1} u\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| > 1 && \frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{csch}^{-1} u\right) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

8.6 The Hanging Cable

- We seek to derive the solution to the following differential equation associated with the hanging cable problem, as described in Section 8.3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

- Since the above equation involves the second derivative, we will have to deal with two constants of integration. Since it doesn't matter where the hanging cable lies in the Cartesian plane, we can choose its location such that the final answer will be as simple as possible.
  – "By choosing the y-axis to be the vertical line through the lowest point of the cable, one condition becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$" (Thomas, 1972, p. 277).
  – "Then we may still move the x-axis up or down to suit our convenience. That is, we let $y = y_0$ when $x = 0$, and we may choose y_0 so as to give us the simplest form in our final answer" (Thomas, 1972, p. 277).
- Let's begin solving the original equation. Since it involves y' and y'' but not y, let $y' = p$ and start by integrating with respect to p.

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{w}{H}\sqrt{1+p^2} \\
\frac{\mathrm{d}p}{\sqrt{1+p^2}} &= \frac{w}{H}\,\mathrm{d}x \\
\int \frac{\mathrm{d}p}{\sqrt{1+p^2}} &= \int \frac{w}{H}\,\mathrm{d}x \\
\sinh^{-1} p &= \frac{w}{H}x + C_1
\end{aligned}$$

- Since $p = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $x = 0$, $C_1 = 0$. Thus,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sinh\left(\frac{w}{H}x\right) \\
\mathrm{d}y &= \sinh\left(\frac{w}{H}x\right)\mathrm{d}x \\
\int \mathrm{d}y &= \int \sinh\left(\frac{w}{H}x\right)\mathrm{d}x \\
y &= \frac{H}{w}\cosh\left(\frac{w}{H}x\right) + C_2
\end{aligned}$$

- Since $y = y_0$ when $x = 0$, $C_2 = y_0 - \frac{H}{w}$. Thus, let $y_0 = \frac{H}{w}$. Therefore, we are finished:

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

Chapter 9

Methods of Integration

9.1 Basic Formulas

6/30: • Useful, abstract info (that I already know) on what makes a student good at integating, e.g., integrating is an exercise in trial-and-error, but there are ways to increase your likelihood of being successful.

9.2 Powers of Trigonometric Functions

- When integrating power functions, look for integral/derivative relationships, which may allow you to substitute u and $\mathrm{d}u$ at the same time.
  - For example, when confronted with $\int \sin^n ax \cos ax \,\mathrm{d}x$, note that $\cos ax$ is almost the derivative of $\sin ax$, and choose $u = \sin ax$ and $\frac{\mathrm{d}u}{a} = \cos ax \,\mathrm{d}x$ to yield $\frac{1}{a} \int u^n \,\mathrm{d}u$.
- When integrating power functions, it may be possible to split the exponent into a product ($u^n = u^a u^b$ where $a + b = n$) and work off of properties of one of the functions raised to a smaller exponent (u^a may have properties that u^n lacks).
  - For example, when confronted with $\int \sin^3 x \,\mathrm{d}x$, recall that $\sin^2 x$ has Pythagorean properties, and split the exponent.

    $$\begin{aligned}\int \sin^3 x \,\mathrm{d}x &= \int \sin^2 x \sin x \,\mathrm{d}x \\ &= \int \left(1 - \cos^2 x\right) \sin x \,\mathrm{d}x\end{aligned}$$

    Now we can use the previous property, since $\sin x$ and $\cos x$ have an integral/derivative relationship.

    $$\begin{aligned}&= -\int \left(1 - u^2\right) \mathrm{d}u \\ &= \int \left(u^2 - 1\right) \mathrm{d}u\end{aligned}$$

  - Note that this technique is applicable whenever an odd power of sine or cosine is to be integrated. For higher powers, consider the following.

    $$\int \cos^{2n+1} x \,\mathrm{d}x = \int \left(\cos^2 x\right)^n \cos x \,\mathrm{d}x = \int \left(1 - \sin^2 x\right)^n \cos x \,\mathrm{d}x = \int \left(1 - u^2\right)^n \mathrm{d}u$$

    Remember that $\left(1 - u^2\right)^n$ can be expanded via the binomial theorem.

- When integrating a composite trigonometric function, consider reducing it to a radical of powers of sines and cosines.
  - For example, $\sec x \tan x = \frac{\sin x}{\cos^2 x}$.
- When integrating positive integer powers of $\tan x$, use either the base cases or the **reduction formula**.
  - Begin by deriving a reduction formula.

$$\begin{aligned}\int \tan^n x \,\mathrm{d}x &= \int \tan^{n-2} x \left(\sec^2 x - 1\right) \mathrm{d}x \\ &= \int \tan^{n-2} \sec^2 x \,\mathrm{d}x - \int \tan^{n-2} x \,\mathrm{d}x \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \,\mathrm{d}x\end{aligned}$$

  Since the reduction formula decreases the exponent by 2, work out two base cases.

$$\int \tan^0 x \,\mathrm{d}x = \int \mathrm{d}x = x + C$$
$$\int \tan^1 x \,\mathrm{d}x = \int \frac{\sin x}{\cos x} \,\mathrm{d}x = -\int \frac{\mathrm{d}u}{u} = -\ln|\cos x| + C$$

  - Note that the impetus for initially deriving such a formula was the search for a way to get $\sec^2 x$ into the integrand, which can be done by splitting the exponent.
  - This method can easily be adjusted to suit negative powers of $\tan x$ (positive powers of $\cot x$).
- When integrating even powers of $\sec x$, either use the secant reduction formula, or split the exponent.
  - We derive the following formula.

$$\begin{aligned}\int \sec^{2n} x \,\mathrm{d}x &= \int \sec^{2n-2} x \sec^2 x \,\mathrm{d}x \\ &= \int \left(1 + \tan^2 x\right)^{n-1} \sec^2 x \,\mathrm{d}x \\ &= \int \left(1 + u^2\right)^{n-1} \mathrm{d}u\end{aligned}$$

- When integrating secant (or cosecant) alone, produce $\frac{u'}{u}$ by multiplying the integrand by a clever form of 1.
  - For example,

$$\begin{aligned}\int \sec x \,\mathrm{d}x &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \,\mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

9.3 Even Powers of Sines and Cosines

- When integrating the product of sines and cosines raised to powers where at least one exponent is a positive odd integer, split the exponent and use u-substitution.
  - In effect, we wish to evaluate $\int \sin^m x \cos^n x \,\mathrm{d}x$ where at least one of m, n is a positive odd integer.

  – For example, when confronted with $\int \cos^{\frac{2}{3}} x \sin^5 x \, \mathrm{d}x$, split the exponent of $\sin^5 x$ and choose $u = \cos x$ and $-\mathrm{d}u = \sin x \, \mathrm{d}x$.

$$\int \cos^{\frac{2}{3}} x \sin^5 x \, \mathrm{d}x = \int \cos^{\frac{2}{3}} x \left(1 - \cos^2 x\right)^2 \sin x \, \mathrm{d}x = \int u^{\frac{2}{3}} \left(u^2 - 1\right) \mathrm{d}u$$

- When integrating the product of sines and cosines raised to powers where both exponents are even integers, begin by transforming it into a sum of either just sines *or* just cosines raised to even integers. Then split the exponents and use one of the following formulas. It may be necessary to use these formulas multiple times. Use them until the problem has been reduced to a sum with only odd exponents.

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \qquad\qquad \cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

  – Note that "these identities may be derived very quickly by adding or subtracting the equations [$\cos^2 u + \sin^2 u = 1$ and $\cos^2 u - \sin^2 u = \cos 2u$] and by dividing by two" (Thomas, 1972, p. 287).
  – For example, when confronted with $\int \sin^2 x \cos^4 x \, \mathrm{d}x$, begin by changing it to a case with only powers of cosine (chose to eliminate the sine function because it is raised to a lower exponent and, thus, will need less binomial expansion).

$$\begin{aligned} \int \sin^2 x \cos^4 x \, \mathrm{d}x &= \int \left(1 - \cos^2 x\right) \cos^4 x \, \mathrm{d}x \\ &= \int \cos^4 x \, \mathrm{d}x - \int \cos^6 x \, \mathrm{d}x \end{aligned}$$

  Now split the exponents.

$$= \int \left(\cos^2 x\right)^2 \mathrm{d}x - \int \left(\cos^2 x\right)^3 \mathrm{d}x$$

  Employ the above formulas and use binomial expansion. If necessary, repeat (split the exponents, employ the above formulas, use binomial expansion) until only odd exponents remain (remember that 1 is an odd exponent).

$$\begin{aligned} &= \int \left(\frac{1}{2}(1 + \cos 2x)\right)^2 \mathrm{d}x - \int \left(\frac{1}{2}(1 + \cos 2x)\right)^3 \mathrm{d}x \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \cos^2 2x\right) \mathrm{d}x \\ &\quad - \frac{1}{8} \int \left(1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x\right) \mathrm{d}x \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) \mathrm{d}x \\ &\quad - \frac{1}{8} \int \left(1 + 3\cos 2x + \frac{3}{2}(1 + \cos 4x) + \cos^3 2x\right) \mathrm{d}x \end{aligned}$$

  These integrals may now be handled using previously discussed techniques.

9.4 Integrals With Terms $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, $\sqrt{u^2 - a^2}$, a^2+u^2, a^2-u^2

- When integrating a radical that resembles the derivative of an inverse trig function, we may factor out the issue so as to make the integral resemble one of the known formulas.

– For example, when confronted with $\int \frac{\mathrm{d}u}{a^2+u^2}$, divide the a^2 term out of the denominator and integrate with respect to $\frac{u}{a}$[1].

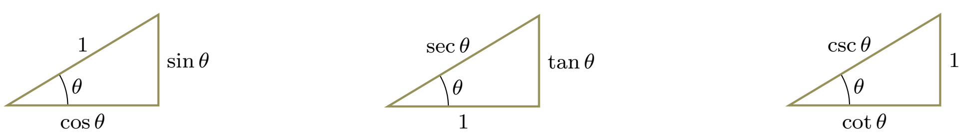
$$\begin{aligned}\int \frac{\mathrm{d}u}{a^2+u^2} &= \frac{1}{a^2}\int \frac{\mathrm{d}u}{1+\left(\frac{u}{a}\right)^2} \\ &= \frac{1}{a^2}\int \frac{a\,\mathrm{d}\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2} \\ &= \frac{1}{a}\tan^{-1}\frac{u}{a}+C\end{aligned}$$

– However, this method is partially flawed in that it relies on having memorized the derivatives of the inverse trig functions, i.e., it is not terribly analytical. This shortcoming will now be addressed with a new, more general technique.

- The new method leans heavily on the following three identities.

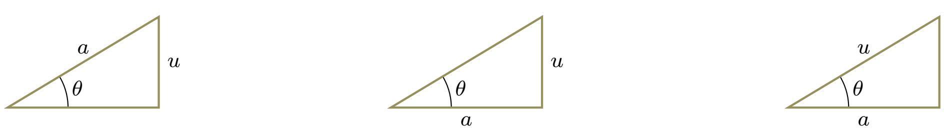
$$1-\sin^2\theta=\cos^2\theta \qquad 1+\tan^2\theta=\sec^2\theta \qquad \sec^2\theta-1=\tan^2\theta$$

– With the help of these identities, it is possible to...
  1. use $u=a\sin\theta$ to replace a^2-u^2 with $a^2\cos^2\theta$;
  2. use $u=a\tan\theta$ to replace a^2+u^2 with $a^2\sec^2\theta$;
  3. use $u=a\sec\theta$ to replace u^2-a^2 with $a^2\tan^2\theta$.



(a) $\cos^2\theta+\sin^2\theta=1$. (b) $1+\tan^2\theta=\sec^2\theta$. (c) $\cot^2\theta+1=\csc^2\theta$.

Figure 9.1: Geometric rationale for the trigonometric identities.



(a) $\sqrt{a^2-u^2}=a\cos\theta$
$u=a\sin\theta$.

(b) $\sqrt{u^2+a^2}=a\sec\theta$
$u=a\tan\theta$.

(c) $\sqrt{u^2-a^2}=a\tan\theta$
$u=a\sec\theta$.

Figure 9.2: Geometric rationale for the trigonometric substitutions.

– These identities and substitutions can be easily remembered by thinking of the Pythagorean theorem in conjunction with Figures 9.1 and 9.2, respectively.

- We may now evaluate inverse trig integrals analytically.

[1]Thomas, 1972 uses differentials with more complex functions than a single variable quite often. It's not something I've seen before, but it's something I should get used to (and it does make sense if you think about it — it's just an extension of the underlying concept of separation of variables integration).

- For example, when confronted with $\int \frac{\mathrm{d}u}{a^2+u^2}$, choose $u = a\tan\theta$ and $\mathrm{d}u = a\sec^2\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int \frac{\mathrm{d}u}{a^2+u^2} &= \int \frac{a\sec^2\theta\,\mathrm{d}\theta}{a^2+(a\tan\theta)^2} \\ &= \int \frac{a\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\,\mathrm{d}\theta \\ &= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta \\ &= \frac{1}{a}\int \mathrm{d}\theta \\ &= \frac{1}{a}\theta + C\end{aligned}$$

  At this point, solve $u = a\tan\theta$ for θ and substitute.

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

- Some integrals will simplify to have a plus/minus in the denominator, leading to two possible solutions. However, there are sometimes ways to isolate a single solution.
  - For example, $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta\,\mathrm{d}\theta}{\pm a\cos\theta} = \pm\theta + C$. However, when we consider the fact that $\theta = \sin^{-1}\frac{u}{a}$, we know that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (because inverse sine is not arcsine, and inverse sine is only defined over the principal branch of sine). Thus, since $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta \in [0,1]$, i.e., is always positive. Thus, we choose $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = +\theta + C = \sin^{-1}\frac{u}{a} + C$ as our one solution.
  - For example, $\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}}$ equals $\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$ or $-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| + C$ depending on whether $\tan\theta$ is positive or negative. But it can be shown algebraically that the two solutions are actually equal:

$$\begin{aligned}-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right| \\ &= \ln\left|\frac{a\left(u+\sqrt{u^2-a^2}\right)}{\left(u-\sqrt{u^2-a^2}\right)\left(u+\sqrt{u^2-a^2}\right)}\right| \\ &= \ln\left|\frac{a\left(u+\sqrt{u^2-a^2}\right)}{a^2}\right| \\ &= \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right|\end{aligned}$$

    Thus, we have $\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$ as the one solution[2].

- Some integrals will have extraneous constants that can be combined with C to simplify the *indefinite* integral.
  - Continuing with the above example,

$$\begin{aligned}\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} &= \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C \\ &= \ln\left|u + \sqrt{u^2-a^2}\right| - \ln|a| + C \\ &= \ln\left|u + \sqrt{u^2-a^2}\right| + C\end{aligned}$$

[2] Note that we could choose to use the other solution, but we choose this one because it's "simpler" (it uses addition instead of subtraction).

- When integrating an inverse trig integral with excess polynomial terms, look to transform it into a (power of a) trig integral problem.
  - For example, when confronted with $\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^2}}$, treat it as a case of a^2-u^2, but substitute the trig expression into the x^2 term in the numerator, too.

$$\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,\mathrm{d}\theta}{3\cos\theta} = 9\int \sin^2\theta\,\mathrm{d}\theta$$

  This integral may now be handled using previously discussed techniques.

- Many inverse trig integrals can also be evaluated hyperbolically, making use of the following three identities.

$$\cosh^2\theta - 1 = \sinh^2\theta \qquad 1-\tanh^2\theta = \operatorname{sech}^2\theta \qquad 1+\sinh^2\theta = \cosh^2\theta$$

  - With the help of these identities, it is possible to...
    1. use $u = a\tanh\theta$ to replace a^2-u^2 with $a^2\operatorname{sech}^2\theta$;
    2. use $u = a\sinh\theta$ to replace a^2+u^2 with $a^2\cosh^2\theta$;
    3. use $u = a\cosh\theta$ to replace u^2-a^2 with $a^2\sinh^2\theta$.

9.5 Integrals With ax^2+bx+c

- When integrating composite functions where the inner function is a binomial, look to factor said binomial.
  - The general quadratic $f(x) = ax^2+bx+c$, $a \neq 0$, can be reduced to the form $a\left(u^2+B\right)$ by completing the square and choosing $u = x+\frac{b}{2a}$ and $B = \frac{4ac-b^2}{4a^2}$:

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a} \\ &= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right) \end{aligned}$$

- When integrating the square root of a binomial, or some similarly tricky function of a binomial, we can transform the binomial into a form such that it will suit one of the inverse trig integrals.
  - Since it would lead to complex numbers, we disregard cases where $a\left(u^2+B\right)$ is negative, i.e., we focus on cases where (1) a is positive, and (2) a, B are both negative.
  - That being said, if it is an odd root ($\sqrt[3]{x}$, $\sqrt[5]{x}$, etc.), the sign doesn't matter.
  - For example, when confronted with $\int \frac{(x+1)\,\mathrm{d}x}{\sqrt{2x^2-6x+4}}$, begin by factoring the binomial[3].

$$2x^2-6x+4 = 2\left(x^2-3x\right)+4 = 2\left(x-\frac{3}{2}\right)^2-\frac{1}{2} = 2\left(u^2-a^2\right)$$

  Note that $u = x-\frac{3}{2}$ and $a = \frac{1}{2}$. We can now return to the integral, which we shall reformulate in terms of u in its entirety.

$$\int \frac{(x+1)\,\mathrm{d}x}{\sqrt{2x^2-6x+4}} = \int \frac{\left(u+\frac{5}{2}\right)\mathrm{d}u}{\sqrt{2\left(u^2-a^2\right)}}$$

[3] Note that, in place of inspection, we could use the general form factorization derived above.

Split it into two separate integrals and factor out the constants.

$$= \frac{1}{\sqrt{2}} \int \frac{u\,\mathrm{d}u}{\sqrt{u^2 - a^2}} + \frac{5}{2\sqrt{2}} \int \frac{\mathrm{d}u}{\sqrt{u^2 - a^2}}$$

The right integral is a straight-up inverse trig integral. The left one, however, needs something special. It could be dealt with as previously discussed by substituting $u = a\tan\theta$ for all instances of u and evaluating it is a more complex trig integral in θ. However, for the sake of showing a different technique, we will choose $z = u^2 - a^2$ and $\frac{1}{2}\mathrm{d}z = u\,\mathrm{d}u$ and treat it as a power function in z.

$$\begin{aligned}
&= \frac{1}{2\sqrt{2}} \int \frac{\mathrm{d}z}{\sqrt{z}} + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2 \\
&= \frac{1}{2\sqrt{2}} \int z^{-\frac{1}{2}}\,\mathrm{d}z + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2 \\
&= \frac{1}{\sqrt{2}} z^{\frac{1}{2}} + C_1 + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2
\end{aligned}$$

Return all of the substitutions and combine the constants of integration.

$$\begin{aligned}
&= \sqrt{\frac{u^2 - a^2}{2}} + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C \\
&= \sqrt{\frac{x^2 - 3x + 2}{2}} + \frac{5}{2\sqrt{2}} \ln\left|x - \frac{3}{2} + \sqrt{x^2 - 3x + 2}\right| + C
\end{aligned}$$

9.6 Integration by the Method of Partial Fractions

7/1:

- **Method of Partial Fractions**: The process of "split[ting] a fraction into a sum of fractions having simpler denominators" (Thomas, 1972, p. 294).
- If we wish to split a rational fraction $\frac{f(x)}{g(x)}$ into a sum of simpler fractions, then...
  - "The degree of $f(x)$ should be less than the degree of $g(x)$. If this is not the case, we first perform a long division, then work with the remainder term. This remainder can always be put into the required form" (Thomas, 1972, p. 294).
  - "The factors of $g(x)$ should be known. Theoretically, any polynomial $g(x)$ with real coefficient can be expressed as a product of real linear and quadratic factors. In practice, it may be difficult to perform the factorization" (Thomas, 1972, p. 294).
- If $x - r$ is a linear factor of $g(x)$ and $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$, then, to this factor, assign the sum of m partial fractions

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}$$

  - If $x^2 + px + q$ is a quadratic factor[4] of $g(x)$ and $\left(x^2 + px + q\right)^n$ is the highest power of $x^2 + px + q$ that divides $g(x)$, then, to this factor, assign the sum of n partial fractions

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{\left(x^2 + px + q\right)^2} + \cdots + \frac{B_nx + C_n}{\left(x^2 + px + q\right)^n}$$

  - Continue, as necessary, to higher degree factors of $g(x)$ (although this caveat is not addressed by Thomas, 1972).

[4] A binomial factor, the factoring of which into linear factors would introduce complex numbers.

  - Notice how the degree of the polynomial in the numerator of the partial fractions will be at most one less than the degree of the denominator.

- Any rational function integrated via the method of partial fractions can be reduced to the problem of evaluating the following two types of integrals.

$$\int \frac{\mathrm{d}x}{(x-r)^h} \qquad\qquad \int \frac{(ax+b)\,\mathrm{d}x}{(x^2+px+q)^k}$$

  - The left integral, with the substitution $z = x - r$ and $\mathrm{d}z = \mathrm{d}x$, becomes a power integral.
  - The right integral, after completing the square in the denominator, substituting, and splitting into two fractions by the numerator, becomes a pair of inverse trig substitution integrals.

- With the method of partial fractions, there is a new way to integrate $\sec\theta$.

$$\begin{aligned}
\int \sec\theta\,\mathrm{d}\theta &= \int \frac{\mathrm{d}\theta}{\cos\theta} \\
&= \int \frac{\cos\theta\,\mathrm{d}\theta}{\cos^2\theta} \\
&= \int \frac{\mathrm{d}x}{1-x^2} \\
&= \int \frac{0.5}{1+x}\,\mathrm{d}x + \int \frac{0.5}{1-x}\,\mathrm{d}x \\
&= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + C \\
&= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C \\
&= \ln\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + C
\end{aligned}$$

  - Note that $\ln\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + C$ is equivalent to the previously derived form:

$$\begin{aligned}
\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} &= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} \\
&= \left|\frac{1+\sin\theta}{\cos\theta}\right| \\
&= |\sec\theta + \tan\theta|
\end{aligned}$$

9.7 Integration by Parts

- This is the second general method of integration (the first being substitution).
- It relies on the following formulas for indefinite and definite integrals, respectively.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u + C \qquad\qquad \int_{(1)}^{(2)} u\,\mathrm{d}v = uv\Big]_{(1)}^{(2)} - \int_{(1)}^{(2)} v\,\mathrm{d}u$$

  - The indefinite integral formula can be derived from the differential of a product rule as follows.

$$\begin{aligned}
\mathrm{d}(uv) &= u\,\mathrm{d}v + v\,\mathrm{d}u \\
u\,\mathrm{d}v &= \mathrm{d}(uv) - v\,\mathrm{d}u \\
\int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u + C
\end{aligned}$$

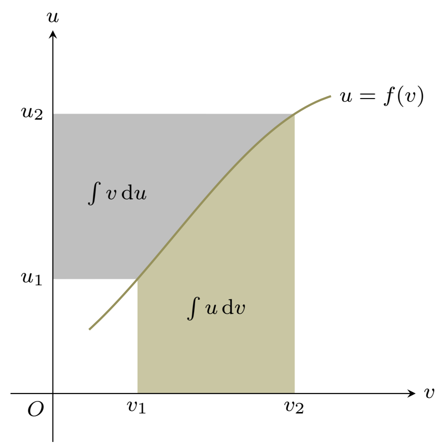


Figure 9.3: Geometric rationale for definite integration by parts.

- The definite integral formula can be thought of as an adjustment of the above, or it can be conceived geometrically: In Figure 9.3, $\int_{(1)}^{(2)} u\,\mathrm{d}v$ is the yellow area, which is clearly equivalent to the total area[5] $uv\big]_{(1)}^{(2)}$ minus the grey area $\int_{(1)}^{(2)} v\,\mathrm{d}u$.

- Since $\int \mathrm{d}v = v + C_1$, $\int u\,\mathrm{d}v$ actually equals $u(v+C_1) - \int (v+C_1)\,\mathrm{d}u$. However, since

$$\begin{aligned} u(v+C_1) - \int (v+C_1)\,\mathrm{d}u &= uv + C_1 u - \int v\,\mathrm{d}u - \int C_1\,\mathrm{d}u \\ &= uv - \int v\,\mathrm{d}u \end{aligned}$$

  it is customary to drop the first constant of integration.

  - That being said, it is sometimes useful — when evaluating $\int \ln(x+1)\,\mathrm{d}x = \ln(x+1)(x+C_1) - \int \frac{x+C_1}{x+1}\,\mathrm{d}x$, being able to choose $C_1 = 1$ greatly simplifies the second integral.

- When integrating an inverse trig function, consider using integration by parts.

  - For example, when confronted with $\int \tan^{-1} x\,\mathrm{d}x$, integration by parts turns it into an inverse trig derivative problem.

$$\int \tan^{-1} x\,\mathrm{d}x = x\tan^{-1} x - \int \frac{x\,\mathrm{d}x}{1+x^2}$$

- When attempting integration by parts, don't be afraid to use it multiple times.

  - For example, when confronted with $\int x^2\mathrm{e}^x\,\mathrm{d}x$, use integration by parts twice.

$$\begin{aligned} \int x^2\mathrm{e}^x\,\mathrm{d}x &= x^2\mathrm{e}^x - \int 2x\mathrm{e}^x\,\mathrm{d}x \\ &= x^2\mathrm{e}^x - \left(2x\mathrm{e}^x - \int 2\mathrm{e}^x\,\mathrm{d}x\right) \end{aligned}$$

- When attempting integration by parts, look for the original integral showing up again — if it does, combine like terms.

- When integrating powers of $\cos x$, consider using a reduction formula.

[5]Note that $uv\big]_{(1)}^{(2)} = u_2v_2 - u_1v_1$, the latter of which, as the difference of two rectangles, clearly represents the total shaded area.

– Begin by deriving a reduction formula (this will involve splitting the exponent!).

$$\begin{aligned}
\int \cos^n x \,\mathrm{d}x &= \int \cos^{n-1} x \cos x \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \,\mathrm{d}x - (n-1) \int \cos^n x \,\mathrm{d}x \\
(1 + (n-1)) \int \cos^n x \,\mathrm{d}x &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \,\mathrm{d}x \\
\int \cos^n x \,\mathrm{d}x &= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \,\mathrm{d}x
\end{aligned}$$

Since the reduction formula decreases the exponent by 2, work out two base cases.

$$\int \cos^0 x \,\mathrm{d}x = x + C$$

$$\int \cos^1 x \,\mathrm{d}x = \sin x + C$$

- When integrating powers of $\sin x$, consider using a reduction formula.

$$\int \sin^n x \,\mathrm{d}x = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \,\mathrm{d}x$$

9.8 Integration of Rational Functions of $\sin x$ and $\cos x$, and Other Trigonometric Integrals

- "It has been discovered that the substitution $z = \tan \frac{x}{2}$ enables one to reduce the problem of integrating any rational function of $\sin x$ and $\cos x$ to a problem involving a rational function of z. This in turn can be integrated by the method of partial fractions" (Thomas, 1972, p. 300).
  – However, this substitution should be used only as a last resort — the associated algebra is often quite cumbersome.
- To increase the ease of use for this substitution, it will help to derive three results.

$$\cos x = \frac{1 - z^2}{1 + z^2} \qquad \sin x = \frac{2z}{1 + z^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}z}{1 + z^2}$$

- "The following types of integrals…arise in connection with alternating-current theory, heat transfer problems, bending of beams, cable stress analysis in suspension bridges, and many other places where trigonometric series (or Fourier series) are applied to problems in mathematics, science, and engineering" (Thomas, 1972, p. 301).

$$\int \sin mx \sin nx \,\mathrm{d}x \qquad \int \sin mx \cos nx \,\mathrm{d}x \qquad \int \cos mx \cos nx \,\mathrm{d}x$$

- When confronted with one of these integrals, integration by parts may be used. However, using one of the following three identities will be more simple.

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}(\cos(m-n)x - \cos(m+n)x) \\
\sin mx \cos nx &= \frac{1}{2}(\sin(m-n)x + \sin(m+n)x) \\
\cos mx \cos nx &= \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)
\end{aligned}$$

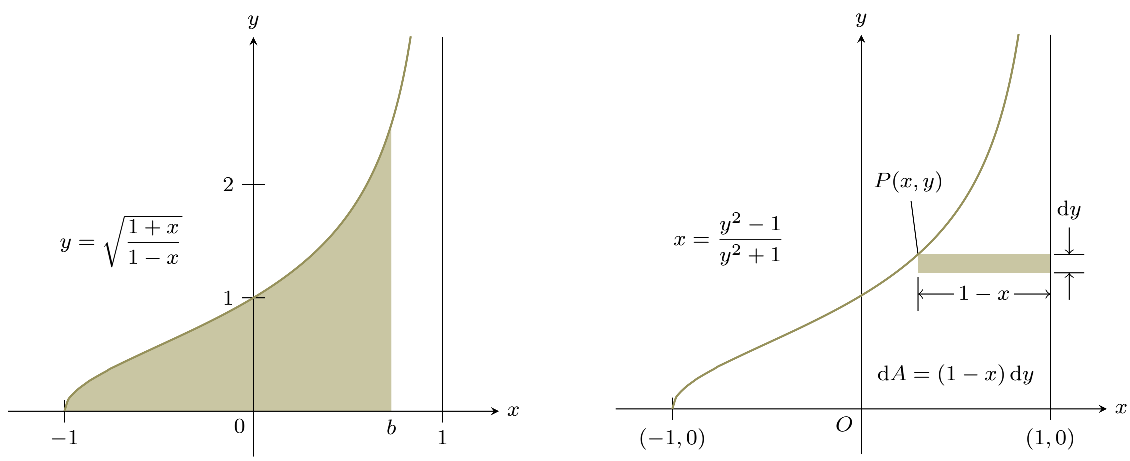
- Note that "these identities follow at once from $\cos(A+B) = \cos A \cos B - \sin A \sin B$, $\cos(A-B) = \cos A \cos B + \sin A \sin B$, and $\sin(A+B) = \sin A \cos B + \cos A \sin B$, $\sin(A-B) = \sin A \cos B - \cos A \sin B$" (Thomas, 1972, p. 301).

9.9 Further Substitutions

- "Some integrals involving fractional powers of the variable x may be simplified by substituting $x = z^n$, where n is the least common multiple of the denominators of the exponents" (Thomas, 1972, p. 302).
  - For example, $\int \frac{\sqrt{x}\,\mathrm{d}x}{1+\sqrt[4]{x}}$ can be simplified by taking $x = z^4$.
- "Even when it is not clear at the start that a substitution will work, it is advisable to try one that seems reasonable and pursue it until it either gives results or appears to make matters worse. In the latter case, try something else! Sometimes a chain of substitutions $u = f(x)$, $v = g(u)$, $z = h(v)$, and so on, will produce results when it is by no means obvious that this will work" (Thomas, 1972, p. 302).
- "The criterion of success is whether the new integrals so obtained appear to be simpler than the original integral. Here it is handy to remember that any rational function of x can be integrated by the method of partial fractions and that any rational function of $\sin x$ and $\cos x$ can be integrated by using the substitution $z = \tan \frac{x}{2}$. If we can reduce a given integral to one of these types, we then know how to finish the job" (Thomas, 1972, p. 302).
- To evaluate a definite integral after a (series of) substitution(s), either return the substitution(s) and keep the bounds or keep the substitution(s) and determine new bounds based on the new variable of integration.
  - For example, if $z^2 = \frac{1+x}{1-x}$ and $x \in [-1, 1]$, then $z \in [0, \infty)$. And if $z = \tan\theta$, then $\theta \in \left[0, \frac{\pi}{2}\right]$.

9.10 Improper Integrals

- **Improper integral**: An integral of the form $\int_a^b f(x)\,\mathrm{d}x$ where some $x \in [a,b]$ is infinite, and/or one or both of a, b are infinite in magnitude.



(a) With respect to x.   (b) With respect to y.

Figure 9.4: Defining improper integrals.

- Say we wish to evaluate $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$, knowing that the integrand approaches ∞ as $x \to 1$ (see Figure 9.4a). Well, if the upper bound b is some value slightly *less than* 1, we *can* evaluate the integral. Thus evaluating the original integral becomes a problem of evaluating

$$\lim_{b\to 1^-} \int_{-1}^{b} \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \lim_{b\to 1^-} \left(\sin^{-1} x - \sqrt{1-x^2} \Big]_{-1}^{b} \right) = \lim_{b\to 1^-} \left(\sin^{-1} b - \sqrt{1-b^2} + \frac{\pi}{2} \right)$$

  - Sometimes such a limit will converge. Sometimes it will not (it will diverge). Either way, it answers the question of the nature of the area under the curve (by yielding some finite value, or the infinite one).
  - Note that the integral works out just the same if we sum vertical elements instead (see Figure 9.4b), evaluating the following.

$$\lim_{c\to\infty} \int_0^c (1-x)\,\mathrm{d}y = \lim_{c\to\infty} \int_0^c \frac{2\,\mathrm{d}y}{y^2+1}$$

- When integrating a function $f(x)$ on $[a,b]$ where $f(x) \to \infty$ at some x-value $c \in (a,b)$, split the integral.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to c^-} \int_a^c f(x)\,\mathrm{d}x + \lim_{c\to c^+} \int_c^b f(x)\,\mathrm{d}x$$

- On determining whether or not an improper integral with a nonintegrable integrand exists, we can sometimes compare it with an integral that we know.
  - For example,

$$\int_1^b \mathrm{e}^{-x^2}\,\mathrm{d}x \leq \int_1^b \mathrm{e}^{-x}$$

    for all $b \geq 1$ since $0 < \mathrm{e}^{-x^2} \leq \mathrm{e}^{-x}$ for all $x \geq 1$. Thus, $\int_1^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x$ evaluates to some finite value.

- Note that some improper integrals diverge by oscillation.
  - For example, $\int_0^\infty \cos x\,\mathrm{d}x$ diverges in this manner.

9.11 Numerical Methods for Approximating Definite Integrals

- One could use the **trapezoidal rule**.
- A better choice, though, is **Simpson's rule**.

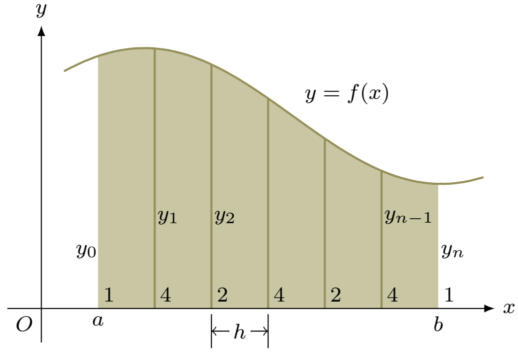


Figure 9.5: Simpson's rule.

- Simpson's rule approximates the curve via parabolas, which can be uniquely defined by three points and have a nice formula for the area underneath them.
- We now derive Simpson's rule.
  - Suppose we wish to approximate the area under the part of the curve from x_i to x_{i+2} in Figure 9.5. We know that there exists some parabola intersecting (x_i, y_0), (x_{i+1}, y_1), and (x_{i+2}, y_2). However, for the sake of simplifying the algebra, we choose to consider the parabola intersecting $(-h, y_0)$, $(0, y_1)$, and (h, y_2) (the area under both parabolas will be equivalent since $x_{i+1} - x_i = h$). Let this translated parabola be called $Ax^2 + Bx + C$ for some $A, B, C \in \mathbb{R}$. Then the area underneath this parabola A_p can be described by the following.

$$\begin{aligned} A_p &= \int_{-h}^{h} \left(Ax^2 + Bx + C\right) \mathrm{d}x \\ &= \frac{2Ah^3}{3} + 2Ch \end{aligned}$$

  - We know that the area underneath this parabola is dependent only on h, y_0, y_1, and y_2. Thus, we look to express A, C from the integral in terms of h, y_0, y_1, y_2. To accomplish this, we will use the facts that

$$\begin{aligned} y_0 &= Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C \end{aligned}$$

    We can now see that $C = y_1$, so all that's left is to solve for A. This can be done by adding the first and third equations, substituting, and solving as follows.

$$\begin{aligned} y_0 + y_2 &= 2Ah^2 + 2C \\ 2Ah^2 &= y_0 + y_2 - 2y_1 \\ A &= \frac{y_0 - 2y_1 + y_2}{2h^2} \end{aligned}$$

  - Thus, we can reformulate the area under the parabola as follows.

$$\begin{aligned} A_p &= \frac{2h^3}{3} \cdot \frac{y_0 - 2y_1 + y_2}{2h^2} + 2y_1 h \\ &= \frac{h}{3}(y_0 + 4y_1 + y_2) \end{aligned}$$

  - "Simpson's rule follows from applying this result to successive pieces of the curve $y = f(x)$ between $x = a$ and $x = b$. Each separate piece of the curve, covering an x-subinterval of width $2h$, is approximated by an arc of a parabola through its ends and its mid-point. The area under each parabolic arc is then given by an expression like [the above] and the results are added to give [the following]" (Thomas, 1972, p. 309).

$$\begin{aligned} A_S &= \frac{h}{3}((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$

- Note that the number of subdivisions must be an even integer.

• To find the error in a Simpson's rule approximation, use the fact[6] that "if f is continuous on $[a, b]$ and four times differentiable on (a, b), then there is a number c between a and b such that [the following holds]" (Thomas, 1972, p. 310).

$$\int_a^b f(x)\,\mathrm{d}x = A_S - \frac{b-a}{180} f^{(4)}(c) \cdot h^4$$

[6] A proof of this fact is a topic best left until Analysis. The framework for such a proof may be found on Olmsted, 1956, p. 146.

References

Olmsted, J. M. H. (1956). *Intermediate analysis: An introduction to the theory of functions of one real variable*. New York: Appleton-Century-Crofts.

Thomas, G. B., Jr. (1972). *Calculus and analytic geometry* (fourth). Addison-Wesley.