Chapter 19

Complex Numbers and Functions

19.1 Inverted Number Systems

9/5:
- Reviews the construction of the real numbers (from sequences of rational numbers).
- **Rational operations**: The operations of addition, subtraction, multiplication, and division, as they pertain to the rational numbers.
- The systems of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ form a hierarchy, both in terms of set inclusion and in terms of an increase in the number of operations that can be performed without going outside the system.
  - Mathematically...
    - In $\mathbb{Z}$, we can solve all equations of the form $x + a = 0$, provided $a \in \mathbb{Z}$.
    - In $\mathbb{Q}$, we can solve all equations of the form $ax + b = 0$, provided $a, b \in \mathbb{Q}$ and $a \neq 0$.
    - In $\mathbb{R}$, we can solve all equations of the form $ax^2 + bx + c = 0$, provided $a, b, c \in \mathbb{R}$, $a \neq 0$, and $b^2 - 4ac \geq 0$.
- However, even in $\mathbb{R}$, we cannot solve equations such as $x^2 + 1 = 0$. Thus, we need $\mathbb{C}$.
- **Complex number system**: The set of all ordered pairs (a, b) of real numbers subject to the laws of equality, addition, and multiplication.

  Equality

  Two complex numbers (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$:

  $$a + ib = c + id \text{ iff } a = c \text{ and } b = d$$

  Addition

  The sum of two complex numbers (a, b) and (c, d) is the complex number $(a + c, b + d)$:

  $$(a + ib) + (c + id) = (a + c) + i(b + d)$$

  Multiplication

  The product of two complex numbers (a, b) and (c, d) is the complex number $(ac - bd, ad + bc)$:

  $$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

  The product of a real number c and the complex number (a, b) is the complex number (ad, bc):

  $$c(a + ib) = ac + i(bc)$$

  - Note that the set of all complex numbers $(a, 0)$ has all the properties of the set of ordinary real numbers a.
    - In particular, $(0, 0)$ is the zero element of $\mathbb{C}$ and $(1, 0)$ is the identity element of $\mathbb{C}$.
- Now that we are equipped with the complex numbers, we can see that $(0, 1)^2 + (1, 0) = 0$, i.e., $(0, 1)$ solves $x^2 + 1 = 0$.

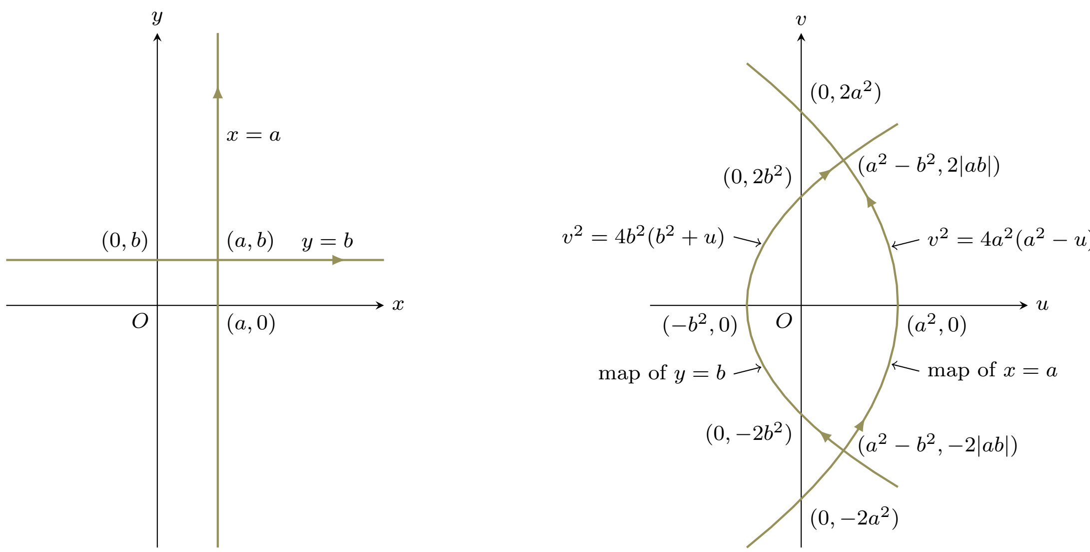
19.2 The Argand Diagram

- Reviews division in $\mathbb{C}$ and complex conjugates.
- **Argand diagram**: A 2D Cartesian coordinate system where the x-axis is the axis of reals and the y-axis is the imaginary axis. *Also known as* $\boldsymbol{z}$**-plane**.
  - In an Argand diagram, we can visualize complex numbers either as points in a 2D plane or as vectors from the origin to the point.
- In polar coordinates, the complex number $z = x + iy$ becomes $r(\cos\theta + i\sin\theta)$.
- **Absolute value** (of $(x+iy) \in \mathbb{C}$): The length r of the vector $\overrightarrow{OP}$ from the origin to $P(x,y)$. *Given by*
$$|x + iy| = \sqrt{x^2 + y^2}$$
- **Argument** (of $z \in \mathbb{C}$): The polar angle θ. *Denoted by* $\mathbf{arg}\,\boldsymbol{z}$.
- **Principal value** (of $\arg z$): The value of $\arg z$ that satisfies $-\pi < \arg z \leq \pi$.
- Note that
$$z \cdot \bar{z} = |z|^2$$
- $\mathbf{cis}\,\boldsymbol{\theta}$: The function given by
$$\operatorname{cis}\theta = \cos\theta + i\sin\theta$$
  - We can prove from basic trigonometric identities that the following properties hold for the cis function.
$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2) \qquad (\operatorname{cis}\theta)^{-1} = \operatorname{cis}(-\theta) \qquad \frac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2} = \operatorname{cis}(\theta_1 - \theta_2)$$
  - It follows that if $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$, then
$$\begin{array}{ccc} z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) & |z_1 z_2| = |z_1| \cdot |z_2| & \arg(z_1 z_2) = \arg z_1 + \arg z_2 \\ \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) & \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} & \arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \\ z_1^n = r_1^n \operatorname{cis} n\theta_1 & |z_1^n| = |z_1|^n & \arg(z_1^n) = n \arg z_1 \end{array}$$
- **De Moivre's theorem**: The statement that
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$
  - De Moivre's theorem allows us to obtain formulas for $\cos n\theta$ and $\sin n\theta$ by expanding the left-hand side of the above equation via the binomial theorem and matching up real and imaginary components (the real component of the expansion will be equal to $\cos n\theta$, and vice versa for the imaginary part).
- With respect to roots of complex numbers, we have that every complex number has n, n^{th} roots. These are given by
$$\sqrt[n]{z} = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)$$
for $k \in \mathbb{Z}$.
  - For convenience, we need only consider $k = 0, \dots, n-1$.
  - Note that "all the n^{th} roots of $r \operatorname{cis}\theta$ lie on a circle centered at the origin O and having radius equal to the real, positive n^{th} root of r. One of them has argument $\alpha = \theta/n$. The others are uniformly spaced around the circumference of the circle, each being separated from its neighbors by an angel equal to $2\pi/n$" (Thomas, 1972, p. 674).

- We need not invent further number systems for $\sqrt[4]{-1}, \sqrt[6]{-1}, \dots$ since these quantities are expressible in terms of the complex numbers.
- **Fundamental Theorem of Algebra**: Every polynomial equation of the form $a_0z^n + z_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$ in which the coefficients $a_0, \dots, a_n$ are any complex numbers, whose degree n is greater than or equal to one, and whose leading coefficient $a_0 \neq 0$ possesses precisely n roots in the complex number system, provided that each multiple root of multiplicity m is counted as m roots.

19.3 The Complex Variable

- A complex, time-dependent variable $z = x + iy$ approaches the **limit** $\alpha = a + ib$ if the **distance** between z and α approaches zero as t approaches some value.
  - Mathematically, $z \to \alpha$ iff $|z - \alpha| \to 0$.
  - If $z \to \alpha$ as $t \to 1$, for example, we write $\lim_{t\to 1} z = \alpha$.
  - Additionally, since $|z - \alpha| \leq |x - a| + |y - b|$, $|x - a| \leq |z - \alpha|$, and $|y - b| \leq |z - \alpha|$, we have that $z \to \alpha$ iff $x \to a$ and $y \to b$.
- **Complex single-valued function**: A function $w : S \to \mathbb{C}$ where $S \subset \mathbb{C}$.
- One way of representing a complex single-valued function graphically is with **mapping**.
- The following example maps the function $w = z^2$.



(a) Lines to be mapped. (b) Mapping.

Figure 19.1: Mapping a complex single-valued function.

  - Since $w = z^2$, we know that $u + iv = (x^2 - y^2) + i(2xy)$.
  - Consider the line $x = a$ in the Argand plane (see Figure 19.1a). As y varies from $-\infty$ to ∞ along this line, it follows from the above equation that u varies as a function of y:

$$u(y) = a^2 - y^2$$

Similarly, v varies as a function of y:

$$v(y) = 2ay$$

  - These functions $u(y)$ and $v(y)$ are a parameterization of a curve in the w-plane in terms of the parameter y. Eliminating y yields the parabola

$$v^2 = 4a^2(a^2 - u)$$

    which we can see graphed in Figure 19.1b.
  - We may do the same with the line $y = b$, yielding the parameterization

$$\left.\begin{aligned} u &= x^2 - b^2 \\ v &= 2bx \end{aligned}\right\} -\infty < x < +\infty$$

    and the parabola

$$v^2 = 4b^2(b^2 + u)$$

  - Note that it is easily seen that the line $x = -a$ maps onto the same parabola as $x = a$, and similarly for $y = -b$. These phenomena are to be expected since $w(-z) = (-z)^2 = z^2 = w(z)$.
- **Continuous** (complex single-valued function at $z = \alpha$): A function $w = f(z)$ defined throughout some neighborhood of $z = \alpha$ such that $|f(z) - f(\alpha)| \to 0$ as $|z - \alpha| \to 0$.

19.4 Derivatives

- **Derivative** (at $z = \alpha$): The number

$$f'(\alpha) = \lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha}$$

  provided that the limit exists.
  - "Since z may approach α from any direction... the existence of such a limit imposes a rather strong restriction on the function $w = f(z)$" (Thomas, 1972, p. 678).
- For example, the function $f(z) = \bar{z}$ has no derivative at any point, as can be shown by considering z approaching an arbitrary α along the lines $x = a$ and $y = b$:

$$\begin{aligned}
\lim_{\substack{x \to a \\ y = b}} \frac{\bar{z} - \bar{\alpha}}{z - \alpha} &= \lim_{\substack{x \to a \\ y = b}} \frac{(x - a) - i(y - b)}{(x - a) + i(y - b)} \\
&= \lim_{x \to a} \frac{(x - a) - i(b - b)}{(x - a) + i(b - b)} \\
&= \lim_{x \to a} 1 \\
&= 1 \\
&\neq -1 \\
&= \lim_{y \to b} -1 \\
&= \lim_{y \to b} \frac{(a - a) - i(y - b)}{(a - a) + i(y - b)} \\
&= \lim_{\substack{x = a \\ y \to b}} \frac{(x - a) - i(y - b)}{(x - a) + i(y - b)} \\
&= \lim_{\substack{x = a \\ y \to b}} \frac{\bar{z} - \bar{\alpha}}{z - \alpha}
\end{aligned}$$

- The constant, sum, product, quotient, chain, and power rules for complex functions are exactly analogous to those for real functions.

- For example, $\mathrm{d}z^3 / \mathrm{d}z = 3z^2$, as we can prove via

$$\begin{aligned} w + \Delta w &= z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 \\ \frac{\Delta w}{\Delta z} &= 3z^2 + 3z(\Delta z) + (\Delta z)^2 \end{aligned}$$

  and

$$\left| \frac{\Delta w}{\Delta z} - 3z^2 \right| = |3z + \Delta z| \cdot |\Delta z| \to 0$$

  as $\Delta z \to 0$.

  - Notice that in this case, it does not matter how $\Delta z \to 0$, only that it does.

19.5 Cauchy-Riemann Differential Equations

- Let $w = u + iv$ be differentiable at $\alpha = a + ib$. Then by making $z \to \alpha$ along the line $x = a$ and along the line $y = b$, we obtain

$$\begin{aligned} f'(\alpha) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \\ &= \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \right) \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right)_{z=\alpha} \end{aligned} \qquad \begin{aligned} f'(\alpha) &= \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \\ &= \lim_{\Delta y \to 0} \left(\frac{1}{i}\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta y} \right) \\ &= \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right)_{z=\alpha} \end{aligned}$$

  - By equating the real and imaginary parts of the above two equations, we obtain the **Cauchy-Riemann differential equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

- "If we take functions which do satisfy the Cauchy-Riemann equations and which, in addition, have continuous partial derivatives u_x, u_y, v_x, v_y, then it is true... that the resulting function $w = u + iv$ is differentiable with respect to z" (Thomas, 1972, p. 680).
- **Analytic** (function in $G \subset \mathbb{C}$): A function $w = f(z)$ that has a derivative at every point of some region G in the z-plane.
- **Analytic** (function in $G \subset \mathbb{C}$) **except** (at α): A function $w = f(z)$ that has a derivative at every point of some region G in the z-plane save $\alpha \in G$.
- **Singular point** (of an analytic function in G except at α): The point α.