Chapter 14

Vector Functions and Their Derivatives

14.1 Introduction

12/14:

- **Vector function** (of h): A function $\mathbf{F}(h)$ with n components where each component is a function. Essentially, $\mathbf{F} = (f_1, f_2, \dots, f_n)$.
- **Limit** (of $\mathbf{F}(h)$ as $h \to a$): If each component $f_1, \dots, f_n$ of $\mathbf{F}$ has a limit $L_1, \dots, L_n$ as $h \to a$, then

$$\lim_{h \to a} \mathbf{F}(h) = (L_1, \dots, L_n)$$

- **Continuous** (vector function $\mathbf{F}$ at a): A vector function $\mathbf{F}$ where for every $\epsilon > 0$, there corresponds a $\delta > 0$ such that

$$|\mathbf{F}(h) - \mathbf{F}(a)| < \epsilon \quad \text{when} \quad |h - a| < \delta$$

  - Thomas (1972) shows that this is equivalent to the requirement that each component of $\mathbf{F}$ is continuous at a.
- **Derivative** (of a vector function at c): The derivative $\mathbf{F}'(c)$ of a vector function $\mathbf{F}$ at c is given by the equation

$$\mathbf{F}'(c) = \lim_{h \to 0} \frac{\mathbf{F}(c+h) - \mathbf{F}(c)}{h}$$

  - It can be proven that $\mathbf{F}$ is differentiable at c if and only if each of its components are differentiable at c, and that if this condition is met,

$$\mathbf{F}'(c) = (f_1'(c), \dots, f_n'(c))$$

14.2 Velocity and Acceleration

- Results from here on out will generally pertain to 2D questions, but these methods can easily be generalized to higher dimensions.
- Applications of vectors to physics problems.
  - To solve **statics** problems, we only need to know the **algebra** of vectors.
  - To solve **dynamics** problems, we also need to know the **calculus** of vectors.
- **Position vector**: The vector from the origin to a point P that moves along a parametrically defined curve. *Denoted by* $\mathbf{R}$.

- **Velocity vector**: The vector tangent to a point P that moves along a parametrically defined curve and with magnitude $|\mathrm{d}s/\mathrm{d}t|$. *Denoted by* $\mathbf{v}$.

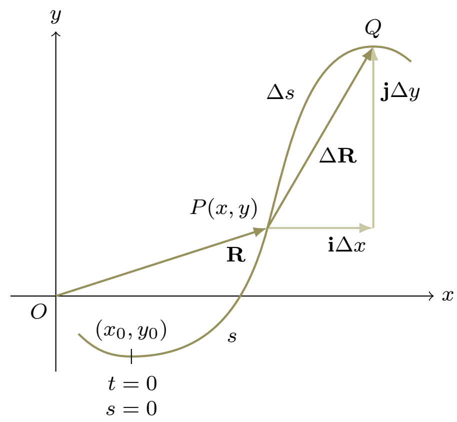


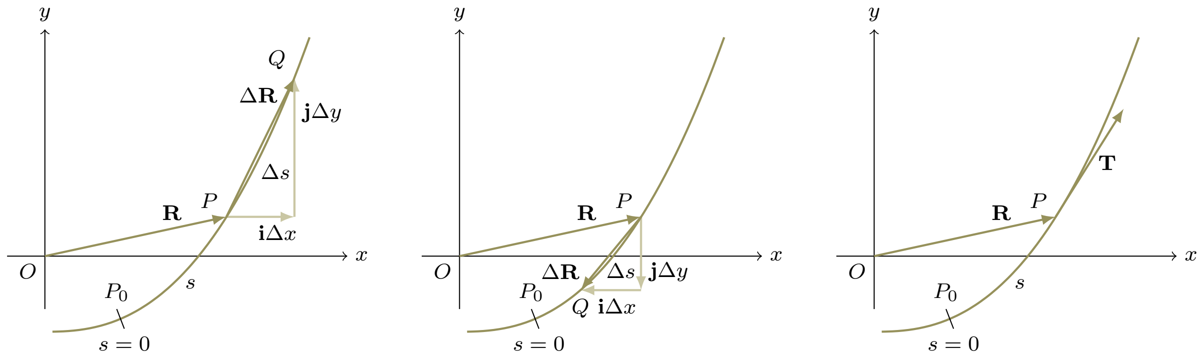
Figure 14.1: Velocity vector.

  - Thomas (1972) semi-rigorously proves from Figure 14.1 that if $\mathbf{R}$ is the position vector, then $\mathrm{d}\mathbf{R}/\mathrm{d}t$ is the velocity vector.
  - Essentially, he proves that
$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}t}$$
    It follows from this that
$$\text{slope of } \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{\mathbf{j}\text{-component}}{\mathbf{i}\text{-component}} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\left|\frac{\mathrm{d}R}{\mathrm{d}t}\right| = \left|\mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}t}\right| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|$$

- **Acceleration vector**: The derivative of the velocity vector and second derivative of the position vector. *Denoted by* $\mathbf{a}$.
$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathbf{j}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$
- Sometimes, we are given a force vector $\mathbf{F} = m\mathbf{a}$ and initial conditions.
  - From these, we can solve for velocity and position vectors via fairly straightforward component integration.
  - Note, however, that constants of integration are now vectors.

14.3 Tangential Vectors

- Let P_0 be a point on a curve. The distance s from P_0 to some point P along the curve is clearly related to the position of P. Thus, we may think of $\mathbf{R}$ as a function of s, and investigate the properties of $\mathrm{d}\mathbf{R}/\mathrm{d}s$.
- **Tangent vector**: The unit vector tangent to a point P along a curve.
  - Since $\Delta\mathbf{R}$ and Δs approach the same quantity as $\Delta s \to 0$, $\Delta\mathbf{R}/\Delta s$ approaches unity, i.e., $|\mathrm{d}\mathbf{R}/\mathrm{d}s| = 1$.
  - Because of the sign change, whether Δs is positive or negative, $\Delta\mathbf{R}/\Delta s$ points in the same general direction for sufficiently small Δs. Indeed, it converges to pointing tangentially.



(a) $\Delta s > 0$. (b) $\Delta s < 0$. (c) $\Delta s \to 0$.

Figure 14.2: Tangent vector.

- Thus,

$$\mathbf{T} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s}$$

- There are two different ways to find $\mathbf{T}$: Straight differentiation combined with manipulations of differentials, and the chain rule combined with the dot product. We will explore each, in turn, with an example.
- "Find the unit vector $\mathbf{T}$ tangent to the circle $x = a\cos\theta$, $y = a\sin\theta$ at any point $P(x,y)$" (Thomas, 1972, p. 471).
  - From the given equations, we have

$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta \qquad \mathrm{d}y = a\cos\theta\,\mathrm{d}\theta \qquad \begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= a^2(\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta^2 \\ &= a^2\,\mathrm{d}\theta^2 \\ \mathrm{d}s &= \pm a\,\mathrm{d}\theta \end{aligned}$$

    - We could alternatively obtain $\mathrm{d}s$ by expressing the arc length formula $S = R\theta$ in terms of differentials.
  - "If we measure arc length in the counterclockwise direction, with $s = 0$ at $(a, 0)$, s will be an increasing function of θ, so the $+$-sign should be taken: $\mathrm{d}s = a\,\mathrm{d}\theta$" (Thomas, 1972, p. 471).
  - Therefore,

$$\begin{aligned} \mathbf{T} &= \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= \mathbf{i}\left(\frac{-a\sin\theta\,\mathrm{d}\theta}{a\,\mathrm{d}\theta}\right) + \mathbf{j}\left(\frac{a\cos\theta\,\mathrm{d}\theta}{a\,\mathrm{d}\theta}\right) \\ &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned}$$

- The equations

$$x = a\cos\omega t \qquad y = a\sin\omega t \qquad z = bt$$

  where a, b, ω are positive constants define a circular helix in E^3[1].

[1] Three-dimensional Euclidean space, equivalent to $\mathbb{R}^3$

- Let $P_0 = (a, 0, 0)$, since this is the point on the locus of the parametric equations where $t = 0$. Additionally, let arc length be measured in the direction in which P moves away from P_0 as t increases from 0.
- Using the chain rule to differentiate, we have

$$\begin{aligned}\mathbf{T} &= \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s} + \mathbf{k}\frac{\mathrm{d}z}{\mathrm{d}s} \\ &= \mathbf{i}\left(-a\omega\sin\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right) + \mathbf{j}\left(a\omega\cos\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right) + \mathbf{k}\left(b\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)\end{aligned}$$

- Since $\mathbf{T}$ is a unit vector, we have $1 = |\mathbf{T}| = |\mathbf{T}|^2 = \mathbf{T}\cdot\mathbf{T}$. Thus,

$$\begin{aligned}1 &= \mathbf{T}\cdot\mathbf{T} \\ &= \mathbf{i}\cdot\mathbf{i}\left(-a\omega\sin\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{j}\cdot\mathbf{j}\left(a\omega\cos\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{k}\cdot\mathbf{k}\left(b\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \\ &= \left(a^2\omega^2 + b^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= \pm\frac{1}{\sqrt{a^2\omega^2 + b^2}}\end{aligned}$$

- We choose the $+$-sign because s should be a positive function of t.
- Putting this all together, we get

$$\mathbf{T} = \frac{a\omega(-\mathbf{i}\sin\omega t + \mathbf{j}\cos\omega t) + \mathbf{k}b}{\sqrt{a^2\omega^2 + b^2}}$$

14.4 Curvature and Normal Vectors

12/15: • **Curvature**: The rate of change of the slope angle ϕ between $\mathbf{T}$ and the x-axis with respect to the arc length s. *Denoted by* κ.

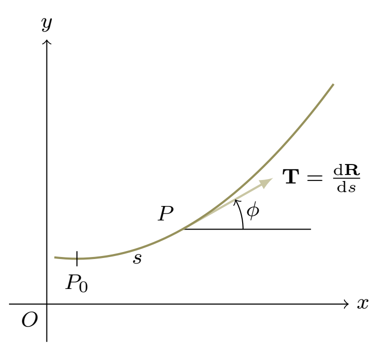


Figure 14.3: Curvature.

  - Measured in radians per unit length.

- From the facts that

$$\kappa = \frac{\mathrm{d}\phi}{\mathrm{d}s} \qquad \tan\phi = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \mathrm{d}s = \pm\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

we can derive a formula for κ in terms of the original function $y = f(x)$ as follows.

$$\phi = \tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} \qquad\qquad \frac{\mathrm{d}s}{\mathrm{d}x} = \pm\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

$$\begin{aligned}\kappa &= \left|\frac{\mathrm{d}\phi}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\phi/\mathrm{d}x}{\mathrm{d}s/\mathrm{d}x}\right| \\ &= \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2}}\end{aligned}$$

- We can similarly derive that

$$\kappa = \frac{\left|\frac{\mathrm{d}^2x}{\mathrm{d}y^2}\right|}{\left[1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2\right]^{3/2}}$$

- If the equations for y and x are given parametrically in terms of t, we have

$$\begin{aligned}\kappa &= \frac{\left|\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|}{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]^{3/2}} \\ &= \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}}\end{aligned}$$

- Naturally, the curvature of a straight line should be 0. Indeed, we find this from the above equations.
- Naturally, the curvature of a circle should be constant, and should somehow decrease as the radius increases. Indeed, we find from the facts that $s = r\theta$ and $\phi = \theta + \frac{\pi}{2}$ that

$$\kappa = \left|\frac{\mathrm{d}\phi}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\theta}{r\,\mathrm{d}\theta}\right| = \frac{1}{r}$$

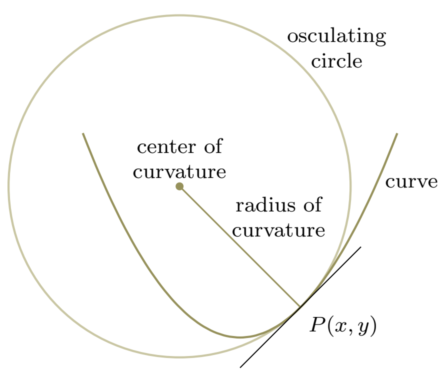


Figure 14.4: Circle, radius, and center of curvature.

- **Circle of curvature** (at P): "The circle that is tangent to a given curve at P, whose center lies on the concave side of the curve and which has the same curvature as the curve has at P" (Thomas, 1972, p. 475). *Also known as* **osculating circle**.
  - Calling the circle of curvature the "osculating circle" refers to the fact that its first and second derivatives at P are equal to the first and second derivatives of the curve at P, meaning that it has a higher degree of contact with the curve at P than any other circle.
- **Radius of curvature** (at P): The radius of the circle of curvature at P. *Denoted by* ρ.

$$\rho = \frac{1}{\kappa} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2}}{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}$$

- **Normal vector**: The unit vector normal to a point P along a curve.
  - Observe that $\mathbf{T}$ can be expressed in terms of the slope angle ϕ:

$$\mathbf{T} = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi$$

  - Since $\mathbf{T}$ can be thought of as a function of ϕ, we can investigate the properties of $\mathrm{d}\mathbf{T}/\mathrm{d}\phi$.
  - Indeed, it is not difficult to show that

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi \qquad \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi}\right| = \sqrt{\sin^2\phi + \cos^2\phi} = 1 \qquad \mathbf{T}\cdot\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} = 0$$

  - Thus,

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$$

- In 3D space, it is harder to define a single normal vector, so we define the...
- **Principal normal vector**: The vector

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|}$$

  - We will soon prove that $\mathbf{T}\cdot\mathrm{d}\mathbf{T}/\mathrm{d}s = 0$.
  - If ϕ is an increasing function of s, then by the chain rule,

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s} = \mathbf{N}\kappa$$

  - Since $\mathbf{N}$ is a unit vector and κ is a constant, κ is equal to the magnitude of $\mathrm{d}\mathbf{T}/\mathrm{d}s$.
  - Thus, we can define the principal normal as above.
- Thomas (1972) uses $\mathrm{d}\mathbf{T}/\mathrm{d}s$ to find both the curvature and principal normal vector of the general circular helix investigated earlier. He also checks limiting cases to rederive the curvature of a circle and of a straight line.
- **Binormal vector**: The vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, as defined by

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

14.5 Differentiation of Products of Vectors

- Let $\mathbf{U}$ and $\mathbf{V}$ be vectors whose components are differentiable functions of t.
- Then we can verify by components that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U}\cdot\mathbf{V}) = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\cdot\mathbf{V} + \mathbf{U}\cdot\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U}\times\mathbf{V}) = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\times\mathbf{V} + \mathbf{U}\times\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}$$

  - Note that we can *derive* the above, too, through the Δ-process.
  - Let $\mathbf{W} = \mathbf{U}\times\mathbf{V}$. Then

$$\begin{aligned}
\mathbf{W} + \Delta\mathbf{W} &= (\mathbf{U}+\Delta\mathbf{U})\times(\mathbf{V}+\Delta\mathbf{V}) \\
&= \mathbf{U}\times\mathbf{V} + \mathbf{U}\times\Delta\mathbf{V} + \Delta\mathbf{U}\times\mathbf{V} + \Delta\mathbf{U}\times\Delta\mathbf{V} \\
\Delta\mathbf{W} &= \mathbf{U}\times\Delta\mathbf{V} + \Delta\mathbf{U}\times\mathbf{V} + \Delta\mathbf{U}\times\Delta\mathbf{V} \\
\frac{\Delta\mathbf{W}}{\Delta t} &= \mathbf{U}\times\frac{\Delta\mathbf{V}}{\Delta t} + \frac{\Delta\mathbf{U}}{\Delta t}\times\mathbf{V} + \frac{\Delta\mathbf{U}}{\Delta t}\times\Delta\mathbf{V} \\
\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} &= \mathbf{U}\times\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\times\mathbf{V}
\end{aligned}$$

- Differentiating the triple scalar product:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U}\cdot\mathbf{V}\times\mathbf{W}) = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\cdot\mathbf{V}\times\mathbf{W} + \mathbf{U}\cdot\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\times\mathbf{W} + \mathbf{U}\cdot\mathbf{V}\times\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t}$$

  - Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}t} & \frac{\mathrm{d}u_2}{\mathrm{d}t} & \frac{\mathrm{d}u_3}{\mathrm{d}t} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ \frac{\mathrm{d}v_1}{\mathrm{d}t} & \frac{\mathrm{d}v_2}{\mathrm{d}t} & \frac{\mathrm{d}v_3}{\mathrm{d}t} \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \frac{\mathrm{d}w_1}{\mathrm{d}t} & \frac{\mathrm{d}w_2}{\mathrm{d}t} & \frac{\mathrm{d}w_3}{\mathrm{d}t} \end{vmatrix}$$

- Differentiating $\mathbf{V}\cdot\mathbf{V} = |\mathbf{V}|^2$ gives

$$\begin{aligned}
\mathbf{V}\cdot\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\cdot\mathbf{V} &= 0 \\
2\mathbf{V}\cdot\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} &= 0
\end{aligned}$$

  - Thus, for any vector function $\mathbf{V}$, there are three cases: (1) $\mathbf{V} = \mathbf{0}$, (2) $\mathrm{d}\mathbf{V}/\mathrm{d}t = \mathbf{0}$, so $\mathbf{V}$ is constant in both direction and magnitude, and (3) $\mathbf{V}$ and $\mathrm{d}\mathbf{V}/\mathrm{d}t$ are perpendicular.
  - Note that this fact allows to verify that $\mathbf{T}\cdot\mathrm{d}\mathbf{T}/\mathrm{d}s = 0$.

- We can use the calculus of tangential and normal vectors to break velocity and acceleration vectors into tangential and normal components.

$$\begin{aligned}
\mathbf{v} &= \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} & \mathbf{a} &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} & &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} & &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \mathbf{N}\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \\
&= \mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}t} & & & &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \mathbf{N}\left(\frac{\mathbf{v}^2}{\rho}\right)
\end{aligned}$$

  - $\mathbf{v}^2/\rho$ is very similar to v^2/r (think about the circle and radius of curvature)!
  - Another important related equation:

$$|\mathbf{a}| = a_T^2 + a_N^2$$

– Lastly, we can derive a formula for the curvature in terms of velocity and acceleration.

$$\begin{aligned}\mathbf{v}\times\mathbf{a} &= \mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}t}\times\left[\mathbf{T}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\mathbf{N}\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right]\\ &= \mathbf{T}\times\mathbf{N}\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\\ |\mathbf{v}\times\mathbf{a}| &= \left|\mathbf{B}\kappa|\mathbf{v}|^3\right|\\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}\end{aligned}$$

14.6 Polar and Cylindrical Coordinates

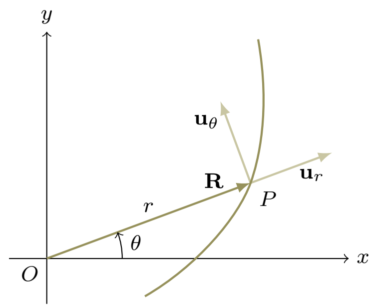


Figure 14.5: Vectors in polar coordinates.

- To analyze polar coordinates, we introduce the unit vectors

$$\mathbf{u}_r = \mathbf{i}\cos\theta+\mathbf{j}\sin\theta \qquad \mathbf{u}_\theta = -\mathbf{i}\sin\theta+\mathbf{j}\cos\theta$$

- Clearly, we have

$$\frac{\mathrm{d}\mathbf{u}_r}{\mathrm{d}\theta}=\mathbf{u}_\theta \qquad \frac{\mathrm{d}\mathbf{u}_\theta}{\mathrm{d}\theta}=-\mathbf{u}_r$$

- Additionally, we can see that

$$\mathbf{R}=r\mathbf{u}_r$$

- The velocity vector can easily be expressed in terms of these quantities (and visualized as such geometrically, as in Figure 14.6).

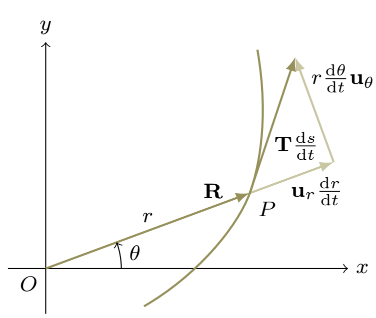


Figure 14.6: Polar velocity vector.

$$\begin{aligned}\mathbf{v} &= \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} \\ &= \mathbf{u}_r \frac{\mathrm{d}r}{\mathrm{d}t} + r\frac{\mathrm{d}\mathbf{u}_r}{\mathrm{d}t} \\ &= \mathbf{u}_r \frac{\mathrm{d}r}{\mathrm{d}t} + r\mathbf{u}_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t}\end{aligned}$$

- The acceleration vector can also be expressed in terms of these quantities (the following can be derived by differentiating the above with respect to t and substituting).

$$\mathbf{a} = \mathbf{u}_r \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right] + \mathbf{u}_\theta \left[r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right]$$

- In three dimensions (esp. for cylindrical coordinates), we have

$$\begin{aligned}\mathbf{R} &= r\mathbf{u}_r + \mathbf{k}z \\ \mathbf{v} &= \mathbf{u}_r \frac{\mathrm{d}r}{\mathrm{d}t} + r\mathbf{u}_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} + \mathbf{k}\frac{\mathrm{d}z}{\mathrm{d}t} \\ \mathbf{a} &= \mathbf{u}_r \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right] + \mathbf{u}_\theta \left[r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right] + \mathbf{k}\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\end{aligned}$$

- Thomas (1972) goes into an lengthy application of the above definitions to deriving Kepler's Laws.