Chapter 20

Differential Equations

20.1 Introduction

9/9:

- **Differential equation**: An equation that involves one or more derivatives, or differentials.
- **Type** (of a differential equation): A differential equation is either an **ordinary differential equation** or a **partial differential equation**.
- **Order** (of a differential equation): The order of the highest-order derivative that occurs in the equation.
- **Degree** (of a differential equation): The exponent of the highest power of the highest-order derivative, after the equation has been cleared of fractions and radicals in the dependent variable and its derivatives.
- **Ordinary differential equation**: A differential equation where the only derivatives that appear are those of a dependent variable y varying as a function of a single independent variable x.
- **Partial differential equation**: A differential equation where partial derivatives appear.
- "For example,

$$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)^2+\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^5+\frac{y}{x^2+1}=\mathrm{e}^x$$

  is an ordinary differential equation, of order three and degree two" (Thomas, 1972, p. 691).
- Thomas (1972) does not include a systematic treatment of partial differential equations, so he recommends Chapter 10 of Kaplan (1952).
- If A,B,C are radioactive substances such that A decomposes into B at a rate proportional to the amount of A present (proportionality constant k_1), B decomposes into C at a rate proportional to the amount of A present (proportionality constant k_2), and C decomposes into A at a rate proportional to the amount of C present (proportionality constant k_3), we can write the following system of differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-k_1x+k_3z \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=k_1x-k_2y \qquad \frac{\mathrm{d}z}{\mathrm{d}t}=k_2y-k_3z$$

  - Since $\mathrm{d}x/\mathrm{d}t+\mathrm{d}y/\mathrm{d}t+\mathrm{d}z/\mathrm{d}t=0$ in this case, our solution is that $x+y+z=C$, i.e., the statement that if the amounts of substances change in this manner, the total amount of substance present remains constant.

20.2 Solutions

- **Solution** (of a differential equation): A function $y = f(x)$ such that the differential equation is identically satisfied when y and its derivatives are replaced throughout by $f(x)$ and its corresponding derivatives.
- Differential equations often have solutions in which certain arbitrary constants occur.
  - However, these constants can often be resolved into specific values with the help of initial conditions.
  - In fact, "a differential equation of order n will in general possess a solution involving n arbitrary constants" (Thomas, 1972, p. 693).
    - There is a more precise theorem that implies this result, but Thomas (1972) neither states nor proves it.
- **General solution**: A solution that still contains all arbitrary constants arising from the solving process.
- Since finding general solutions requires calculus and finding specific solutions from general solutions and initial conditions only requires algebra, we will focus on finding general solutions.
- Note that this is only an introduction; for a more exhaustive treatment of differential equations, refer to Martin and Reissner (1961).

20.3 First-Order Equations with Variables Separable

- If it is possible to collect all y-terms with $\mathrm{d}y$ and all x-terms with $\mathrm{d}x$, i.e., if it is possible to write the equation in the form
$$f(y)\,\mathrm{d}y + g(x)\,\mathrm{d}x = 0$$
then the general solution is
$$\int f(y)\,\mathrm{d}y + \int g(x)\,\mathrm{d}x = C$$
where C is an arbitrary constant.

20.4 First-Order Homogeneous Equations

- **Homogeneous** (differential equation): A differential equation that can be put into the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(y/x)$$
where $F(y/x)$ is some function of y/x.
- Solving a first-order homogeneous differential equation.
  - We use u-substitution.
  - If we let $u = y/x$, then $\mathrm{d}y/\mathrm{d}x = u + x\ \mathrm{d}u/\mathrm{d}x$ by the product rule.
  - Thus, the differential equation can be rewritten in the form $u + x\ \mathrm{d}u/\mathrm{d}x = F(u)$, which can be solved in terms of u and x via separation of variables as follows.
$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}u}{u - F(u)} = 0$$
  - Once the above is solved, we can return the substitution to obtain our final solution.

- "Show that the equation $(x^2 + y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0$ is homogeneous, and solve it" (Thomas, 1972, p. 694).
  - Via basic algebra, we can rewrite the above in the form

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x^2 + y^2}{2xy}\\ &= -\frac{1 + (y/x)^2}{2(y/x)}\end{aligned}$$

    implying that it is homogeneous with $F(u) = -(1 + u^2)/(2u)$.
  - Therefore, the only remaining task is to solve

$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}u}{u - [-(1 + u^2)/(2u)]} = 0$$

    via separation of variables integration.
  - After doing so, we obtain

$$\begin{aligned}\ln|x| + \frac{1}{3}\ln\bigl(1 + 3u^2\bigr) &= \frac{1}{3}\ln C\\ x^3(1 + 3u^2) &= \pm C\end{aligned}$$

  - Therefore, returning our substitution, we have that

$$x(x^2 + 3y^2) = C$$

    is our solution.

20.5 First-Order Linear Equations

- **Degree** (of a term in a differential equation): The sum of the exponents of the dependent variable and any of its derivatives in a given term.
  - For example, $(\,\mathrm{d}^2y/\mathrm{d}x^2\,)$ is of degree one but $y(\,\mathrm{d}y/\mathrm{d}x\,)$ is of degree two.
- **Linear differential equation**: A differential equation such that every term is of degree zero or degree one.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

  - A linear differential equation of first order can always be put into the above form, where P, Q are functions of x.
- **Integrating factor**: A function ρ of the independent variable x such that if the differential equation at hand is multiplied by ρ, it will compress into a form that is easier to integrate.
  - For first-order linear differential equations, the function ρ that we seek makes the left-hand side becomes the derivative of the product ρy.
- Deriving ρ in terms of the values given in a general first-order linear differential equation.
  - Multiplying by ρ, we have

$$\rho\frac{\mathrm{d}y}{\mathrm{d}x} + \rho Py = \rho Q$$

- Thus, since we want
$$\rho\frac{\mathrm{d}y}{\mathrm{d}x}+\rho Py=\frac{\mathrm{d}}{\mathrm{d}x}(\rho y)=\rho\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}\rho}{\mathrm{d}x}y$$
we must have by comparing terms that
$$\frac{\mathrm{d}\rho}{\mathrm{d}x}=\rho P$$
- It follows by separation of variables integration that
$$\begin{aligned}\frac{\mathrm{d}\rho}{\rho}&=P\,\mathrm{d}x\\ \ln|\rho|&=\int P\,\mathrm{d}x+\ln C\\ \rho&=\pm C\mathrm{e}^{\int P\mathrm{d}x}\\ \rho&=\mathrm{e}^{\int P\mathrm{d}x}\end{aligned}$$
  - Note that we can choose $\pm C=1$ since in the equation $\rho\,\mathrm{d}y/\mathrm{d}x+\rho Py=\rho Q$, any C term can be divided out of both sides anyways.

- With the help of the integrating factor, we can now derive the general solution to a first-order linear differential equation as follows.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}+Py&=Q\\ \rho\frac{\mathrm{d}y}{\mathrm{d}x}+\rho Py&=\rho Q\\ \frac{\mathrm{d}}{\mathrm{d}x}(\rho y)&=\rho Q\\ \rho y&=\int\rho Q\,\mathrm{d}x+C\\ y&=\frac{1}{\mathrm{e}^{\int P\mathrm{d}x}}\left(\int\mathrm{e}^{\int P\mathrm{d}x}Q\,\mathrm{d}x+C\right)\end{aligned}$$
- Note that a first-order linear differential equation may also be separable, or homogeneous. In such cases, we have a choice of solution methods.

20.6 First-Order Equations With Exact Differentials

- Refer to Section 15.13 for the method of solving exact differentials.
- Every first-order differential equation $P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y=0$ can be made exact by multiplication by an integrating factor $\rho(x,y)$ having the property that
$$\frac{\partial}{\partial y}[\rho(x,y)P(x,y)]=\frac{\partial}{\partial x}[\rho(x,y)Q(x,y)]$$
  - "It is not easy to determine ρ from this equation. However, one can often recognize certain combinations of differentials that can be made exact by 'ingenious devices'" (Thomas, 1972, p. 696).
- For example, consider $x\,\mathrm{d}y-y\,\mathrm{d}x=xy^2\,\mathrm{d}x$.
  - We may solve this differential equation by recognizing that it can be rewritten as
$$-x\,\mathrm{d}x=\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2}=\mathrm{d}(x/y)$$
  - Alternatively, we can multiply by the integrating factor $1/y^2$.

20.7 Special Types of Second-Order Equations

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- "Certain types of second-order differential equations, of which the general form is

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$$

can be reduced to first-order equations by a suitable change of variables" (Thomas, 1972, p. 697).

- Equations with the dependent variable missing:

$$F\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$$

  - Equations of the above form can be reduced to a first-order equation by substituting $p = \mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}p/\mathrm{d}x = \mathrm{d}^2y/\mathrm{d}x^2$.
  - Then if we can solve for $p(x, C_1)$, we can integrate again to solve for y.

- Equations with the independent variable missing:

$$F\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$$

  - Equations of the above form can be reduced to a first-order equation by substituting $p = \mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}^2y/\mathrm{d}x^2 = p\,\mathrm{d}p/\mathrm{d}x$.
  - Then if we solve for $p(y, C_1)$, we can further solve for y.

- For example, consider

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$$

  - By the second method above, we get

$$\begin{aligned} 0 &= p\frac{\mathrm{d}p}{\mathrm{d}y} + y \\ &= p\,\mathrm{d}p + y\,\mathrm{d}y \\ \frac{C_1^2}{2} &= \frac{p^2}{2} + \frac{y^2}{2} \end{aligned}$$

  - Returning the substitutions, we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\sqrt{C_1^2 - y^2} \\ \frac{\mathrm{d}y}{\sqrt{C_1^2 - y^2}} &= \pm\,\mathrm{d}x \\ \sin^{-1}\left(\frac{y}{C_1}\right) &= \pm(x + C_2) \\ y &= \pm C_1 \sin(x + C_2) \end{aligned}$$

20.8 Linear Equations With Constant Coefficients

- **Linear differential equation of order n**: An equation of the form

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + a_2\frac{\mathrm{d}^{n-2}y}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} + a_ny = F(x)$$

which is linear in y and its derivatives, where $a_1, \dots, a_n$ are functions of x.

- **Homogeneous** (linear differential equation of order n): A linear differential equation of order n such that $F(x) = 0$.
- **Nonhomogeneous** (linear differential equation of order n): A linear differential equation of order n that is not homogeneous, i.e., such that $F(x) \neq 0$.
- Let $Df(x) = (\mathrm{d}/\mathrm{d}x)f(x)$. We similarly denote higher order derivatives with $D^n f(x) = (\mathrm{d}^n/\mathrm{d}x^n)f(x)$.
- **Linear differential operator**: A polynomial in D, meant to be interpreted as an operator which, when applied to $f(x)$, produces a linear combination of f and its successive derivatives. *Denoted by* $\boldsymbol{L}$.
  - For example, $(D^2 + D - 2)f(x) = \mathrm{d}^2 f(x)/\mathrm{d}x^2 + \mathrm{d}f(x)/\mathrm{d}x - 2f(x)$.
- Linear differential operators are additive and multiplicative:

$$(L_1 + L_2)f(x) = L_1 f(x) + L_2 f(x) \qquad\qquad L_1 L_2 f(x) = L_1(L_2 f(x))$$

  - Thus, they satisfy basic algebraic laws, so we may express $D^2 - D - 2$ as $(D+2)(D-2)$ for example, making the differential equation easier to solve.

20.9 Homogeneous Linear Second-Order Differential Equations With Constant Coefficients

- **Characteristic equation** (of a homogeneous linear differential equation with constant coefficients): The polynomial obtained by substituting r^n for each $\mathrm{d}^n y/\mathrm{d}x^n$ term. Note that we substitute $1 = r^0 = \mathrm{d}^0 y/\mathrm{d}x^0 = y$ for y, as well.
- Suppose we wish to solve an equation of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2a\frac{\mathrm{d}y}{\mathrm{d}x} + by = 0$$

  - Then in the notation of the last section, $(D^2 + 2aD + b)y = (D - r_1)(D - r_2)y = 0$, where r_1, r_2 are the roots of the polynomial $r^2 + 2ar + b$.
  - Thus, we can solve this differential equation by solving the sub-equations

$$(D - r_2)y = u \qquad\qquad (D - r_1)u = 0$$

  - It follows that $u = C_1 \mathrm{e}^{r_1 x}$, meaning that to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - r_2 y = C_1 \mathrm{e}^{r_1 x}$$

    as a first-order homogeneous equation with integrating factor $\rho = \mathrm{e}^{-r_2 x}$, we need to divide into two cases ($r_1 \neq r_2$ and $r_1 = r_2$).
    - If $r_1 \neq r_2$, then
$$y = C_1 \mathrm{e}^{r_1 x} + C_2 \mathrm{e}^{r_2 x}$$
    - If $r_1 = r_2$, then
$$y = (C_1 x + C_2)\mathrm{e}^{r_2 x}$$
- Imaginary roots.
  - Suppose the polynomial has roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$.
  - If $\beta = 0$, then the roots are real and we just use the $r_1 = r_2$ formula above.

- However, if $\beta \neq 0$, then the roots are complex and we can modify the $r_1 \neq r_2$ solution above into a more convenient form.

$$\begin{aligned}
y &= c_1 \mathrm{e}^{(\alpha+i\beta)x} + c_2 \mathrm{e}^{(\alpha-i\beta)x} \\
&= \mathrm{e}^{\alpha x}[c_1 \mathrm{e}^{i\beta x} + c_2 \mathrm{e}^{-i\beta x}] \\
&= \mathrm{e}^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\
&= \mathrm{e}^{\alpha x}[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x] \\
&= \mathrm{e}^{\alpha x}[C_1\cos\beta x + C_2\sin\beta x]
\end{aligned}$$

  - Note that C_1 and C_2 will generally be real.

20.10 Nonhomogeneous Linear Second-Order Differential Equations With Constant Coefficients

- Suppose we wish to solve an equation of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2a\frac{\mathrm{d}y}{\mathrm{d}x} + by = F(x)$$

  - First, find the general solution y_h to the homogeneous form of the above equation, $\mathrm{d}^2y/\mathrm{d}x^2 + 2a\ \mathrm{d}y/\mathrm{d}x + by = 0$, i.e.,

$$y_h = C_1 u_1(x) + C_2 u_2(x)$$

    where u_1, u_2 equal the appropriate corresponding functions, as derived in Section 20.9.
  - There are two possibilities from here.
  - Inspection.
    - We can find by inspection one particular function $y = y_p(x)$ which satisfies the original differential equation, in which case

$$y = y_h(x) + y_p(x)$$

  - Variation of parameters.
    - Replace C_1, C_2 with v_1, v_2, which we let be functions of x.
    - The goal is now just to impose conditions that allow us to solve for v_1, v_2.
    - As a first condition, require that

$$v_1' u_1 + v_2' u_2 = 0$$

    - Thus, using the substitutions of v_1, v_2 and the above condition, we have that

$$y = v_1u_1 + v_2u_2 \qquad \begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= v_1u_1' + v_1'u_1 + v_2u_2' + v_2'u_2 \\ &= v_1u_1' + v_2u_2'\end{aligned} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = v_1u_1'' + v_1'u_1' + v_2u_2'' + v_2'u_2'$$

    - It follows by substitution into the original equation that

$$\begin{aligned}
F(x) &= (v_1u_1'' + v_1'u_1' + v_2u_2'' + v_2'u_2') + 2a(v_1u_1' + v_2u_2') + b(v_1u_1 + v_2u_2) \\
&= v_1\left(\frac{\mathrm{d}^2u_1}{\mathrm{d}x^2} + 2a\frac{\mathrm{d}u_1}{\mathrm{d}x} + bu_1\right) + v_2\left(\frac{\mathrm{d}^2u_2}{\mathrm{d}x^2} + 2a\frac{\mathrm{d}u_2}{\mathrm{d}x} + bu_2\right) + v_1'u_1' + v_2'u_2'
\end{aligned}$$

    - But since u_1, u_2 are solutions to the homogeneous form of the original equation, the terms in the parentheses above equal zero.
    - Thus, as a second condition, we require that

$$v_1'u_1' + v_2'u_2' = F(x)$$

- In summary, to solve for v_1, v_2 after making the appropriate substitutions, solve

$$v_1'u_1 + v_2'u_2 = 0 \qquad\qquad v_1'u_1' + v_2'u_2' = F(x)$$

as a two-variable system of equations in v_1', v_2'. Then recover v_1, v_2 with integration.

- For example, consider the following equation.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 6$$

  - The characteristic equation of the homogeneous form is $r^2 + 2r - 3 = (r-1)(r+3) = 0$.
  - Thus, $y_h = C_1\mathrm{e}^{-3x} + C_2\mathrm{e}^x$.
  - Substitute $u_1(x) = \mathrm{e}^{-3x}$ and $u_2(x) = \mathrm{e}^x$.
  - Consequently, our two-variable system of equations is

$$v_1'\mathrm{e}^{-3x} + v_2'\mathrm{e}^x = 0 \qquad\qquad v_1'(-3\mathrm{e}^{-3x}) + v_2'\mathrm{e}^x = 6$$

  - Solving the above yields

$$v_1' = -\frac{3}{2}\mathrm{e}^{3x} \qquad\qquad v_2' = \frac{3}{2}\mathrm{e}^{-x}$$

  - It follows that

$$v_1 = \int -\frac{3}{2}\mathrm{e}^{3x}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{3x} + c_1 \qquad\qquad v_2 = \int \frac{3}{2}\mathrm{e}^{-x}\,\mathrm{d}x = -\frac{3}{2}\mathrm{e}^{-x} + c_2$$

  - Therefore, we have that

$$\begin{aligned}
y &= v_1u_1 + v_2u_2 \\
&= \left(-\frac{1}{2}\mathrm{e}^{3x} + c_1\right)\mathrm{e}^{-3x} + \left(-\frac{3}{2}\mathrm{e}^{-x} + c_2\right)\mathrm{e}^x \\
&= -2 + c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^x
\end{aligned}$$

20.11 Higher-Order Linear Differential Equations With Constant Coefficients

- The methods of Sections 20.9-20.10 extend to equations of higher order.
- If the roots $r_1, \dots, r_n$ of the characteristic polynomial are all distinct, then the solution of the homogenous equation is

$$y_h = c_1\mathrm{e}^{r_1x} + \cdots + c_n\mathrm{e}^{r_nx}$$

- If the roots of the characteristic polynomial are not all distinct, then the portion of y_h corresponding to a root r of multiplicity m is to be replaced by

$$(C_1x^{m-1} + C_2x^{m-2} + \cdots + C_m)\mathrm{e}^{rx}$$

- If the general solution of the homogeneous equation is $y_h = C_1u_1 + \cdots + C_nu_n$, then $y = v_1u_1 + \cdots + v_nu_n$ will be a solution of the nonhomogeneous equation if and only if

$$\begin{aligned}
v_1'u_1 + \cdots + v_n'u_n &= 0 \\
v_1'\frac{\mathrm{d}u_1}{\mathrm{d}x} + \cdots + v_n'\frac{\mathrm{d}u_n}{\mathrm{d}x} &= 0 \\
&\vdots \\
v_1'\frac{\mathrm{d}^{n-2}u_1}{\mathrm{d}x^{n-2}} + \cdots + v_n'\frac{\mathrm{d}^{n-2}u_n}{\mathrm{d}x^{n-2}} &= 0 \\
v_1'\frac{\mathrm{d}^{n-1}u_1}{\mathrm{d}x^{n-1}} + \cdots + v_n'\frac{\mathrm{d}^{n-1}u_n}{\mathrm{d}x^{n-1}} &= F(x)
\end{aligned}$$

- These equations can be solved for $v_1', \dots, v_n'$ and then integrated.

20.12 Vibrations

- Thomas (1972) uses differential equations and Newton's Laws to derive the undamped and damped motion of a system vibrating under a linear restoring force.