Chapter 15

Partial Differentiation

15.1 Functions of Two or More Variables

12/16:

- **Function** (from D to E^1): A mapping that assigns a unique number w to each point $(x_1, \dots, x_n) \in D \subset E^n$.
  - We write $w = f(x_1, \dots, x_n)$ and say that w is the value of the function f at $(x_1, \dots, x_n)$.
- **Continuous** (function $f(x, y)$): A function $f(x, y)$ such that $w \to w_0 = f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$.

15.2 The Directional Derivative: Special Cases

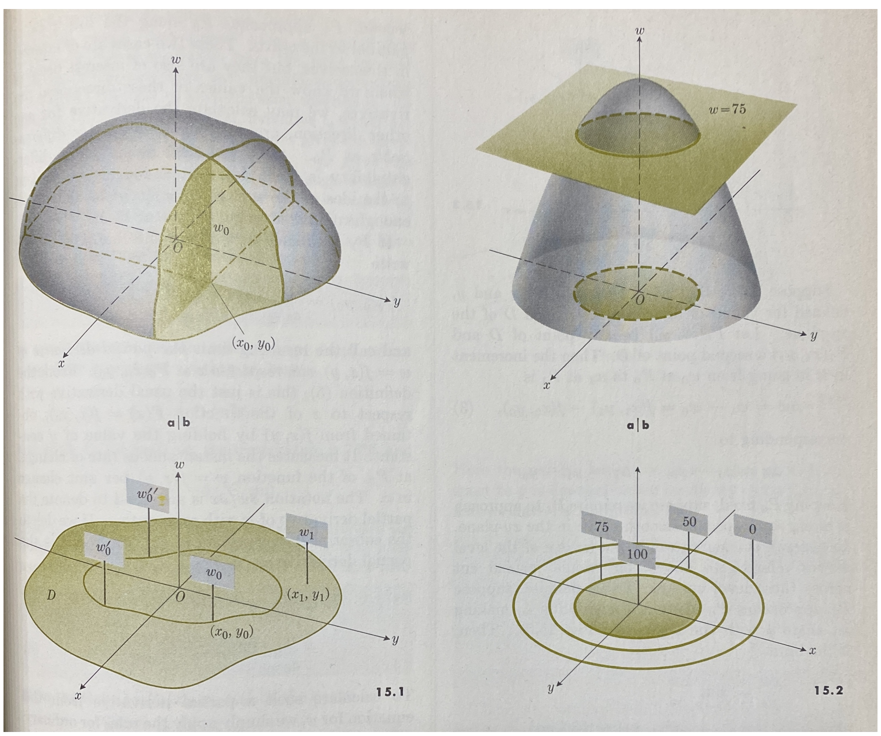

Figure 15.1: Surface plots and contour maps of 2D functions.

- The equation $w = f(x, y)$ can be interpreted as representing a surface in xyw-space, or as a base region D in the xy-plane with a marker bearing a corresponding w-value attached to each point.

- To introduce order into the second interpretation, we can construct a **contour map** with a number of **contour curves**.

- **Contour curve**: A curve consisting of points $(x, y) \in D$ with equal w-values.
  - The formula for such a curve can be derived by setting $w_0 = f(x, y)$, where $w_0 \in R_f$.

- **Directional derivative** (of $f(x, y)$ at (x_0, y_0) in the ϕ-direction): The limit

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s} = \lim_{P_1 \to P_0} \frac{f(x_1, y_1) - f(x_0, y_0)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

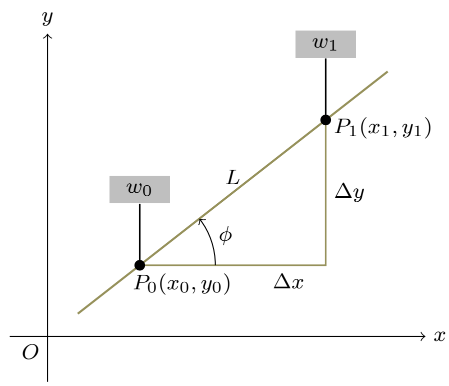

Figure 15.2: The directional derivative.

  - Basically, we let P_1 approach P_0 along a smooth curve (the line L connecting P_1 and P_0 for simplicity and to be definite; L makes an angle ϕ with the x-axis) and watch how $\Delta w = w_1 - w_0 = f(x_1, y_1) - f(x_0, y_0)$, $\Delta x = x_1 - x_0$, and $\Delta y = y_1 - y_0$ change.
  - Note that the directional derivative does depend on the *direction* from which P_1 approaches P_0, not just the absolute distance between P_1 and P_0.

- We now consider two special cases: When "P_1 approaches P_0 along the line $y = y_0$ parallel to the x-axis, [and when] P_1 approaches P_0 along the line $x = x_0$ parallel to the y-axis" (Thomas, 1972, p. 498).
  - These cases are important because if $f(x, y)$ is **differentiable** at P_0, we can calculate the directional derivative in any direction from them.

- **Partial derivative** (of $f(x, y)$ with respect to x at $P_0(x_0, y_0)$): The value

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

  - Essentially, this is the derivative with respect to x of the function $g(x) = f(x, y)$ with y held constant.
  - It measures "the instantaneous rate of change, at P_0, of the function $[f(x, y)]$ per unit change in x" (Thomas, 1972, p. 498).

- **Partial derivative** (of $w = f(x, y)$ with respect to x): The function

$$\frac{\partial w}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

  - To evaluate this, we apply the ordinary rules of differentiation, treating y as a constant.

- In either of the partial derivative definitions, Δx can be positive or negative. However, if we take the directional derivative in the positive x direction (for example), then Δx in the partial derivative definitions can only be positive.
  - Similarly, if f_x exists, it gives the directional derivative in the positive x-direction, whereas $-f_x$ is the directional derivative in the negative x-direction.

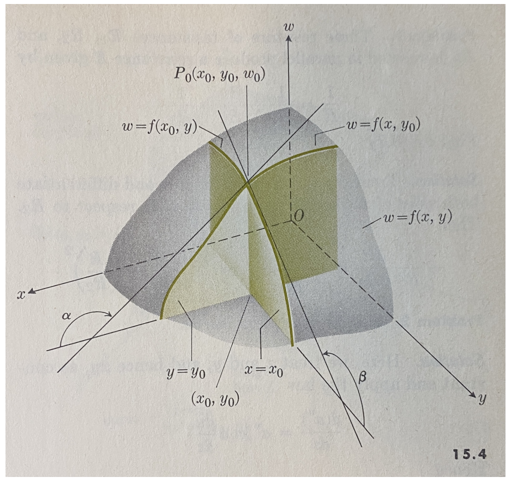

Figure 15.3: Geometric interpretation of the partial derivative.

- As in Figure 15.3, the geometric interpretation of the partial derivative (wrt. x) at a point $P(x_0, y_0, w_0)$ is as the slope of the curve $f(x, y_0)$, and symetrically wrt. y.
- We can define the partial derivative with respect to y similarly to how it is defined for x.

$$\frac{\partial w}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

- With higher order derivatives $\partial w/\partial z$, $\partial w/\partial u$, $\partial w/\partial v$, and more as in $w = f(x, y, z, u, v)$, we evaluate by holding all but the variable of interest constant.
- To denote the partial derivative at a point, we have two notations:

$$\left(\frac{\partial w}{\partial x}\right)_{(x_0, y_0)} \qquad\qquad f_x(x_0, y_0)$$

15.3 Tangent Plane and Normal Line

- **Tangent plane** (to $w = f(x, y)$ at $P_0(x_0, y_0, w_0)$): A plane T such that for any point P on the surface described by $f(x, y)$, as $P \to P_0$, the angle between T and $\overline{PP_0}$ approaches 0.
- **Normal line** (to $w = f(x, y)$ at $P_0(x_0, y_0, w_0)$): The line through P_0 which is normal to the tangent plane to $f(x, y)$ at P_0.
- The tangent plane is determined by the lines L_1 and L_2 tangent to the curves $C_1 : w = f(x_0, y)$ and $C_2 : w = f(x, y_0)$; the slopes of these lines are given by $\partial w/\partial y$ and $\partial w/\partial x$, respectively.
- Formulae for the tangent plane and normal line follow easily after finding a normal vector $\mathbf{N}$ to the plane of L_1 and L_2. To find $\mathbf{N}$, we can use the cross product of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ lying along L_1 and L_2, respectively.

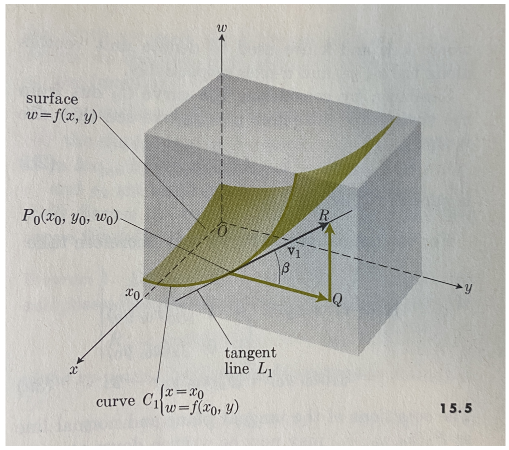

Figure 15.4: Deriving formulae for the tangent plane and normal line.

- From Figure 15.4, we can see that

$$\mathbf{v}_1 = \mathbf{j} + f_y(x_0, y_0)\mathbf{k} \qquad\qquad \mathbf{v}_2 = \mathbf{i} + f_x(x_0, y_0)\mathbf{k}$$

- Thus,

$$\mathbf{N} = \mathbf{i}f_x(x_0, y_0) + \mathbf{j}f_y(x_0, y_0) - \mathbf{k}$$

- Therefore, the formulae for the tangent plane and normal line, respectively, are

$$A(x - x_0) + B(y - y_0) + C(w - w_0) = 0 \qquad\qquad (x, y, w) = (x_0, y_0, w_0) + t(A, B, C)$$

where $A = f_x(x_0, y_0)$, $B = f_y(x_0, y_0)$, $C = -1$, and $t \in (-\infty, \infty)$.

- In vector form, if $\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ and $\mathbf{R}_0 = \mathbf{i}x_0 + \mathbf{j}y_0 + \mathbf{k}z_0$, then

$$\mathbf{N} = \mathbf{i}f_x(x_0, y_0) + \mathbf{j}f_y(x_0, y_0) - \mathbf{k} \qquad\qquad \mathbf{N} \cdot (\mathbf{R} - \mathbf{R}_0) = 0 \qquad\qquad \mathbf{R} = \mathbf{R}_0 + t\mathbf{N}$$

15.4 Approximate Value of Δw

- **Linearization** (of f at P_0): The function (based off of the tangent plane)

$$w = f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$$

- Note that

$$\Delta w_{\text{tan}} = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

meaning that to calculate Δw_{tan}, we need only add the tangential components; no other interaction term is needed.

- Important results:

Theorem 15.1. Let the function $w = f(x, y)$ be continuous and possess partial derivatives f_x, f_y throughout a region $R : |x - x_0| < h,\ |y - y_0| < k$ of the xy-plane. Let f_x and f_y be continuous at (x_0, y_0). Let $\Delta w = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. Then

$$\Delta w = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$.

Corollary 15.1. *Let $w = f(x, y)$ be continuous in a region $R : |x - x_0| < h,\ |y - y_0| < k$. Let f_x and f_y exist in R and be continuous at (x_0, y_0). Then the surface $w = f(x, y)$ has a tangent plane at $P_0(x_0, y_0, w_0)$, where $w_0 = f(x_0, y_0)$.*

- These results extend into finitely higher dimensions.

15.5 The Directional Derivative: General Case

12/17: • We first prove that the directional derivative can be expressed in terms of the partial derivatives.

Theorem 15.2. Let $w = f(x, y)$ be continuous and possess partial derivatives f_x, f_y throughout some neighborhood of the point $P_0(x_0, y_0)$. Let f_x and f_y be continuous at P_0. Then the directional derivative at P_0 exists for any direction angle ϕ and is given by

$$\frac{\mathrm{d}w}{\mathrm{d}s} = f_x(x_0, y_0)\cos\phi + f_y(x_0, y_0)\sin\phi$$

Proof. By Theorem 15.1, we know that $\Delta w = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Thus,

$$\frac{\Delta w}{\Delta s} = f_x(x_0, y_0)\frac{\Delta x}{\Delta s} + f_y(x_0, y_0)\frac{\Delta y}{\Delta s} + \epsilon_1\frac{\Delta x}{\Delta s} + \epsilon_2\frac{\Delta y}{\Delta s}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Hence, since $\lim \frac{\Delta w}{\Delta s} = \frac{\mathrm{d}w}{\mathrm{d}s}$, $\lim \frac{\Delta x}{\Delta s} = \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\phi$, and $\lim \frac{\Delta y}{\Delta s} = \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\phi$, we have

$$\frac{\mathrm{d}w}{\mathrm{d}s} = f_x(x_0, y_0)\cos\phi + f_y(x_0, y_0)\sin\phi$$

as desired. □

- Note that $\phi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ lead to $\mathrm{d}w/\mathrm{d}s = \partial w/\partial x$, $\partial w/\partial y$, $-\partial w/\partial x$, $-\partial w/\partial y$, respectively.

• The directional derivative in three dimensions is given by

$$\frac{\mathrm{d}w}{\mathrm{d}s} = f_x(x_0, y_0, z_0)\cos\alpha + f_y(x_0, y_0, z_0)\cos\beta + f_z(x_0, y_0, z_0)\cos\gamma$$

- If we have a direction vector $\mathbf{u} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$ and another one that depends on the function and P_0 (the **gradient**) given by $\mathbf{v} = \mathbf{i}f_x(x_0, y_0, z_0) + \mathbf{j}f_y(x_0, y_0, z_0) + \mathbf{k}f_z(x_0, y_0, z_0)$, then

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \mathbf{u} \cdot \mathbf{v}$$

15.6 The Gradient

• **Gradient** (of w): The vector function

$$\operatorname{grad} w = \nabla w = \mathbf{i}\frac{\partial w}{\partial x} + \mathbf{j}\frac{\partial w}{\partial y} + \mathbf{k}\frac{\partial w}{\partial z}$$

• The inverted capital delta is the **del operator**. In its own right, it is defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

• Recall that the directional derivative at $P_0(x_0, y_0, z_0)$ can be written as

$$\left(\frac{\mathrm{d}w}{\mathrm{d}s}\right)_0 = (\nabla w)_0 \cdot \mathbf{u}$$

- The significance of this is that it implies that

$$\left(\frac{\mathrm{d}w}{\mathrm{d}s}\right)_0 = |(\nabla w)_0|\,|u|\,\cos\theta = |(\nabla w)_0|\,\cos\theta$$

which means that the directional derivative is the "scalar projection of $\operatorname{grad} w$ at P_0, onto the direction $\mathbf{u}$" (Thomas, 1972, p. 511).

  - Since $\mathrm{d}w/\mathrm{d}s$ is maximized when $\cos\theta = 1$, it must be that "the function $w = f(x, y, z)$ changes most rapidly in the direction given by the vector grad w itself. Moreover, the directional derivative in this direction is equal to the magnitude of the gradient" (Thomas, 1972, p. 511).
- The gradient vector at $P_0(x_0, y_0, z_0)$, where $w_0 = f(x_0, y_0, z_0)$ is also normal to the contour surface consisting of all points $P(x, y, z)$ for which $f(x, y, z) = w_0$.
  - We can prove this from the fact that the directional derivative in the direction of any line tangent to P_0 along the contour surface will be 0. But since $(\nabla w)_0 \neq \mathbf{0}$, we must have $\cos\theta = 0$, meaning that ∇w is perpendicular to any line tangent to P_0. It follows that it is normal to the contour surface, itself.
- Be careful with dimensions: The 3D vector grad w is points in the direction in 3-space of greatest change for a function with a 3D domain (a function best graphed in 4D), but is normal to a 2D surface that is a subset of this 3D domain.

15.7 The Chain Rule for Partial Derivatives

- Let $w = f(x, y, z)$ be a function with continuous partial derivatives f_x, f_y, f_z throughout some region R of xyz-space. If C is a curve lying in R defined by the parameterization $x = x(t)$, $y = y(t)$, and $z = z(t)$, then
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$
  - By generalizing Theorem 15.1 to three dimensions and dividing by Δt, we have
$$\frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x}\right)_0 \frac{\Delta x}{\Delta t} + \left(\frac{\partial w}{\partial t}\right)_0 \frac{\Delta y}{\Delta t} + \left(\frac{\partial w}{\partial z}\right)_0 \frac{\Delta z}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t} + \epsilon_3\frac{\Delta z}{\Delta t}$$
    where $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ as $\Delta x, \Delta y, \Delta z \to 0$.
  - Now if C is differentiable in R, too, (i.e., $\mathrm{d}x/\mathrm{d}t$, $\mathrm{d}y/\mathrm{d}t$, $\mathrm{d}z/\mathrm{d}t$ all exist) then $\Delta x, \Delta y, \Delta z \to 0$ as $\Delta t \to 0$.
  - Therefore, if we take the limit as $\Delta t \to 0$ of the above equation, we get the desired result.
- Note that we can also write
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \nabla w \cdot \mathbf{v}$$
  where $\mathbf{v}$ is the velocity vector along the curve C.
- We can also consider the behavior of a function along a surface S lying in R parameterized by $x = x(r, s)$, $y = y(r, s)$, and $z = z(r, s)$. In this case we can take
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$
  - We can also expand this to many more dimensions. This idea is best summarized in matrix form (for a function $w = f(x_1, \dots, x_n)$ parameterized by $x_1 = x_1(y_1, \dots, y_m)$, $\dots$, $x_n = (y_1, \dots, y_m)$):
$$\begin{bmatrix} \frac{\partial w}{\partial y_1} & \frac{\partial w}{\partial y_2} & \cdots & \frac{\partial w}{\partial y_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \cdots & \frac{\partial w}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_m} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_m} \end{bmatrix}$$
- To clarify the above results, we investigate a few problems.

- "Suppose that $w = r^2 \cos 2\theta$, where $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, [and] $\theta = \tan^{-1}(y/x)$. Find $\partial w/\partial x$ and $\partial w/\partial y$" (Thomas, 1972, p. 516).
  - We use the matrix methods to get an equation for the desired results.

$$\begin{aligned}
\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} &= \begin{bmatrix} \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \end{bmatrix} \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{bmatrix} \\
&= \begin{bmatrix} 2x\cos 2\theta + 2y \sin 2\theta & 2y\cos 2\theta - 2x\sin 2\theta \end{bmatrix}
\end{aligned}$$

  - Now, we can use some substitutions to put the above results in terms of the variable with respect to which we are differentiating.

$$\begin{aligned}
\frac{\partial w}{\partial x} &= 2(x\cos 2\theta + y\sin 2\theta) \\
&= 2r(\cos\theta\cos 2\theta + \sin\theta \sin 2\theta) \\
&= 2r\cos(2\theta - \theta) \\
&= 2x
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial w}{\partial y} &= 2(y\cos 2\theta - x\sin 2\theta) \\
&= 2r(\sin\theta\cos 2\theta - \cos\theta \sin 2\theta) \\
&= 2r\sin(\theta - 2\theta) \\
&= -2y
\end{aligned}$$

- "Show that the change of variables from x and y to $r = y - ax$ [and] $s = y + ax$ transforms the differential equation

$$\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = 0$$

into a form that is more easily solved, and solve it. (Here a is a constant.)" (Thomas, 1972, p. 516).
  - We divide into two cases ($a \neq 0$ and $a = 0$), beginning with the former.
  - Imagine that $w = f(x, y)$ is transformed into $w = \tilde{f}(r, s)$ via substitutions which can be derived by treating the definitions of r and s as a two-variable system of equations and solving for x and y.

$$x = \frac{1}{2a}(s - r) \qquad y = \frac{1}{2}(r + s)$$

  - Now to find $\partial w/\partial x$ and $\partial w/\partial y$, we use the chain rule.

$$\begin{aligned}
\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} &= \begin{bmatrix} \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} \end{bmatrix} \begin{bmatrix} -a & 1 \\ a & 1 \end{bmatrix} \\
&= \begin{bmatrix} -a\frac{\partial w}{\partial r} + a\frac{\partial w}{\partial s} & \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \end{bmatrix}
\end{aligned}$$

  - By substituting into the original differential equation, we get $-2a\ \partial w/\partial r = 0$, or $\frac{\partial w}{\partial r} = 0$.
  - Thus, this differential equation is easy to solve — we need only require that w is constant in the r-direction. Indeed, w can be any function of s. Therefore, the solution is

$$w = \phi(s) = \phi(y + ax)$$

where $\phi(s)$ is any differentiable function of s, whatsoever.

- If $a = 0$, then we have the similar case $\partial w/\partial x = 0$.
- Note that in solving an ordinary differential equation, we often get constants of integration. In solving a partial differentiable equation, arbitrary functions (such as $\phi(s)$) are analogous to these constants of integration. Extending the analogy, they can sometimes be solved for with "initial conditions," as in the next problem.

- Find an explicit formula for w in the above problem if its values along the x-axis are given by $w = \sin x$ and if $a \neq 0$.
  - The general solution is $w = f(x, y) = \phi(y + ax)$.
  - We are given $f(x, 0) = \phi(0 + ax) = \sin x$.
  - Thus, if we let $u = ax$, we have $\phi(u) = \sin \frac{u}{a}$, meaning that $w = \phi(y + ax) = \sin \frac{y+ax}{a}$.

15.8 The Total Differential

- **Partial differential** (of $w = f(x, y, z)$ with respect to x): The infinitesimal value

$$\frac{\partial w}{\partial x}\,\mathrm{d}x$$

  - There also exist symmetric partial differentials with respect to y and z.

- **Total differential** (of $w = f(x, y, z)$): The infinitesimal value

$$\mathrm{d}w = \frac{\partial w}{\partial x}\,\mathrm{d}x + \frac{\partial w}{\partial y}\,\mathrm{d}y + \frac{\partial w}{\partial z}\,\mathrm{d}z$$

  that is the sum of the partial differentials of w.

- If x, y, z are given as functions of a single variable t, we have

$$\mathrm{d}x = x'(t)\,\mathrm{d}t \qquad \mathrm{d}y = y'(t)\,\mathrm{d}t \qquad \mathrm{d}z = z'(t)\,\mathrm{d}t$$

- If x, y, z are given as functions of two variables r, s, their total differentials are

$$\mathrm{d}x = \frac{\partial x}{\partial r}\,\mathrm{d}r + \frac{\partial x}{\partial s}\,\mathrm{d}s \qquad \mathrm{d}y = \frac{\partial y}{\partial r}\,\mathrm{d}r + \frac{\partial y}{\partial s}\,\mathrm{d}s \qquad \mathrm{d}z = \frac{\partial z}{\partial r}\,\mathrm{d}r + \frac{\partial z}{\partial s}\,\mathrm{d}s$$

  - If we take the perspective that $w = f[x(r, s), y(r, s), z(r, s)] = \tilde{f}(r, s)$ is a function of two variables, then we should have

$$\mathrm{d}w = \frac{\partial w}{\partial r}\,\mathrm{d}r + \frac{\partial w}{\partial s}\,\mathrm{d}s$$

  - Now this $\mathrm{d}w$ is the same as the one given earlier with respect to x, y, z, as a consequence of the chain rule. Indeed,

$$\begin{aligned}
\mathrm{d}w &= \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\,\mathrm{d}r + \frac{\partial x}{\partial s}\,\mathrm{d}s\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r}\,\mathrm{d}r + \frac{\partial y}{\partial s}\,\mathrm{d}s\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial r}\,\mathrm{d}r + \frac{\partial z}{\partial s}\,\mathrm{d}s\right) \\
&= \left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\right)\mathrm{d}r + \left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\right)\mathrm{d}s \\
&= \frac{\partial w}{\partial r}\,\mathrm{d}r + \frac{\partial w}{\partial s}\,\mathrm{d}s
\end{aligned}$$

  - Note that r and s are the independent variables here, so while we can approximate $\mathrm{d}r$ and $\mathrm{d}s$ with Δr and Δs, respectively, we should not approximate $\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z$ with $\Delta x, \Delta y, \Delta z$. Indeed, we should use the differentials $\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z$ as defined in terms of $\mathrm{d}r\,, \mathrm{d}s$ to approximate $\Delta x, \Delta y, \Delta z$.

- These results can be generalized to higher dimensions.
- The following example should make clear the power of these differential definitions: "Consider the function $w = x^2 + y^2 + z^2$ with $x = r\cos s$, $y = r\sin s$, [and] $z = r$" (Thomas, 1972, p. 519).
  - The total differential is $\mathrm{d}w = 2(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)$.
  - The differentials in terms of r and s are $\mathrm{d}x = \cos s\,\mathrm{d}r - r\sin s\,\mathrm{d}s$, $\mathrm{d}y = \sin s\,\mathrm{d}r + r\cos s\,\mathrm{d}s$, and $\mathrm{d}z = \mathrm{d}r$.
  - Hence, the total differential can also be written as

$$
\begin{aligned}
\mathrm{d}w &= 2(x\cos s + y\sin s + z)\,\mathrm{d}r + 2(-xr\sin s + yr\cos s)\,\mathrm{d}s \\
&= 2(r\cos^2 s + r\sin^2 s + r)\,\mathrm{d}r + 2(-r^2\cos s\sin s + r^2\sin s\cos s)\,\mathrm{d}s \\
&= 4r\,\mathrm{d}r
\end{aligned}
$$

  - Integrating, the above yields $w = 2r^2$.
  - Critically, we can also derive this formula for w from the original function and parameterization since $w = r^2\cos^2 s + r^2\sin^2 s + r^2 = 2r^2$.
- If $F(x,y) = 0$, then for the plane curve generated, $\mathrm{d}y/\mathrm{d}x = -F_x(x,y)/F_y(x,y)$ if $F_y(x,y) \neq 0$.
  - We can also derive this result by implicitly differentiating $F(x,y) = 0$, as we are allowed to by the **implicit function theorem**.