# Chapter 14

# Vector Functions and Their Derivatives

## 14.1 Introduction

- **Vector function** (of $h$): A function $\mathbf{F}(h)$ with $n$ components where each component is a function. Essentially, $\mathbf{F} = (f_1, f_2, \ldots, f_n)$.

- **Limit** (of $\mathbf{F}(h)$ as $h \to a$): If each component $f_1, \ldots, f_n$ of $\mathbf{F}$ has a limit $L_1, \ldots, L_n$ as $h \to a$, then

$$\lim_{h \to a} \mathbf{F}(h) = (L_1, \ldots, L_n)$$

- **Continuous** (vector function $\mathbf{F}$ at $a$): A vector function $\mathbf{F}$ where for every $\epsilon > 0$, there corresponds a $\delta > 0$ such that
$$|\mathbf{F}(h) - \mathbf{F}(a)| < \epsilon \quad \text{when} \quad |h - a| < \delta$$

  – Thomas, 1972 shows that this is equivalent to the requirement that each component of $\mathbf{F}$ is continuous at $a$.

- **Derivative** (of a vector function at $c$): The derivative $\mathbf{F}'(c)$ of a vector function $\mathbf{F}$ at $c$ is given by the equation
$$\mathbf{F}'(c) = \lim_{h \to 0} \frac{\mathbf{F}(c+h) - \mathbf{F}(c)}{h}$$

  – It can be proven that $\mathbf{F}$ is differentiable at $c$ if and only if each of its components are differentiable at $c$, and that if this condition is met,

$$\mathbf{F}'(c) = (f_1'(c), \ldots, f_n'(c))$$

## 14.2 Velocity and Acceleration

- Results from here on out will generally pertain to 2D questions, but these methods can easily be generalized to higher dimensions.

- Applications of vectors to physics problems.

  – To solve **statics** problems, we only need to know the **algebra** of vectors.
  – To solve **dynamics** problems, we also need to know the **calculus** of vectors.

- **Position vector**: The vector from the origin to a point $P$ that moves along a parametrically defined curve. *Denoted by* $\mathbf{R}$.

- **Velocity vector**: The vector tangent to a point $P$ that moves along a parametrically defined curve and with magnitude $|\mathrm{d}s/\mathrm{d}t|$. *Denoted by* $\mathbf{v}$.
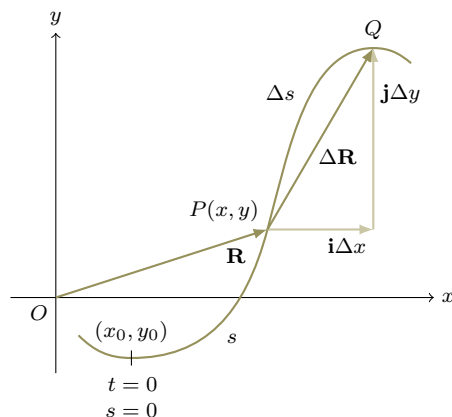


Figure 14.1: Velocity vector.

- Thomas, 1972 semi-rigorously proves from Figure 14.1 that if $\mathbf{R}$ is the position vector, then $\mathrm{d}\mathbf{R}/\mathrm{d}t$ is the velocity vector.
- Essentially, he proves that

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}t}$$

  It follows from this that

$$\text{slope of } \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{\mathbf{j}\text{-component}}{\mathbf{i}\text{-component}} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left|\frac{\mathrm{d}R}{\mathrm{d}t}\right| = \left|\mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}t}\right| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|$$

- **Acceleration vector**: The derivative of the velocity vector and second derivative of the position vector. *Denoted by* $\mathbf{a}$.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathbf{j}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

- Sometimes, we are given a force vector $\mathbf{F} = m\mathbf{a}$ and initial conditions.

  - From these, we can solve for velocity and position vectors via fairly straightforward component integration.
  - Note, however, that constants of integration are now vectors.

## 14.3    Tangential Vectors

- Let $P_0$ be a point on a curve. The distance $s$ from $P_0$ to some point $P$ along the curve is clearly related to the position of $P$. Thus, we may think of $\mathbf{R}$ as a function of $s$, and investigate the properties of $\mathrm{d}\mathbf{R}/\mathrm{d}s$.

- **Tangent vector**: The unit vector tangent to a point $P$ along a curve.

  - Since $\Delta\mathbf{R}$ and $\Delta s$ approach the same quantity as $\Delta s \to 0$, $\Delta\mathbf{R}/\Delta s$ approaches unity, i.e., $|\mathrm{d}\mathbf{R}/\mathrm{d}s| = 1$.
  - Because of the sign change, whether $\Delta s$ is positive or negative, $\Delta\mathbf{R}/\Delta s$ points in the same general direction for sufficiently small $\Delta s$. Indeed, it converges to pointing tangentially.
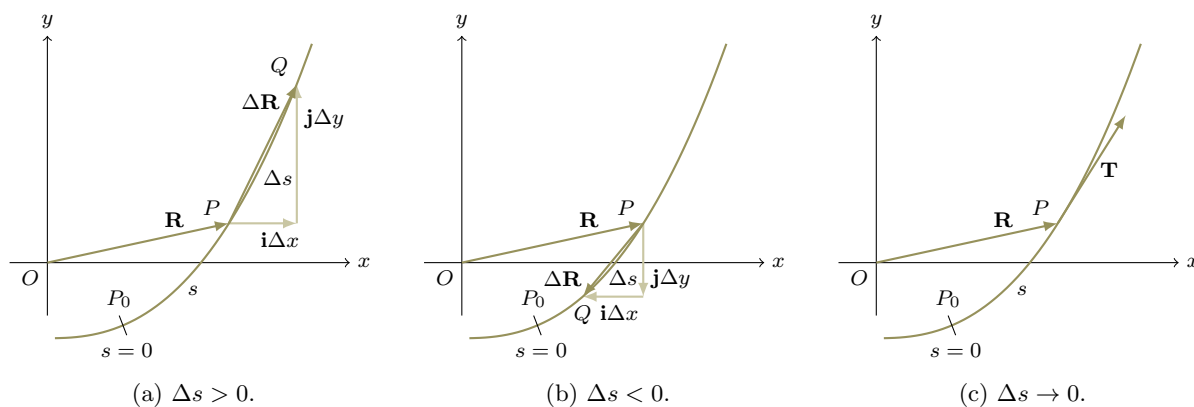
(a) $\Delta s > 0$.

(b) $\Delta s < 0$.

(c) $\Delta s \to 0$.

Figure 14.2: Tangent vector.

- Thus,

$$\mathbf{T} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s}$$

- There are two different ways to find $\mathbf{T}$: Straight differentiation combined with manipulations of differentials, and the chain rule combined with the dot product. We will explore each, in turn, with an example.

- "Find the unit vector $\mathbf{T}$ tangent to the circle $x = a\cos\theta$, $y = a\sin\theta$ at any point $P(x,y)$" (Thomas, 1972, p. 471).

  - From the given equations, we have

$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta \qquad \mathrm{d}y = a\cos\theta\,\mathrm{d}\theta \qquad \begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= a^2(\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta^2 \\ &= a^2\,\mathrm{d}\theta^2 \\ \mathrm{d}s &= \pm a\,\mathrm{d}\theta \end{aligned}$$

    - ■ We could alternatively obtain $\mathrm{d}s$ by expressing the arc length formula $S = R\theta$ in terms of differentials.

  - "If we measure arc length in the counterclockwise direction, with $s = 0$ at $(a,0)$, $s$ will be an increasing function of $\theta$, so the +-sign should be taken: $\mathrm{d}s = a\,\mathrm{d}\theta$" (Thomas, 1972, p. 471).

  - Therefore,

$$\begin{aligned} \mathbf{T} &= \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= \mathbf{i}\left(\frac{-a\sin\theta\,\mathrm{d}\theta}{a\,\mathrm{d}\theta}\right) + \mathbf{j}\left(\frac{a\cos\theta\,\mathrm{d}\theta}{a\,\mathrm{d}\theta}\right) \\ &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned}$$

- The equations

$$x = a\cos\omega t \qquad\qquad y = a\sin\omega t \qquad\qquad z = bt$$

where $a, b, \omega$ are positive constants define a circular helix in $E^{3}$[1].

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[1]Three-dimensional Euclidean space, equivalent to $\mathbb{R}^3$

- Let $P_0 = (a, 0, 0)$, since this is the point on the locus of the parametric equations where $t = 0$. Additionally, let arc length be measured in the direction in which $P$ moves away from $P_0$ as $t$ increases from 0.

- Using the chain rule to differentiate, we have

$$\mathbf{T} = \mathbf{i}\frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{j}\frac{\mathrm{d}y}{\mathrm{d}s} + \mathbf{k}\frac{\mathrm{d}z}{\mathrm{d}s}$$
$$= \mathbf{i}\left(-a\omega\sin\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right) + \mathbf{j}\left(a\omega\cos\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right) + \mathbf{k}\left(b\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)$$

- Since $\mathbf{T}$ is a unit vector, we have $1 = |\mathbf{T}| = |\mathbf{T}|^2 = \mathbf{T}\cdot\mathbf{T}$. Thus,

$$1 = \mathbf{T}\cdot\mathbf{T}$$
$$= \mathbf{i}\cdot\mathbf{i}\left(-a\omega\sin\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{j}\cdot\mathbf{j}\left(a\omega\cos\omega t\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{k}\cdot\mathbf{k}\left(b\,\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$$
$$= \left(a^2\omega^2 + b^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$$
$$\frac{\mathrm{d}t}{\mathrm{d}s} = \pm\frac{1}{\sqrt{a^2\omega^2 + b^2}}$$

- We choose the +-sign because $s$ should be a positive function of $t$.

- Putting this all together, we get

$$\mathbf{T} = \frac{a\omega(-\mathbf{i}\sin\omega t + \mathbf{j}\cos\omega t) + \mathbf{k}b}{\sqrt{a^2\omega^2 + b^2}}$$

## 14.4    Curvature and Normal Vectors

12/15:    • **Curvature**: The rate of change of the slope angle $\phi$ between $\mathbf{T}$ and the $x$-axis with respect to the arc length $s$. *Denoted by* $\kappa$.
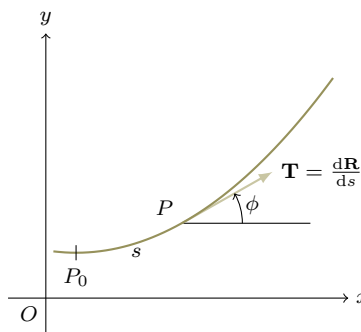


Figure 14.3: Curvature.

- Measured in radians per unit length.

• From the facts that

$$\kappa = \frac{\mathrm{d}\phi}{\mathrm{d}s} \qquad\qquad \tan\phi = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad\qquad \mathrm{d}s = \pm\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

we can derive a formula for $\kappa$ in terms of the original function $y = f(x)$ as follows.

$$\phi = \tan^{-1}\frac{dy}{dx} \qquad\qquad \frac{ds}{dx} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d\phi}{dx} = \frac{\frac{d^2 y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi/dx}{ds/dx}\right|$$

$$= \frac{\left|\frac{d^2 y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

- We can similarly derive that

$$\kappa = \frac{\left|\frac{d^2 x}{dy^2}\right|}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}$$

- If the equations for $y$ and $x$ are given parametrically in terms of $t$, we have

$$\kappa = \frac{\left|\frac{dx}{dt}\frac{d^2 y}{dt^2} - \frac{dy}{dt}\frac{d^2 x}{dt^2}\right|}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}$$

$$= \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

- Naturally, the curvature of a straight line should be 0. Indeed, we find this from the above equations.

- Naturally, the curvature of a circle should be constant, and should somehow decrease as the radius increases. Indeed, we find from the facts that $s = r\theta$ and $\phi = \theta + \frac{\pi}{2}$ that

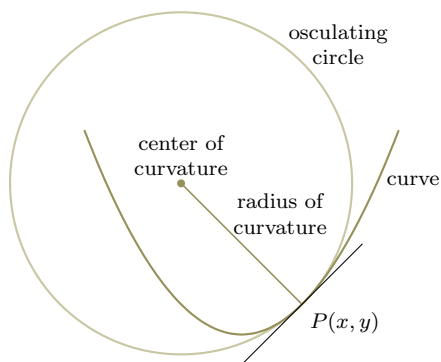$$\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{d\theta}{r\,d\theta}\right| = \frac{1}{r}$$



Figure 14.4: Circle, radius, and center of curvature.

- **Circle of curvature** (at $P$): "The circle that is tangent to a given curve at $P$, whose center lies on the concave side of the curve and which has the same curvature as the curve has at $P$" (Thomas, 1972, p. 475). *Also known as* **osculating circle**.

    - Calling the circle of curvature the "osculating circle" refers to the fact that its first and second derivatives at $P$ are equal to the first and second derivatives of the curve at $P$, meaning that it has a higher degree of contact with the curve at $P$ than any other circle.

- **Radius of curvature** (at $P$): The radius of the circle of curvature at $P$. *Denoted by $\rho$.*

$$\rho = \frac{1}{\kappa} = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2}}{\left|\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|}$$

- **Normal vector**: The unit vector normal to a point $P$ along a curve.

    - Observe that $\mathbf{T}$ can be expressed in terms of the slope angle $\phi$:

    $$\mathbf{T} = \mathbf{i} \cos \phi + \mathbf{j} \sin \phi$$

    - Since $\mathbf{T}$ can be thought of as a function of $\phi$, we can investigate the properties of $\mathrm{d}\mathbf{T}/\mathrm{d}\phi$.
    - Indeed, it is not difficult to show that

    $$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} = -\mathbf{i} \sin \phi + \mathbf{j} \cos \phi \qquad \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi}\right| = \sqrt{\sin^2 \phi + \cos^2 \phi} = 1 \qquad \mathbf{T} \cdot \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} = 0$$

    - Thus,

    $$\mathbf{N} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} = -\mathbf{i} \sin \phi + \mathbf{j} \cos \phi$$

- In 3D space, it is harder to define a single normal vector, so we define the. . .

- **Principal normal vector**: The vector

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|}$$

    - We will soon prove that $\mathbf{T} \cdot \mathrm{d}\mathbf{T}/\mathrm{d}s = 0$.
    - If $\phi$ is an increasing function of $s$, then by the chain rule,

    $$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\phi} \frac{\mathrm{d}\phi}{\mathrm{d}s} = \mathbf{N}\kappa$$

    - Since $\mathbf{N}$ is a unit vector and $\kappa$ is a constant, $\kappa$ is equal to the magnitude of $\mathrm{d}\mathbf{T}/\mathrm{d}s$.
    - Thus, we can define the principal normal as above.

- Thomas, 1972 uses $\mathrm{d}\mathbf{T}/\mathrm{d}s$ to find both the curvature and principal normal vector of the general circular helix investigated earlier. He also checks limiting cases to rederive the curvature of a circle and of a straight line.

- **Binormal vector**: The vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, as defined by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

## 14.5    Differentiation of Products of Vectors

- Let $\mathbf{U}$ and $\mathbf{V}$ be vectors whose components are differentiable functions of $t$.

- Then we can verify by components that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U} \cdot \mathbf{V}) = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \cdot \mathbf{V} + \mathbf{U} \cdot \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U} \times \mathbf{V}) = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \times \mathbf{V} + \mathbf{U} \times \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}$$

  - Note that we can *derive* the above, too, through the $\Delta$-process.
  - Let $\mathbf{W} = \mathbf{U} \times \mathbf{V}$. Then

$$\begin{aligned}
\mathbf{W} + \Delta\mathbf{W} &= (\mathbf{U} + \Delta\mathbf{U}) \times (\mathbf{V} + \Delta\mathbf{V}) \\
&= \mathbf{U} \times \mathbf{V} + \mathbf{U} \times \Delta\mathbf{V} + \Delta\mathbf{U} \times \mathbf{V} + \Delta\mathbf{U} \times \Delta\mathbf{V} \\
\Delta\mathbf{W} &= \mathbf{U} \times \Delta\mathbf{V} + \Delta\mathbf{U} \times \mathbf{V} + \Delta\mathbf{U} \times \Delta\mathbf{V} \\
\frac{\Delta\mathbf{W}}{\Delta t} &= \mathbf{U} \times \frac{\Delta\mathbf{V}}{\Delta t} + \frac{\Delta\mathbf{U}}{\Delta t} \times \mathbf{V} + \frac{\Delta\mathbf{U}}{\Delta t} \times \Delta\mathbf{V} \\
\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} &= \mathbf{U} \times \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \times \mathbf{V}
\end{aligned}$$

- Differentiating the triple scalar product:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U} \cdot \mathbf{V} \times \mathbf{W}) = \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \cdot \mathbf{V} \times \mathbf{W} + \mathbf{U} \cdot \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \times \mathbf{W} + \mathbf{U} \cdot \mathbf{V} \times \frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t}$$

  - Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}t} & \frac{\mathrm{d}u_2}{\mathrm{d}t} & \frac{\mathrm{d}u_3}{\mathrm{d}t} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ \frac{\mathrm{d}v_1}{\mathrm{d}t} & \frac{\mathrm{d}v_2}{\mathrm{d}t} & \frac{\mathrm{d}v_3}{\mathrm{d}t} \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \frac{\mathrm{d}w_1}{\mathrm{d}t} & \frac{\mathrm{d}w_2}{\mathrm{d}t} & \frac{\mathrm{d}w_3}{\mathrm{d}t} \end{vmatrix}$$

- Differentiating $\mathbf{V} \cdot \mathbf{V} = |\mathbf{V}|^2$ gives

$$\mathbf{V} \cdot \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \cdot \mathbf{V} = 0$$

$$2\mathbf{V} \cdot \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = 0$$

  - Thus, for any vector function $\mathbf{V}$, there are three cases: (1) $\mathbf{V} = \mathbf{0}$, (2) $\mathrm{d}\mathbf{V}/\mathrm{d}t = \mathbf{0}$, so $\mathbf{V}$ is constant in both direction and magnitude, and (3) $\mathbf{V}$ and $\mathrm{d}\mathbf{V}/\mathrm{d}t$ are perpendicular.
  - Note that this fact allows to verify that $\mathbf{T} \cdot \mathrm{d}\mathbf{T}/\mathrm{d}s = 0$.

- We can use the calculus of tangential and normal vectors to break velocity and acceleration vectors into tangential and normal components.

$$\begin{aligned}
\mathbf{v} &= \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} & \mathbf{a} &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} & &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} & &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \mathbf{N}\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \\
&= \mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}t} & & & &= \mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \mathbf{N}\left(\frac{\mathbf{v}^2}{\rho}\right)
\end{aligned}$$

  - $\mathbf{v}^2/\rho$ is very similar to $v^2/r$ (think about the circle and radius of curvature)!
  - Another important related equation:
$$|\mathbf{a}| = a_T^2 + a_N^2$$

– Lastly, we can derive a formula for the curvature in terms of velocity and acceleration.

$$\mathbf{v} \times \mathbf{a} = \mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}t} \times \left[\mathbf{T}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \mathbf{N}\kappa \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right]$$

$$= \mathbf{T} \times \mathbf{N}\kappa \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3$$

$$|\mathbf{v} \times \mathbf{a}| = \left|\mathbf{B}\kappa|\mathbf{v}|^3\right|$$

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$
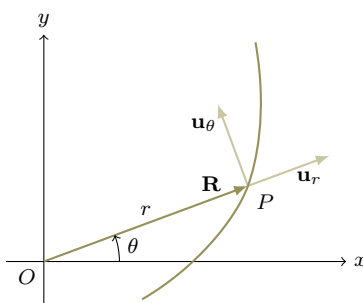
## 14.6   Polar and Cylindrical Coordinates



Figure 14.5: Vectors in polar coordinates.

- To analyze polar coordinates, we introduce the unit vectors

$$\mathbf{u}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \qquad\qquad \mathbf{u}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

- Clearly, we have

$$\frac{\mathrm{d}\mathbf{u}_r}{\mathrm{d}\theta} = \mathbf{u}_\theta \qquad\qquad \frac{\mathrm{d}\mathbf{u}_\theta}{\mathrm{d}\theta} = -\mathbf{u}_r$$

- Additionally, we can see that

$$\mathbf{R} = r\mathbf{u}_r$$

- The velocity vector can easily be expressed in terms of these quantities (and visualized as such geometrically, as in Figure 14.6).
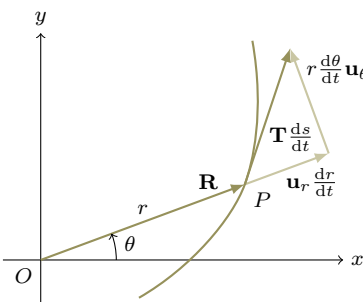


Figure 14.6: Polar velocity vector.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}$$
$$= \mathbf{u}_r \frac{\mathrm{d}r}{\mathrm{d}t} + r \frac{\mathrm{d}\mathbf{u}_r}{\mathrm{d}t}$$
$$= \mathbf{u}_r \frac{\mathrm{d}r}{\mathrm{d}t} + r\mathbf{u}_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

- The acceleration vector can also be expressed in terms of these quantities (the following can be derived by differentiating the above with respect to $t$ and substituting).

$$\mathbf{a} = \mathbf{u}_r \left[ \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r \left( \frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 \right] + \mathbf{u}_\theta \left[ r \frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} + 2 \frac{\mathrm{d}r}{\mathrm{d}t} \frac{\mathrm{d}\theta}{\mathrm{d}t} \right]$$

- In three dimensions (esp. for cylindrical coordinates), we have

$$\mathbf{R} = r\mathbf{u}_r + \mathbf{k}z$$
$$\mathbf{v} = \mathbf{u}_r \frac{\mathrm{d}r}{\mathrm{d}t} + r\mathbf{u}_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} + \mathbf{k}\frac{\mathrm{d}z}{\mathrm{d}t}$$
$$\mathbf{a} = \mathbf{u}_r \left[ \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r \left( \frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 \right] + \mathbf{u}_\theta \left[ r \frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} + 2 \frac{\mathrm{d}r}{\mathrm{d}t} \frac{\mathrm{d}\theta}{\mathrm{d}t} \right] + \mathbf{k}\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}$$

- Thomas, 1972 goes into an lengthy application of the above definitions to deriving Kepler's Laws.