Calculus and Analytic Geometry (Thomas) Notes

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Contents

List of Figures

Chapter 8

Hyperbolic Functions

8.1 Introduction

6/24: • **Hyperbolic functions**: Certain combinations of e^x and e^{-x} that are used to solve certain engineering problems (the hanging cable) and are useful in connection with differential equations.

8.2 Definitions and Identities

- Let
$$\cosh u = \frac{1}{2}(e^u + e^{-u}) \qquad\qquad \sinh u = \frac{1}{2}(e^u - e^{-u})$$
  - These combinations of exponentials occur sufficiently frequently that we give a special name to them.
  - Although the names may seem random, $\sinh u$ and $\cosh u$ do share many analogous properties with $\sin u$ and $\cos u$.
  - Pronounced to rhyme with "gosh you" and as "cinch you," respectively.
- Like $x = \cos u$ and $y = \sin u$ are associated with the point (x, y) on the unit circle $x^2 + y^2 = 1$, $x = \cosh u$ and $y = \sinh u$ are associated with the point (x, y) on the unit hyperbola $x^2 - y^2 = 1$.
  - Note that $x = \cosh u$ and $y = \sinh u$ are associated with the *right-hand* branch of the unit hyperbola.
  - Also note that sine and cosine are sometimes referred to as the **circular functions**.
- Analogous to sine and cosine, we have the identity
$$\cosh^2 u - \sinh^2 u = 1$$
- We define the remaining hyperbolic trig functions as would be expected.
$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} \qquad\qquad \operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}}$$
$$\coth u = \frac{\cosh u}{\sinh u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} \qquad\qquad \operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}}$$
- Since $\cosh u + \sinh u = e^u$, we can replace any combination of exponentials with hyperbolic sines and cosines and vice versa.
- Note that the hyperbolic functions are *not* periodic.
  - This does mean, though, that they have more easily defined properties at infinity.
- "Practically all the circular trigonometric identities have hyperbolic analogies" (Thomas, 1972, p. 267).

8.3 Derivatives and Integrals

6/25: - Derivatives of the hyperbolic functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = \cosh u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh u) = \sinh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh u) = \operatorname{sech}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth u) = -\operatorname{csch}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{\mathrm{d}u}{\mathrm{d}x}$$

  - Note that these are almost exact analogs of the formulas for the corresponding circular functions, the exception being that the negative signs are not associated with the cofunctions but with the latter three.

- We now introduce the hanging cable problem, deriving the differential equation that represents the condition for equilibrium of forces acting on a section AP of a hanging cable (Figure 8.1).

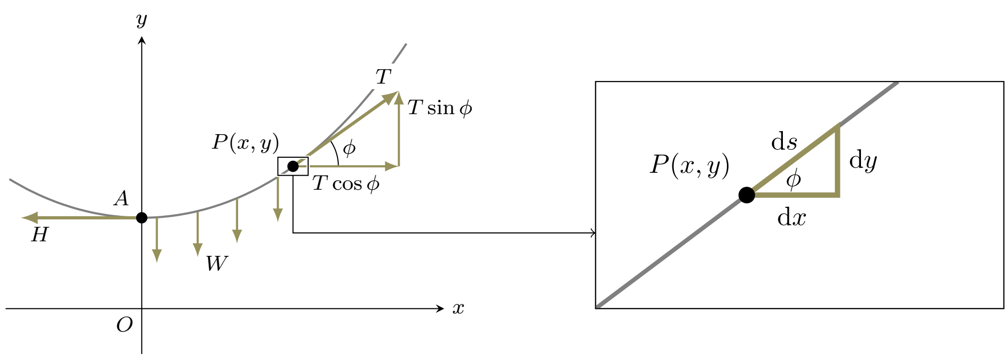


Figure 8.1: A section AP of a hanging cable.

- Let point A to be the lowest point in the arc of the hanging cable, and let it be at $(0, y_0)$ in the Cartesian plane.
- Continue along the right arc of the cable until arriving at some point $P(x, y)$.
- We wish to consider only segment AP, so we need to anchor points A and P as if the rest of the cable were still there. Now every infinitesimal sliver of the cable is being pulled (downward) slightly by gravity, but significantly (tangentially) by the rest of the cable. Thus, we can compensate at point A by pulling it tangentially left with some force H, and at point P by pulling it tangentially up and to the right with some force T.
- Since the cable is at equilibrium, the three forces acting on the cable as a whole (T, H, and W) are balanced. Thus,

$$T \sin\phi = W$$
$$T \cos\phi = H$$

- Combining these two equations gives an important result:

$$\frac{T\sin\phi}{T\cos\phi} = \frac{W}{H}$$
$$\tan\phi = \frac{W}{H}$$

- Now $\tan\phi$ is a particularly important piece of the puzzle, because it actually equals $\frac{\mathrm{d}y}{\mathrm{d}x}$ (see the zoomed-in section of Figure 8.1). Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{W}{H}$$
- Contrary to how it may look, W is actually not a constant — the weight of section AP is dependent on P (i.e., is dependent on how long the section is). If we assume that the cable has a uniform weight per length ratio w and that section AP is s units long, then we have $W = ws$. Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ws}{H}$$
- s is just arc length. Thus, $s = \int_A^P \sqrt{1+(\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$. However, because we cannot have an integral in a differential equation, we differentiate to find $\mathrm{d}s = \sqrt{1+(\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$.
  - Note that this expression for $\mathrm{d}s$ makes sense because, by the zoomed-in section of Figure 8.1, $\mathrm{d}s = \sqrt{\mathrm{d}x^2+\mathrm{d}y^2} = \sqrt{(\mathrm{d}x^2/\mathrm{d}x^2+\mathrm{d}y^2/\mathrm{d}x^2)\mathrm{d}x^2} = \sqrt{1+(\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$.
- If we differentiate $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ws}{H}$, we can get $\mathrm{d}s$ into the equation and substitute, as follows, to yield the final differential equation.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\frac{\mathrm{d}s}{\mathrm{d}x}$$
$$\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}$$

8.4 Geometric Meaning of the Hyperbolic Radian

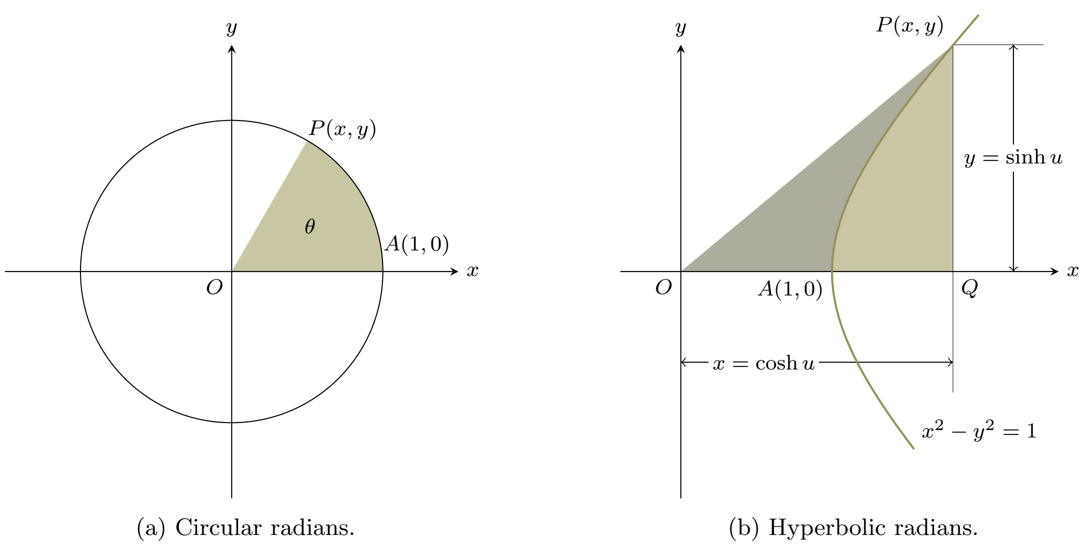


Figure 8.2: Geometric meaning of radians.

- For circular sine and cosine, the "meaning of the variable θ in the equations $x = \cos\theta$, $y = \sin\theta$ as they relatee to the point $P(x, y)$ on the unit circle $x^2 + y^2 = 1$ [is] the radian measure of the angle AOP in [Figure 8.2a], that is $\theta = \frac{\text{arc } AP}{\text{radius } OA}$" (Thomas, 1972, p. 271).
  - However, since $A = \frac{1}{2}r^2\theta = \frac{\theta}{2}$ for $r = 1$, θ also equals twice the area of the sector AOP.

- To understand the meaning of the variable u, calculate the area of the sector AOP in Figure 8.2b as an analog to circular area.

$$\begin{aligned} A_{AOP} &= A_{OQP} - A_{AQP} \\ &= \frac{1}{2}bh - \int_A^P y\,\mathrm{d}x \end{aligned}$$

$$y = \sinh u, \quad x = \cosh u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \sinh u \Rightarrow \mathrm{d}x = \sinh u\,\mathrm{d}u$$

$$\begin{aligned} &= \frac{1}{2}xy - \int_A^P \sinh^2 u\,\mathrm{d}u \\ &= \frac{1}{2}\cosh u \sinh u - \frac{1}{2}\int_A^P (\cosh 2u - 1)\,\mathrm{d}u \\ &= \frac{1}{2}\sinh u \cosh u - \frac{1}{2}\left[\frac{1}{2}\sinh 2u - u\right]_{A(u=0)}^{P(u=u)} \\ &= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{4}\sinh 2u - \frac{1}{2}u\right) \\ &= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{2}\sinh u \cosh u - \frac{1}{2}u\right) \\ &= \frac{1}{2}u \end{aligned}$$

  - This implies that u also equals twice the area of the sector AOP (the hyperbolic sector, that is).
  - This means, for example, that "$\cosh 2$ and $\sinh 2$ may be interpreted as the coordinates of P when the area of the sector AOP is just equal to the area of a square having OA as side" (Thomas, 1972, p. 272).

8.5 The Inverse Hyperbolic Functions

6/26:

- The inverse of $x = \sinh y$ is $y = \sinh^{-1} x$.
  - Since there is a one-to-one correspondence between x and y values for the inverse hyperbolic sine function, there is no need to define a principal branch (as there was with circular sine).
- The inverse of $x = \cosh y$ is $y = \cosh^{-1} x$, where $y \geq 0$ and $x \geq 1$.
  - Since there is a two-to-one correspondence between x and y values this time, we choose the positive values to be the principal branch and let the negative values be defined by the function $y = -\cosh^{-1} x$.
- Note that the only other inverse hyperbolic trig function that needs a principal branch is (rather appropriately) $\operatorname{sech} x$. Likewise, the positive values make up the principal branch.
- Like the hyperbolic trig functions have exponential forms, the inverse hyperbolic trig functions have logarithmic forms.

- For example,

$$\begin{aligned}
y &= \sinh^{-1} x \\
\sinh y &= x \\
\frac{1}{2}(\mathrm{e}^y - \mathrm{e}^{-y}) &= x \\
\mathrm{e}^y - \frac{1}{\mathrm{e}^y} &= 2x \\
\mathrm{e}^{2y} - 1 &= 2x\mathrm{e}^y \\
0 &= \mathrm{e}^{2y} - 2x\mathrm{e}^y - 1 \\
\mathrm{e}^y &= x \pm \sqrt{x^2+1} \\
y &= \ln\left(x + \sqrt{x^2+1}\right)
\end{aligned}$$

- Like the inverse circular trig functions, the inverse hyperbolic functions are quite useful as the end results of integration of radicals. First, however, we must derive their derivatives.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} u\right) = \frac{1}{\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} && \frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} u\right) = \frac{1}{\sqrt{u^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} u\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| < 1 && \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{sech}^{-1} u\right) = \frac{-1}{u\sqrt{1-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\coth^{-1} u\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u| > 1 && \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{csch}^{-1} u\right) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

8.6 The Hanging Cable

- We seek to derive the solution to the following differential equation associated with the hanging cable problem, as described in Section 8.3.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{w}{H}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

- Since the above equation involves the second derivative, we will have to deal with two constants of integration. Since it doesn't matter where the hanging cable lies in the Cartesian plane, we can choose its location such that the final answer will be as simple as possible.
  - "By choosing the y-axis to be the vertical line through the lowest point of the cable, one condition becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$" (Thomas, 1972, p. 277).
  - "Then we may still move the x-axis up or down to suit our convenience. That is, we let $y = y_0$ when $x = 0$, and we may choose y_0 so as to give us the simplest form in our final answer" (Thomas, 1972, p. 277).
- Let's begin solving the original equation. Since it involves y' and y'' but not y, let $y' = p$ and start by integrating with respect to p.

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{w}{H}\sqrt{1+p^2} \\
\frac{\mathrm{d}p}{\sqrt{1+p^2}} &= \frac{w}{H}\,\mathrm{d}x \\
\int \frac{\mathrm{d}p}{\sqrt{1+p^2}} &= \int \frac{w}{H}\,\mathrm{d}x \\
\sinh^{-1} p &= \frac{w}{H}x + C_1
\end{aligned}$$

- Since $p = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $x = 0$, $C_1 = 0$. Thus,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \sinh\left(\frac{w}{H}x\right) \\ \mathrm{d}y &= \sinh\left(\frac{w}{H}x\right)\mathrm{d}x \\ \int \mathrm{d}y &= \int \sinh\left(\frac{w}{H}x\right)\mathrm{d}x \\ y &= \frac{H}{w}\cosh\left(\frac{w}{H}x\right) + C_2 \end{aligned}$$

- Since $y = y_0$ when $x = 0$, $C_2 = y_0 - \frac{H}{w}$. Thus, let $y_0 = \frac{H}{w}$. Therefore, we are finished:

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

Chapter 9

Methods of Integration

9.1 Basic Formulas

6/30:

- Useful, abstract info (that I already know) on what makes a student good at integating, e.g., integrating is an exercise in trial-and-error, but there are ways to increase your likelihood of being successful.

9.2 Powers of Trigonometric Functions

- When integrating power functions, look for integral/derivative relationships, which may allow you to substitute u and $\mathrm{d}u$ at the same time.
  - For example, when confronted with $\int \sin^n ax \cos ax \, \mathrm{d}x$, note that $\cos ax$ is almost the derivative of $\sin ax$, and choose $u = \sin ax$ and $\frac{\mathrm{d}u}{a} = \cos ax \, \mathrm{d}x$ to yield $\frac{1}{a} \int u^n \, \mathrm{d}u$.
- When integrating power functions, it may be possible to split the exponent into a product ($u^n = u^a u^b$ where $a + b = n$) and work off of properties of one of the functions raised to a smaller exponent (u^a may have properties that u^n lacks).
  - For example, when confronted with $\int \sin^3 x \, \mathrm{d}x$, recall that $\sin^2 x$ has Pythagorean properties, and split the exponent.

$$\begin{aligned}
\int \sin^3 x \, \mathrm{d}x &= \int \sin^2 x \sin x \, \mathrm{d}x \\
&= \int \left(1 - \cos^2 x\right) \sin x \, \mathrm{d}x
\end{aligned}$$

Now we can use the previous property, since $\sin x$ and $\cos x$ have an integral/derivative relationship.

$$\begin{aligned}
&= -\int \left(1 - u^2\right) \mathrm{d}u \\
&= \int \left(u^2 - 1\right) \mathrm{d}u
\end{aligned}$$

  - Note that this technique is applicable whenever an odd power of sine or cosine is to be integrated. For higher powers, consider the following.

$$\int \cos^{2n+1} x \, \mathrm{d}x = \int \left(\cos^2 x\right)^n \cos x \, \mathrm{d}x = \int \left(1 - \sin^2 x\right)^n \cos x \, \mathrm{d}x = \int \left(1 - u^2\right)^n \mathrm{d}u$$

Remember that $\left(1 - u^2\right)^n$ can be expanded via the binomial theorem.

- When integrating a composite trigonometric function, consider reducing it to a radical of powers of sines and cosines.
  - For example, $\sec x \tan x = \frac{\sin x}{\cos^2 x}$.
- When integrating positive integer powers of $\tan x$, use either the base cases or the **reduction formula**.
  - Begin by deriving a reduction formula.

$$\begin{aligned}\int \tan^n x \,\mathrm{d}x &= \int \tan^{n-2} x \left(\sec^2 x - 1\right) \mathrm{d}x \\ &= \int \tan^{n-2} \sec^2 x \,\mathrm{d}x - \int \tan^{n-2} x \,\mathrm{d}x \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \,\mathrm{d}x\end{aligned}$$

Since the reduction formula decreases the exponent by 2, work out two base cases.

$$\int \tan^0 x \,\mathrm{d}x = \int \mathrm{d}x = x + C$$
$$\int \tan^1 x \,\mathrm{d}x = \int \frac{\sin x}{\cos x} \,\mathrm{d}x = -\int \frac{\mathrm{d}u}{u} = -\ln|\cos x| + C$$

  - Note that the impetus for initially deriving such a formula was the search for a way to get $\sec^2 x$ into the integrand, which can be done by splitting the exponent.
  - This method can easily be adjusted to suit negative powers of $\tan x$ (positive powers of $\cot x$).
- When integrating even powers of $\sec x$, either use the secant reduction formula, or split the exponent.
  - We derive the following formula.

$$\begin{aligned}\int \sec^{2n} x \,\mathrm{d}x &= \int \sec^{2n-2} x \sec^2 x \,\mathrm{d}x \\ &= \int \left(1 + \tan^2 x\right)^{n-1} \sec^2 x \,\mathrm{d}x \\ &= \int \left(1 + u^2\right)^{n-1} \mathrm{d}u\end{aligned}$$

- When integrating secant (or cosecant) alone, produce $\frac{u'}{u}$ by multiplying the integrand by a clever form of 1.
  - For example,

$$\begin{aligned}\int \sec x \,\mathrm{d}x &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \,\mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

9.3 Even Powers of Sines and Cosines

- When integrating the product of sines and cosines raised to powers where at least one exponent is a positive odd integer, split the exponent and use u-substitution.
  - In effect, we wish to evaluate $\int \sin^m x \cos^n x \,\mathrm{d}x$ where at least one of m, n is a positive odd integer.

 - For example, when confronted with $\int \cos^{\frac{2}{3}} x \sin^5 x \, \mathrm{d}x$, split the exponent of $\sin^5 x$ and choose $u = \cos x$ and $-\mathrm{d}u = \sin x \, \mathrm{d}x$.

$$\int \cos^{\frac{2}{3}} x \sin^5 x \, \mathrm{d}x = \int \cos^{\frac{2}{3}} x \left(1 - \cos^2 x\right)^2 \sin x \, \mathrm{d}x = \int u^{\frac{2}{3}} \left(u^2 - 1\right) \mathrm{d}u$$

- When integrating the product of sines and cosines raised to powers where both exponents are even integers, begin by transforming it into a sum of either just sines *or* just cosines raised to even integers. Then split the exponents and use one of the following formulas. It may be necessary to use these formulas multiple times. Use them until the problem has been reduced to a sum with only odd exponents.

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \qquad \cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

 - Note that "these identities may be derived very quickly by adding or subtracting the equations [$\cos^2 u + \sin^2 u = 1$ and $\cos^2 u - \sin^2 u = \cos 2u$] and by dividing by two" (Thomas, 1972, p. 287).
 - For example, when confronted with $\int \sin^2 x \cos^4 x \, \mathrm{d}x$, begin by changing it to a case with only powers of cosine (chose to eliminate the sine function because it is raised to a lower exponent and, thus, will need less binomial expansion).

$$\begin{aligned}\int \sin^2 x \cos^4 x \, \mathrm{d}x &= \int \left(1 - \cos^2 x\right) \cos^4 x \, \mathrm{d}x \\ &= \int \cos^4 x \, \mathrm{d}x - \int \cos^6 x \, \mathrm{d}x\end{aligned}$$

 Now split the exponents.

$$= \int \left(\cos^2 x\right)^2 \mathrm{d}x - \int \left(\cos^2 x\right)^3 \mathrm{d}x$$

 Employ the above formulas and use binomial expansion. If necessary, repeat (split the exponents, employ the above formulas, use binomial expansion) until only odd exponents remain (remember that 1 is an odd exponent).

$$\begin{aligned} &= \int \left(\frac{1}{2}(1 + \cos 2x)\right)^2 \mathrm{d}x - \int \left(\frac{1}{2}(1 + \cos 2x)\right)^3 \mathrm{d}x \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \cos^2 2x\right) \mathrm{d}x \\ &\quad - \frac{1}{8} \int \left(1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x\right) \mathrm{d}x \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) \mathrm{d}x \\ &\quad - \frac{1}{8} \int \left(1 + 3\cos 2x + \frac{3}{2}(1 + \cos 4x) + \cos^3 2x\right) \mathrm{d}x \end{aligned}$$

 These integrals may now be handled using previously discussed techniques.

9.4 Integrals With Terms $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, $\sqrt{u^2 - a^2}$, $a^2 + u^2$, $a^2 - u^2$

- When integrating a radical that resembles the derivative of an inverse trig function, we may factor out the issue so as to make the integral resemble one of the known formulas.

- For example, when confronted with $\int \frac{\mathrm{d}u}{a^2+u^2}$, divide the a^2 term out of the denominator and integrate with respect to $\frac{u}{a}$[1].

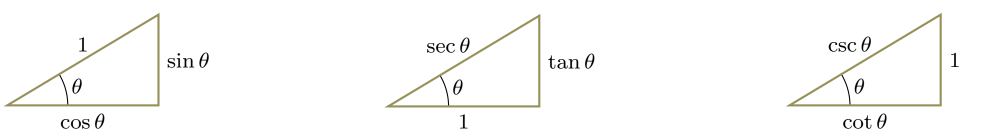
$$\begin{aligned}\int \frac{\mathrm{d}u}{a^2+u^2} &= \frac{1}{a^2}\int \frac{\mathrm{d}u}{1+\left(\frac{u}{a}\right)^2}\\ &= \frac{1}{a^2}\int \frac{a\,\mathrm{d}\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2}\\ &= \frac{1}{a}\tan^{-1}\frac{u}{a}+C\end{aligned}$$

- However, this method is partially flawed in that it relies on having memorized the derivatives of the inverse trig functions, i.e., it is not terribly analytical. This shortcoming will now be addressed with a new, more general technique.

- The new method leans heavily on the following three identities.

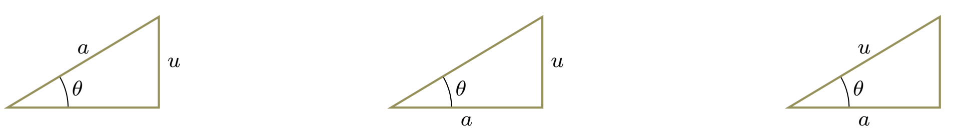
$$1-\sin^2\theta = \cos^2\theta \qquad 1+\tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

  - With the help of these identities, it is possible to...
    1. use $u = a\sin\theta$ to replace a^2-u^2 with $a^2\cos^2\theta$;
    2. use $u = a\tan\theta$ to replace a^2+u^2 with $a^2\sec^2\theta$;
    3. use $u = a\sec\theta$ to replace u^2-a^2 with $a^2\tan^2\theta$.



(a) $\cos^2\theta+\sin^2\theta=1$. (b) $1+\tan^2\theta=\sec^2\theta$. (c) $\cot^2\theta+1=\csc^2\theta$.

Figure 9.1: Geometric rationale for the trigonometric identities.



(a) $\sqrt{a^2-u^2}=a\cos\theta$
$u=a\sin\theta$.

(b) $\sqrt{u^2+a^2}=a\sec\theta$
$u=a\tan\theta$.

(c) $\sqrt{u^2-a^2}=a\tan\theta$
$u=a\sec\theta$.

Figure 9.2: Geometric rationale for the trigonometric substitutions.

  - These identities and substitutions can be easily remembered by thinking of the Pythagorean theorem in conjunction with Figures 9.1 and 9.2, respectively.

- We may now evaluate inverse trig integrals analytically.

[1]Thomas, 1972 uses differentials with more complex functions than a single variable quite often. It's not something I've seen before, but it's something I should get used to (and it does make sense if you think about it — it's just an extension of the underlying concept of separation of variables integration).

– For example, when confronted with $\int \frac{\mathrm{d}u}{a^2+u^2}$, choose $u = a\tan\theta$ and $\mathrm{d}u = a\sec^2\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int \frac{\mathrm{d}u}{a^2+u^2} &= \int \frac{a\sec^2\theta\,\mathrm{d}\theta}{a^2+(a\tan\theta)^2}\\ &= \int \frac{a\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\,\mathrm{d}\theta\\ &= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta\\ &= \frac{1}{a}\int \mathrm{d}\theta\\ &= \frac{1}{a}\theta + C\end{aligned}$$

At this point, solve $u = a\tan\theta$ for θ and substitute.

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

– Some integrals will simplify to have a plus/minus in the denominator, leading to two possible solutions. However, there are sometimes ways to isolate a single solution.

- For example, $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta\,\mathrm{d}\theta}{\pm a\cos\theta} = \pm\theta + C$. However, when we consider the fact that $\theta = \sin^{-1}\frac{u}{a}$, we know that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (because inverse sine is not arcsine, and inverse sine is only defined over the principal branch of sine). Thus, since $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta \in [0,1]$, i.e., is always positive. Thus, we choose $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = +\theta + C = \sin^{-1}\frac{u}{a} + C$ as our one solution.
- For example, $\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}}$ equals $\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$ or $-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| + C$ depending on whether $\tan\theta$ is positive or negative. But it can be shown algebraically that the two solutions are actually equal:

$$\begin{aligned}-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right|\\ &= \ln\left|\frac{a\left(u+\sqrt{u^2-a^2}\right)}{\left(u-\sqrt{u^2-a^2}\right)\left(u+\sqrt{u^2-a^2}\right)}\right|\\ &= \ln\left|\frac{a\left(u+\sqrt{u^2-a^2}\right)}{a^2}\right|\\ &= \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right|\end{aligned}$$

Thus, we have $\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$ as the one solution[2].

– Some integrals will have extraneous constants that can be combined with C to simplify the *indefinite* integral.

- Continuing with the above example,

$$\begin{aligned}\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} &= \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C\\ &= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln|a| + C\\ &= \ln\left|u+\sqrt{u^2-a^2}\right| + C\end{aligned}$$

[2]Note that we could choose to use the other solution, but we choose this one because it's "simpler" (it uses addition instead of subtraction).

- When integrating an inverse trig integral with excess polynomial terms, look to transform it into a (power of a) trig integral problem.
  - For example, when confronted with $\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^2}}$, treat it as a case of a^2-u^2, but substitute the trig expression into the x^2 term in the numerator, too.

$$\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,\mathrm{d}\theta}{3\cos\theta} = 9\int \sin^2\theta\,\mathrm{d}\theta$$

    This integral may now be handled using previously discussed techniques.

- Many inverse trig integrals can also be evaluated hyperbolically, making use of the following three identities.

$$\cosh^2\theta - 1 = \sinh^2\theta \qquad 1-\tanh^2\theta = \operatorname{sech}^2\theta \qquad 1+\sinh^2\theta = \cosh^2\theta$$

  - With the help of these identities, it is possible to...
    1. use $u = a\tanh\theta$ to replace a^2-u^2 with $a^2\operatorname{sech}^2\theta$;
    2. use $u = a\sinh\theta$ to replace a^2+u^2 with $a^2\cosh^2\theta$;
    3. use $u = a\cosh\theta$ to replace u^2-a^2 with $a^2\sinh^2\theta$.

9.5 Integrals With ax^2+bx+c

- When integrating composite functions where the inner function is a binomial, look to factor said binomial.
  - The general quadratic $f(x) = ax^2+bx+c$, $a \neq 0$, can be reduced to the form $a\left(u^2+B\right)$ by completing the square and choosing $u = x+\frac{b}{2a}$ and $B = \frac{4ac-b^2}{4a^2}$:

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a} \\ &= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right) \end{aligned}$$

- When integrating the square root of a binomial, or some similarly tricky function of a binomial, we can transform the binomial into a form such that it will suit one of the inverse trig integrals.
  - Since it would lead to complex numbers, we disregard cases where $a\left(u^2+B\right)$ is negative, i.e., we focus on cases where (1) a is positive, and (2) a, B are both negative.
  - That being said, if it is an odd root ($\sqrt[3]{x}$, $\sqrt[5]{x}$, etc.), the sign doesn't matter.
  - For example, when confronted with $\int \frac{(x+1)\,\mathrm{d}x}{\sqrt{2x^2-6x+4}}$, begin by factoring the binomial[3].

$$2x^2-6x+4 = 2\left(x^2-3x\right)+4 = 2\left(x-\frac{3}{2}\right)^2-\frac{1}{2} = 2\left(u^2-a^2\right)$$

    Note that $u = x-\frac{3}{2}$ and $a = \frac{1}{2}$. We can now return to the integral, which we shall reformulate in terms of u in its entirety.

$$\int \frac{(x+1)\,\mathrm{d}x}{\sqrt{2x^2-6x+4}} = \int \frac{\left(u+\frac{5}{2}\right)\mathrm{d}u}{\sqrt{2\left(u^2-a^2\right)}}$$

[3]Note that, in place of inspection, we could use the general form factorization derived above.

Split it into two separate integrals and factor out the constants.

$$= \frac{1}{\sqrt{2}} \int \frac{u\,\mathrm{d}u}{\sqrt{u^2 - a^2}} + \frac{5}{2\sqrt{2}} \int \frac{\mathrm{d}u}{\sqrt{u^2 - a^2}}$$

The right integral is a straight-up inverse trig integral. The left one, however, needs something special. It could be dealt with as previously discussed by substituting $u = a\tan\theta$ for all instances of u and evaluating it is a more complex trig integral in θ. However, for the sake of showing a different technique, we will choose $z = u^2 - a^2$ and $\frac{1}{2}\mathrm{d}z = u\,\mathrm{d}u$ and treat it as a power function in z.

$$\begin{aligned}
&= \frac{1}{2\sqrt{2}} \int \frac{\mathrm{d}z}{\sqrt{z}} + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2 \\
&= \frac{1}{2\sqrt{2}} \int z^{-\frac{1}{2}}\,\mathrm{d}z + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2 \\
&= \frac{1}{\sqrt{2}} z^{\frac{1}{2}} + C_1 + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2
\end{aligned}$$

Return all of the substitutions and combine the constants of integration.

$$\begin{aligned}
&= \sqrt{\frac{u^2 - a^2}{2}} + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C \\
&= \sqrt{\frac{x^2 - 3x + 2}{2}} + \frac{5}{2\sqrt{2}} \ln\left|x - \frac{3}{2} + \sqrt{x^2 - 3x + 2}\right| + C
\end{aligned}$$

9.6 Integration by the Method of Partial Fractions

7/1:

- **Method of Partial Fractions**: The process of "split[ting] a fraction into a sum of fractions having simpler denominators" (Thomas, 1972, p. 294).
- If we wish to split a rational fraction $\frac{f(x)}{g(x)}$ into a sum of simpler fractions, then...
  - "The degree of $f(x)$ should be less than the degree of $g(x)$. If this is not the case, we first perform a long division, then work with the remainder term. This remainder can always be put into the required form" (Thomas, 1972, p. 294).
  - "The factors of $g(x)$ should be known. Theoretically, any polynomial $g(x)$ with real coefficient can be expressed as a product of real linear and quadratic factors. In practice, it may be difficult to perform the factorization" (Thomas, 1972, p. 294).
- If $x - r$ is a linear factor of $g(x)$ and $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$, then, to this factor, assign the sum of m partial fractions

$$\frac{A_1}{x - r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

  - If $x^2 + px + q$ is a quadratic factor[4] of $g(x)$ and $\left(x^2 + px + q\right)^n$ is the highest power of $x^2 + px + q$ that divides $g(x)$, then, to this factor, assign the sum of n partial fractions

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{\left(x^2 + px + q\right)^2} + \cdots + \frac{B_n x + C_n}{\left(x^2 + px + q\right)^n}$$

  - Continue, as necessary, to higher degree factors of $g(x)$ (although this caveat is not addressed by Thomas, 1972).

[4] A binomial factor, the factoring of which into linear factors would introduce complex numbers.

  - Notice how the degree of the polynomial in the numerator of the partial fractions will be at most one less than the degree of the denominator.

- Any rational function integrated via the method of partial fractions can be reduced to the problem of evaluating the following two types of integrals.

$$\int \frac{\mathrm{d}x}{(x-r)^h} \qquad\qquad \int \frac{(ax+b)\,\mathrm{d}x}{(x^2+px+q)^k}$$

  - The left integral, with the substitution $z = x - r$ and $\mathrm{d}z = \mathrm{d}x$, becomes a power integral.
  - The right integral, after completing the square in the denominator, substituting, and splitting into two fractions by the numerator, becomes a pair of inverse trig substitution integrals.

- With the method of partial fractions, there is a new way to integrate $\sec\theta$.

$$\begin{aligned}
\int \sec\theta\,\mathrm{d}\theta &= \int \frac{\mathrm{d}\theta}{\cos\theta} \\
&= \int \frac{\cos\theta\,\mathrm{d}\theta}{\cos^2\theta} \\
&= \int \frac{\mathrm{d}x}{1-x^2} \\
&= \int \frac{0.5}{1+x}\,\mathrm{d}x + \int \frac{0.5}{1-x}\,\mathrm{d}x \\
&= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + C \\
&= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C \\
&= \ln\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + C
\end{aligned}$$

  - Note that $\ln\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + C$ is equivalent to the previously derived form:

$$\begin{aligned}
\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} &= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} \\
&= \left|\frac{1+\sin\theta}{\cos\theta}\right| \\
&= |\sec\theta + \tan\theta|
\end{aligned}$$

9.7 Integration by Parts

- This is the second general method of integration (the first being substitution).
- It relies on the following formulas for indefinite and definite integrals, respectively.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u + C \qquad\qquad \int_{(1)}^{(2)} u\,\mathrm{d}v = uv\Big]_{(1)}^{(2)} - \int_{(1)}^{(2)} v\,\mathrm{d}u$$

  - The indefinite integral formula can be derived from the differential of a product rule as follows.

$$\begin{aligned}
\mathrm{d}(uv) &= u\,\mathrm{d}v + v\,\mathrm{d}u \\
u\,\mathrm{d}v &= \mathrm{d}(uv) - v\,\mathrm{d}u \\
\int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u + C
\end{aligned}$$

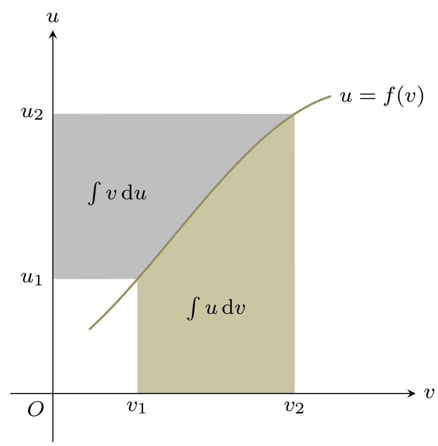


Figure 9.3: Geometric rationale for definite integration by parts.

  - The definite integral formula can be thought of as an adjustment of the above, or it can be conceived geometrically: In Figure 9.3, $\int_{(1)}^{(2)} u\,\mathrm{d}v$ is the yellow area, which is clearly equivalent to the total area[5] $uv\big]_{(1)}^{(2)}$ minus the grey area $\int_{(1)}^{(2)} v\,\mathrm{d}u$.

- Since $\int \mathrm{d}v = v + C_1$, $\int u\,\mathrm{d}v$ actually equals $u(v + C_1) - \int (v + C_1)\,\mathrm{d}u$. However, since

$$\begin{aligned} u(v + C_1) - \int (v + C_1)\,\mathrm{d}u &= uv + C_1 u - \int v\,\mathrm{d}u - \int C_1\,\mathrm{d}u \\ &= uv - \int v\,\mathrm{d}u \end{aligned}$$

  it is customary to drop the first constant of integration.

  - That being said, it is sometimes useful — when evaluating $\int \ln(x+1)\,\mathrm{d}x = \ln(x+1)(x+C_1) - \int \frac{x+C_1}{x+1}\,\mathrm{d}x$, being able to choose $C_1 = 1$ greatly simplifies the second integral.

- When integrating an inverse trig function, consider using integration by parts.

  - For example, when confronted with $\int \tan^{-1} x\,\mathrm{d}x$, integration by parts turns it into an inverse trig derivative problem.

$$\int \tan^{-1} x\,\mathrm{d}x = x\tan^{-1} x - \int \frac{x\,\mathrm{d}x}{1+x^2}$$

- When attempting integration by parts, don't be afraid to use it multiple times.

  - For example, when confronted with $\int x^2 \mathrm{e}^x\,\mathrm{d}x$, use integration by parts twice.

$$\begin{aligned} \int x^2 \mathrm{e}^x\,\mathrm{d}x &= x^2\mathrm{e}^x - \int 2x\mathrm{e}^x\,\mathrm{d}x \\ &= x^2\mathrm{e}^x - \left(2x\mathrm{e}^x - \int 2\mathrm{e}^x\,\mathrm{d}x\right) \end{aligned}$$

- When attempting integration by parts, look for the original integral showing up again — if it does, combine like terms.

- When integrating powers of $\cos x$, consider using a reduction formula.

[5]Note that $uv\big]_{(1)}^{(2)} = u_2v_2 - u_1v_1$, the latter of which, as the difference of two rectangles, clearly represents the total shaded area.

- Begin by deriving a reduction formula (this will involve splitting the exponent!).

$$
\begin{aligned}
\int \cos^n x \,\mathrm{d}x &= \int \cos^{n-1} x \cos x \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \,\mathrm{d}x - (n-1) \int \cos^n x \,\mathrm{d}x \\
(1+(n-1)) \int \cos^n x \,\mathrm{d}x &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \,\mathrm{d}x \\
\int \cos^n x \,\mathrm{d}x &= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \,\mathrm{d}x
\end{aligned}
$$

Since the reduction formula decreases the exponent by 2, work out two base cases.

$$
\begin{aligned}
\int \cos^0 x \,\mathrm{d}x &= x + C \\
\int \cos^1 x \,\mathrm{d}x &= \sin x + C
\end{aligned}
$$

- When integrating powers of $\sin x$, consider using a reduction formula.

$$
\int \sin^n x \,\mathrm{d}x = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \,\mathrm{d}x
$$

9.8 Integration of Rational Functions of $\sin x$ and $\cos x$, and Other Trigonometric Integrals

- "It has been discovered that the substitution $z = \tan \frac{x}{2}$ enables one to reduce the problem of integrating any rational function of $\sin x$ and $\cos x$ to a problem involving a rational function of z. This in turn can be integrated by the method of partial fractions" (Thomas, 1972, p. 300).
  - However, this substitution should be used only as a last resort — the associated algebra is often quite cumbersome.
- To increase the ease of use for this substitution, it will help to derive three results.

$$
\cos x = \frac{1-z^2}{1+z^2} \qquad \sin x = \frac{2z}{1+z^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}z}{1+z^2}
$$

- "The following types of integrals... arise in connection with alternating-current theory, heat transfer problems, bending of beams, cable stress analysis in suspension bridges, and many other places where trigonometric series (or Fourier series) are applied to problems in mathematics, science, and engineering" (Thomas, 1972, p. 301).

$$
\int \sin mx \sin nx \,\mathrm{d}x \qquad \int \sin mx \cos nx \,\mathrm{d}x \qquad \int \cos mx \cos nx \,\mathrm{d}x
$$

- When confronted with one of these integrals, integration by parts may be used. However, using one of the following three identities will be more simple.

$$
\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}(\cos(m-n)x - \cos(m+n)x) \\
\sin mx \cos nx &= \frac{1}{2}(\sin(m-n)x + \sin(m+n)x) \\
\cos mx \cos nx &= \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)
\end{aligned}
$$

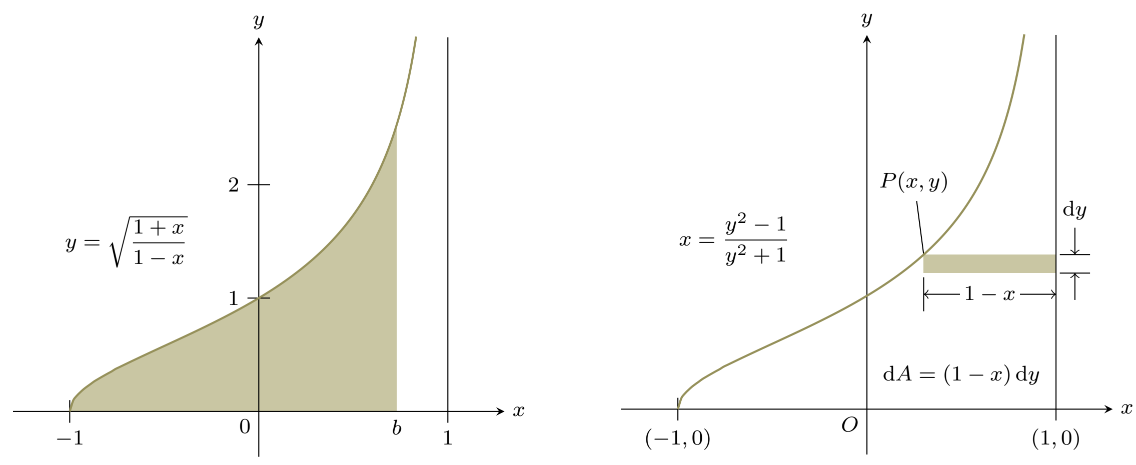
- Note that "these identities follow at once from $\cos(A+B) = \cos A \cos B - \sin A \sin B$, $\cos(A-B) = \cos A \cos B + \sin A \sin B$, and $\sin(A+B) = \sin A \cos B + \cos A \sin B$, $\sin(A-B) = \sin A \cos B - \cos A \sin B$" (Thomas, 1972, p. 301).

9.9 Further Substitutions

- "Some integrals involving fractional powers of the variable x may be simplified by substituting $x = z^n$, where n is the least common multiple of the denominators of the exponents" (Thomas, 1972, p. 302).
  - For example, $\int \frac{\sqrt{x}\,\mathrm{d}x}{1+\sqrt[4]{x}}$ can be simplified by taking $x = z^4$.
- "Even when it is not clear at the start that a substitution will work, it is advisable to try one that seems reasonable and pursue it until it either gives results or appears to make matters worse. In the latter case, try something else! Sometimes a chain of substitutions $u = f(x)$, $v = g(u)$, $z = h(v)$, and so on, will produce results when it is by no means obvious that this will work" (Thomas, 1972, p. 302).
- "The criterion of success is whether the new integrals so obtained appear to be simpler than the original integral. Here it is handy to remember that any rational function of x can be integrated by the method of partial fractions and that any rational function of $\sin x$ and $\cos x$ can be integrated by using the substitution $z = \tan \frac{x}{2}$. If we can reduce a given integral to one of these types, we then know how to finish the job" (Thomas, 1972, p. 302).
- To evaluate a definite integral after a (series of) substitution(s), either return the substitution(s) and keep the bounds or keep the substitution(s) and determine new bounds based on the new variable of integration.
  - For example, if $z^2 = \frac{1+x}{1-x}$ and $x \in [-1, 1]$, then $z \in [0, \infty)$. And if $z = \tan\theta$, then $\theta \in \left[0, \frac{\pi}{2}\right]$.

9.10 Improper Integrals

- **Improper integral**: An integral of the form $\int_a^b f(x)\,\mathrm{d}x$ where some $x \in [a,b]$ is infinite, and/or one or both of a, b are infinite in magnitude.



(a) With respect to x. (b) With respect to y.

Figure 9.4: Defining improper integrals.

- Say we wish to evaluate $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$, knowing that the integrand approaches ∞ as $x \to 1$ (see Figure 9.4a). Well, if the upper bound b is some value slightly *less than* 1, we *can* evaluate the integral. Thus evaluating the original integral becomes a problem of evaluating

$$\lim_{b\to 1^-} \int_{-1}^{b} \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \lim_{b\to 1^-} \left(\sin^{-1} x - \sqrt{1-x^2} \Big]_{-1}^{b} \right) = \lim_{b\to 1^-} \left(\sin^{-1} b - \sqrt{1-b^2} + \frac{\pi}{2} \right)$$

  - Sometimes such a limit will converge. Sometimes it will not (it will diverge). Either way, it answers the question of the nature of the area under the curve (by yielding some finite value, or the infinite one).
  - Note that the integral works out just the same if we sum vertical elements instead (see Figure 9.4b), evaluating the following.

$$\lim_{c\to\infty} \int_0^c (1-x)\,\mathrm{d}y = \lim_{c\to\infty} \int_0^c \frac{2\,\mathrm{d}y}{y^2+1}$$

- When integrating a function $f(x)$ on $[a,b]$ where $f(x)\to\infty$ at some x-value $c \in (a,b)$, split the integral.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to c^-} \int_a^c f(x)\,\mathrm{d}x + \lim_{c\to c^+} \int_c^b f(x)\,\mathrm{d}x$$

- On determining whether or not an improper integral with a nonintegrable integrand exists, we can sometimes compare it with an integral that we know.
  - For example,

$$\int_1^b \mathrm{e}^{-x^2}\,\mathrm{d}x \leq \int_1^b \mathrm{e}^{-x}$$

    for all $b \geq 1$ since $0 < \mathrm{e}^{-x^2} \leq \mathrm{e}^{-x}$ for all $x \geq 1$. Thus, $\int_1^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x$ evaluates to some finite value.

- Note that some improper integrals diverge by oscillation.
  - For example, $\int_0^\infty \cos x\,\mathrm{d}x$ diverges in this manner.

9.11 Numerical Methods for Approximating Definite Integrals

- One could use the **trapezoidal rule**.
- A better choice, though, is **Simpson's rule**.

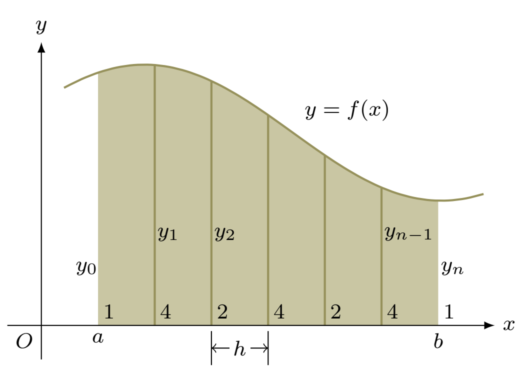


Figure 9.5: Simpson's rule.

- Simpson's rule approximates the curve via parabolas, which can be uniquely defined by three points and have a nice formula for the area underneath them.
- We now derive Simpson's rule.
  - Suppose we wish to approximate the area under the part of the curve from x_i to x_{i+2} in Figure 9.5. We know that there exists some parabola intersecting (x_i, y_0), (x_{i+1}, y_1), and (x_{i+2}, y_2). However, for the sake of simplifying the algebra, we choose to consider the parabola intersecting $(-h, y_0)$, $(0, y_1)$, and (h, y_2) (the area under both parabolas will be equivalent since $x_{i+1} - x_i = h$). Let this translated parabola be called $Ax^2 + Bx + C$ for some $A, B, C \in \mathbb{R}$. Then the area underneath this parabola A_p can be described by the following.

$$\begin{aligned} A_p &= \int_{-h}^{h} \left(Ax^2 + Bx + C\right) \mathrm{d}x \\ &= \frac{2Ah^3}{3} + 2Ch \end{aligned}$$

  - We know that the area underneath this parabola is dependent only on h, y_0, y_1, and y_2. Thus, we look to express A, C from the integral in terms of h, y_0, y_1, y_2. To accomplish this, we will use the facts that

$$\begin{aligned} y_0 &= Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C \end{aligned}$$

    We can now see that $C = y_1$, so all that's left is to solve for A. This can be done by adding the first and third equations, substituting, and solving as follows.

$$\begin{aligned} y_0 + y_2 &= 2Ah^2 + 2C \\ 2Ah^2 &= y_0 + y_2 - 2y_1 \\ A &= \frac{y_0 - 2y_1 + y_2}{2h^2} \end{aligned}$$

  - Thus, we can reformulate the area under the parabola as follows.

$$\begin{aligned} A_p &= \frac{2h^3}{3} \cdot \frac{y_0 - 2y_1 + y_2}{2h^2} + 2y_1 h \\ &= \frac{h}{3}(y_0 + 4y_1 + y_2) \end{aligned}$$

  - "Simpson's rule follows from applying this result to successive pieces of the curve $y = f(x)$ between $x = a$ and $x = b$. Each separate piece of the curve, covering an x-subinterval of width $2h$, is approximated by an arc of a parabola through its ends and its mid-point. The area under each parabolic arc is then given by an expression like [the above] and the results are added to give [the following]" (Thomas, 1972, p. 309).

$$\begin{aligned} A_S &= \frac{h}{3}((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$

- Note that the number of subdivisions must be an even integer.

- To find the error in a Simpson's rule approximation, use the fact[6] that "if f is continuous on $[a, b]$ and four times differentiable on (a, b), then there is a number c between a and b such that [the following holds]" (Thomas, 1972, p. 310).

$$\int_a^b f(x)\,\mathrm{d}x = A_S - \frac{b-a}{180} f^{(4)}(c) \cdot h^4$$

[6] A proof of this fact is a topic best left until Analysis. The framework for such a proof may be found on Olmsted, 1956, p. 146.

References

Olmsted, J. M. H. (1956). *Intermediate analysis: An introduction to the theory of functions of one real variable*. New York: Appleton-Century-Crofts.

Thomas, G. B., Jr. (1972). *Calculus and analytic geometry* (fourth). Addison-Wesley.