Chapter 17

Vector Analysis

17.1 Introduction: Vector Fields

12/29: • In this chapter, we will consider vector functions of several variables, such as the function giving the velocity $\mathbf{v} = \mathbf{F}(x, y, z, t)$ of a particle in a fluid located at position (x, y, z) at time t.

- **Steady-state flow**: A flow for which the velocity function does not depend on the time t.
- **Vector field**: The collection of all vectors $\mathbf{F}(P)$ assigned to each point P in a region G.
- **Gradient field**: The vector field defined for points in the domain G of a scalar function T such that $\mathbf{F}(P) = \nabla T(P)$.

17.2 Surface Integrals

12/30: • Just like we have $\mathrm{d}s = \sqrt{1 + f_x^2}\,\mathrm{d}x$, we have

$$\mathrm{d}\sigma = g(x, y)\,\mathrm{d}A$$

where $\mathrm{d}\sigma$ is "an element of surface area in the tangent plane that approximates the corresponding portion $\Delta\sigma$ of the surface itself" (Thomas, 1972, p. 581) and $g(x, y) = \sqrt{1 + f_x^2 + f_y^2}$.

- Thus, we can think of surface area as either the lefthand or righthand side of the below equation.

$$\iint_\Sigma \mathrm{d}\sigma = \iint_R g(x, y)\,\mathrm{d}A$$

  - The lefthand interpretation sums infinitely many, infinitely small pieces $\mathrm{d}\sigma$ of the surface Σ.
  - The righthand interpretation sums infinitely many, infinitely small pieces $\mathrm{d}A$ of the shadow R of the surface Σ on the xy-plane, adjusted by the factor $g(x, y)$.

- These formulations are important because sometimes we want to conceive and evaluate an integral of the form $\iint_\Sigma h(x, y, z)\,\mathrm{d}\sigma$.
- **Surface integral** (of $h(x, y, z)$ over the surface Σ): The limit as $\Delta\sigma \to 0$ of the sum of every $\Delta\sigma_k$ (composing Σ) times $h(x, y, z)$ for some $(x, y, z) \in \Delta\sigma_k$. Mathematically,

$$\iint_\Sigma h(x, y, z)\,\mathrm{d}\sigma = \lim_{\Delta\sigma \to 0} \sum_{k=1}^{n} h(x_k, y_k, z_k)\,\Delta\sigma_k$$

- Consider a surface Σ consisting of all points $P(x,y,z)$ satisfying $z=f(x,y)$ for $(x,y)\in R$, where R is a closed, bounded region of the xy-plane and f,f_x,f_y are continuous throughout R and its boundary.
- Approximate R by dividing it into n rectangles using lines parallel to the y-axis spaced Δx apart and lines parallel to the x-axis spaced Δy apart.
- Let the part of Σ above each rectangle be denoted by $\Delta\sigma_k$ for some $1\leq k\leq n$.
- Now if $P_k(x_k,y_k,z_k)$ is a point in $\Delta\sigma_k$, we can consider the above sum and take its limit.

12/31:

- To evaluate the surface integral, we substitute $\Delta\sigma_k=g(x_k,y_k)\,\Delta x\,\Delta y$ and $z_k=f(x_k,y_k)$ in the sum, and take iterated integrals over R (the shadow of Σ on the xy-plane) instead of Σ.

$$\iint\limits_{\Sigma} h(x,y,z)\,\mathrm{d}\sigma=\iint\limits_{R} h[x,y,f(x,y)]g(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

- We now explore a useful surface integration technique through a problem.
- Evaluate $\iint(x^2+y^2)\,\mathrm{d}\sigma$ over the hemisphere Σ described by $z=\sqrt{a^2-x^2-y^2}$.
  - Because of a *sphere* 2Σ of radius a's high degree of symmetry,

$$\iint\limits_{2\Sigma} x^2\,\mathrm{d}\sigma=\iint\limits_{2\Sigma} y^2\,\mathrm{d}\sigma=\iint\limits_{2\Sigma} z^2\,\mathrm{d}\sigma=\frac{1}{3}\iint\limits_{2\Sigma}(x^2+y^2+z^2)\,\mathrm{d}\sigma=\frac{1}{3}\iint\limits_{2\Sigma} a^2\,\mathrm{d}\sigma$$

  Thus, for the *hemisphere* Σ,

$$\begin{aligned}
\iint\limits_{\Sigma}(x^2+y^2)\,\mathrm{d}\sigma &= \frac{1}{2}\iint\limits_{2\Sigma}(x^2+y^2)\,\mathrm{d}\sigma\\
&= \frac{1}{2}\left(\iint\limits_{2\Sigma} x^2\,\mathrm{d}\sigma+\iint\limits_{2\Sigma} y^2\,\mathrm{d}\sigma\right)\\
&= \frac{1}{2}\left(\frac{1}{3}\iint\limits_{2\Sigma} a^2\,\mathrm{d}\sigma+\frac{1}{3}\iint\limits_{2\Sigma} a^2\,\mathrm{d}\sigma\right)\\
&= \frac{a^2}{3}\iint\limits_{2\Sigma}\mathrm{d}\sigma\\
&= \frac{a^2}{3}\cdot 4\pi a^2\\
&= \frac{4}{3}\pi a^4
\end{aligned}$$

- Alternate formulations of $\mathrm{d}\sigma$.
  - Let the surface Σ be defined by the equation $F(x,y,z)=0$.
  - For the same reasons discussed in Chapter 17,

$$\mathrm{d}\sigma=\frac{\mathrm{d}A}{\cos\phi}$$

  where ϕ is the angle between $\mathbf{N}=\nabla F$ and the unit vector normal to the plane onto which Σ is projected, which we will take to be the xy-plane at first (this means that this normal vector is $\mathbf{k}$).

- Since

$$\cos\phi = \frac{\mathbf{N}\cdot\mathbf{k}}{|\mathbf{N}|\,|\mathbf{k}|} = \frac{|F_z|}{\sqrt{F_x^2+F_y^2+F_z^2}}$$

we thus have that

$$\mathrm{d}\sigma = \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}\,\mathrm{d}x\,\mathrm{d}y$$

- Note that if we project Σ onto a different plane, an analog to the above can easily be derived.

17.3 Line Integrals

- **Line integral** (of $w(x,y,z)$ along the curve C from A to B): The limit as $\Delta s \to 0$ of the sum of every Δs_k (composing the section of C between points A and B along C) times $w(x,y,z)$ for some $(x,y,z) \in \Delta s_k$. Mathematically,

$$\int_C w\,\mathrm{d}s = \lim_{\Delta s\to 0}\sum_{k=1}^{n} w(x_k,y_k,z_k)\,\Delta s_k$$

  - Suppose that C is a directed curve in three-space from A to B. Let $w(x,y,z)$ be a scalar function of position that is continuous in a region D containing C.
  - Divide C into n segments, and let $P_k(x_k,y_k,z_k)$ be an arbitrary point on the kth subarc.
  - If the above sum has a limit as $n\to\infty$ and the largest $\Delta s_k \to 0$, and if this limit is the same for all ways of subdividing C and all choices of the points P_k, then we call this limit the line integral.
- If C is parameterized by the functions $x=f(t)$, $y=g(t)$, and $z=h(t)$ for $t_A \le t \le t_B$, where f,g,h are continuous and have bounded and piecewise-continuous first derivatives on $[t_A,t_B]$, then we may evaluate the line integral of $w(x,y,z)$ along C from A to B with the following formula.

$$\int_C w\,\mathrm{d}s = \int_{t_A}^{t_B} w[f(t),g(t),h(t)]\sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

  - Note that the line integral is the same for any appropriate parameterization of C, or no parameterization.