# Chapter 14

# Vector Functions and Their Derivatives

## 14.1 Introduction

12/14:

- **Vector function** (of $h$): A function $\mathbf{F}(h)$ with $n$ components where each component is a function. Essentially, $\mathbf{F} = (f_1, f_2, \dots, f_n)$.
- **Limit** (of $\mathbf{F}(h)$ as $h \to a$): If each component $f_1, \dots, f_n$ of $\mathbf{F}$ has a limit $L_1, \dots, L_n$ as $h \to a$, then

$$\lim_{h \to a} \mathbf{F}(h) = (L_1, \dots, L_n)$$

- **Continuous** (vector function $\mathbf{F}$ at $a$): A vector function $\mathbf{F}$ where for every $\epsilon > 0$, there corresponds a $\delta > 0$ such that

$$|\mathbf{F}(h) - \mathbf{F}(a)| < \epsilon \quad \text{when} \quad |h - a| < \delta$$

  - Thomas, 1972 shows that this is equivalent to the requirement that each component of $\mathbf{F}$ is continuous at $a$.
- **Derivative** (of a vector function at $c$): The derivative $\mathbf{F}'(c)$ of a vector function $\mathbf{F}$ at $c$ is given by the equation

$$\mathbf{F}'(c) = \lim_{h \to 0} \frac{\mathbf{F}(c+h) - \mathbf{F}(c)}{h}$$

  - It can be proven that $\mathbf{F}$ is differentiable at $c$ if and only if each of its components are differentiable at $c$, and that if this condition is met,

$$\mathbf{F}'(c) = (f_1'(c), \dots, f_n'(c))$$

## 14.2 Velocity and Acceleration

- Results from here on out will generally pertain to 2D questions, but these methods can easily be generalized to higher dimensions.
- Applications of vectors to physics problems.
  - To solve **statics** problems, we only need to know the **algebra** of vectors.
  - To solve **dynamics** problems, we also need to know the **calculus** of vectors.
- **Position vector**: The vector from the origin to a point $P$ that moves along a parametrically defined curve. *Denoted by* $\mathbf{R}$.

- **Velocity vector**: The vector tangent to a point $P$ that moves along a parametrically defined curve and with magnitude $|\mathrm{d}s/\mathrm{d}t|$. *Denoted by* $\mathbf{v}$.

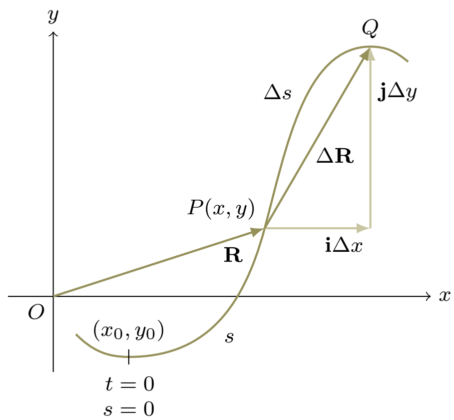



Figure 14.1: Velocity vector.

  - Thomas, 1972 semi-rigorously proves from Figure 14.1 that if $\mathbf{R}$ is the position vector, then $\mathrm{d}\mathbf{R}/\mathrm{d}t$ is the velocity vector.
  - Essentially, he proves that

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}t}$$

It follows from this that

$$\text{slope of } \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{\mathbf{j}\text{-component}}{\mathbf{i}\text{-component}} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left|\frac{\mathrm{d}R}{\mathrm{d}t}\right| = \left|\mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}t}\right| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|$$

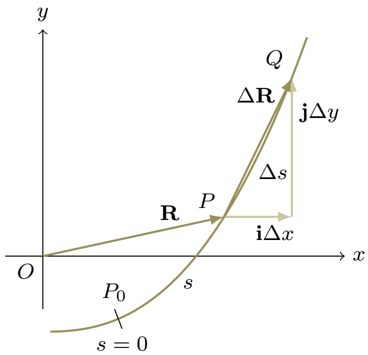
- **Acceleration vector**: The derivative of the velocity vector and second derivative of the position vector. *Denoted by* $\mathbf{a}$.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathbf{j}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

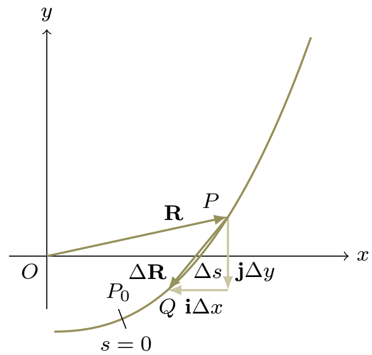
- Sometimes, we are given a force vector $\mathbf{F} = m\mathbf{a}$ and initial conditions.
  - From these, we can solve for velocity and position vectors via fairly straightforward component integration.
  - Note, however, that constants of integration are now vectors.

## 14.3 Tangential Vectors

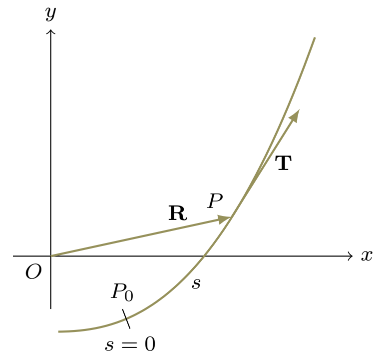
- Let $P_0$ be a point on a curve. The distance $s$ from $P_0$ to some point $P$ along the curve is clearly related to the position of $P$. Thus, we may think of $\mathbf{R}$ as a function of $s$, and investigate the properties of $\mathrm{d}\mathbf{R}/\mathrm{d}s$.
- **Tangent vector**: The unit vector tangent to a point $P$ along a curve.
  - Since $\Delta\mathbf{R}$ and $\Delta s$ approach the same quantity as $\Delta s \to 0$, $\Delta\mathbf{R}/\Delta s$ approaches unity, i.e., $|\mathrm{d}\mathbf{R}/\mathrm{d}s| = 1$.
  - Because of the sign change, whether $\Delta s$ is positive or negative, $\Delta\mathbf{R}/\Delta s$ points in the same general direction for sufficiently small $\Delta s$. Indeed, it converges to pointing tangentially.




(a) $\Delta s > 0$.




(b) $\Delta s < 0$.




(c) $\Delta s \to 0$.

Figure 14.2: Tangent vector.

- Thus,

$$\mathbf{T} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s}$$

- There are two different ways to find $\mathbf{T}$: Straight differentiation combined with manipulations of differentials, and the chain rule combined with the dot product. We will explore each, in turn, with an example.
- "Find the unit vector $\mathbf{T}$ tangent to the circle $x = a\cos\theta$, $y = a\sin\theta$ at any point $P(x, y)$" (Thomas, 1972, p. 471).
  - From the given equations, we have

$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta \qquad \mathrm{d}y = a\cos\theta\,\mathrm{d}\theta \qquad \begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= a^2(\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta^2 \\ &= a^2\,\mathrm{d}\theta^2 \\ \mathrm{d}s &= \pm a\,\mathrm{d}\theta \end{aligned}$$

    - We could alternatively obtain $\mathrm{d}s$ by expressing the arc length formula $S = R\theta$ in terms of differentials.
  - "If we measure arc length in the counterclockwise direction, with $s = 0$ at $(a, 0)$, $s$ will be an increasing function of $\theta$, so the $+$-sign should be taken: $\mathrm{d}s = a\,\mathrm{d}\theta$" (Thomas, 1972, p. 471).
  - Therefore,

$$\begin{aligned} \mathbf{T} &= \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= \mathbf{i}\left(\frac{-a\sin\theta\,\mathrm{d}\theta}{a\,\mathrm{d}\theta}\right) + \mathbf{j}\left(\frac{a\cos\theta\,\mathrm{d}\theta}{a\,\mathrm{d}\theta}\right) \\ &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned}$$

- The equations

$$x = a\cos\omega t \qquad y = a\sin\omega t \qquad z = bt$$

  where $a, b, \omega$ are positive constants define a circular helix in $E^3$[1].

[1] Three-dimensional Euclidean space, equivalent to $\mathbb{R}^3$

- Let $P_0 = (a, 0, 0)$, since this is the point on the locus of the parametric equations where $t = 0$. Additionally, let arc length be measured in the direction in which $P$ moves away from $P_0$ as $t$ increases from 0.
- Using the chain rule to differentiate, we have

$$\begin{aligned}\mathbf{T} &= \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s} + \mathbf{k}\frac{\mathrm{d}z}{\mathrm{d}s} \\ &= \mathbf{i}\left(-a\omega \sin \omega t \, \frac{\mathrm{d}t}{\mathrm{d}s}\right) + \mathbf{j}\left(a\omega \cos \omega t \, \frac{\mathrm{d}t}{\mathrm{d}s}\right) + \mathbf{k}\left(b \, \frac{\mathrm{d}t}{\mathrm{d}s}\right)\end{aligned}$$

- Since $\mathbf{T}$ is a unit vector, we have $1 = |\mathbf{T}| = |\mathbf{T}|^2 = \mathbf{T} \cdot \mathbf{T}$. Thus,

$$\begin{aligned}1 &= \mathbf{T} \cdot \mathbf{T} \\ &= \mathbf{i} \cdot \mathbf{i}\left(-a\omega \sin \omega t \, \frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{j} \cdot \mathbf{j}\left(a\omega \cos \omega t \, \frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{k} \cdot \mathbf{k}\left(b \, \frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \\ &= \left(a^2\omega^2 + b^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= \pm\frac{1}{\sqrt{a^2\omega^2 + b^2}}\end{aligned}$$

- We choose the +-sign because $s$ should be a positive function of $t$.
- Putting this all together, we get

$$\mathbf{T} = \frac{a\omega(-\mathbf{i}\sin \omega t + \mathbf{j}\cos \omega t) + \mathbf{k}b}{\sqrt{a^2\omega^2 + b^2}}$$