Calculus and Analytic Geometry (Thomas) Notes

Steven Labalme

July 1, 2020

Contents

List of Figures

Chapter 8

Hyperbolic Functions

8.1 Introduction

6/24: • **Hyperbolic functions**: Certain combinations of e^x and e^{-x} that are used to solve certain engineering problems (the hanging cable) and are useful in connection with differential equations.

8.2 Definitions and Identities

- Let

$$\cosh u = \frac{1}{2}(e^u + e^{-u}) \qquad \sinh u = \frac{1}{2}(e^u - e^{-u})$$

  - These combinations of exponentials occur sufficiently frequently that we give a special name to them.
  - Although the names may seem random, $\sinh u$ and $\cosh u$ do share many analogous properties with $\sin u$ and $\cos u$.
  - Pronounced to rhyme with "gosh you" and as "cinch you," respectively.
- Like $x = \cos u$ and $y = \sin u$ are associated with the point (x, y) on the unit circle $x^2 + y^2 = 1$, $x = \cosh u$ and $y = \sinh u$ are associated with the point (x, y) on the unit hyperbola $x^2 - y^2 = 1$.
  - Note that $x = \cosh u$ and $y = \sinh u$ are associated with the *right-hand* branch of the unit hyperbola.
  - Also note that sine and cosine are sometimes referred to as the **circular functions**.
- Analogous to sine and cosine, we have the identity

$$\cosh^2 u - \sinh^2 u = 1$$

- We define the remaining hyperbolic trig functions as would be expected.

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} \qquad \operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}}$$
$$\coth u = \frac{\cosh u}{\sinh u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} \qquad \operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}}$$

- Since $\cosh u + \sinh u = e^u$, we can replace any combination of exponentials with hyperbolic sines and cosines and vice versa.
- Note that the hyperbolic functions are *not* periodic.
  - This does mean, though, that they have more easily defined properties at infinity.
- "Practically all the circular trigonometric identities have hyperbolic analogies" (Thomas, 1972, p. 267).

8.3 Derivatives and Integrals

6/25:

- Derivatives of the hyperbolic functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = \cosh u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh u) = \sinh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh u) = \operatorname{sech}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth u) = -\operatorname{csch}^2 u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{\mathrm{d}u}{\mathrm{d}x}$$

  - Note that these are almost exact analogs of the formulas for the corresponding circular functions, the exception being that the negative signs are not associated with the cofunctions but with the latter three.

- We now introduce the hanging cable problem, deriving the differential equation that represents the condition for equilibrium of forces acting on a section AP of a hanging cable (Figure 8.1).

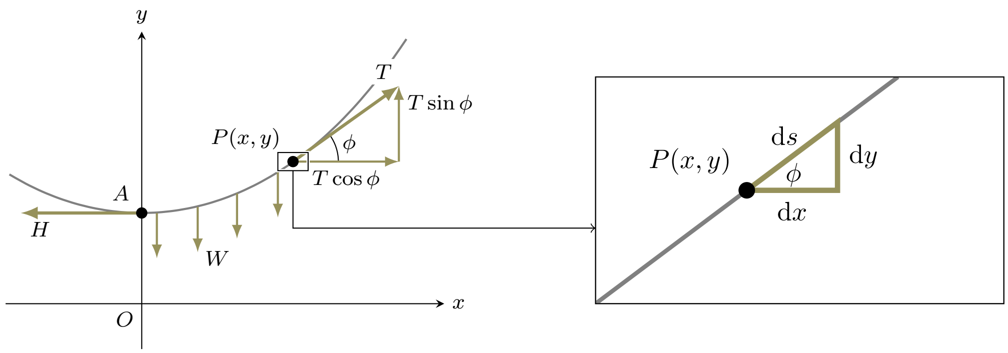


Figure 8.1: A section AP of a hanging cable.

- Let point A to be the lowest point in the arc of the hanging cable, and let it be at $(0, y_0)$ in the Cartesian plane.
- Continue along the right arc of the cable until arriving at some point $P(x, y)$.
- We wish to consider only segment AP, so we need to anchor points A and P as if the rest of the cable were still there. Now every infinitesimal sliver of the cable is being pulled (downward) slightly by gravity, but significantly (tangentially) by the rest of the cable. Thus, we can compensate at point A by pulling it tangentially left with some force H, and at point P by pulling it tangentially up and to the right with some force T.
- Since the cable is at equilibrium, the three forces acting on the cable as a whole (T, H, and W) are balanced. Thus,

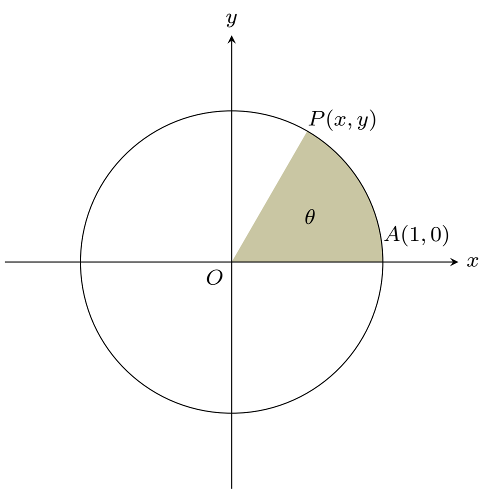
$$T \sin\phi = W$$
$$T \cos\phi = H$$

- Combining these two equations gives an important result:

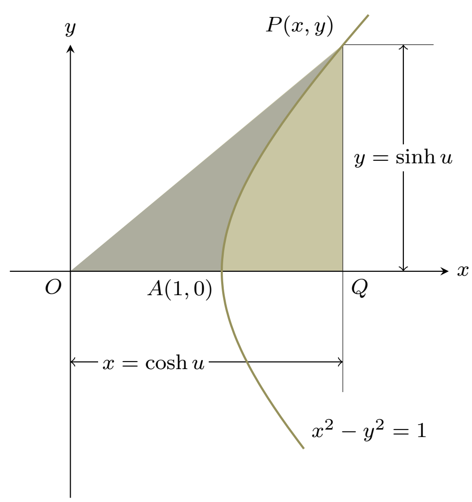
$$\frac{T\sin\phi}{T\cos\phi} = \frac{W}{H}$$
$$\tan\phi = \frac{W}{H}$$

- Now $\tan\phi$ is a particularly important piece of the puzzle, because it actually equals $\frac{\mathrm{d}y}{\mathrm{d}x}$ (see the zoomed-in section of Figure 8.1). Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{W}{H}$$
- Contrary to how it may look, W is actually not a constant — the weight of section AP is dependent on P (i.e., is dependent on how long the section is). If we assume that the cable has a uniform weight per length ratio w and that section AP is s units long, then we have $W = ws$. Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ws}{H}$$
- s is just arc length. Thus, $s = \int_A^P \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$. However, because we cannot have an integral in a differential equation, we differentiate to find $\mathrm{d}s = \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$.
  - Note that this expression for $\mathrm{d}s$ makes sense because, by the zoomed-in section of Figure 8.1, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{(\mathrm{d}x^2/\mathrm{d}x^2 + \mathrm{d}y^2/\mathrm{d}x^2)\mathrm{d}x^2} = \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x$.
- If we differentiate $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ws}{H}$, we can get $\mathrm{d}s$ into the equation and substitute, as follows, to yield the final differential equation.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\frac{\mathrm{d}s}{\mathrm{d}x}$$
$$\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}$$

8.4 Geometric Meaning of the Hyperbolic Radian



(a) Circular radians.



(b) Hyperbolic radians.

Figure 8.2: Geometric meaning of radians.

- For circular sine and cosine, the "meaning of the variable θ in the equations $x = \cos\theta$, $y = \sin\theta$ as they relatee to the point $P(x, y)$ on the unit circle $x^2 + y^2 = 1$ [is] the radian measure of the angle AOP in [Figure 8.2a], that is $\theta = \frac{\text{arc } AP}{\text{radius } OA}$" (Thomas, 1972, p. 271).
  - However, since $A = \frac{1}{2}r^2\theta = \frac{\theta}{2}$ for $r = 1$, θ also equals twice the area of the sector AOP.

- To understand the meaning of the variable u, calculate the area of the sector AOP in Figure 8.2b as an analog to circular area.

$$\begin{aligned} A_{AOP} &= A_{OQP} - A_{AQP} \\ &= \frac{1}{2}bh - \int_A^P y\,\mathrm{d}x \end{aligned}$$

$$y = \sinh u, \quad x = \cosh u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \sinh u \Rightarrow \mathrm{d}x = \sinh u\,\mathrm{d}u$$

$$\begin{aligned} &= \frac{1}{2}xy - \int_A^P \sinh^2 u\,\mathrm{d}u \\ &= \frac{1}{2}\cosh u \sinh u - \frac{1}{2}\int_A^P (\cosh 2u - 1)\,\mathrm{d}u \\ &= \frac{1}{2}\sinh u \cosh u - \frac{1}{2}\left[\frac{1}{2}\sinh 2u - u\right]_{A(u=0)}^{P(u=u)} \\ &= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{4}\sinh 2u - \frac{1}{2}u\right) \\ &= \frac{1}{2}\sinh u \cosh u - \left(\frac{1}{2}\sinh u \cosh u - \frac{1}{2}u\right) \\ &= \frac{1}{2}u \end{aligned}$$

  - This implies that u also equals twice the area of the sector AOP (the hyperbolic sector, that is).
  - This means, for example, that "$\cosh 2$ and $\sinh 2$ may be interpreted as the coordinates of P when the area of the sector AOP is just equal to the area of a square having OA as side" (Thomas, 1972, p. 272).

8.5 The Inverse Hyperbolic Functions

6/26:

- The inverse of $x = \sinh y$ is $y = \sinh^{-1} x$.
  - Since there is a one-to-one correspondence between x and y values for the inverse hyperbolic sine function, there is no need to define a principal branch (as there was with circular sine).
- The inverse of $x = \cosh y$ is $y = \cosh^{-1} x$, where $y \geq 0$ and $x \geq 1$.
  - Since there is a two-to-one correspondence between x and y values this time, we choose the positive values to be the principal branch and let the negative values be defined by the function $y = -\cosh^{-1} x$.
- Note that the only other inverse hyperbolic trig function that needs a principal branch is (rather appropriately) $\operatorname{sech} x$. Likewise, the positive values make up the principal branch.
- Like the hyperbolic trig functions have exponential forms, the inverse hyperbolic trig functions have logarithmic forms.

- For example,

$$\begin{aligned}
y &= \sinh^{-1} x \\
\sinh y &= x \\
\frac{1}{2}(\mathrm{e}^y - \mathrm{e}^{-y}) &= x \\
\mathrm{e}^y - \frac{1}{\mathrm{e}^y} &= 2x \\
\mathrm{e}^{2y} - 1 &= 2x\mathrm{e}^y \\
0 &= \mathrm{e}^{2y} - 2x\mathrm{e}^y - 1 \\
\mathrm{e}^y &= x \pm \sqrt{x^2+1} \\
y &= \ln\left(x + \sqrt{x^2+1}\right)
\end{aligned}$$

- Like the inverse circular trig functions, the inverse hyperbolic functions are quite useful as the end results of integration of radicals. First, however, we must derive their derivatives.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} u\right) = \frac{1}{\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} && \frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} u\right) = \frac{1}{\sqrt{u^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} u\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u|<1 && \frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{sech}^{-1} u\right) = \frac{-1}{u\sqrt{1-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\coth^{-1} u\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x}, \quad |u|>1 && \frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{csch}^{-1} u\right) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

8.6 The Hanging Cable

- We seek to derive the solution to the following differential equation associated with the hanging cable problem, as described in Section 8.3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{w}{H}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

- Since the above equation involves the second derivative, we will have to deal with two constants of integration. Since it doesn't matter where the hanging cable lies in the Cartesian plane, we can choose its location such that the final answer will be as simple as possible.
  - "By choosing the y-axis to be the vertical line through the lowest point of the cable, one condition becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$" (Thomas, 1972, p. 277).
  - "Then we may still move the x-axis up or down to suit our convenience. That is, we let $y = y_0$ when $x = 0$, and we may choose y_0 so as to give us the simplest form in our final answer" (Thomas, 1972, p. 277).
- Let's begin solving the original equation. Since it involves y' and y'' but not y, let $y' = p$ and start by integrating with respect to p.

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{w}{H}\sqrt{1+p^2} \\
\frac{\mathrm{d}p}{\sqrt{1+p^2}} &= \frac{w}{H}\,\mathrm{d}x \\
\int \frac{\mathrm{d}p}{\sqrt{1+p^2}} &= \int \frac{w}{H}\,\mathrm{d}x \\
\sinh^{-1} p &= \frac{w}{H}x + C_1
\end{aligned}$$

- Since $p = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $x = 0$, $C_1 = 0$. Thus,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sinh\left(\frac{w}{H}x\right) \\
\mathrm{d}y &= \sinh\left(\frac{w}{H}x\right)\mathrm{d}x \\
\int \mathrm{d}y &= \int \sinh\left(\frac{w}{H}x\right)\mathrm{d}x \\
y &= \frac{H}{w}\cosh\left(\frac{w}{H}x\right) + C_2
\end{aligned}$$

- Since $y = y_0$ when $x = 0$, $C_2 = y_0 - \frac{H}{w}$. Thus, let $y_0 = \frac{H}{w}$. Therefore, we are finished:

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

Chapter 9

Methods of Integration

9.1 Basic Formulas

6/30:
- Useful, abstract info (that I already know) on what makes a student good at integating, e.g., integrating is an exercise in trial-and-error, but there are ways to increase your likelihood of being successful.

9.2 Powers of Trigonometric Functions

- When integrating power functions, look for integral/derivative relationships, which may allow you to substitute u and $\mathrm{d}u$ at the same time.
  - For example, when confronted with $\int \sin^n ax \cos ax \, \mathrm{d}x$, note that $\cos ax$ is almost the derivative of $\sin ax$, and choose $u = \sin ax$ and $\frac{\mathrm{d}u}{a} = \cos ax \, \mathrm{d}x$ to yield $\frac{1}{a} \int u^n \, \mathrm{d}u$.
- When integrating power functions, it may be possible to split the exponent into a product ($u^n = u^a u^b$ where $a + b = n$) and work off of properties of one of the functions raised to a smaller exponent (u^a may have properties that u^n lacks).
  - For example, when confronted with $\int \sin^3 x \, \mathrm{d}x$, recall that $\sin^2 x$ has Pythagorean properties, and split the exponent.

$$\begin{aligned}\int \sin^3 x \, \mathrm{d}x &= \int \sin^2 x \sin x \, \mathrm{d}x \\ &= \int \left(1 - \cos^2 x\right) \sin x \, \mathrm{d}x\end{aligned}$$

    Now we can use the previous property, since $\sin x$ and $\cos x$ have an integral/derivative relationship.

$$\begin{aligned}&= -\int \left(1 - u^2\right) \mathrm{d}u \\ &= \int \left(u^2 - 1\right) \mathrm{d}u\end{aligned}$$

  - Note that this technique is applicable whenever an odd power of sine or cosine is to be integrated. For higher powers, consider the following.

$$\int \cos^{2n+1} x \, \mathrm{d}x = \int \left(\cos^2 x\right)^n \cos x \, \mathrm{d}x = \int \left(1 - \sin^2 x\right)^n \cos x \, \mathrm{d}x = \int \left(1 - u^2\right)^n \mathrm{d}u$$

    Remember that $\left(1 - u^2\right)^n$ can be expanded via the binomial theorem.

- When integrating a composite trigonometric function, consider reducing it to a radical of powers of sines and cosines.
  - For example, $\sec x \tan x = \frac{\sin x}{\cos^2 x}$.
- When integrating positive integer powers of $\tan x$, use either the base cases or the **reduction formula**.
  - Begin by deriving a reduction formula.

$$\begin{aligned}\int \tan^n x \, \mathrm{d}x &= \int \tan^{n-2} x \left(\sec^2 x - 1\right) \mathrm{d}x \\ &= \int \tan^{n-2} \sec^2 x \, \mathrm{d}x - \int \tan^{n-2} x \, \mathrm{d}x \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, \mathrm{d}x\end{aligned}$$

  Since the reduction formula decreases the exponent by 2, we must work out two base cases:

$$\int \tan^0 x \, \mathrm{d}x = \int \mathrm{d}x = x + C$$

$$\int \tan^1 x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x = -\int \frac{\mathrm{d}u}{u} = -\ln|\cos x| + C$$

  - Note that the impetus for initially deriving such a formula was the search for a way to get $\sec^2 x$ into the integrand, which can be done by splitting the exponent.
  - This method can easily be adjusted to suit negative powers of $\tan x$ (positive powers of $\cot x$).
- When integrating even powers of $\sec x$, either use the secant reduction formula, or split the exponent.
  - We derive the following formula.

$$\begin{aligned}\int \sec^{2n} x \, \mathrm{d}x &= \int \sec^{2n-2} x \sec^2 x \, \mathrm{d}x \\ &= \int \left(1 + \tan^2 x\right)^{n-1} \sec^2 x \, \mathrm{d}x \\ &= \int \left(1 + u^2\right)^{n-1} \mathrm{d}u\end{aligned}$$

- When integrating secant (or cosecant) alone, produce $\frac{u'}{u}$ by multiplying the integrand by a clever form of 1.
  - For example,

$$\begin{aligned}\int \sec x \, \mathrm{d}x &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

9.3 Even Powers of Sines and Cosines

- When integrating the product of sines and cosines raised to powers where at least one exponent is a positive odd integer, split the exponent and use u-substitution.
  - In effect, we wish to evaluate $\int \sin^m x \cos^n x \, \mathrm{d}x$ where at least one of m, n is a positive odd integer.

- For example, when confronted with $\int \cos^{\frac{2}{3}} x \sin^5 x \, \mathrm{d}x$, split the exponent of $\sin^5 x$ and choose $u = \cos x$ and $-\mathrm{d}u = \sin x \, \mathrm{d}x$.

$$\int \cos^{\frac{2}{3}} x \sin^5 x \, \mathrm{d}x = \int \cos^{\frac{2}{3}} x \left(1 - \cos^2 x\right)^2 \sin x \, \mathrm{d}x = \int u^{\frac{2}{3}} \left(u^2 - 1\right) \mathrm{d}u$$

- When integrating the product of sines and cosines raised to powers where both exponents are even integers, begin by transforming it into a sum of either just sines *or* just cosines raised to even integers. Then split the exponents and use one of the following formulas. It may be necessary to use these formulas multiple times. Use them until the problem has been reduced to a sum with only odd exponents.

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \qquad\qquad \cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

  - Note that "these identities may be derived very quickly by adding or subtracting the equations [$\cos^2 u + \sin^2 u = 1$ and $\cos^2 u - \sin^2 u = \cos 2u$] and by dividing by two" (Thomas, 1972, p. 287).
  - For example, when confronted with $\int \sin^2 x \cos^4 x \, \mathrm{d}x$, begin by changing it to a case with only powers of cosine (chose to eliminate the sine function because it is raised to a lower exponent and, thus, will need less binomial expansion).

$$\begin{aligned} \int \sin^2 x \cos^4 x \, \mathrm{d}x &= \int \left(1 - \cos^2 x\right) \cos^4 x \, \mathrm{d}x \\ &= \int \cos^4 x \, \mathrm{d}x - \int \cos^6 x \, \mathrm{d}x \end{aligned}$$

Now split the exponents.

$$= \int \left(\cos^2 x\right)^2 \mathrm{d}x - \int \left(\cos^2 x\right)^3 \mathrm{d}x$$

Employ the above formulas and use binomial expansion. If necessary, repeat (split the exponents, employ the above formulas, use binomial expansion) until only odd exponents remain (remember that 1 is an odd exponent).

$$\begin{aligned} &= \int \left(\frac{1}{2}(1 + \cos 2x)\right)^2 \mathrm{d}x - \int \left(\frac{1}{2}(1 + \cos 2x)\right)^3 \mathrm{d}x \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \cos^2 2x\right) \mathrm{d}x \\ &\quad - \frac{1}{8} \int \left(1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x\right) \mathrm{d}x \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) \mathrm{d}x \\ &\quad - \frac{1}{8} \int \left(1 + 3\cos 2x + \frac{3}{2}(1 + \cos 4x) + \cos^3 2x\right) \mathrm{d}x \end{aligned}$$

These integrals may now be handled using previously discussed techniques.

9.4 Integrals With Terms $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, $\sqrt{u^2 - a^2}$, $a^2 + u^2$, $a^2 - u^2$

- When integrating a radical that resembles the derivative of an inverse trig function, we may factor out the issue so as to make the integral resemble one of the known formulas.

– For example, when confronted with $\int \frac{\mathrm{d}u}{a^2+u^2}$, divide the a^2 term out of the denominator and integrate with respect to $\frac{u}{a}$[1].

$$\begin{aligned}
\int \frac{\mathrm{d}u}{a^2+u^2} &= \frac{1}{a^2}\int \frac{\mathrm{d}u}{1+\left(\frac{u}{a}\right)^2} \\
&= \frac{1}{a^2}\int \frac{a\,\mathrm{d}\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2} \\
&= \frac{1}{a}\tan^{-1}\frac{u}{a}+C
\end{aligned}$$

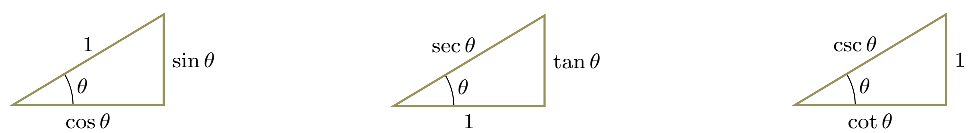
– However, this method is partially flawed in that it relies on having memorized the derivatives of the inverse trig functions, i.e., it is not terribly analytical. This shortcoming will now be addressed with a new, more general technique.

- The new method leans heavily on the following three identities.

$$1-\sin^2\theta=\cos^2\theta \qquad 1+\tan^2\theta=\sec^2\theta \qquad \sec^2\theta-1=\tan^2\theta$$

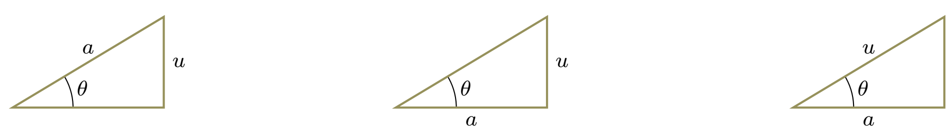
– With the help of these identities, it is possible to...

1. use $u=a\sin\theta$ to replace a^2-u^2 with $a^2\cos^2\theta$;
2. use $u=a\tan\theta$ to replace a^2+u^2 with $a^2\sec^2\theta$;
3. use $u=a\sec\theta$ to replace u^2-a^2 with $a^2\tan^2\theta$.



(a) $\cos^2\theta+\sin^2\theta=1$. (b) $1+\tan^2\theta=\sec^2\theta$. (c) $\cot^2\theta+1=\csc^2\theta$.

Figure 9.1: Geometric rationale for the trigonometric identities.



(a) $\sqrt{a^2-u^2}=a\cos\theta$, $u=a\sin\theta$. (b) $\sqrt{u^2+a^2}=a\sec\theta$, $u=a\tan\theta$. (c) $\sqrt{u^2-a^2}=a\tan\theta$, $u=a\sec\theta$.

Figure 9.2: Geometric rationale for the trigonometric substitutions.

– These identities and substitutions can be easily remembered by thinking of the Pythagorean theorem in conjunction with Figures 9.1 and 9.2, respectively.

- We may now evaluate inverse trig integrals analytically.

[1]Thomas, 1972 uses differentials with more complex functions than a single variable quite often. It's not something I've seen before, but it's something I should get used to (and it does make sense if you think about it — it's just an extension of the underlying concept of separation of variables integration).

– For example, when confronted with $\int \frac{\mathrm{d}u}{a^2+u^2}$, choose $u = a\tan\theta$ and $\mathrm{d}u = a\sec^2\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int \frac{\mathrm{d}u}{a^2+u^2} &= \int \frac{a\sec^2\theta\,\mathrm{d}\theta}{a^2+(a\tan\theta)^2} \\ &= \int \frac{a\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\,\mathrm{d}\theta \\ &= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta \\ &= \frac{1}{a}\int \mathrm{d}\theta \\ &= \frac{1}{a}\theta + C\end{aligned}$$

At this point, solve $u = a\tan\theta$ for θ and substitute.

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

– Some integrals will simplify to have a plus/minus in the denominator, leading to two possible solutions. However, there are sometimes ways to isolate a single solution.

- For example, $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta\mathrm{d}\theta}{\pm a\cos\theta} = \pm\theta + C$. However, when we consider the fact that $\theta = \sin^{-1}\frac{u}{a}$, we know that $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (because inverse sine is not arcsine, and inverse sine is only defined over the principal branch of sine). Thus, since $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\cos\theta \in [0,1]$, i.e., is always positive. Thus, we choose $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = +\theta + C = \sin^{-1}\frac{u}{a} + C$ as our one solution.
- For example, $\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}}$ equals $\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$ or $-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| + C$ depending on whether $\tan\theta$ is positive or negative. But it can be shown algebraically that the two solutions are actually equal:

$$\begin{aligned}-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2-a^2}}{a}\right| &= \ln\left|\frac{a}{u-\sqrt{u^2-a^2}}\right| \\ &= \ln\left|\frac{a\left(u+\sqrt{u^2-a^2}\right)}{\left(u-\sqrt{u^2-a^2}\right)\left(u+\sqrt{u^2-a^2}\right)}\right| \\ &= \ln\left|\frac{a\left(u+\sqrt{u^2-a^2}\right)}{a^2}\right| \\ &= \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right|\end{aligned}$$

Thus, we have $\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$ as the one solution[2].

– Some integrals will have extraneous constants that can be combined with C to simplify the *indefinite* integral.

- Continuing with the above example,

$$\begin{aligned}\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} &= \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C \\ &= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln|a| + C \\ &= \ln\left|u+\sqrt{u^2-a^2}\right| + C\end{aligned}$$

[2]Note that we could choose to use the other solution, but we choose this one because it's "simpler" (it uses addition instead of subtraction).

- When integrating an inverse trig integral with excess polynomial terms, look to transform it into a (power of a) trig integral problem.
  - For example, when confronted with $\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^2}}$, treat it as a case of $a^2 - u^2$, but substitute the trig expression into the x^2 term in the numerator, too.

$$\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,\mathrm{d}\theta}{3\cos\theta} = 9\int \sin^2\theta\,\mathrm{d}\theta$$

    This integral may now be handled using previously discussed techniques.

- Many inverse trig integrals can also be evaluated hyperbolically, making use of the following three identities.

$$\cosh^2\theta - 1 = \sinh^2\theta \qquad 1 - \tanh^2\theta = \operatorname{sech}^2\theta \qquad 1 + \sinh^2\theta = \cosh^2\theta$$

  - With the help of these identities, it is possible to...
    1. use $u = a\tanh\theta$ to replace $a^2 - u^2$ with $a^2\operatorname{sech}^2\theta$;
    2. use $u = a\sinh\theta$ to replace $a^2 + u^2$ with $a^2\cosh^2\theta$;
    3. use $u = a\cosh\theta$ to replace $u^2 - a^2$ with $a^2\sinh^2\theta$.

9.5 Integrals With $ax^2 + bx + c$

- When integrating composite functions where the inner function is a binomial, look to factor said binomial.
  - The general quadratic $f(x) = ax^2 + bx + c$, $a \neq 0$, can be reduced to the form $a\left(u^2 + B\right)$ by completing the square and choosing $u = x + \frac{b}{2a}$ and $B = \frac{4ac-b^2}{4a^2}$:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right) \end{aligned}$$

- When integrating the square root of a binomial, or some similarly tricky function of a binomial, we can transform the binomial into a form such that it will suit one of the inverse trig integrals.
  - Since it would lead to complex numbers, we disregard cases where $a\left(u^2 + B\right)$ is negative, i.e., we focus on cases where (1) a is positive, and (2) a, B are both negative.
  - That being said, if it is an odd root ($\sqrt[3]{x}$, $\sqrt[5]{x}$, etc.), the sign doesn't matter.
  - For example, when confronted with $\int \frac{(x+1)\,\mathrm{d}x}{\sqrt{2x^2-6x+4}}$, begin by factoring the binomial[3].

$$2x^2 - 6x + 4 = 2\left(x^2 - 3x\right) + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2} = 2\left(u^2 - a^2\right)$$

    Note that $u = x - \frac{3}{2}$ and $a = \frac{1}{2}$. We can now return to the integral, which we shall reformulate in terms of u in its entirety.

$$\int \frac{(x+1)\,\mathrm{d}x}{\sqrt{2x^2-6x+4}} = \int \frac{\left(u + \frac{5}{2}\right)\mathrm{d}u}{\sqrt{2\left(u^2 - a^2\right)}}$$

[3]Note that, in place of inspection, we could use the general form factorization derived above.

Split it into two separate integrals and factor out the constants.

$$= \frac{1}{\sqrt{2}} \int \frac{u\,\mathrm{d}u}{\sqrt{u^2 - a^2}} + \frac{5}{2\sqrt{2}} \int \frac{\mathrm{d}u}{\sqrt{u^2 - a^2}}$$

The right integral is a straight-up inverse trig integral. The left one, however, needs something special. It could be dealt with as previously discussed by substituting $u = a\tan\theta$ for all instances of u and evaluating it is a more complex trig integral in θ. However, for the sake of showing a different technique, we will choose $z = u^2 - a^2$ and $\frac{1}{2}\mathrm{d}z = u\,\mathrm{d}u$ and treat it as a power function in z.

$$\begin{aligned}
&= \frac{1}{2\sqrt{2}} \int \frac{\mathrm{d}z}{\sqrt{z}} + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2 \\
&= \frac{1}{2\sqrt{2}} \int z^{-\frac{1}{2}}\,\mathrm{d}z + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2 \\
&= \frac{1}{\sqrt{2}} z^{\frac{1}{2}} + C_1 + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2
\end{aligned}$$

Return all of the substitutions and combine the constants of integration.

$$\begin{aligned}
&= \sqrt{\frac{u^2 - a^2}{2}} + \frac{5}{2\sqrt{2}} \ln\left|u + \sqrt{u^2 - a^2}\right| + C \\
&= \sqrt{\frac{x^2 - 3x + 2}{2}} + \frac{5}{2\sqrt{2}} \ln\left|x - \frac{3}{2} + \sqrt{x^2 - 3x + 2}\right| + C
\end{aligned}$$

References

Thomas, G. B., Jr. (1972). *Calculus and analytic geometry* (fourth). Addison-Wesley Publishing Company.