

## Chapter 4

# Applications

### 4.1 Increasing or Decreasing Functions: The Sign of $dy/dx$

- 7/8:
- **Increasing** (function  $f$  on  $[a, b]$ ): A function  $f$  such that  $f(x_1) > f(x_2)$  when  $x_1 > x_2$  for all  $x_1, x_2$  in the interval  $[a, b]$ . *Also known as rising.*
  - **Decreasing** (function  $f$  on  $[a, b]$ ): A function  $f$  such that  $f(x_1) < f(x_2)$  for  $a \leq x_2 < x_1 \leq b$ . *Also known as falling.*
    - Sometimes, we consider functions that increase or decrease on open or half-open intervals.
  - **Increasing** (function  $f$  at a point  $c$ ): A function  $f$  such that in some neighborhood  $N$  of  $c$ ,  $x > c \Rightarrow f(x) > f(c)$  and  $x < c \Rightarrow f(x) < f(c)$  for all  $x \in N$ .
  - **Decreasing** (function  $f$  at a point  $c$ ): A function  $f$  such that in some neighborhood  $N$  of  $c$ ,  $x > c \Rightarrow f(x) < f(c)$  and  $x < c \Rightarrow f(x) > f(c)$  for all  $x \in N$ .
    - As an odd example,  $\operatorname{sgn} x$  is increasing at  $x = 0$ .
  - A function may oscillate sufficiently fast at a point to be neither increasing nor decreasing.
    - For example, for

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

“no matter how small a neighborhood of zero  $N$  may be, there are  $x$ ’s in  $N$  for which  $f(x)$  is positive and those for which it is negative. This function oscillates infinitely often between positive and negative values in every neighborhood of  $x = 0$ ” (Thomas, 1972, p. 107).

- When  $dy/dx > 0$ ,  $y$  is increasing. When  $dy/dx < 0$ ,  $y$  is decreasing. When  $dy/dx = 0$ ,  $y$  may be increasing (consider  $y = x^3$ ), decreasing (consider  $y = -x^3$ ), or neither (consider  $y = x^2$ ).
- There is a relation between increasing and decreasing points, and positive and negative slopes, respectively, of the tangent lines to those points.
- Knowing where a function is increasing or decreasing can help in sketching the curve.

### 4.2 Related Rates

- In certain physical settings, we must consider not only quantities but the rates at which those quantities are changing to answer questions.

- For a **problem in related rates**, it is typical that “(a) certain variables are related in a definite way for all value of  $t$  under consideration, (b) the values of some or all of these variables and the rates of change of some of them are given at some particular instant, and (c) it is required to find the rate of change of one or more of them at this instant” (Thomas, 1972, p. 110).
  - “The variables may then all be considered to be functions of time, and if the equations which relate them for all values of  $t$  are differentiated with respect to  $t$ , the new equations so obtained will tell how their rates of change are related” (Thomas, 1972, p. 110).
- We explore three examples to illustrate the most common techniques used.
- Suppose (see Figure 4.1a) there is a “rope running through a pulley at  $P$ , bearing a weight  $W$  at one end. The other end is held in a man’s hand  $M$  at a distance of 5 feet above the ground as he walks in a straight line at the rate of 6 [ft/s]” (Thomas, 1972, p. 108). Additionally (see Figure 4.1b), “suppose that the pulley is 25 ft above the ground, the rope is 45 ft long, and at a given instant the distance  $x$  is 15 ft and the man is walking away from the pulley. How fast is the weight being raised at this particular instant?” (Thomas, 1972, p. 109).

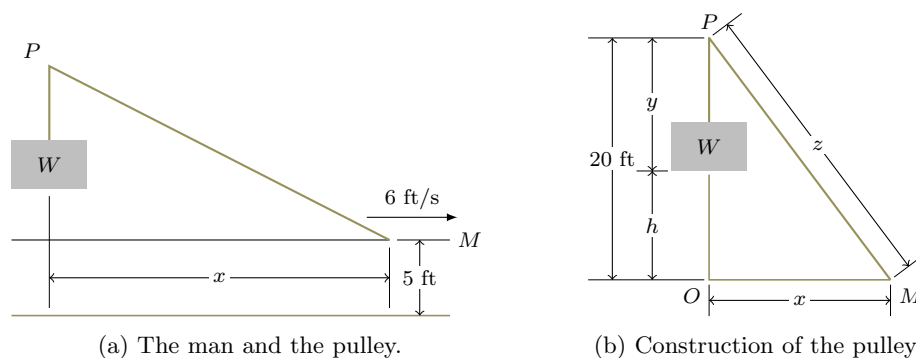


Figure 4.1: Related rates: The pulley.

- We begin by assessing what is given and what we want to find.

We are given...

- (a) Relationships between the variables which are to hold for all instants of time:

$$y + z = 45$$

$$h + y = 20$$

$$20^2 + x^2 = z^2$$

- (b) Quantities at a given instant in time, which we may take to be  $t = 0$ :

$$x = 15$$

$$\frac{dx}{dt} = 6$$

We want to find...

$$\frac{dh}{dt}$$

at the instant  $t = 0$ .

- We obtain a relationship between  $x$  (whose rate is given) and  $h$  (whose rate we want).

$$y = 20 - h$$

$$z = 45 - (20 - h) = 25 + h$$

$$20^2 + x^2 = (25 + h)^2$$

- We now implicitly differentiate the above equation with respect to  $t$  and solve for  $dh/dt$ .

$$\begin{aligned}\frac{d}{dt}(20^2 + x^2) &= \frac{d}{dt}(25 + h)^2 \\ 0 + 2x \frac{dx}{dt} &= 2(25 + h) \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{x}{25 + h} \frac{dx}{dt}\end{aligned}$$

- We see that we will need the value of  $h$  at  $t = 0$ . This can be found via the equation  $20^2 + x^2 = (25 + h)^2$  since we know the value of  $x$  at  $t = 0$ .

$$\begin{aligned}(25 + h)^2 &= 20^2 + (15)^2 \\ h &= 0\end{aligned}$$

- Since we now have every value that we have set equal to  $dh/dt$ , all that is left is to plug and chug.

$$\begin{aligned}\frac{dh}{dt} &= \frac{x}{25 + h} \frac{dx}{dt} \\ &= \frac{15}{25 + 0} \cdot 6 \\ &= \frac{18}{5} \text{ ft/s}\end{aligned}$$

- Suppose (see Figure 4.2) there is a “ladder 26 ft long which leans against a vertical wall. At a particular instant, the foot of the ladder is 10 ft out from the base of the wall and is being drawn away from the wall at the rate of 4 [ft/s]. How fast is the top of the ladder moving down the wall at this instant?” (Thomas, 1972, p. 110).

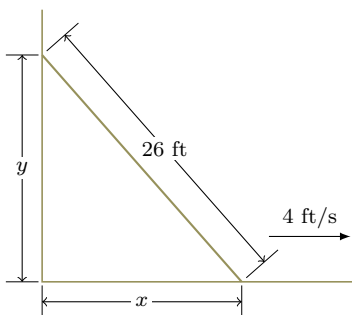


Figure 4.2: Related rates: The ladder.

- Symbolically, the problem is asking this: given

$$x^2 + y^2 = 26^2$$

$$x = 10$$

$$\frac{dx}{dt} = 4$$

find

$$\frac{dy}{dt}$$

- As before, differentiate and solve for  $dy/dt$ .

$$\begin{aligned}\frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}(26^2) \\ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{x}{y} \frac{dx}{dt}\end{aligned}$$

- Now find  $y$  and substitute.

$$\begin{aligned} 10^2 + y^2 &= 26^2 \\ y &= 24 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= -\frac{x}{y} \frac{dx}{dt} \\ &= -\frac{10}{24} \cdot 4 \\ &= -\frac{5}{3} \text{ ft/s} \end{aligned}$$

- Note that the negative sign indicates that  $y$  is decreasing; that the top of the ladder is moving *down* at  $5/3$  ft/s (or up at  $-5/3$  ft/s).
- Suppose there is an inverted right “conical reservoir [of height 10 ft and base radius 5 ft] into which water runs at the constant rate of  $2 \text{ ft}^3$  per minute. How fast is the water level rising when it is 6 ft deep?” (Thomas, 1972, p. 111).
  - Let  $h$  be the height (in ft) of the reservoir,  $r$  be the base radius (in ft) of the reservoir,  $x$  be the radius (in ft) of the section of the cone at the water line at time  $t$  (in min),  $y$  be the depth (in ft) of water in the tank at time  $t$  (in min), and  $v$  be the volume (in  $\text{ft}^3$ ) of water in the tank at time  $t$  (in min).
  - Thus, the problem is asking this: given

$$v = \frac{1}{3}\pi x^2 y \qquad \frac{x}{y} = \frac{r}{h}$$

$$h = 10 \qquad r = 5 \qquad y = 6 \qquad \frac{dv}{dt} = 2$$

find

$$\frac{dy}{dt}$$

- Like with the pulley, we need to find an equation relating just  $v$  and  $y$ . Use a substitution based on similar triangles.

$$\begin{aligned} v &= \frac{1}{3}\pi x^2 y \\ &= \frac{1}{3}\pi \left(\frac{ry}{h}\right)^2 y \\ &= \frac{\pi r^2}{3h^2} y^3 \end{aligned}$$

- Differentiate, solve, and substitute.

$$\begin{aligned} \frac{dv}{dt} &= \frac{\pi r^2}{h^2} y^2 \frac{dy}{dt} \\ \frac{dy}{dt} &= \frac{h^2}{\pi r^2 y^2} \frac{dv}{dt} \\ &= \frac{10^2}{\pi 5^2 6^2} \cdot 2 \\ &= \frac{2}{9\pi} \approx 0.071 \text{ ft/min} \end{aligned}$$

### 4.3 Significance of the Sign of the Second Derivative

- Note: if  $dy/dx$  fails to exist at some point  $P$ , *but*  $dx/dy = 0$ , the tangent to  $P$  is vertical.
  - On obtaining  $dx/dy$ <sup>[1]</sup>:

$$\frac{dx}{dy} = \left( \frac{dy}{dx} \right)^{-1}$$

- “The sign of the second derivative tells whether the graph of  $y = f(x)$  is concave upward ( $y''$  positive) or downward ( $y''$  negative)” (Thomas, 1972, p. 113).
- **Point of inflection:** “A point where the curve changes the direction of its concavity from downward to upward or vice versa [that is not a **cusp**]” (Thomas, 1972, p. 114). *Also known as inflection point.*
  - Inflection points occur where  $y''$  changes sign. This can happen when  $y'' = 0$  or when  $y''$  fails to exist.
- **Cusp:** A sharp corner on a graph (a place where  $y''$  fails to exist).

### 4.4 Curve Plotting

- When sketching curves given the equation, use the following procedure.
  - “Calculate  $dy/dx$  and  $d^2y/dx^2$ ” (Thomas, 1972, p. 115).
  - “Find the values of  $x$  for which  $dy/dx$  is positive and for which it is negative. Calculate  $y$  and  $d^2y/dx^2$  at the points of transition between positive and negative values of  $dy/dx$ . These may give maximum or minimum points on the curve” (Thomas, 1972, p. 115).
  - “Find the values of  $x$  for which  $d^2y/dx^2$  is positive and for which it is negative. Calculate  $y$  and  $dy/dx$  at the points of transition between positive and negative values of  $d^2y/dx^2$ . These may give points of inflection of the curve” (Thomas, 1972, p. 115).
  - “Plot a few additional points. In particular, points which lie between the transition points already determined or points which lie to the left and to the right of all of them will ordinarily be useful. The nature of the curve for large values of  $|x|$  should also be indicated” (Thomas, 1972, p. 115).
  - “Sketch a smooth curve through the points found above, unless there are discontinuities in the curve or its slope. Have the curve pass through its points rising or falling as indicated by the sign of  $dy/dx$ , and concave upward or downward as indicated by the sign of  $d^2y/dx^2$ ” (Thomas, 1972, p. 115).
- As you plot points, consider sketching their tangents, too.
- Consider making a table with columns of significant  $x$  values, their assigned  $y$ ,  $y'$ , and  $y''$  values, and any important remarks before starting to draw.
- If  $f(x) = \frac{P(x)}{Q(x)}$ , solve  $Q(x) = 0$  to find vertical asymptotes.

### 4.5 Maxima and Minima: Theory

- **Relative maximum** (of  $f$ ): A point  $(a, f(a))$  of a function  $f$  such that  $f(a) \geq f(a+h)$  for all positive and negative values of  $h$  sufficiently near zero. *Also known as local maximum.*
- **Relative minimum** (of  $f$ ): A point  $(b, f(b))$  of a function  $f$  such that  $f(b) \leq f(x)$  for all  $x$  in some neighborhood of  $a$ . *Also known as local minimum.*

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<sup>1</sup>This is another place where Leibniz’s notation is particularly useful.

- **Absolute maximum** (of  $f$ ): A point  $(a, f(a))$  of a function  $f$  such that  $f(a) \geq f(x)$  for all  $x \in D_f$ .
- **Absolute minimum** (of  $f$ ): A point  $(b, f(b))$  of a function  $f$  such that  $f(b) \leq f(x)$  for all  $x \in D_f$ .
- We now prove a relationship between  $f'$  and the maxima and minima of  $f$ .

**Theorem 4.1.** Let the function  $f$  be defined for  $a \leq x \leq b$  and have a relative maximum or minimum at  $x = c$ , where  $a < c < b$ . If the derivative  $f'(x)$  exists as a finite number at  $x = c$ , then  $f'(c) = 0$ .

*Proof.* If  $f'(c)$  were positive, then  $f$  would be increasing. But  $f$  is neither increasing nor decreasing at  $c$  because  $f$  has a local maximum or minimum at  $c$ . Hence,  $f'(c)$  cannot be positive. Likewise,  $f'(c)$  cannot be negative. Therefore,  $f'(c) = 0$ .  $\square$

- Note that the theorem does not pertain to cases where  $f'(c)$  does not exist, nor does it pertain to cases where  $c$  is at one of the endpoints of the interval  $[a, b]$ .
- Also note that the converse of the theorem does not hold.
- The inverse of an increasing function is increasing. This also holds for decreasing functions.
- If  $f$  is only defined on  $[a, b]$ , then  $f'(a)$  and  $f'(b)$  do not exist (because the limit is different on both sides). However,

$$f'(a^+) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \qquad f'(b^-) = \lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h}$$

may exist. These are called the **right-hand derivative** and **left-hand derivative** respectively.

- It's imprecise to say that  $f$  is differentiable at these endpoints, but many mathematicians will allow it<sup>[2]</sup> as they expect us to just know that what is really meant is it is differentiable on  $(a, b)$  and one-side differentiable at the endpoints.
- “If the domain of  $f$  is the bounded, closed interval  $[a, b]$  and if  $f'(a^+)$  and  $f'(b^-)$  exist, then it is easy to verify that  $f$  has a local [maximum or minimum] at  $a$  if  $[f'(a^+) < 0$  or  $f'(a^+) > 0$ , respectively] and  $f$  has a local [minimum or maximum] at  $b$  if  $[f'(b^-) < 0$  or  $f'(b^-) > 0]$ ” (Thomas, 1972, p. 121).
- Maxima and minima are more generally referred to as **critical points** or **extrema**.
- Candidates for extrema exist at points where (1) the derivative is zero, (2) the derivative fails to exist, and (3) the domain of the function has an end.

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<sup>2</sup>Really?