

## Chapter 9

# Methods of Integration

### 9.1 Basic Formulas

- 6/30:
- Useful, abstract info (that I already know) on what makes a student good at integrating, e.g., integrating is an exercise in trial-and-error, but there are ways to increase your likelihood of being successful.

### 9.2 Powers of Trigonometric Functions

- When integrating power functions, look for integral/derivative relationships, which may allow you to substitute  $u$  and  $du$  at the same time.
  - For example, when confronted with  $\int \sin^n ax \cos ax \, dx$ , note that  $\cos ax$  is almost the derivative of  $\sin ax$ , and choose  $u = \sin ax$  and  $\frac{du}{a} = \cos ax \, dx$  to yield  $\frac{1}{a} \int u^n \, du$ .
- When integrating power functions, it may be possible to split the exponent into a product ( $u^n = u^a u^b$  where  $a + b = n$ ) and work off of properties of one of the functions raised to a smaller exponent ( $u^a$  may have properties that  $u^n$  lacks).
  - For example, when confronted with  $\int \sin^3 x \, dx$ , recall that  $\sin^2 x$  has Pythagorean properties, and split the exponent.

$$\begin{aligned}\int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \sin x \, dx\end{aligned}$$

Now we can use the previous property, since  $\sin x$  and  $\cos x$  have an integral/derivative relationship.

$$\begin{aligned}&= - \int (1 - u^2) \, du \\ &= \int (u^2 - 1) \, du\end{aligned}$$

- Note that this technique is applicable whenever an odd power of sine or cosine is to be integrated. For higher powers, consider the following.

$$\int \cos^{2n+1} x \, dx = \int (\cos^2 x)^n \cos x \, dx = \int (1 - \sin^2 x)^n \cos x \, dx = \int (1 - u^2)^n \, du$$

Remember that  $(1 - u^2)^n$  can be expanded via the binomial theorem.

- When integrating a composite trigonometric function, consider reducing it to a radical of powers of sines and cosines.
  - For example,  $\sec x \tan x = \frac{\sin x}{\cos^2 x}$ .
- When integrating positive integer powers of  $\tan x$ , use either the base cases or the **reduction formula**.
  - Begin by deriving a reduction formula.

$$\begin{aligned}\int \tan^n x \, dx &= \int \tan^{n-2} x (\sec^2 x - 1) \, dx \\ &= \int \tan^{n-2} \sec^2 x \, dx - \int \tan^{n-2} x \, dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx\end{aligned}$$

Since the reduction formula decreases the exponent by 2, work out two base cases.

$$\begin{aligned}\int \tan^0 x \, dx &= \int dx = x + C \\ \int \tan^1 x \, dx &= \int \frac{\sin x}{\cos x} \, dx = - \int \frac{du}{u} = -\ln |\cos x| + C\end{aligned}$$

- Note that the impetus for initially deriving such a formula was the search for a way to get  $\sec^2 x$  into the integrand, which can be done by splitting the exponent.
- This method can easily be adjusted to suit negative powers of  $\tan x$  (positive powers of  $\cot x$ ).
- When integrating even powers of  $\sec x$ , either use the secant reduction formula, or split the exponent.
  - We derive the following formula.

$$\begin{aligned}\int \sec^{2n} x \, dx &= \int \sec^{2n-2} x \sec^2 x \, dx \\ &= \int (1 + \tan^2 x)^{n-1} \sec^2 x \, dx \\ &= \int (1 + u^2)^{n-1} \, du\end{aligned}$$

- When integrating secant (or cosecant) alone, produce  $\frac{u'}{u}$  by multiplying the integrand by a clever form of 1.
  - For example,

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \\ &= \ln |\sec x + \tan x| + C\end{aligned}$$

### 9.3 Even Powers of Sines and Cosines

- When integrating the product of sines and cosines raised to powers where at least one exponent is a positive odd integer, split the exponent and use  $u$ -substitution.
  - In effect, we wish to evaluate  $\int \sin^m x \cos^n x \, dx$  where at least one of  $m, n$  is a positive odd integer.

- For example, when confronted with  $\int \cos^{\frac{2}{3}} x \sin^5 x \, dx$ , split the exponent of  $\sin^5 x$  and choose  $u = \cos x$  and  $-du = \sin x \, dx$ .

$$\int \cos^{\frac{2}{3}} x \sin^5 x \, dx = \int \cos^{\frac{2}{3}} x (1 - \cos^2 x)^2 \sin x \, dx = \int u^{\frac{2}{3}} (u^2 - 1) \, du$$

- When integrating the product of sines and cosines raised to powers where both exponents are even integers, begin by transforming it into a sum of either just sines *or* just cosines raised to even integers. Then split the exponents and use one of the following formulas. It may be necessary to use these formulas multiple times. Use them until the problem has been reduced to a sum with only odd exponents.

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \qquad \cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

- Note that “these identities may be derived very quickly by adding or subtracting the equations  $[\cos^2 u + \sin^2 u = 1$  and  $\cos^2 u - \sin^2 u = \cos 2u]$  and by dividing by two” (Thomas, 1972, p. 287).
- For example, when confronted with  $\int \sin^2 x \cos^4 x \, dx$ , begin by changing it to a case with only powers of cosine (chose to eliminate the sine function because it is raised to a lower exponent and, thus, will need less binomial expansion).

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx &= \int (1 - \cos^2 x) \cos^4 x \, dx \\ &= \int \cos^4 x \, dx - \int \cos^6 x \, dx \end{aligned}$$

Now split the exponents.

$$= \int (\cos^2 x)^2 \, dx - \int (\cos^2 x)^3 \, dx$$

Employ the above formulas and use binomial expansion. If necessary, repeat (split the exponents, employ the above formulas, use binomial expansion) until only odd exponents remain (remember that 1 is an odd exponent).

$$\begin{aligned} &= \int \left( \frac{1}{2}(1 + \cos 2x) \right)^2 \, dx - \int \left( \frac{1}{2}(1 + \cos 2x) \right)^3 \, dx \\ &= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) \, dx \\ &\quad - \frac{1}{8} \int (1 + 3 \cos 2x + 3 \cos^2 2x + \cos^3 2x) \, dx \\ &= \frac{1}{4} \int \left( 1 + 2 \cos 2x + \frac{1}{2}(1 + \cos 4x) \right) \, dx \\ &\quad - \frac{1}{8} \int \left( 1 + 3 \cos 2x + \frac{3}{2}(1 + \cos 4x) + \cos^3 2x \right) \, dx \end{aligned}$$

These integrals may now be handled using previously discussed techniques.

## 9.4 Integrals With Terms $\sqrt{a^2 - u^2}$ , $\sqrt{a^2 + u^2}$ , $\sqrt{u^2 - a^2}$ , $a^2 + u^2$ , $a^2 - u^2$

- When integrating a radical that resembles the derivative of an inverse trig function, we may factor out the issue so as to make the integral resemble one of the known formulas.

- For example, when confronted with  $\int \frac{du}{a^2+u^2}$ , divide the  $a^2$  term out of the denominator and integrate with respect to  $\frac{u}{a}$ <sup>1</sup>.

$$\begin{aligned}\int \frac{du}{a^2+u^2} &= \frac{1}{a^2} \int \frac{du}{1+\left(\frac{u}{a}\right)^2} \\ &= \frac{1}{a^2} \int \frac{a d\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2} \\ &= \frac{1}{a} \tan^{-1} \frac{u}{a} + C\end{aligned}$$

- However, this method is partially flawed in that it relies on having memorized the derivatives of the inverse trig functions, i.e., it is not terribly analytical. This shortcoming will now be addressed with a new, more general technique.
- The new method leans heavily on the following three identities.

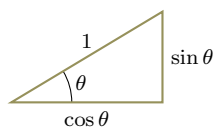
$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

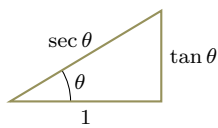
$$\sec^2 \theta - 1 = \tan^2 \theta$$

- With the help of these identities, it is possible to...

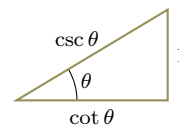
1. use  $u = a \sin \theta$  to replace  $a^2 - u^2$  with  $a^2 \cos^2 \theta$ ;
2. use  $u = a \tan \theta$  to replace  $a^2 + u^2$  with  $a^2 \sec^2 \theta$ ;
3. use  $u = a \sec \theta$  to replace  $u^2 - a^2$  with  $a^2 \tan^2 \theta$ .



(a)  $\cos^2 \theta + \sin^2 \theta = 1$ .

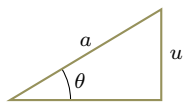


(b)  $1 + \tan^2 \theta = \sec^2 \theta$ .

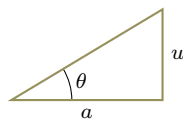


(c)  $\cot^2 \theta + 1 = \csc^2 \theta$ .

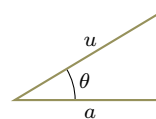
Figure 9.1: Geometric rationale for the trigonometric identities.



(a)  $\sqrt{a^2 - u^2} = a \cos \theta$   
 $u = a \sin \theta$ .



(b)  $\sqrt{u^2 + a^2} = a \sec \theta$   
 $u = a \tan \theta$ .



(c)  $\sqrt{u^2 - a^2} = a \tan \theta$   
 $u = a \sec \theta$ .

Figure 9.2: Geometric rationale for the trigonometric substitutions.

- These identities and substitutions can be easily remembered by thinking of the Pythagorean theorem in conjunction with Figures 9.1 and 9.2, respectively.
- We may now evaluate inverse trig integrals analytically.

<sup>1</sup>Thomas, 1972 uses differentials with more complex functions than a single variable quite often. It's not something I've seen before, but it's something I should get used to (and it does make sense if you think about it — it's just an extension of the underlying concept of separation of variables integration).

- For example, when confronted with  $\int \frac{du}{a^2+u^2}$ , choose  $u = a \tan \theta$  and  $du = a \sec^2 \theta d\theta$ .

$$\begin{aligned} \int \frac{du}{a^2+u^2} &= \int \frac{a \sec^2 \theta d\theta}{a^2 + (a \tan \theta)^2} \\ &= \int \frac{a \sec^2 \theta}{a^2 (1 + \tan^2 \theta)} d\theta \\ &= \frac{1}{a} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta \\ &= \frac{1}{a} \int d\theta \\ &= \frac{1}{a} \theta + C \end{aligned}$$

At this point, solve  $u = a \tan \theta$  for  $\theta$  and substitute.

$$= \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

- Some integrals will simplify to have a plus/minus in the denominator, leading to two possible solutions. However, there are sometimes ways to isolate a single solution.

- For example,  $\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a \cos \theta d\theta}{\pm a \cos \theta} = \pm \theta + C$ . However, when we consider the fact that  $\theta = \sin^{-1} \frac{u}{a}$ , we know that  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  (because inverse sine is not arcsine, and inverse sine is only defined over the principal branch of sine). Thus, since  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ,  $\cos \theta \in [0, 1]$ , i.e., is always positive. Thus, we choose  $\int \frac{du}{\sqrt{a^2-u^2}} = +\theta + C = \sin^{-1} \frac{u}{a} + C$  as our one solution.
- For example,  $\int \frac{du}{\sqrt{u^2-a^2}}$  equals  $\ln \left| \frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a} \right| + C$  or  $-\ln \left| \frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a} \right| + C$  depending on whether  $\tan \theta$  is positive or negative. But it can be shown algebraically that the two solutions are actually equal:

$$\begin{aligned} -\ln \left| \frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a} \right| &= \ln \left| \frac{a}{u - \sqrt{u^2-a^2}} \right| \\ &= \ln \left| \frac{a(u + \sqrt{u^2-a^2})}{(u - \sqrt{u^2-a^2})(u + \sqrt{u^2-a^2})} \right| \\ &= \ln \left| \frac{a(u + \sqrt{u^2-a^2})}{a^2} \right| \\ &= \ln \left| \frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a} \right| \end{aligned}$$

Thus, we have  $\ln \left| \frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a} \right| + C$  as the one solution<sup>[2]</sup>.

- Some integrals will have extraneous constants that can be combined with  $C$  to simplify the *indefinite* integral.
- Continuing with the above example,

$$\begin{aligned} \int \frac{du}{\sqrt{u^2-a^2}} &= \ln \left| \frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a} \right| + C \\ &= \ln \left| u + \sqrt{u^2-a^2} \right| - \ln |a| + C \\ &= \ln \left| u + \sqrt{u^2-a^2} \right| + C \end{aligned}$$

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<sup>2</sup>Note that we could choose to use the other solution, but we choose this one because it's "simpler" (it uses addition instead of subtraction).

- When integrating an inverse trig integral with excess polynomial terms, look to transform it into a (power of a) trig integral problem.
  - For example, when confronted with  $\int \frac{x^2 dx}{\sqrt{9-x^2}}$ , treat it as a case of  $a^2 - u^2$ , but substitute the trig expression into the  $x^2$  term in the numerator, too.

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{3 \cos \theta} = 9 \int \sin^2 \theta d\theta$$

This integral may now be handled using previously discussed techniques.

- Many inverse trig integrals can also be evaluated hyperbolically, making use of the following three identities.

$$\cosh^2 \theta - 1 = \sinh^2 \theta \qquad 1 - \tanh^2 \theta = \operatorname{sech}^2 \theta \qquad 1 + \sinh^2 \theta = \cosh^2 \theta$$

- With the help of these identities, it is possible to...
  1. use  $u = a \tanh \theta$  to replace  $a^2 - u^2$  with  $a^2 \operatorname{sech}^2 \theta$ ;
  2. use  $u = a \sinh \theta$  to replace  $a^2 + u^2$  with  $a^2 \cosh^2 \theta$ ;
  3. use  $u = a \cosh \theta$  to replace  $u^2 - a^2$  with  $a^2 \sinh^2 \theta$ .

## 9.5 Integrals With $ax^2 + bx + c$

- When integrating composite functions where the inner function is a binomial, look to factor said binomial.
  - The general quadratic  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$ , can be reduced to the form  $a(u^2 + B)$  by completing the square and choosing  $u = x + \frac{b}{2a}$  and  $B = \frac{4ac-b^2}{4a^2}$ :

$$\begin{aligned} ax^2 + bx + c &= a \left( x^2 + \frac{b}{a}x \right) + c \\ &= a \left( x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) + c - \frac{b^2}{4a} \\ &= a \left( \left( x + \frac{b}{2a} \right)^2 + \frac{4ac-b^2}{4a^2} \right) \end{aligned}$$

- When integrating the square root of a binomial, or some similarly tricky function of a binomial, we can transform the binomial into a form such that it will suit one of the inverse trig integrals.
  - Since it would lead to complex numbers, we disregard cases where  $a(u^2 + B)$  is negative, i.e., we focus on cases where (1)  $a$  is positive, and (2)  $a, B$  are both negative.
  - That being said, if it is an odd root ( $\sqrt[3]{x}$ ,  $\sqrt[5]{x}$ , etc.), the sign doesn't matter.
  - For example, when confronted with  $\int \frac{(x+1)dx}{\sqrt{2x^2-6x+4}}$ , begin by factoring the binomial<sup>[3]</sup>.

$$2x^2 - 6x + 4 = 2(x^2 - 3x) + 4 = 2 \left( x - \frac{3}{2} \right)^2 - \frac{1}{2} = 2(u^2 - a^2)$$

Note that  $u = x - \frac{3}{2}$  and  $a = \frac{1}{2}$ . We can now return to the integral, which we shall reformulate in terms of  $u$  in its entirety.

$$\int \frac{(x+1)dx}{\sqrt{2x^2-6x+4}} = \int \frac{(u+\frac{5}{2})du}{\sqrt{2(u^2-a^2)}}$$

<sup>3</sup>Note that, in place of inspection, we could use the general form factorization derived above.

Split it into two separate integrals and factor out the constants.

$$= \frac{1}{\sqrt{2}} \int \frac{u \, du}{\sqrt{u^2 - a^2}} + \frac{5}{2\sqrt{2}} \int \frac{du}{\sqrt{u^2 - a^2}}$$

The right integral is a straight-up inverse trig integral. The left one, however, needs something special. It could be dealt with as previously discussed by substituting  $u = a \tan \theta$  for all instances of  $u$  and evaluating it is a more complex trig integral in  $\theta$ . However, for the sake of showing a different technique, we will choose  $z = u^2 - a^2$  and  $\frac{1}{2}dz = u \, du$  and treat it as a power function in  $z$ .

$$\begin{aligned} &= \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{z}} + \frac{5}{2\sqrt{2}} \ln |u + \sqrt{u^2 - a^2}| + C_2 \\ &= \frac{1}{2\sqrt{2}} \int z^{-\frac{1}{2}} dz + \frac{5}{2\sqrt{2}} \ln |u + \sqrt{u^2 - a^2}| + C_2 \\ &= \frac{1}{\sqrt{2}} z^{\frac{1}{2}} + C_1 + \frac{5}{2\sqrt{2}} \ln |u + \sqrt{u^2 - a^2}| + C_2 \end{aligned}$$

Return all of the substitutions and combine the constants of integration.

$$\begin{aligned} &= \sqrt{\frac{u^2 - a^2}{2}} + \frac{5}{2\sqrt{2}} \ln |u + \sqrt{u^2 - a^2}| + C \\ &= \sqrt{\frac{x^2 - 3x + 2}{2}} + \frac{5}{2\sqrt{2}} \ln \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + 2} \right| + C \end{aligned}$$

## 9.6 Integration by the Method of Partial Fractions

- 7/1:
- **Method of Partial Fractions:** The process of “split[ting] a fraction into a sum of fractions having simpler denominators” (Thomas, 1972, p. 294).
  - If we wish to split a rational fraction  $\frac{f(x)}{g(x)}$  into a sum of simpler fractions, then...
    - “The degree of  $f(x)$  should be less than the degree of  $g(x)$ . If this is not the case, we first perform a long division, then work with the remainder term. This remainder can always be put into the required form” (Thomas, 1972, p. 294).
    - “The factors of  $g(x)$  should be known. Theoretically, any polynomial  $g(x)$  with real coefficient can be expressed as a product of real linear and quadratic factors. In practice, it may be difficult to perform the factorization” (Thomas, 1972, p. 294).
  - If  $x - r$  is a linear factor of  $g(x)$  and  $(x - r)^m$  is the highest power of  $x - r$  that divides  $g(x)$ , then, to this factor, assign the sum of  $m$  partial fractions

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}$$

- If  $x^2 + px + q$  is a quadratic factor<sup>[4]</sup> of  $g(x)$  and  $(x^2 + px + q)^n$  is the highest power of  $x^2 + px + q$  that divides  $g(x)$ , then, to this factor, assign the sum of  $n$  partial fractions

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}$$

- Continue, as necessary, to higher degree factors of  $g(x)$  (although this caveat is not addressed by Thomas, 1972).

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<sup>4</sup>A binomial factor, the factoring of which into linear factors would introduce complex numbers.

- Notice how the degree of the polynomial in the numerator of the partial fractions will be at most one less than the degree of the denominator.
- Any rational function integrated via the method of partial fractions can be reduced to the problem of evaluating the following two types of integrals.

$$\int \frac{dx}{(x-r)^h} \qquad \int \frac{(ax+b) dx}{(x^2+px+q)^k}$$

- The left integral, with the substitution  $z = x - r$  and  $dz = dx$ , becomes a power integral.
- The right integral, after completing the square in the denominator, substituting, and splitting into two fractions by the numerator, becomes a pair of inverse trig substitution integrals.
- With the method of partial fractions, there is a new way to integrate  $\sec \theta$ .

$$\begin{aligned} \int \sec \theta d\theta &= \int \frac{d\theta}{\cos \theta} \\ &= \int \frac{\cos \theta d\theta}{\cos^2 \theta} \\ &= \int \frac{dx}{1-x^2} \\ &= \int \frac{0.5}{1+x} dx + \int \frac{0.5}{1-x} dx \\ &= \frac{1}{2} \ln |1+x| - \frac{1}{2} \ln |1-x| + C \\ &= \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C \\ &= \ln \sqrt{\frac{1+\sin \theta}{1-\sin \theta}} + C \end{aligned}$$

- Note that  $\ln \sqrt{\frac{1+\sin \theta}{1-\sin \theta}} + C$  is equivalent to the previously derived form:

$$\begin{aligned} \sqrt{\frac{1+\sin \theta}{1-\sin \theta}} &= \sqrt{\frac{(1+\sin \theta)^2}{1-\sin^2 \theta}} \\ &= \left| \frac{1+\sin \theta}{\cos \theta} \right| \\ &= |\sec \theta + \tan \theta| \end{aligned}$$

## 9.7 Integration by Parts

- This is the second general method of integration (the first being substitution).
- It relies on the following formulas for indefinite and definite integrals, respectively.

$$\int u dv = uv - \int v du + C \qquad \int_{(1)}^{(2)} u dv = uv \Big|_{(1)}^{(2)} - \int_{(1)}^{(2)} v du$$

- The indefinite integral formula can be derived from the differential of a product rule as follows.

$$\begin{aligned} d(uv) &= u dv + v du \\ u dv &= d(uv) - v du \\ \int u dv &= uv - \int v du + C \end{aligned}$$



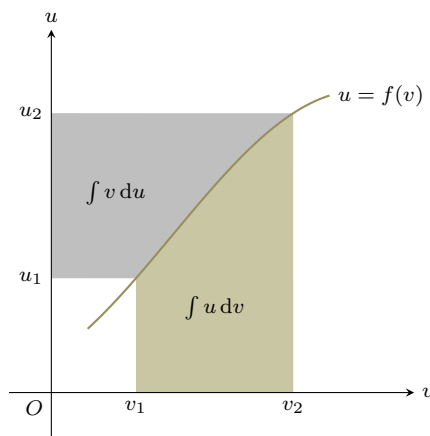


Figure 9.3: Geometric rationale for definite integration by parts.

- The definite integral formula can be thought of as an adjustment of the above, or it can be conceived geometrically: In Figure 9.3,  $\int_{(1)}^{(2)} u \, dv$  is the yellow area, which is clearly equivalent to the total area<sup>[5]</sup>  $uv \Big|_{(1)}^{(2)}$  minus the grey area  $\int_{(1)}^{(2)} v \, du$ .
- Since  $\int dv = v + C_1$ ,  $\int u \, dv$  actually equals  $u(v + C_1) - \int (v + C_1) \, du$ . However, since

$$\begin{aligned} u(v + C_1) - \int (v + C_1) \, du &= uv + C_1 u - \int v \, du - \int C_1 \, du \\ &= uv - \int v \, du \end{aligned}$$

it is customary to drop the first constant of integration.

- That being said, it is sometimes useful — when evaluating  $\int \ln(x+1) \, dx = \ln(x+1)(x + C_1) - \int \frac{x+C_1}{x+1} \, dx$ , being able to choose  $C_1 = 1$  greatly simplifies the second integral.
- When integrating an inverse trig function, consider using integration by parts.
  - For example, when confronted with  $\int \tan^{-1} x \, dx$ , integration by parts turns it into an inverse trig derivative problem.

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x \, dx}{1 + x^2}$$

- When attempting integration by parts, don't be afraid to use it multiple times.
  - For example, when confronted with  $\int x^2 e^x \, dx$ , use integration by parts twice.

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - \left( 2x e^x - \int 2e^x \, dx \right) \end{aligned}$$

- When attempting integration by parts, look for the original integral showing up again — if it does, combine like terms.
- When integrating powers of  $\cos x$ , consider using a reduction formula.

<sup>5</sup>Note that  $uv \Big|_{(1)}^{(2)} = u_2 v_2 - u_1 v_1$ , the latter of which, as the difference of two rectangles, clearly represents the total shaded area.

- Begin by deriving a reduction formula (this will involve splitting the exponent!).

$$\begin{aligned}
 \int \cos^n x \, dx &= \int \cos^{n-1} x \cos x \, dx \\
 &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\
 &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \\
 (1 + (n-1)) \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx \\
 \int \cos^n x \, dx &= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx
 \end{aligned}$$

Since the reduction formula decreases the exponent by 2, work out two base cases.

$$\begin{aligned}
 \int \cos^0 x \, dx &= x + C \\
 \int \cos^1 x \, dx &= \sin x + C
 \end{aligned}$$

- When integrating powers of  $\sin x$ , consider using a reduction formula.

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

## 9.8 Integration of Rational Functions of $\sin x$ and $\cos x$ , and Other Trigonometric Integrals

- “It has been discovered that the substitution  $z = \tan \frac{x}{2}$  enables one to reduce the problem of integrating any rational function of  $\sin x$  and  $\cos x$  to a problem involving a rational function of  $z$ . This in turn can be integrated by the method of partial fractions” (Thomas, 1972, p. 300).

- However, this substitution should be used only as a last resort — the associated algebra is often quite cumbersome.

- To increase the ease of use for this substitution, it will help to derive three results.

$$\cos x = \frac{1 - z^2}{1 + z^2} \qquad \sin x = \frac{2z}{1 + z^2} \qquad dx = \frac{2 \, dz}{1 + z^2}$$

- “The following types of integrals...arise in connection with alternating-current theory, heat transfer problems, bending of beams, cable stress analysis in suspension bridges, and many other places where trigonometric series (or Fourier series) are applied to problems in mathematics, science, and engineering” (Thomas, 1972, p. 301).

$$\int \sin mx \sin nx \, dx \qquad \int \sin mx \cos nx \, dx \qquad \int \cos mx \cos nx \, dx$$

- When confronted with one of these integrals, integration by parts may be used. However, using one of the following three identities will be more simple.

$$\begin{aligned}
 \sin mx \sin nx &= \frac{1}{2}(\cos(m-n)x - \cos(m+n)x) \\
 \sin mx \cos nx &= \frac{1}{2}(\sin(m-n)x + \sin(m+n)x) \\
 \cos mx \cos nx &= \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)
 \end{aligned}$$

- Note that “these identities follow at once from  $\cos(A + B) = \cos A \cos B - \sin A \sin B$ ,  $\cos(A - B) = \cos A \cos B + \sin A \sin B$ , and  $\sin(A + B) = \sin A \cos B + \cos A \sin B$ ,  $\sin(A - B) = \sin A \cos B - \cos A \sin B$ ” (Thomas, 1972, p. 301).

## 9.9 Further Substitutions

- “Some integrals involving fractional powers of the variable  $x$  may be simplified by substituting  $x = z^n$ , where  $n$  is the least common multiple of the denominators of the exponents” (Thomas, 1972, p. 302).
  - For example,  $\int \frac{\sqrt{x} dx}{1 + \sqrt[4]{x}}$  can be simplified by taking  $x = z^4$ .
- “Even when it is not clear at the start that a substitution will work, it is advisable to try one that seems reasonable and pursue it until it either gives results or appears to make matters worse. In the latter case, try something else! Sometimes a chain of substitutions  $u = f(x)$ ,  $v = g(u)$ ,  $z = h(v)$ , and so on, will produce results when it is by no means obvious that this will work” (Thomas, 1972, p. 302).
- “The criterion of success is whether the new integrals so obtained appear to be simpler than the original integral. Here it is handy to remember that any rational function of  $x$  can be integrated by the method of partial fractions and that any rational function of  $\sin x$  and  $\cos x$  can be integrated by using the substitution  $z = \tan \frac{x}{2}$ . If we can reduce a given integral to one of these types, we then know how to finish the job” (Thomas, 1972, p. 302).
- To evaluate a definite integral after a (series of) substitution(s), either return the substitution(s) and keep the bounds or keep the substitution(s) and determine new bounds based on the new variable of integration.
  - For example, if  $z^2 = \frac{1+x}{1-x}$  and  $x \in [-1, 1]$ , then  $z \in [0, \infty)$ . And if  $z = \tan \theta$ , then  $\theta \in [0, \frac{\pi}{2}]$ .

## 9.10 Improper Integrals

- **Improper integral:** An integral of the form  $\int_a^b f(x) dx$  where some  $x \in [a, b]$  is infinite, and/or one or both of  $a, b$  are infinite in magnitude.

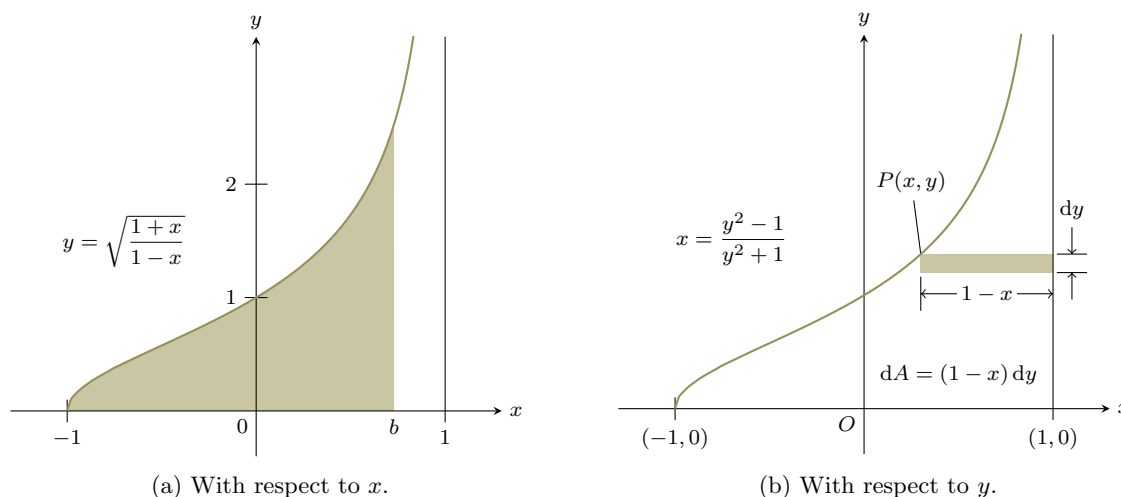


Figure 9.4: Defining improper integrals.

- Say we wish to evaluate  $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx$ , knowing that the integrand approaches  $\infty$  as  $x \rightarrow 1$  (see Figure 9.4a). Well, if the upper bound  $b$  is some value slightly *less than* 1, we *can* evaluate the integral. Thus evaluating the original integral becomes a problem of evaluating

$$\lim_{b \rightarrow 1^-} \int_{-1}^b \sqrt{\frac{1+x}{1-x}} dx = \lim_{b \rightarrow 1^-} \left( \sin^{-1} x - \sqrt{1-x^2} \right)_{-1}^b = \lim_{b \rightarrow 1^-} \left( \sin^{-1} b - \sqrt{1-b^2} + \frac{\pi}{2} \right)$$

- Sometimes such a limit will converge. Sometimes it will not (it will diverge). Either way, it answers the question of the nature of the area under the curve (by yielding some finite value, or the infinite one).
- Note that the integral works out just the same if we sum vertical elements instead (see Figure 9.4b), evaluating the following.

$$\lim_{c \rightarrow \infty} \int_0^c (1-x) dy = \lim_{c \rightarrow \infty} \int_0^c \frac{2 dy}{y^2 + 1}$$

- When integrating a function  $f(x)$  on  $[a, b]$  where  $f(x) \rightarrow \infty$  at some  $x$ -value  $c \in (a, b)$ , split the integral.

$$\int_a^b f(x) dx = \lim_{c \rightarrow c^-} \int_a^c f(x) dx + \lim_{c \rightarrow c^+} \int_c^b f(x) dx$$

- On determining whether or not an improper integral with a nonintegrable integrand exists, we can sometimes compare it with an integral that we know.

- For example,

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx$$

for all  $b \geq 1$  since  $0 < e^{-x^2} \leq e^{-x}$  for all  $x \geq 1$ . Thus,  $\int_1^\infty e^{-x^2} dx$  evaluates to some finite value.

- Note that some improper integrals diverge by oscillation.

- For example,  $\int_0^\infty \cos x dx$  diverges in this manner.

## 9.11 Numerical Methods for Approximating Definite Integrals

- One could use the **trapezoidal rule**.
- A better choice, though, is **Simpson's rule**.

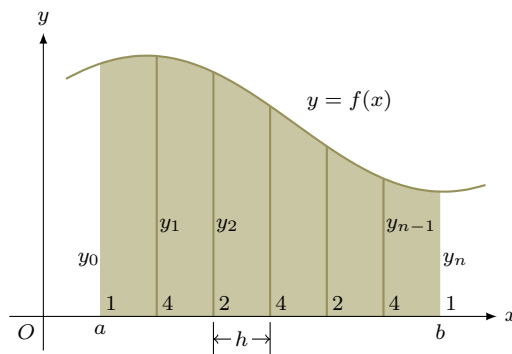


Figure 9.5: Simpson's rule.

- Simpson’s rule approximates the curve via parabolas, which can be uniquely defined by three points and have a nice formula for the area underneath them.
- We now derive Simpson’s rule.
  - Suppose we wish to approximate the area under the part of the curve from  $x_i$  to  $x_{i+2}$  in Figure 9.5. We know that there exists some parabola intersecting  $(x_i, y_0)$ ,  $(x_{i+1}, y_1)$ , and  $(x_{i+2}, y_2)$ . However, for the sake of simplifying the algebra, we choose to consider the parabola intersecting  $(-h, y_0)$ ,  $(0, y_1)$ , and  $(h, y_2)$  (the area under both parabolas will be equivalent since  $x_{i+1} - x_i = h$ ). Let this translated parabola be called  $Ax^2 + Bx + C$  for some  $A, B, C \in \mathbb{R}$ . Then the area underneath this parabola  $A_p$  can be described by the following.

$$\begin{aligned} A_p &= \int_{-h}^h (Ax^2 + Bx + C) dx \\ &= \frac{2Ah^3}{3} + 2Ch \end{aligned}$$

- We know that the area underneath this parabola is dependent only on  $h, y_0, y_1$ , and  $y_2$ . Thus, we look to express  $A, C$  from the integral in terms of  $h, y_0, y_1, y_2$ . To accomplish this, we will use the facts that

$$\begin{aligned} y_0 &= Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C \end{aligned}$$

We can now see that  $C = y_1$ , so all that’s left is to solve for  $A$ . This can be done by adding the first and third equations, substituting, and solving as follows.

$$\begin{aligned} y_0 + y_2 &= 2Ah^2 + 2C \\ 2Ah^2 &= y_0 + y_2 - 2y_1 \\ A &= \frac{y_0 - 2y_1 + y_2}{2h^2} \end{aligned}$$

- Thus, we can reformulate the area under the parabola as follows.

$$\begin{aligned} A_p &= \frac{2h^3}{3} \cdot \frac{y_0 - 2y_1 + y_2}{2h^2} + 2y_1h \\ &= \frac{h}{3}(y_0 + 4y_1 + y_2) \end{aligned}$$

- “Simpson’s rule follows from applying this result to successive pieces of the curve  $y = f(x)$  between  $x = a$  and  $x = b$ . Each separate piece of the curve, covering an  $x$ -subinterval of width  $2h$ , is approximated by an arc of a parabola through its ends and its mid-point. The area under each parabolic arc is then given by an expression like [the above] and the results are added to give [the following]” (Thomas, 1972, p. 309).

$$\begin{aligned} A_S &= \frac{h}{3}((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$

- Note that the number of subdivisions must be an even integer.

- To find the error in a Simpson’s rule approximation, use the fact<sup>[6]</sup> that “if  $f$  is continuous on  $[a, b]$  and four times differentiable on  $(a, b)$ , then there is a number  $c$  between  $a$  and  $b$  such that [the following holds]” (Thomas, 1972, p. 310).

$$\int_a^b f(x) dx = A_S - \frac{b-a}{180} f^{(4)}(c) \cdot h^4$$

<sup>6</sup>A proof of this fact is a topic best left until Analysis. The framework for such a proof may be found on Olmsted, 1956, p. 146.