Chapter 8

Hyperbolic Functions

8.1 Introduction

6/24: • **Hyperbolic functions**: Certain combinations of e^x and e^{-x} that are used to solve certain engineering problems (the hanging cable) and are useful in connection with differential equations.

8.2 Definitions and Identities

- Let

$$\cosh u = \frac{1}{2}(e^u + e^{-u}) \qquad \sinh u = \frac{1}{2}(e^u - e^{-u})$$

  - These combinations of exponentials occur sufficiently frequently that we give a special name to them.
  - Although the names may seem random, $\sinh u$ and $\cosh u$ do share many analogous properties with $\sin u$ and $\cos u$.
  - Pronounced to rhyme with "gosh you" and as "cinch you," respectively.

- Like $x = \cos u$ and $y = \sin u$ are associated with the point (x, y) on the unit circle $x^2 + y^2 = 1$, $x = \cosh u$ and $y = \sinh u$ are associated with the point (x, y) on the unit hyperbola $x^2 - y^2 = 1$.
  - Note that $x = \cosh u$ and $y = \sinh u$ are associated with the *right-hand* branch of the unit hyperbola.
  - Also note that sine and cosine are sometimes referred to as the **circular functions**.

- Analogous to sine and cosine, we have the identity

$$\cosh^2 u - \sinh^2 u = 1$$

- We define the remaining hyperbolic trig functions as would be expected.

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} \qquad \operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}}$$

$$\coth u = \frac{\cosh u}{\sinh u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} \qquad \operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}}$$

- Since $\cosh u + \sinh u = e^u$, we can replace any combination of exponentials with hyperbolic sines and cosines and vice versa.

- Note that the hyperbolic functions are *not* periodic.
  - This does mean, though, that they have more easily defined properties at infinity.

- "Practically all the circular trigonometric identities have hyperbolic analogies" (Thomas, 1972, p. 267).

8.3 Derivatives and Integrals

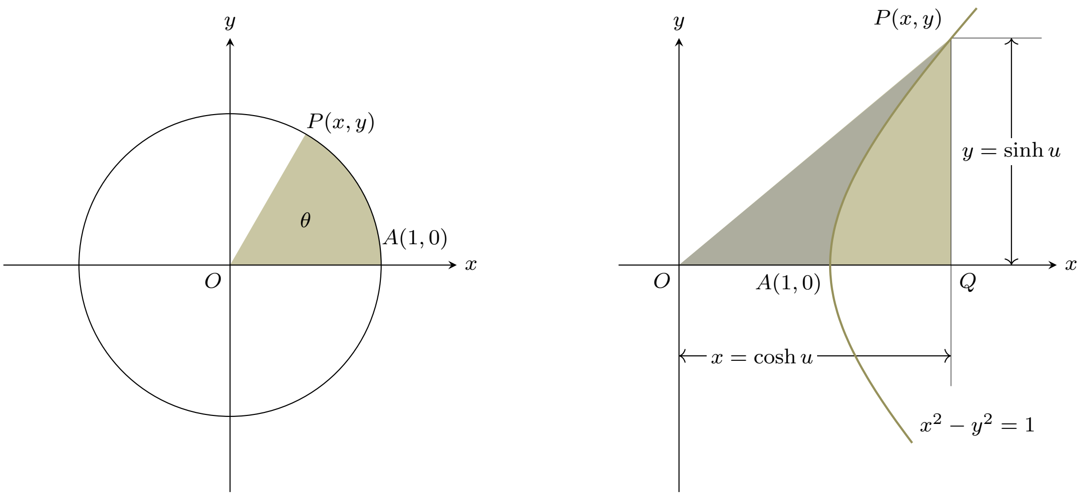
6/25:
- Derivatives of the hyperbolic functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = \cosh u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh u) = \sinh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh u) = \operatorname{sech}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth u) = -\operatorname{csch}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{\mathrm{d}u}{\mathrm{d}x}$$

  - Note that these are almost exact analogs of the formulas for the corresponding circular functions, the exception being that the negative signs are not associated with the cofunctions but with the latter three.

- Lots of intro to the hanging cable problem.

8.4 Geometric Meaning of the Hyperbolic Radian



(a) Circular radians. (b) Hyperbolic radians.

Figure 8.1: Geometric meaning of radians.

- For circular sine and cosine, the "meaning of the variable θ in the equations $x = \cos\theta$, $y = \sin\theta$ as they relatee to the point $P(x, y)$ on the unit circle $x^2 + y^2 = 1$ [is] the radian measure of the angle AOP in [Figure 8.1a], that is $\theta = \frac{\text{arc } AP}{\text{radius } OA}$" (Thomas, 1972, p. 271).
  - However, since $A = \frac{1}{2}r^2\theta = \frac{\theta}{2}$ for $r = 1$, θ also equals twice the area of the sector AOP.
- To understand the meaning of the variable u, calculate the area of the sector AOP in Figure 8.1b as an analog to circular area.

$$\begin{aligned} A_{AOP} &= A_{OQP} - A_{AQP} \\ &= \frac{1}{2}bh - \int_A^P y \, \mathrm{d}x \end{aligned}$$

$$y = \sinh u, \quad x = \cosh u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \sinh u \Rightarrow \mathrm{d}x = \sinh u \, \mathrm{d}u$$

$$= \frac{1}{2}xy - \int_A^P \sinh^2 u \, \mathrm{d}u$$

$$
\begin{aligned}
&= \frac{1}{2}\cosh u \sinh u - \frac{1}{2}\int_A^P (\cosh 2u - 1)\,\mathrm{d}u \\
&= \frac{1}{2}\sinh u \cosh u - \frac{1}{2}\left[\frac{1}{2}\sinh 2u - u\right]_{A(u=0)}^{P(u=u)} \\
&= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{4}\sinh 2u - \frac{1}{2}u\right) \\
&= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{2}\sinh u \cosh u - \frac{1}{2}u\right) \\
&= \frac{1}{2}u
\end{aligned}
$$

- This implies that u also equals twice the area of the sector AOP (the hyperbolic sector, that is).
- This means, for example, that "$\cosh 2$ and $\sinh 2$ may be interpreted as the coordinates of P when the area of the sector AOP is just equal to the area of a square having OA as side" (Thomas, 1972, p. 272).