Chapter 17

Vector Analysis

17.1 Introduction: Vector Fields

12/29:

- In this chapter, we will consider vector functions of several variables, such as the function giving the velocity $\mathbf{v} = \mathbf{F}(x, y, z, t)$ of a particle in a fluid located at position (x, y, z) at time t.
- **Steady-state flow**: A flow for which the velocity function does not depend on the time t.
- **Vector field**: The collection of all vectors $\mathbf{F}(P)$ assigned to each point P in a region G.
- **Gradient field**: The vector field defined for points in the domain G of a scalar function T such that $\mathbf{F}(P) = \nabla T(P)$.

17.2 Surface Integrals

12/30:

- Just like we have $\mathrm{d}s = \sqrt{1 + f_x^2}\,\mathrm{d}x$, we have

$$\mathrm{d}\sigma = g(x, y)\,\mathrm{d}A$$

  where $\mathrm{d}\sigma$ is "an element of surface area in the tangent plane that approximates the corresponding portion $\Delta\sigma$ of the surface itself" (Thomas, 1972, p. 581) and $g(x, y) = \sqrt{1 + f_x^2 + f_y^2}$.
- Thus, we can think of surface area as either the lefthand or righthand side of the below equation.

$$\iint\limits_{\Sigma} \mathrm{d}\sigma = \iint\limits_{R} g(x, y)\,\mathrm{d}A$$

  - The lefthand interpretation sums infinitely many, infinitely small pieces $\mathrm{d}\sigma$ of the surface Σ.
  - The righthand interpretation sums infinitely many, infinitely small pieces $\mathrm{d}A$ of the shadow R of the surface Σ on the xy-plane, adjusted by the factor $g(x, y)$.
- These formulations are important because sometimes we want to conceive and evaluate an integral of the form $\iint_\Sigma h(x, y, z)\,\mathrm{d}\sigma$.
- **Surface integral** (of $h(x, y, z)$ over the surface Σ): The limit as $\Delta\sigma \to 0$ of the sum of every $\Delta\sigma_k$ (composing Σ) times $h(x, y, z)$ for some $(x, y, z) \in \Delta\sigma_k$. Mathematically,

$$\iint\limits_{\Sigma} h(x, y, z)\,\mathrm{d}\sigma = \lim_{\Delta\sigma \to 0} \sum_{k=1}^{n} h(x_k, y_k, z_k)\,\Delta\sigma_k$$

- Consider a surface Σ consisting of all points $P(x, y, z)$ satisfying $z = f(x, y)$ for $(x, y) \in R$, where R is a closed, bounded region of the xy-plane and f, f_x, f_y are continuous throughout R and its boundary.
- Approximate R by dividing it into n rectangles using lines parallel to the y-axis spaced Δx apart and lines parallel to the x-axis spaced Δy apart.
- Let the part of Σ above each rectangle be denoted by $\Delta\sigma_k$ for some $1 \leq k \leq n$.
- Now if $P_k(x_k, y_k, z_k)$ is a point in $\Delta\sigma_k$, we can consider the above sum and take its limit.