Chapter 4

Applications

4.1 Increasing or Decreasing Functions: The Sign of $\mathrm{d}y/\mathrm{d}x$

7/8:

- **Increasing** (function f on $[a, b]$): A function f such that $f(x_1) > f(x_2)$ when $x_1 > x_2$ for all x_1, x_2 in the interval $[a, b]$. *Also known as* **rising**.
- **Decreasing** (function f on $[a, b]$): A function f such that $f(x_1) < f(x_2)$ for $a \leq x_2 < x_1 \leq b$. *Also known as* **falling**.
  - Sometimes, we consider functions that increase or decrease on open or half-open intervals.
- **Increasing** (function f at a point c): A function f such that in some neighborhood N of c, $x > c \Rightarrow f(x) > f(c)$ and $x < c \Rightarrow f(x) < f(c)$ for all $x \in N$.
- **Decreasing** (function f at a point c): A function f such that in some neighborhood N of c, $x > c \Rightarrow f(x) < f(c)$ and $x < c \Rightarrow f(x) > f(c)$ for all $x \in N$.
  - As a strange example, $\operatorname{sgn} x$ is increasing at $x = 0$.
- A function may oscillate sufficiently fast at a point to be neither increasing nor decreasing.
  - For example, for

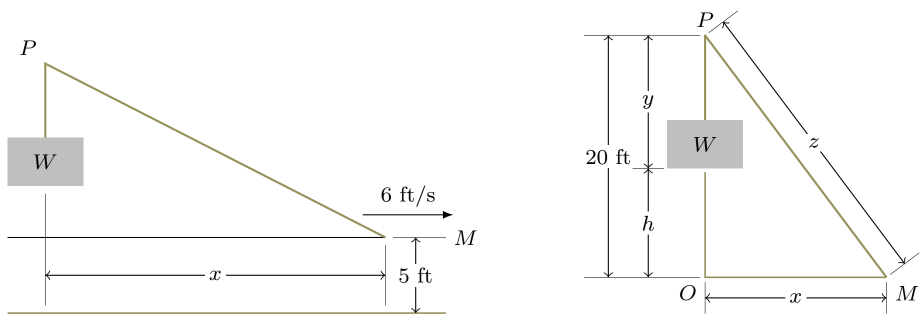
    $$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

    "no matter how small a neighborhood of zero N may be, there are x's in N for which $f(x)$ is positive and those for which it is negative. This function oscillates infinitely often between positive and negative values in ever neighborhood of $x = 0$" (Thomas, 1972, p. 107).
- When $\mathrm{d}y/\mathrm{d}x > 0$, y is increasing. When $\mathrm{d}y/\mathrm{d}x < 0$, y is decreasing. When $\mathrm{d}y/\mathrm{d}x = 0$, y may be increasing (consider $y = x^3$), decreasing (consider $y = -x^3$), or neither (consider $y = x^2$).
- There is a relation between increasing and decreasing points, and positive and negative slopes, respectively, of the tangent lines to those points.
- Knowing where a function is increasing or decreasing can help in sketching the curve.

4.2 Related Rates

- In certain physical settings, we must consider not only quantities but the rates at which those quantities are changing to answer questions.

- For a **problem in related rates**, it is typical that "(a) certain variables are related in a definite way for all values of t under consideration, (b) the values of some or all of these variables and the rates of change of some of them are given at some particular instant, and (c) it is required to find the rate of change of one or more of them at this instant" (Thomas, 1972, p. 110).
  - "The variables may then all be considered to be functions of time, and if the equations which relate them for all values of t are differentiated with respect to t, the new equations so obtained will tell how their rates of change are related" (Thomas, 1972, p. 110).
- We explore three examples to illustrate the most common techniques used.
- Suppose (see Figure 4.1a) there is a "rope running through a pulley at P, bearing a weight W at one end. The other end is held in a man's hand M at a distance of 5 feet above the ground as he walks in a straight line at the rate of 6 [ft/s]" (Thomas, 1972, p. 108). Additionally (see Figure 4.1b), "suppose that the pulley is 25 ft above the ground, the rope is 45 ft long, and at a given instant the distance x is 15 ft and the man is walking away from the pulley. How fast is the weight being raised at this particular instant?" (Thomas, 1972, p. 109).



(a) The man and the pulley. (b) Construction of the pulley.

Figure 4.1: Related rates: The pulley.

  - We begin by assessing what is given and what we want to find.
    We are given...
    (a) Relationships between the variables which are to hold for all instants of time:
    $$y + z = 45 \qquad h + y = 20 \qquad 20^2 + x^2 = z^2$$
    (b) Quantities at a given instant in time, which we may take to be $t = 0$:
    $$x = 15 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 6$$
    We want to find...
    $$\frac{\mathrm{d}h}{\mathrm{d}t}$$
    at the instant $t = 0$.
  - We obtain a relationship between x (whose rate is given) and h (whose rate we want).
    $$\begin{gathered} y = 20 - h \\ z = 45 - (20 - h) = 25 + h \\ 20^2 + x^2 = (25 + h)^2 \end{gathered}$$

– We now implicitly differentiate the above equation with respect to t and solve for $\mathrm{d}h/\mathrm{d}t$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(20^2+x^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(25+h)^2\\ 0+2x\frac{\mathrm{d}x}{\mathrm{d}t} &= 2(25+h)\frac{\mathrm{d}h}{\mathrm{d}t}\\ \frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{x}{25+h}\frac{\mathrm{d}x}{\mathrm{d}t}\end{aligned}$$

– We see that we will need the value of h at $t=0$. This can be found via the equation $20^2+x^2=(25+h)^2$ since we know the value of x at $t=0$.

$$\begin{aligned}(25+h)^2 &= 20^2+(15)^2\\ h &= 0\end{aligned}$$

– Since we now have every value that we have set equal to $\mathrm{d}h/\mathrm{d}t$, all that is left is to plug and chug.

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{x}{25+h}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{15}{25+0}\cdot 6\\ &= \frac{18}{5}\ \text{ft/s}\end{aligned}$$

- Suppose (see Figure 4.2) there is a "ladder 26 ft long which leans against a vertical wall. At a particular instant, the foot of the ladder is 10 ft out from the base of the wall and is being drawn away from the wall at the rate of 4 [ft/s]. How fast is the top of the ladder moving down the wall at this instant?" (Thomas, 1972, p. 110).

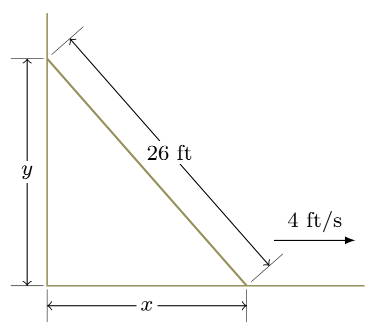


Figure 4.2: Related rates: The ladder.

– Symbolically, the problem is asking this: given

$$x^2+y^2=26^2 \qquad x=10 \qquad \frac{\mathrm{d}x}{\mathrm{d}t}=4$$

find

$$\frac{\mathrm{d}y}{\mathrm{d}t}$$

– As before, differentiate and solve for $\mathrm{d}y/\mathrm{d}t$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2+y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(26^2\right)\\ 2x\frac{\mathrm{d}x}{\mathrm{d}t}+2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}\end{aligned}$$

- Now find y and substitute.

$$\begin{aligned} 10^2 + y^2 &= 26^2 \\ y &= 24 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -\frac{10}{24} \cdot 4 \\ &= -\frac{5}{3} \text{ ft/s} \end{aligned}$$

- Note that the negative sign indicates that y is decreasing; that the top of the ladder is moving *down* at 5/3 ft/s (or up at $-5/3$ ft/s).

- Suppose there is an inverted right "conical reservoir [of height 10 ft and base radius 5 ft] into which water runs at the constant rate of 2 ft^3 per minute. How fast is the water level rising when it is 6 ft deep?" (Thomas, 1972, p. 111).
  - Let h be the height (in ft) of the reservoir, r be the base radius (in ft) of the reservoir, x be the radius (in ft) of the section of the cone at the water line at time t (in min), y be the depth (in ft) of water in the tank at time t (in min), and v be the volume (in ft^3) of water in the tank at time t (in min).
  - Thus, the problem is asking this: given

$$v = \frac{1}{3}\pi x^2 y \qquad \frac{x}{y} = \frac{r}{h}$$

$$h = 10 \qquad r = 5 \qquad y = 6 \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = 2$$

find

$$\frac{\mathrm{d}y}{\mathrm{d}t}$$

  - Like with the pulley, we need to find an equation relating just v and y. Use a substitution based on similar triangles.

$$\begin{aligned} v &= \frac{1}{3}\pi x^2 y \\ &= \frac{1}{3}\pi \left(\frac{ry}{h}\right)^2 y \\ &= \frac{\pi r^2}{3h^2} y^3 \end{aligned}$$

  - Differentiate, solve, and substitute.

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\pi r^2}{h^2} y^2 \frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{h^2}{\pi r^2 y^2}\frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{10^2}{\pi 5^2 6^2} \cdot 2 \\ &= \frac{2}{9\pi} \approx 0.071 \text{ ft/min} \end{aligned}$$

4.3 Significance of the Sign of the Second Derivative

- Note: if $\mathrm{d}y/\mathrm{d}x$ fails to exist at some point P, *but* $\mathrm{d}x/\mathrm{d}y = 0$, the tangent to P is vertical.
  - On obtaining $\mathrm{d}x/\mathrm{d}y$[1]:
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$$
- "The sign of the second derivative tells whether the graph of $y = f(x)$ is concave upward (y'' positive) or downward (y'' negative)" (Thomas, 1972, p. 113).
- **Point of inflection**: "A point where the curve changes the direction of its concavity from downward to upward or vice versa [that is not a **cusp**]" (Thomas, 1972, p. 114). *Also known as* **inflection point**.
  - Inflection points occur where y'' changes sign. This can happen when $y'' = 0$ or when y'' fails to exist.
- **Cusp**: A sharp corner on a graph (a place where y' fails to exist).

4.4 Curve Plotting

- When sketching curves given the equation, use the following procedure.
  A. "Calculate $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}^2y/\mathrm{d}x^2$" (Thomas, 1972, p. 115).
  B. "Find the values of x for which $\mathrm{d}y/\mathrm{d}x$ is positive and for which it is negative. Calculate y and $\mathrm{d}^2y/\mathrm{d}x^2$ at the points of transition between positive and negative values of $\mathrm{d}y/\mathrm{d}x$. These may give maximum or minimum points on the curve" (Thomas, 1972, p. 115).
  C. "Find the values of x for which $\mathrm{d}^2y/\mathrm{d}x^2$ is positive and for which it is negative. Calculate y and $\mathrm{d}y/\mathrm{d}x$ at the points of transition between positive and negative values of $\mathrm{d}^2y/\mathrm{d}x^2$. These may give points of inflection of the curve" (Thomas, 1972, p. 115).
  D. "Plot a few additional points. In particular, points which lie between the transition points already determined or points which lie to the left and to the right of all of them will ordinarily be useful. The nature of the curve for large values of $|x|$ should also be indicated" (Thomas, 1972, p. 115).
  E. "Sketch a smooth curve through the points found above, unless there are discontinuitites in the curve or its slope. Have the curve pass through its points rising or falling as indicated by the sign of $\mathrm{d}y/\mathrm{d}x$, and concave upward or downward as indicated by the sign of $\mathrm{d}^2y/\mathrm{d}x^2$" (Thomas, 1972, p. 115).
- As you plot points, consider sketching their tangents, too.
- Consider making a table with columns of significant x values, their assigned y, y', and y'' values, and any important remarks before starting to draw.
- If $f(x) = \frac{P(x)}{Q(x)}$, solve $Q(x) = 0$ to find vertical asymptotes.

4.5 Maxima and Minima: Theory

- **Relative maximum** (of f): A point $(a, f(a))$ of a function f such that $f(a) \geq f(a+h)$ for all positive and negative values of h sufficiently near zero. *Also known as* **local maximum**.
- **Relative minimum** (of f): A point $(b, f(b))$ of a function f such that $f(b) \leq f(x)$ for all x in some neighborhood of a. *Also known as* **local minimum**.

[1]This is another place where Leibniz's notation is particularly useful.

- **Absolute maximum** (of f): A point $(a, f(a))$ of a function f such that $f(a) \geq f(x)$ for all $x \in D_f$.
- **Absolute minimum** (of f): A point $(b, f(b))$ of a function f such that $f(b) \geq f(x)$ for all $x \in D_f$.
- We now prove a relationship between f' and the maxima and minima of f.

  Theorem 4.1. Let the function f be defined for $a \leq x \leq b$ and have a relative maximum or minimum at $x = c$, where $a < c < b$. If the derivative $f'(x)$ exists as a finite number at $x = c$, then $f'(c) = 0$.

  Proof. If $f'(c)$ were positive, then f would be increasing. But f is neither increasing nor decreasing at c because f has a local maximum or minimum at c. Hence, $f'(c)$ cannot be positive. Likewise, $f'(c)$ cannot be negative. Therefore, $f'(c) = 0$. □

  - Note that the theorem does not pertain to cases where $f'(c)$ does not exist, nor does it pertain to cases where c is at one of the endpoints of the interval $[a, b]$.
  - Also note that the converse of the theorem does not hold.
- The inverse of an increasing function is increasing. This also holds for decreasing functions.
- If f is only defined on $[a, b]$, then $f'(a)$ and $f'(b)$ do not exist (because the limit is different on both sides). However,

  $$f'(a^+) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \qquad f'(b^-) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

  may exist. These are called the **right-hand derivative** and **left-hand derivative** respectively.

  - It's imprecise to say that f is differentiable at these endpoints, but many mathematicians will allow it[2] as they expect us to just know that what is really meant is it is differentiable on (a, b) and one-side differentiable at the endpoints.
- "If the domain of f is the bounded, closed interval $[a, b]$ and if $f'(a^+)$ and $f'(b^-)$ exist, then it is easy to verify that f has a local [maximum or minimum] at a if [$f'(a^+) < 0$ or $f'(a^+) > 0$, respectively] and f has a local [minimum or maximum] at b if [$f'(b^-) < 0$ or $f'(b^-) > 0$]" (Thomas, 1972, p. 121).
- Maxima and minima are more generally referred to as **critical points** or **extrema**.
- Candidates for extrema exist at points where (1) the derivative is zero, (2) the derivative fails to exist, and (3) the domain of the function has an end.

4.6 Maxima and Minima: Problems

7/9:

- Basically optimization: Maximizing or minimizing functions using differential calculus.
- We explore a number of problems to illustrate the most common techniques used.
- "Find two positive numbers whose sum is 20 and such that their product is as large as possible" (Thomas, 1972, p. 122).
  - We want $x + y = 20$ and $x \cdot y = \max$. Thus, we need to use calculus on xy. But xy is a function of multiple variables. However, using the substitution, $y = 20 - x$, we can change xy into $20x - x^2$ and optimize that.
  - Since x, y are positive, $x \geq 0$ and $20 - x = y \geq 0 \Rightarrow x \leq 20$. Thus, the domain of $20x - x^2$ is $[0, 20]$.

[2]Really?

 - Critical points exist where $0 = \mathrm{d}y/\mathrm{d}x = 20 - 2x$, or where $x = 10$, and at the endpoints. Since $\mathrm{d}^2y/\mathrm{d}x^2 = -2$ (is negative), we know that the point at $x = 10$ is a maximum. Since $20(10) - (10)^2 > 20(0) - (0)^2 = 20(20) - 20^2$, the point at $x = 10$ is *the* maximum that we're looking for.
 - Thus, $x = 10$, $y = 20 - 10 = 10$ are the two numbers whose sum is 20 and whose product is as large as possible.

- "A square sheet of tin a inches on a side is to be used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner for the box to have as large a volume as possible?" (Thomas, 1972, p. 122).
 - Suppose we cut a square of $x \times x$ in^2 from each corner of the tin sheet. Then the base of the box, when folded up, would be $a - 2x \times a - 2x$ in^2, and the height would be x in. Thus, the volume v of the box as a function of the side length x of one of the squares removed is

$$v(x) = x(a - 2x)^2$$

 - Since we cannot remove a negative area, $x \geq 0$. Furthermore, since we cannot remove more area than exists, $a - 2x \geq 0 \Rightarrow x \leq a/2$. Thus, $D_v = [0, a/2]$.
 - Critical points exist where $0 = \mathrm{d}y/\mathrm{d}x = 12x^2 - 8ax + a^2 = (2x - a)(6x - a)$, or where $x = \frac{a}{2}, \frac{a}{6}$, and at the left endpoint (the right endpoint is already one of the critical points indicated by the first derivative). Since

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{a}{6}} = -4a \qquad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{a}{2}} = 4a$$

 we know that only the point at $x = \frac{a}{6}$ is a maximum. In comparison with the point at $x = 0$, since $v\left(\frac{a}{6}\right) > v(0)$, the point at $x = \frac{a}{6}$ is *the* maximum.
 - Thus, every corner square removed should have dimensions $\frac{a}{6} \times \frac{a}{6}$ in^2.

- "An oil can is to be made in the form of a right circular cylinder and is to contain one quart of oil. What dimensions of the can will require the least amount of material?" (Thomas, 1972, p. 123).
 - There are 57.75 in^3 in a quart. Thus, we require that $57.75 = V = \pi r^2 h$.
 - We interpret "least amount of material" to mean "cylinder with the smallest surface area." Thus, we seek to minimize $A = 2\pi r^2 + 2\pi rh$.
 - We need to substitute either r or h from the volume equation into the area equation. Since $h = \frac{V}{\pi r^2}$ is comparatively simpler algebraically than $r = \sqrt{\frac{V}{\pi h}}$, choose to substitute out the h in the area equation, giving

$$A = 2\pi r^2 + \frac{2V}{r}$$

 - Considering the physical situation, we find that $r \in (0, \infty)$.
 - We differentiate A with respect to r and find the points where $\mathrm{d}A/\mathrm{d}r$ is equal to zero. Note that in this case, there are no endpoints to consider.

$$\begin{aligned} 0 &= \frac{\mathrm{d}A}{\mathrm{d}r} \\ &= 4\pi r - 2Vr^{-2} \\ 4\pi r^3 &= 2V \\ r &= \sqrt[3]{\frac{V}{2\pi}} \end{aligned}$$

- At this value of r,
$$\left.\frac{\mathrm{d}^2 A}{\mathrm{d}r^2}\right|_{r=\sqrt[3]{V/2\pi}} = 4\pi + 4Vr^{-3}\big|_{r=\sqrt[3]{V/2\pi}} = 12\pi$$
Thus, the point at $r = \sqrt[3]{\frac{V}{2\pi}}$ is a relative minimum. Since $\frac{\mathrm{d}^2 A}{\mathrm{d}r^2}$ is positive for all $r \in D_A$, the point at $r = \sqrt[3]{\frac{V}{2\pi}}$ is the absolute minimum.
- Therefore, the radius and height of the desired oil can are given by
$$r = \sqrt[3]{\frac{V}{2\pi}} \approx 2.09 \text{ in} \qquad\qquad h = 2\sqrt[3]{\frac{V}{2\pi}} \approx 4.19 \text{ in}$$
- Note that there is an alternate method of solving problems of this type: Related rates.
  - We have $V = \pi r^2 h$ and $A = 2\pi r^2 + 2\pi rh$. Since V is a constant, $\mathrm{d}V/\mathrm{d}r = 0$. Since we want to find critical points of A, we set $\mathrm{d}A/\mathrm{d}r = 0$.
  - We find that
$$\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r + 2\pi\left(h + r\frac{\mathrm{d}h}{\mathrm{d}r}\right)$$ [3]
  - We can find $\mathrm{d}h/\mathrm{d}r$ by implicitly differentiating $V = \pi r^2 h$.
$$\frac{\mathrm{d}V}{\mathrm{d}r} = 2\pi rh + \pi r^2 \frac{\mathrm{d}h}{\mathrm{d}r}$$
Thus,
$$0 = 2\pi rh + \pi r^2 \frac{\mathrm{d}h}{\mathrm{d}r}$$
$$\frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{2h}{r}$$
  - Substituting, we find that
$$\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - 2\pi h$$
which implies (since we only care about the above equation when $\mathrm{d}A/\mathrm{d}r = 0$) that
$$h = 2r$$
  - Bringing back $V = \pi r^2 h$, we can now find that
$$V = \pi r^2 (2r) \qquad\qquad V = \pi\left(\frac{h}{2}\right)^2 h$$
$$r = \sqrt[3]{\frac{V}{2\pi}} \qquad\qquad h = 2\sqrt[3]{\frac{V}{2\pi}}$$
  - Since
$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = 4\pi + \frac{4\pi h}{r}$$
the second derivative is positive for all permissible values of r, h. Thus, the values that we have found are minimums.

- "A wire of length L is to be cut into two pieces, one of which is bent to form a circle and the other to form a square. How should the wire be cut if the sum of the areas enclosed by the two pieces is to be a maximum?" (Thomas, 1972, p. 125).

[3] Remember product rule implicit differentiation!

– Note: choose your variable names wisely. One could choose $x + y = L \Rightarrow \frac{x^2}{16} + \frac{y^2}{4\pi} = \text{max}$, *or* one could choose $2\pi r + 4x = L \Rightarrow \pi r^2 + x^2 = \text{max}$.

– We use related rates, as in the second answer to the previous problem. Because of the similarity in method, I will transcribe the intro math alone, without prose.

$$L = 2\pi r + 4x \qquad\qquad A = \pi r^2 + x^2$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 2\pi + 4\frac{\mathrm{d}x}{\mathrm{d}r} \qquad\qquad \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r - \pi x$$

$$\frac{\mathrm{d}x}{\mathrm{d}r} = -\frac{\pi}{2} \qquad\qquad \frac{\mathrm{d}^2A}{\mathrm{d}r^2} = 2\pi + \frac{\pi^2}{2}$$

$$x = 2r \Rightarrow \begin{cases} r = \frac{L}{2\pi+8} \\ x = \frac{L}{\pi+4} \end{cases}$$

– Now is where it gets interesting. Since the second derivative is always positive, the r, x values above represent the *minimum* area, not the *maximum*. Thus, in this case, it is actually *necessary* to consider the endpoints.

– Let $r \in [0, L/2\pi]$. Thus,

$$A = \frac{1}{16}L^2,\ r = 0 \qquad A = \frac{1}{4\pi + 16}L^2,\ r = \frac{L}{2\pi + 8} \qquad A = \frac{1}{4\pi}L^2,\ r = \frac{L}{2\pi}$$

It is now clear that $A = \text{max}$ at the right endpoint. Thus, to maximize the enclosed area, dedicate the entirety of the wire to making a circle.

- "Fermat's principle in optics states that light travels from a point A to a point B along that path for which the time of travel is a minimum. Let us find the path that a ray of light will follow in going from a point A in a medium where the velocity of light is c_1 to a point B in a second medium where the velocity of light is c_2, when both points lie in the xy-plane and the x-axis separates the two media" (Thomas, 1972, p. 125).

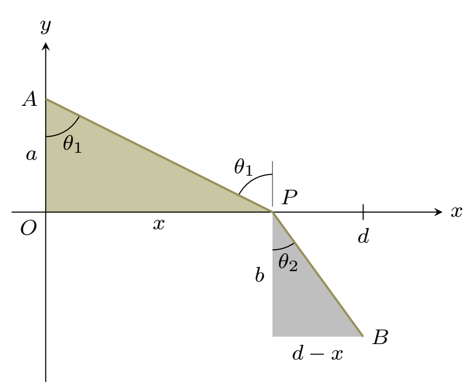


Figure 4.3: Optimization: Fermat's principle (optics).

– WLOG, let point A lie on the positive y-axis, as in Figure 4.3.

– In either medium, light will travel in a straight line since "shortest time" and "shortest path" are equivalent statements. Thus, the path will consist of a straight line segment from A to some point P along the x-axis, and then another straight line segment from P to B.

– Since $v = \frac{\Delta x}{\Delta t}$ relates velocity, distance, and time, and we are looking to minimize time, we observe that

$$t_{AP} = \frac{\sqrt{a^2 + x^2}}{c_1} \qquad\qquad t_{PB} = \frac{\sqrt{b^2 + (d - x)^2}}{c_2}$$

  - We want to collectively minimize $t = t_{AP} + t_{PB}$, a function of x. Thus, we find

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{d-x}{c_2\sqrt{b^2+(d-x)^2}}$$

  - As it so happens, from Figure 4.3, we can see that

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}$$

  - Thus, $\mathrm{d}t/\mathrm{d}x = 0$ when

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$$

    and when θ_1, θ_2 are in domains that makes sense for the physical problem.
  - Note that "instead of determining this value of x explicitly, it is customary to characterize the path followed by the ray of light by leaving the equation for $\mathrm{d}t/\mathrm{d}x = 0$ in the above form[4]" (Thomas, 1972, p. 126).

- "Suppose a manufacturer can sell x items per week at a price $P = 200 - 0.01x$ cents, and that it costs $y = 50x + 20000$ cents to produce the x items. What is the production level for maximum profits?" (Thomas, 1972, p. 127).
  - The total revenue per week on x items is $xP = 200x - 0.01x^2$.
  - The total profit T per week on x items is $T = xP - y = -0.01x^2 + 150x - 20000$.
  - T maximizes when

$$\begin{aligned} 0 &= \frac{\mathrm{d}T}{\mathrm{d}x} \\ &= -0.02x + 150 \\ x &= 7500 \text{ units} \end{aligned}$$

  - These units should be sold at $1.25 per item.

- We conclude by outlining a general procedure to be followed for optimization questions.
  1. "When possible, draw a figure to illustrate the problem and label those parts that are important in the problem. Constants and variables should be clearly distinguished" (Thomas, 1972, p. 126).
  2. "Write an equation for the quantity that is to be a maximum or a minimum. If this quantity is denoted by y, it is desirable to express it in terms of a single independent variable x. This may require some algebraic manipulation to make use of auxiliary conditions of the problem" (Thomas, 1972, p. 127).
  3. "If $y = f(x)$ is the quantity to be a maximum or a minimum, find those values of x for which… $f'(x) = 0$" (Thomas, 1972, p. 127).
  4. "Test each value of x for which $f'(x) = 0$ to determine wheether it provides a maximum or minimum or neither. The usual tests are:
     (a) If $\mathrm{d}^2y/\mathrm{d}x^2$ is positive when $\mathrm{d}y/\mathrm{d}x = 0$, then y is a minimum.
     If $\mathrm{d}^2y/\mathrm{d}x^2$ is negative when $\mathrm{d}y/\mathrm{d}x = 0$, then y is a minimum.
     If $\mathrm{d}^2y/\mathrm{d}x^2 = 0$ when $\mathrm{d}y/\mathrm{d}x = 0$, then the test fails.

[4] This is known as the law of refraction or Snell's law. More can be read about this law on Sears (1949, p. 27).

(b) If

$$\frac{dy}{dx} \text{ is } \begin{cases} \text{positive} & \text{for } x < x_c \\ \text{zero} & \text{for } x = x_c \\ \text{negative} & \text{for } x > x_c \end{cases}$$

then a maximum occurs at x_c. But if dy/dx changes from negative to zero to positive as x advances through x_c, there is a minimum. If dy/dx does not change its sign, neither a maximum or a minimum need occur" (Thomas, 1972, p. 127).

5. "If the derivative fails to exist at some point, examine this point as possible maximum or minimum" (Thomas, 1972, p. 127).
6. "If the function $y = f(x)$ is defined for only a limited range of values $a \leq x \leq b$, examine $x = a$ and $x = b$ for possible extreme values of y" (Thomas, 1972, p. 127).

- Note that it is sometimes acceptable to forego the second-derivative test (as it was in the last problem, above): "It is often obvious from the formulas, or from physical conditions, that we have a continuous and everywhere-differentiable function that does not attain its maximum at an end point. Hence it has at least one maximum at an interior point, at which its derivative must be zero. So if we find just one zero for the derivative, we have the maximum without any appeal to second-derivative or other tests" (Thomas, 1972, p. 127).

4.7 Rolle's Theorem

- Rolle's Theorem formalizes the idea that between two point where a smooth[5] curve crosses the x-axis, there should be at least one point where the tangent to the curve is flat.

- We now formally state and prove Rolle's Theorem.

Theorem 4.2 (Rolle's Theorem)**.** Let the function f be defined and continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Furthermore, let $f(a) = f(b) = 0$. Then there is at least one number c between a and b where $f'(x)$ is zero; that is, $f'(c) = 0$ for some c in (a, b).

Proof. We use casework.

Case 1 ($f(x) = 0$ for all $x \in [a, b]$): Thus, $f'(x) = 0$ for all $x \in (a, b)$, and the theorem holds in this case.

Case 2 ($f(x) \neq 0$ for all $x \in [a, b]$): Thus, $f(x)$ is positive or negative somewhere on the interval. In any case, it will have a maximum positive or minimum negative value $f(c)$ at some point $x = c$ on the interval. As a positive or negative value, $f(c) \neq 0$. Thus, $f(c) \neq f(a)$ and $f(c) \neq f(b)$. Therefore, by Theorem 4.1, $f'(c) = 0$, and the theorem holds in this case, too. □

- As a corollary: "Suppose a and b are two real numbers such that (a) $f(x)$ is continuous on $[a, b]$ and its first derivative $f'(x)$ exists on (a, b), (b) $f(a)$ and $f(b)$ have opposite signs, and (c) $f'(x)$ is different from zero for all values of x in (a, b). Then there is one and only one real root of the equation $f(x) = 0$ between a and b" (Thomas, 1972, p. 130).

4.8 The Mean Value Theorem

- We now look to generalize Rolle's Theorem.

- This theorem considers a "function $y = f(x)$ which is continuous on $[a, b]$ and which has a nonvertical tangent at each point between $A(a, f(a))$ and $B(b, f(b))$, although the tangent may be vertical at one or both of the end points A and B" (Thomas, 1972, p. 131).

[5]A function with a cusp could clearly disobey this rule.

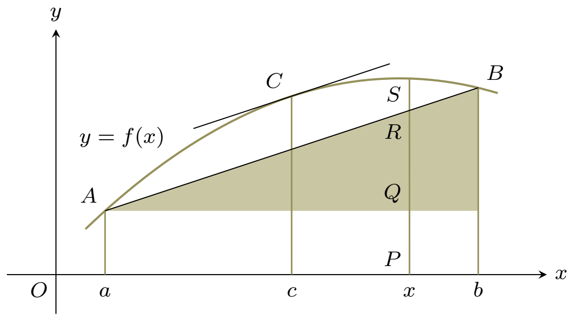


Figure 4.4: The mean value theorem.

- "Geometrically, the Mean Value Theorem states that if the function f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) where the tangent to the curve is parallel to the chord through A and B" (Thomas, 1972, p. 131).
  - This intuitively makes sense — consider moving the chord AB vertically upward or downward until it intersects only 1 point of the curve (as opposed to 0 or 2 [or, in theory, more than 2]). In Figure 4.4, this happens at C.
  - Note that this one point will occur where the vertical distance between AB and the curve is maximized.
- The idea of vertical distance is actually key to analytically proving the Mean Value Theorem.
  - The vertical distance between the chord and the curve is equal to the length of RS in Figure 4.4.
  - Also from Figure 4.4, it is clear that $RS = PS - PR$.
  - Now the length of PS is equal to $f(x)$, and the length of PR is equal to the following.

$$PR = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

  - Thus, the length $F(x)$ of RS at x is given by

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

  - We prove the Mean Value Theorem by applying Rolle's Theorem to $F(x)$.
  - Indeed, Rolle's Theorem guarantees that $F'(c) = 0$ for some $c \in (a, b)$. Since $F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, $f'(c) = \frac{f(b)-f(a)}{b-a}$. Thus, the slope of f is indeed equal to the slope of the chord AB for at least one point in (a, b).
- We now formally state the Mean Value Theorem.

  Theorem 4.3 (The Mean Value Theorem)**.** Let $y = f(x)$ be continuous on $[a, b]$ and be differentiable in the open interval (a, b). Then there is at least one number c between a and b such that

$$f(b) - f(a) = f'(c)(b - a)$$

- Obviously, we can use differential calculus to pinpoint the values of c that satisfy the Mean Value Theorem.
- Applied to kinematics, the Mean Value Theorem tells us that on any interval where position is described by a continuous, differentiable function of time, the instantaneous velocity is equal to the average velocity at least once.

- With the Mean Value Theorem, we can prove some corollaries.

  Corollary 4.1. *If a function F has a derivative which is equal to zero for all values of x in an interval (a,b), that is, if $F'(x) = 0$ for $x \in (a,b)$, then the function is constant throughout the interval: $F(x) =$ constant for $x \in (a,b)$.*

  Proof. Suppose for the sake of contradiction that x_1, x_2 are two distinct elements of the interval (a,b) for which $F(x_1) \neq F(x_2)$. WLOG, let $x_1 < x_2$. Since F is differentiable everywhere on (a,b), the Mean Value Theorem applies. Therefore, there exists some number $c \in (a,b)$ such that $F(x_1) - F(x_2) = F'(c)(x_1 - x_2)$. Since $F'(c) = 0$ everywhere on the interval by the hypothesis, $F(x_1) = F(x_2)$, a contradiction. Thus, the value of F at x_1 is the same as its value at x_2 for all x_1, x_2 in (a,b). □

  Corollary 4.2. *If F_1 and F_2 are two functions each of which has its derivative equal to $f(x)$ for $a < x < b$, that is, if $\mathrm{d}F_1/\mathrm{d}x = \mathrm{d}F_2/\mathrm{d}x = f(x)$ for $a < x < b$, then $F_1(x) - F_2(x) =$ constant for all $x \in (a,b)$.*

  Proof. Apply Corollary 4.1 to $F(x) = F_1(x) - F_2(x)$ (the derivative of this function F is equal to zero everywhere on the interval since $F'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$). □

  Corollary 4.3. *Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x)$ is positive throughout (a,b), then f is an increasing function on $[a,b]$, and if $f'(x)$ is negative throughout (a,b), then f is decreasing on $[a,b]$.*

  Proof. Let x_1 and x_2 be any two numbers in $[a,b]$, such that $x_1 < x_2$. By the Mean Value Theorem, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$. Since $(x_2 - x_1) > 0$, $f'(c)(x_2 - x_1)$ has the same sign as $f'(c)$. Thus, $f(x_2) > f(x_1)$ if $f'(x)$ is positive on (a,b), and $f(x_2) < f(x_1)$ if $f'(x)$ is negative on (a,b). □

4.9 Extension of the Mean Value Theorem

- We extend the Mean Value Theorem to prove a result about using the tangent line to a function to approximate future values of it.

  Theorem 4.4 (Extended Mean Value Theorem)**.** Let $f(x)$ and its first derivative $f'(x)$ be continuous on the closed interval $[a,b]$, and suppose its second derivative $f''(x)$ exists in the open interval (a,b). Then there is a number c_2 between a and b such that the following holds.

  $$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(c_2)(b-a)^2$$

  Proof. Let K be the number defined by the following equation.

  $$f(b) = f(a) + f'(a)(b-a) + K(b-a)^2 \tag{4.1}$$

  Let $F(x)$ be the function defined by replacing every instance of b in Equation 4.1 with x and subtracting the right-hand side from the left; that is,

  $$F(x) = f(x) - f(a) - f'(a)(x-a) + K(x-a)^2 \tag{4.2}$$

  By Equation 4.2, we have $F(a) = 0$. By Equation 4.1, we have $F(b) = 0$. Moreover, F and F' are continuous on $[a,b]$, and

  $$F'(x) = f'(x) - f'(a) - 2K(x-a)$$

  Therefore, F satisfies the hypotheses of Rolle's Theorem, i.e., there exists a number $c_1 \in (a,b)$ such that $F'(c_1) = 0$. This, coupled with the facts that $F'(a) = 0$ and F'' is continuous on (a, c_1), proves that there exists a number $c_2 \in (a, c_1)$ such that $F''(c_2) = 0$. Since

  $$F''(x) = f''(x) - 2K$$

  we have $K = \frac{1}{2}f''(c_2)$. Substituting this result into Equation 4.1 gives the desired result. □

- There is an even more general version of the Extended Mean Value Theorem available as an exercise. This serves as the beginning of the calculus of sequences and series.
- Let's consider an application of the Extended Mean Value theorem: "Use the linearization of $f(x) = \sqrt{x}$ at $a = 4$ to approximate $\sqrt{5}$, and estimate the size of the error in the approximation" (Thomas, 1972, p. 135).
  - The linearization of f at a is $L_a(x) = f(a) + f'(a)(x - a)$.
  - Thus, for $f(x) = \sqrt{x}$, $L_4(x) = 2 + \frac{1}{4}(x - 4)$. Therefore, we find $\sqrt{5} \approx 2.25$.
  - By the Extended Mean Value Theorem, $f(b) - L_a(b) = \frac{1}{2} f''(c_2)(b - a)^2$ for some $c_2 \in (a, b)$.
  - Applied to this problem, the above means that the error equals $-\frac{1}{8x^{3/2}}(5-4)^2$ for some $x \in (4, 5)$. Inputting the bounds on the interval into the error function, we find that the error is between -0.011 and -0.016, i.e., $\sqrt{5} = 2.25 - e$ for some $e \in (0.011, 0.016)$.
  - Thus, even on the upper end of possible error, we have less than 1% error.
  - Practically, we'd correct our estimate to between 2.234 and 2.239, which is extremely close to the actual three-decimal value of 2.236.
- Note that "if the hypotheses of the Extended Mean Value Theorem are satisfied on $[a, b]$, then they also hold on $[a, x]$ for any $x \in (a, b)$" (Thomas, 1972, p. 135).
  - For values of x close to a, we can reasonably approximate f with the quadratic function

    $$Q_a(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$$

    This is also closely related to the beginning of sequences and series.
- Further evidence that the sign of the second derivative indicates concavity: If f'' is continuous and positive at $x = a$, then it is positive in any sufficiently small neighborhood of a. Then by the Extended Mean Value Theorem, the graph of f near a lies above the tangent at a.
  - A similar argument holds for when f'' is continuous and negative at a.
  - This algebraic argument analytically supersedes our previous geometric argument.