Chapter 1

The Rate of Change of a Function

1.1 Introduction

7/3: • Discusses the importance of calculus, when it should be used, and why one should study it.

- **Analytic geometry**: "Uses algebraic methods and equations to study geometric problems. Conversely, it permits us to visualize algebraic equations in terms of geometric curves" (Thomas, 1972, p. 2).

1.2 Coordinates

- "The basic idea in analytic geometry is the establishment of a one-to-one correspondence between the points of a plane on the one hand and pairs of numbers (x, y) on the other hand" (Thomas, 1972, p. 2).
- Such a correspondence is most commonly established as follows.

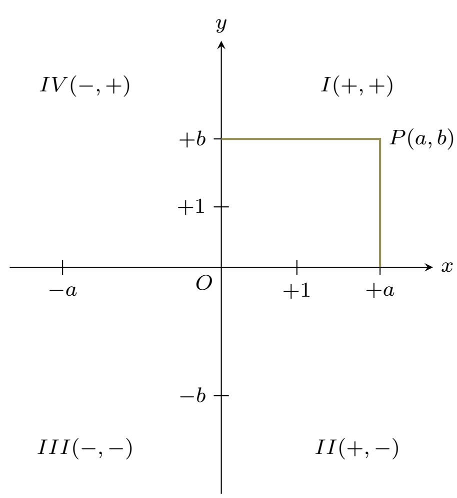


Figure 1.1: Cartesian coordinates.

  - "A horizontal line in the plane, extending indefinitely to the left and to the right, is chosen as the x-axis or axis of **abscissas**. A reference point O on this line and a unit of length are then chosen. The axis is scaled off in terms of this unit of length in such a way that the number zero is attached to O, the number $+a$ is attached to the point which is a units to the right of O, and $-a$ is attached to the symmetrically located point to the left of O. In this way, a one-to-one correspondence is established between points of the x-axis and the set of all **real numbers**" (Thomas, 1972, pp. 2–3).

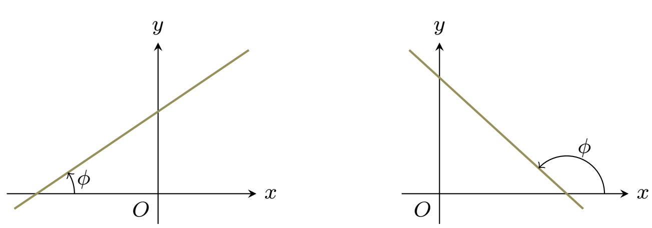
- "Now through O take a second, vertical line in the plane, extending indefinitely up and down. This line becomes the y-axis, or axis of **ordinates**. The unit of length used to represent $+1$ on the y-axis need not be the same as the unit of length used to represent $+1$ on the x-axis. The y-axis is scaled off in terms of the unit of length adopted for it, with the positive number $+b$ attached to the point b units above O and negative number $-b$ attached to the symmetrically located point b units below O" (Thomas, 1972, p. 3).
- "If a line parallel to the y-axis is drawn through the point marked a on the x-axis, and a line parallel to the x-axis is drawn through the point marked b on the y-axis, their point of intersection P is to be labeled $P(a, b)$. Thus, given the pair of real numbers a and b, we find one and only one point with abscissa a and ordinate b, and this point we denote by $P(a, b)$" (Thomas, 1972, p. 3).
- "Conversely, if we start with any point P in the plane, we may draw lines through it parallel to the coordinate axes. If these lines intersect the x-axis at a and the y-axis at b, we then regard the pair of numbers (a, b) as corresponding to the point P. We say that the coordinates of P are (a, b)" (Thomas, 1972, p. 3).
- "The two axes divide the plane into four quadrants, called the first quadrant, second quadrant, and so on, and labeled I, II, III, IV in [Figure 1.1]. Points in the first quadrant have both coordinates positive, and in the second quadrant the x-coordinate (abscissa) is negative and the y-coordinate (ordinate) is positive. The notations $(-, -)$ and $(+, -)$ in quadrants III and IV of [Figure 1.1] represent the signs of the coordinates of points in these quadrants" (Thomas, 1972, p. 3).

1.3 Increments

- **Increments**: The values $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ concerning a particle, the initial position of which is $P_1(x_1, y_1)$ and the terminal position of which is $P_2(x_2, y_2)$.
- If the unit of measurement for both axes is the same, then we may express distances in the plane in terms of this unit using the Pythagorean theorem.

1.4 Slope of a Straight Line

- Let L be a straight line not parallel to the y-axis intersecting distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Then L has a **rise**, **run**, and **slope**.
- **Rise**: The increment Δy.
- **Run**: The increment Δx.
- **Slope**: The rate of rise per run $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. *Also known as* **inclination**.



(a) Negative x-intercept. (b) Positive x-intercept.

Figure 1.2: The slope and the angle of inclination.

- If we chose different distinct points, the slope would be same because the triangles in the Cartesian plane would be similar.
- Δy is proportional to Δx with m as the proportionality factor.
- On interpolation: If we're given the values of a function at (x_1, y_1) and (x_2, y_2), then we may approximate the function by a straight line L passing through those two points and approximate the value $f(x)$ for any $x_1 \leq x \leq x_2$.
- If the scales on both axes are equal, then the slope of L is equal to the tangent of the **angle of inclination** that L makes with the positive x-axis. That is, $m = \tan\phi$ (see Figure 1.2).

- **Parallel** (lines): Two lines with equal inclinations ($m_1 = m_2$).
- **Perpendicular** (lines): Two lines with inclinations that differ by 90° ($m_1 = -\frac{1}{m_2}$).
  - Note that we can prove the relation between the slopes using the angles of inclination as follows.

$$\begin{aligned} m_1 &= \tan\phi_1 \\ &= \tan(\phi_2 + 90^\circ) \\ &= -\cot\phi_2 \\ &= -\frac{1}{\tan\phi_2} \\ &= -\frac{1}{m_2} \end{aligned}$$