Chapter 1

The Rate of Change of a Function

1.1 Introduction

7/3:

- Discusses the importance of calculus, when it should be used, and why one should study it.
- **Analytic geometry**: "Uses algebraic methods and equations to study geometric problems. Conversely, it permits us to visualize algebraic equations in terms of geometric curves" (Thomas, 1972, p. 2).

1.2 Coordinates

- "The basic idea in analytic geometry is the establishment of a one-to-one correspondence between the points of a plane on the one hand and pairs of numbers (x, y) on the other hand" (Thomas, 1972, p. 2).
- Such a correspondence is most commonly established as follows.

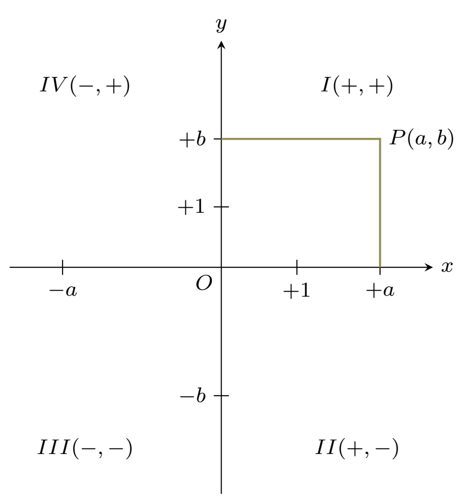


Figure 1.1: Cartesian coordinates.

  - "A horizontal line in the plane, extending indefinitely to the left and to the right, is chosen as the x-axis or axis of **abscissas**. A reference point O on this line and a unit of length are then chosen. The axis is scaled off in terms of this unit of length in such a way that the number zero is attached to O, the number $+a$ is attached to the point which is a units to the right of O, and $-a$ is attached to the symmetrically located point to the left of O. In this way, a one-to-one correspondence is established between points of the x-axis and the set of all **real numbers**" (Thomas, 1972, pp. 2–3).

– "Now through O take a second, vertical line in the plane, extending indefinitely up and down. This line becomes the y-axis, or axis of **ordinates**. The unit of length used to represent $+1$ on the y-axis need not be the same as the unit of length used to represent $+1$ on the x-axis. The y-axis is scaled off in terms of the unit of length adopted for it, with the positive number $+b$ attached to the point b units above O and negative number $-b$ attached to the symmetrically located point b units below O" (Thomas, 1972, p. 3).

– "If a line parallel to the y-axis is drawn through the point marked a on the x-axis, and a line parallel to the x-axis is drawn through the point marked b on the y-axis, their point of intersection P is to be labeled $P(a, b)$. Thus, given the pair of real numbers a and b, we find one and only one point with abscissa a and ordinate b, and this point we denote by $P(a, b)$" (Thomas, 1972, p. 3).

– "Conversely, if we start with any point P in the plane, we may draw lines through it parallel to the coordinate axes. If these lines intersect the x-axis at a and the y-axis at b, we then regard the pair of numbers (a, b) as corresponding to the point P. We say that the coordinates of P are (a, b)" (Thomas, 1972, p. 3).

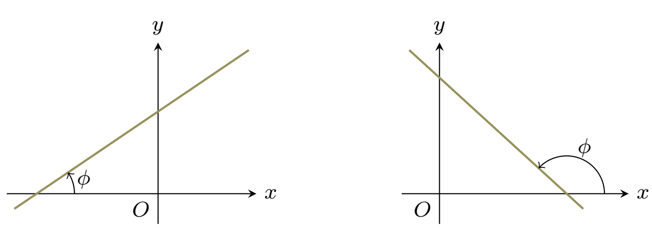
– "The two axes divide the plane into four quadrants, called the first quadrant, second quadrant, and so on, and labeled I, II, III, IV in [Figure 1.1]. Points in the first quadrant have both coordinates positive, and in the second quadrant the x-coordinate (abscissa) is negative and the y-coordinate (ordinate) is positive. The notations $(-, -)$ and $(+, -)$ in quadrants III and IV of [Figure 1.1] represent the signs of the coordinates of points in these quadrants" (Thomas, 1972, p. 3).

1.3 Increments

- **Increments**: The values $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ concerning a particle, the initial position of which is $P_1(x_1, y_1)$ and the terminal position of which is $P_2(x_2, y_2)$.
- If the unit of measurement for both axes is the same, then we may express distances in the plane in terms of this unit using the Pythagorean theorem.

1.4 Slope of a Straight Line

- Let L be a straight line not parallel to the y-axis intersecting distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Then L has a **rise**, **run**, and **slope**.
- **Rise**: The increment Δy.
- **Run**: The increment Δx.
- **Slope**: The rate of rise per run $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. *Also known as* **inclination**.



(a) Negative x-intercept. (b) Positive x-intercept.

Figure 1.2: The slope and the angle of inclination.

  - If we chose different distinct points, the slope would be same because the triangles in the Cartesian plane would be similar.
  - Δy is proportional to Δx with m as the proportionality factor.
  - On interpolation: If we're given the values of a function at (x_1, y_1) and (x_2, y_2), then we may approximate the function by a straight line L passing through those two points and approximate the value $f(x)$ for any $x_1 \leq x \leq x_2$.
  - If the scales on both axes are equal, then the slope of L is equal to the tangent of the **angle of inclination** that L makes with the positive x-axis. That is, $m = \tan\phi$ (see Figure 1.2).
- **Parallel** (lines): Two lines with equal inclinations ($m_1 = m_2$).
- **Perpendicular** (lines): Two lines with inclinations that differ by 90° ($m_1 = -\frac{1}{m_2}$).
  - Note that we can prove the relation between the slopes using the angles of inclination as follows.

$$\begin{aligned}
m_1 &= \tan\phi_1 \\
&= \tan(\phi_2 + 90^\circ) \\
&= -\cot\phi_2 \\
&= -\frac{1}{\tan\phi_2} \\
&= -\frac{1}{m_2}
\end{aligned}$$

1.5 Equations of a Straight Line

7/5: - How do you know if $P(x, y)$ is a point on the line P_1P_2 through distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$?
  - If $x_1 = x_2$, then P_1P_2 is vertical and P lies on P_1P_2 iff $x = x_1$.
  - If $x_1 \neq x_2$, then the slope of P_1P_2 $m_{P_1P_2} = \frac{y_2 - y_1}{x_2 - x_1}$. Thus, P lies on P_1P_2 iff $P = P_1$ or, for the line PP_1 through P and P_1, $m_{P_1P_2} = m_{PP_1} = \frac{y - y_1}{x - x_1}$. In other words, the coordinates x, y of P must satisfy $y - y_1 = m_{P_1P_2}(x - x_1)$.
- Thomas, 1972 calls the above equation the **point-slope form**.
- **Variable**: "A symbol, such as x, which may take on any value in some set of numbers" (Thomas, 1972, p. 10).
- **Slope-intercept form**: $y = mx + b$.
- **General form**: $Ax + By + C = 0$.
  - Such an equation (one that contains only first powers of x and y and constants) is said to be **linear in *x* and *y***.
  - "Every straight line in the plane is represented by a linear equation and, conversely, every linear equation represents a straight line" (Thomas, 1972, p. 10).
- ***y*-intercept**: The constant b in the above equation.
- Let L be a line with the equation $Ax + By + C = 0$. The shortest distance d from a point $P_1(x_1, y_1)$ not on L to L is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

  - Derive by finding a line perpendicular to L through P_1.

1.6 Functions and Graphs

- **Domain** (of a variable x): "The set of numbers over which x may vary" (Thomas, 1972, p. 12).
- Defines **open intervals**, **half-open intervals**, and **closed intervals**.

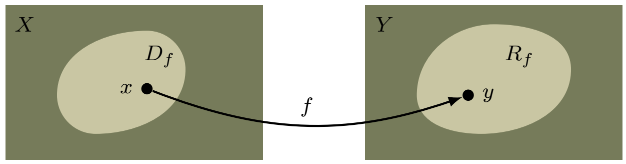


Figure 1.3: A function f maps the domain D_f onto the range R_f. The image of x is $y = f(x)$.

- **Function**: For two nonempty sets X, Y, the collection f of ordered pairs (x, y) with $x \in X$ and $y \in Y$ that assigns to every $x \in X$ a unique $y \in Y$. *Also known as* **mapping** (from X to Y), $\boldsymbol{y = f(x)}$, $\boldsymbol{f : x \to y}$[1].
  - When using the latter notation, it is understood that the domain is $\mathbb{R}$ unless this is impossible (e.g., $f : x \to \frac{1}{x}$ must exclude 0 from the domain).
- **Domain** (of a function f): "The collection of all first elements x of the pairs (x, y) in f" (Thomas, 1972, p. 13). *Also known as* $\boldsymbol{D_f}$.
- **Range** (of a function f): "The set of all second elements y of the pairs (x, y) in f" (Thomas, 1972, p. 13). *Also known as* $\boldsymbol{R_f}$.
- **Image** (of x): The value y to which a function maps x.
- Thomas, 1972 considers functions from the reals to the reals, but also more abstract functions.
  - For example, it considers the function from all triangles (a set of decidedly nonnumerical objects) to their enclosed areas (the set of positive real numbers).

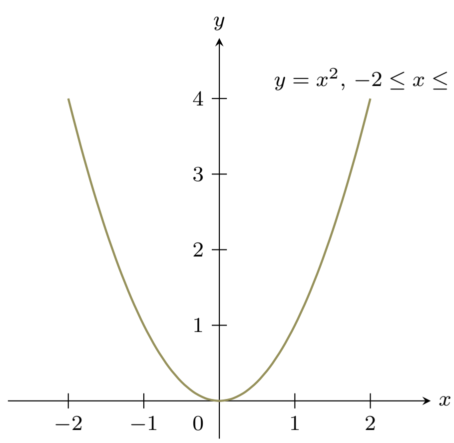


Figure 1.4: Graph of a function.

- **Graph** (of a function): "The set of points which correspond to members of the function" (Thomas, 1972, p. 14).

[1] "eff sends ex into wy"

  - For example, let X be the closed interval $[-2,2]$. To each $x \in X$, assign the number x^2. This describes the function
$$f = \left\{(x,y) : -2 \leq x \leq 2,\ y = x^2\right\}$$
    The graph of f can be seen in Figure 1.4.
- **Independent variable**: The first variable x in the ordered pair (x,y). *Also known as* **argument**.
- **Dependent variable**: The second variable y in the ordered pair (x,y).
- **Real-valued function of a real variable**: "A function f whose domain and range are sets of real numbers" (Thomas, 1972, p. 14).
  - As a general rule, *function* indicates a real-valued function of a real variable for the first seven chapters of Thomas, 1972.
- f can be represented by...
  - A table of corresponding values (this will be incomplete, though).
  - Corresponding numerical scales, as on a slide rule (this will be incomplete, though).
  - A simple formula, such as $f(x) = x^2$ (this may be less exact than ordered pairs, but it is more easily understood/applicable/complete).
  - A graph (for any value x in the domain, begin x units from the origin along the x-axis, move vertically until intersecting the curve, and then move horizontally until intersecting the image y on the y-axis).
- Some mappings cannot be expressed in terms of algebraic operations on x.
  - For example, the **greatest-integer function** "maps any real number x onto that unique integer which is the largest among all integers that are less than or equal to x" (Thomas, 1972, p. 15).
    - The image of x is represented by $[x]$, and the function by $f : x \to [x]$.
    - An example of a **step function**.
    - It exhibits points of **discontinuity**.
- Note: The fact that a one-to-one mapping exists between the points in the interval $(0,1]$ and $[1,\infty)$ (namely, $f : x \to \frac{1}{x}$) proves that there are equally many points in both intervals.
- The absolute value function can be geometrically interpreted in the context of distance from a point. As such, it is useful in describing **neighborhoods**.

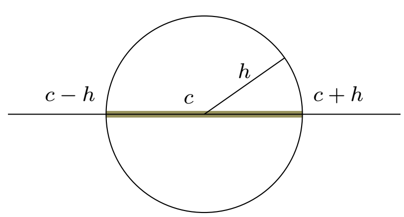


Figure 1.5: The symmetric neighborhood $N_h(c)$, centered at c, with radius h.

- **Symmetric neighborhood** (of a point c): "The open interval $(c-h, c+h)$, where h may be any positive number" (Thomas, 1972, p. 17). *Also known as* $\boldsymbol{N_h(c)}$.
- **Radius** (of a symmetric neighborhood): The value h (see Figure 1.5).
- **Neighborhood** (of a point c): The open interval $(c-h, c+k)$, where h,k may be any positive numbers.
  - Like requiring that $|x-c|$ is small.

- **Deleted neighborhood** (of a point c): "A neighborhood of c from which c itself has been removed" (Thomas, 1972, p. 17).
  - Like requiring that $|x-c|>0$.
- **Intersection** (of the neighborhoods $(c-h_1, c+k_1)$ and $(c-h_2, c+k_2)$): The neighborhood $(c-h, c+k)$, where $h=\min(h_1,h_2)$ and $k=\min(k_1,k_2)$.
  - "The intersection of two neighborhoods of c is a neighborhood of c, and the intersection of two deleted neighborhoods of c is a deleted neighborhood of c" (Thomas, 1972, p. 18).
- Let A be a neighborhood of c. Then denote the deleted neighborhood equivalent to A with c removed by A^-.