Chapter 17

Vector Analysis

17.1 Introduction: Vector Fields

12/29:

- In this chapter, we will consider vector functions of several variables, such as the function giving the velocity $\mathbf{v} = \mathbf{F}(x, y, z, t)$ of a particle in a fluid located at position (x, y, z) at time t.
- **Steady-state flow**: A flow for which the velocity function does not depend on the time t.
- **Vector field**: The collection of all vectors $\mathbf{F}(P)$ assigned to each point P in a region G.
- **Gradient field**: The vector field defined for points in the domain G of a scalar function T such that $\mathbf{F}(P) = \nabla T(P)$.

17.2 Surface Integrals

12/30:

- Just like we have $\mathrm{d}s = \sqrt{1 + f_x^2}\,\mathrm{d}x$, we have

$$\mathrm{d}\sigma = g(x, y)\,\mathrm{d}A$$

  where $\mathrm{d}\sigma$ is "an element of surface area in the tangent plane that approximates the corresponding portion $\Delta\sigma$ of the surface itself" (Thomas, 1972, p. 581) and $g(x, y) = \sqrt{1 + f_x^2 + f_y^2}$.
- Thus, we can think of surface area as either the lefthand or righthand side of the below equation.

$$\iint\limits_{\Sigma} \mathrm{d}\sigma = \iint\limits_{R} g(x, y)\,\mathrm{d}A$$

  - The lefthand interpretation sums infinitely many, infinitely small pieces $\mathrm{d}\sigma$ of the surface Σ.
  - The righthand interpretation sums infinitely many, infinitely small pieces $\mathrm{d}A$ of the shadow R of the surface Σ on the xy-plane, adjusted by the factor $g(x, y)$.
- These formulations are important because sometimes we want to conceive and evaluate an integral of the form $\iint_{\Sigma} h(x, y, z)\,\mathrm{d}\sigma$.
- **Surface integral** (of $h(x, y, z)$ over the surface Σ): The limit as $\Delta\sigma \to 0$ of the sum of every $\Delta\sigma_k$ (composing Σ) times $h(x, y, z)$ for some $(x, y, z) \in \Delta\sigma_k$. Mathematically,

$$\iint\limits_{\Sigma} h(x, y, z)\,\mathrm{d}\sigma = \lim_{\Delta\sigma \to 0} \sum_{k=1}^{n} h(x_k, y_k, z_k)\,\Delta\sigma_k$$

- Consider a surface Σ consisting of all points $P(x,y,z)$ satisfying $z=f(x,y)$ for $(x,y)\in R$, where R is a closed, bounded region of the xy-plane and f,f_x,f_y are continuous throughout R and its boundary.
- Approximate R by dividing it into n rectangles using lines parallel to the y-axis spaced Δx apart and lines parallel to the x-axis spaced Δy apart.
- Let the part of Σ above each rectangle be denoted by $\Delta\sigma_k$ for some $1\leq k\leq n$.
- Now if $P_k(x_k,y_k,z_k)$ is a point in $\Delta\sigma_k$, we can consider the above sum and take its limit.

12/31:

- To evaluate the surface integral, we substitute $\Delta\sigma_k=g(x_k,y_k)\,\Delta x\,\Delta y$ and $z_k=f(x_k,y_k)$ in the sum, and take iterated integrals over R (the shadow of Σ on the xy-plane) instead of Σ.

$$\iint\limits_{\Sigma} h(x,y,z)\,\mathrm{d}\sigma=\iint\limits_{R} h[x,y,f(x,y)]g(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

- We now explore a useful surface integration technique through a problem.
- Evaluate $\iint(x^2+y^2)\,\mathrm{d}\sigma$ over the hemisphere Σ described by $z=\sqrt{a^2-x^2-y^2}$.
  - Because of a *sphere* 2Σ of radius a's high degree of symmetry,

$$\iint\limits_{2\Sigma} x^2\,\mathrm{d}\sigma=\iint\limits_{2\Sigma} y^2\,\mathrm{d}\sigma=\iint\limits_{2\Sigma} z^2\,\mathrm{d}\sigma=\frac{1}{3}\iint\limits_{2\Sigma}(x^2+y^2+z^2)\,\mathrm{d}\sigma=\frac{1}{3}\iint\limits_{2\Sigma} a^2\,\mathrm{d}\sigma$$

Thus, for the *hemisphere* Σ,

$$\begin{aligned}\iint\limits_{\Sigma}(x^2+y^2)\,\mathrm{d}\sigma&=\frac{1}{2}\iint\limits_{2\Sigma}(x^2+y^2)\,\mathrm{d}\sigma\\&=\frac{1}{2}\left(\iint\limits_{2\Sigma} x^2\,\mathrm{d}\sigma+\iint\limits_{2\Sigma} y^2\,\mathrm{d}\sigma\right)\\&=\frac{1}{2}\left(\frac{1}{3}\iint\limits_{2\Sigma} a^2\,\mathrm{d}\sigma+\frac{1}{3}\iint\limits_{2\Sigma} a^2\,\mathrm{d}\sigma\right)\\&=\frac{a^2}{3}\iint\limits_{2\Sigma}\mathrm{d}\sigma\\&=\frac{a^2}{3}\cdot 4\pi a^2\\&=\frac{4}{3}\pi a^4\end{aligned}$$

- Alternate formulations of $\mathrm{d}\sigma$.
  - Let the surface Σ be defined by the equation $F(x,y,z)=0$.
  - For the same reasons discussed in Chapter 17,

$$\mathrm{d}\sigma=\frac{\mathrm{d}A}{\cos\phi}$$

where ϕ is the angle between $\mathbf{N}=\nabla F$ and the unit vector normal to the plane onto which Σ is projected, which we will take to be the xy-plane at first (this means that this normal vector is $\mathbf{k}$).

- Since
$$\cos\phi = \frac{\mathbf{N}\cdot\mathbf{k}}{|\mathbf{N}|\,|\mathbf{k}|} = \frac{|F_z|}{\sqrt{F_x^2+F_y^2+F_z^2}}$$
we thus have that
$$\mathrm{d}\sigma = \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}\,\mathrm{d}x\,\mathrm{d}y$$
- Note that if we project Σ onto a different plane, an analog to the above can easily be derived.

17.3 Line Integrals

- **Line integral** (of $w(x,y,z)$ along the curve C from A to B): The limit as $\Delta s\to 0$ of the sum of every Δs_k (composing the section of C between points A and B along C) times $w(x,y,z)$ for some $(x,y,z)\in\Delta s_k$. Mathematically,
$$\int_C w\,\mathrm{d}s = \lim_{\Delta s\to 0}\sum_{k=1}^{n} w(x_k,y_k,z_k)\,\Delta s_k$$
  - Suppose that C is a directed curve in three-space from A to B. Let $w(x,y,z)$ be a scalar function of position that is continuous in a region D containing C.
  - Divide C into n segments, and let $P_k(x_k,y_k,z_k)$ be an arbitrary point on the kth subarc.
  - If the above sum has a limit as $n\to\infty$ and the largest $\Delta s_k\to 0$, and if this limit is the same for all ways of subdividing C and all choices of the points P_k, then we call this limit the line integral.
- If C is parameterized by the functions $x=f(t)$, $y=g(t)$, and $z=h(t)$ for $t_A\leq t\leq t_B$, where f,g,h are continuous and have bounded and piecewise-continuous first derivatives on $[t_A,t_B]$, then we may evaluate the line integral of $w(x,y,z)$ along C from A to B with the following formula.
$$\int_C w\,\mathrm{d}s = \int_{t_A}^{t_B} w[f(t),g(t),h(t)]\sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$
  - Note that the line integral is the same for any appropriate parameterization of C, or no parameterization.

9/2:

- The line integral can be geometrically interpreted as the area of the region R that lies above the curve C, offset by distance w.
- If C is a straight line, we can take the line integral over it directly wrt. s by expressing f in terms of s and rewriting the limits:
- For example, "let C be the line segment from $A(0,0)$ to $B(1,1)$ and let $w=x+y^2$. Evaluate $\int_C w\,\mathrm{d}s$" (Thomas, 1972, p. 585).
  - Let $x=t$ and $y=t$ for $0\leq t\leq 1$. Then
$$\begin{aligned}\int_C w\,\mathrm{d}s &= \int_0^1 (t+t^2)\sqrt{1+1}\,\mathrm{d}t\\ &= \sqrt{2}\left[\frac{t^2}{2}+\frac{t^3}{3}\right]_0^1\\ &= \frac{5\sqrt{2}}{6}\end{aligned}$$

- By the Pythagorean theorem, $s = \sqrt{x^2 + y^2} = \sqrt{2x^2} = x\sqrt{2}$. Thus, $w = s/\sqrt{2} + s^2/2$. Additionally, as $0 \leq x \leq 1$, $0 \leq s \leq \sqrt{2}$. Therefore,

$$\begin{aligned}
\int_C w \, \mathrm{d}s &= \int_0^{\sqrt{2}} \left(\frac{s}{\sqrt{2}} + \frac{s^2}{2} \right) \mathrm{d}s \\
&= \left[\frac{s^2}{2\sqrt{2}} + \frac{s^3}{6} \right]_0^{\sqrt{2}} \\
&= \frac{5\sqrt{2}}{6}
\end{aligned}$$

- To generalize the above notion, we can always think of w as a function $\phi(s)$, where s is arc length.
- "If the point of application of a force $\mathbf{F} = \mathbf{i}M(x, y, z) + \mathbf{j}N(x, y, z) + \mathbf{k}P(x, y, z)$ moves along a curve C from a point $A(a_1, a_2, a_3)$ to a point $B(b_1, b_2, b_3)$, then the work done by the force is

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{R}$$

where $\mathbf{R}$ [is the position vector]" (Thomas, 1972, p. 586).

  - Since $\mathrm{d}\mathbf{R} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} \mathrm{d}s$ and $\mathrm{d}\mathbf{R}/\mathrm{d}s = \mathbf{T}$, the work can also be thought of as "the value of the line integral along C of the tangential component of the force field $\mathbf{F}$" (Thomas, 1972, p. 587).

- The line integral between two points A and B is independent of the path C joining them if and only if the force field $\mathbf{F}$ is a **gradient field**, that is, if

$$\mathbf{F}(x, y, z) = \nabla f$$

for some differentiable function f.

  - Thomas (1972) proves this.
  - If $\mathbf{F}$ is a gradient field, then

$$\int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{R} = \int_A^B \nabla f \cdot \mathrm{d}\mathbf{R} = f(B) - f(A)$$

  - Furthermore, from $\mathbf{F}$, we define f by

$$f(x', y', z') = \int_A^{(x', y', z')} \mathbf{F} \cdot \mathrm{d}\mathbf{R}$$

- "Find a function f such that if $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, then $\mathbf{F} = \nabla f$" (Thomas, 1972, p. 589).
  - Choose $A = (0, 0, 0)$ to simplify calculations.
  - Assume that $\mathbf{F}$ is a gradient field, i.e., that evaluating $\int_A^{(x', y', z')} \mathbf{F} \cdot \mathrm{d}\mathbf{R}$ along any path will yield the same result.
  - Thus, choose to evaluate the line integral along the line segment from A to (x', y', z'), which we may define by the parameterization $x = x't$, $y = y't$, $z = z't$ for $0 \leq t \leq 1$.
  - Therefore,

$$\begin{aligned}
f(x', y', z') &= \int_{(0,0,0)}^{(x', y', z')} \mathbf{F} \cdot \mathrm{d}\mathbf{R} \\
&= \int_{(0,0,0)}^{(x', y', z')} (2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \cdot (\mathbf{i} \, \mathrm{d}x + \mathbf{j} \, \mathrm{d}y + \mathbf{k} \, \mathrm{d}z)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (2x't\mathbf{i} + 2y't\mathbf{j} + 2z't\mathbf{k}) \cdot (\mathbf{i}x'\,\mathrm{d}t + \mathbf{j}y'\,\mathrm{d}t + \mathbf{k}z'\,\mathrm{d}t) \\
&= \int_0^1 (2x'^2t\,\mathrm{d}t + 2y'^2t\,\mathrm{d}t + 2z'^2t\,\mathrm{d}t) \\
&= (x'^2 + y'^2 + z'^2)\int_0^1 2t\,\mathrm{d}t \\
&= x'^2 + y'^2 + z'^2
\end{aligned}
$$

- **Conservative** (force field): A force field $\mathbf{F}$ such that the work integral from A to B is the same for all paths joining them.
- Another criterion besides $\mathbf{F} = \nabla f$ for some differentiable f is that $\mathrm{d}f = \mathbf{F} \cdot \mathrm{d}\mathbf{R} = M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$ is an exact differential.
  - By an extension of Theorem 15.4, we know that $M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$ is an exact differential if and only if
    $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$
- "Suppose $\mathbf{F} = \mathbf{i}(\mathrm{e}^x\cos y + yz) + \mathbf{j}(xz - \mathrm{e}^x\sin y) + \mathbf{k}(xy + z)$. Is $\mathbf{F}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$" (Thomas, 1972, p. 591).
  - Apply the exact differential test:
    $$\frac{\partial M}{\partial y} = -\mathrm{e}^x\sin y + z = \frac{\partial N}{\partial x} \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial z} = x = \frac{\partial P}{\partial y}$$
    - Therefore, $\mathbf{F}$ is conservative.
  - To calculate f, we integrate the system of equations
    $$\frac{\partial f}{\partial x} = \mathrm{e}^x\cos y + yz \qquad \frac{\partial f}{\partial y} = xz - \mathrm{e}^x\sin y \qquad \frac{\partial f}{\partial z} = xy + z$$
    - Starting with the first one, we obtain
      $$f = \mathrm{e}^x\cos y + xyz + g(y,z)$$
      where $g(y,z)$ is a function of integration.
    - Differentiating wrt. y, we get
      $$
      \begin{aligned}
      xz - \mathrm{e}^x\sin y = \frac{\partial f}{\partial y} &= -e[x]\sin y + xz + \frac{\partial g}{\partial y} \\
      \frac{\partial g}{\partial y} &= 0 \\
      g(y,z) &= h(z)
      \end{aligned}
      $$
    - Differentiating wrt. z, we get
      $$
      \begin{aligned}
      xy + z = \frac{\partial f}{\partial z} &= xy + \frac{\partial h}{\partial z} \\
      \frac{\partial h}{\partial z} &= z \\
      h(z) &= \frac{1}{2}z^2 + C
      \end{aligned}
      $$
    - Therefore,
      $$f(x,y,z) = \mathrm{e}^x\cos y + xyz + \frac{1}{2}z^2 + C$$
- **Potential function**: A function $f(x,y,z)$ which has the property that its gradient gives the force vector $\mathbf{F}$.

17.4 Two-Dimensional Fields: Line Integrals in the Plane and Their Relation to Surface Integrals on Cylinders

- Features of a two-dimensional field $\mathbf{F}$:
  1. The vectors in $\mathbf{F}$ are all parallel to one plane, which we have taken to be the xy-plane.
     - Mathematically, the vectors have no $\mathbf{k}$ component.
  2. In every plane parallel to the xy-plane, the field is the same as it is in that plane.
     - Mathematically, the vectors do not depend on z.
- Imagine fluid of planar mass density δ flowing out from the origin with velocity defined by a vector velocity function $\mathbf{v}$.

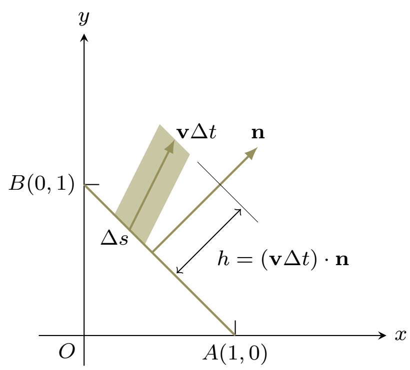

Figure 17.1: Fluid flowing over a line segment.

  - Then the flow rate $\mathrm{d}M/\mathrm{d}t$ over a curve C in the plane is given by
$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_C \delta(\mathbf{v} \cdot \mathbf{n})\,\mathrm{d}s$$
- **Flux** (of $\mathbf{F} = \delta\mathbf{v}$ across C): The quantity
$$\int_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s$$
- If C is a closed curve, we canonically choose $\mathbf{n}$ to point outwards and the orientation to be in the counterclockwise direction.
  - We also choose $\mathbf{n} = \mathbf{T} \times \mathbf{k}$.
- With these conventions, if we let $\mathbf{F}(x, y) = \mathbf{i}M(x, y) + \mathbf{j}N(x, y)$, then
$$\begin{aligned}
\text{flux} &= \int_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s \\
&= \int_C \mathbf{F} \cdot (\mathbf{T} \times \mathbf{k})\,\mathrm{d}s \\
&= \int_C \mathbf{F} \cdot \left(\left(\frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{j}\right) \times \mathbf{k}\right)\mathrm{d}s \\
&= \int_C \mathbf{F} \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{i} - \frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{j}\right)\mathrm{d}s \\
&= \int_C \left(M\frac{\mathrm{d}y}{\mathrm{d}s} - N\frac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s \\
&= \int_C (M\,\mathrm{d}y - N\,\mathrm{d}x)
\end{aligned}$$

- Any flux integral can be reinterpreted as a work integral on a related field and vice versa: If $\mathbf{F}(x, y) = \mathbf{i}M + \mathbf{j}N$ and $\mathbf{G}(x, y) = -\mathbf{i}N + \mathbf{j}M$, then

$$\int_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \int_C \mathbf{G} \cdot \mathbf{T} \, \mathrm{d}s$$

17.5 Green's Theorem

- The following is a formal statement of **Green's theorem**.

Theorem 17.1 (Green's theorem)**.** Let C be a simple closed curve in the xy-plane such that a line parallel to either axis cuts C in at most two points. Let M, N, $\partial N/\partial x$, and $\partial M/\partial y$ be continuous functions of (x, y) inside and on C. Let R be the region inside C. Then

$$\oint_C M \, \mathrm{d}x + N \, \mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y$$ [1]

Proof. We will prove that $\iint_R (-\partial M/\partial y) \, \mathrm{d}x \, \mathrm{d}y = \oint_C M \, \mathrm{d}x$. It will follow by a symmetric argument that $\iint_R (\partial N/\partial x) \, \mathrm{d}x \, \mathrm{d}y = \oint_C N \, \mathrm{d}y$. The sum of these two qualities will yield Green's theorem. Let's begin.

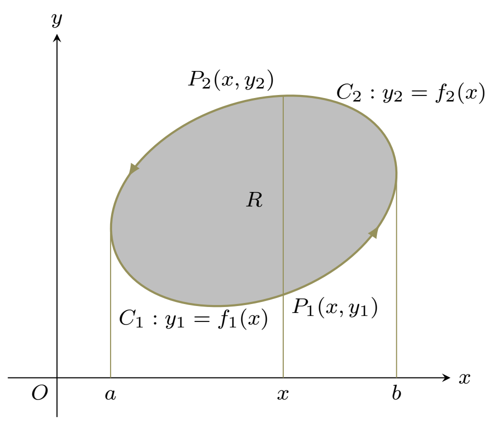

Figure 17.2: Proving Green's theorem.

Consider the curve C enclosing a region R. We divide C into a lower boundary curve C_1 and an upper boundary curve C_2, both of which are functions of x (the constraint that a line parallel to either axis cuts C in at most two points allows us to do this).

Since $\partial M/\partial y$ is continuous, it is integrable, meaning that at any $x \in [a, b]$, we can determine that

$$\begin{aligned} \int_{y_1}^{y_2} \frac{\partial M}{\partial y} \, \mathrm{d}y &= [M(x, y)]_{y=f_1(x)}^{y=f_2(x)} \\ &= M(x, f_2(x)) - M(x, f_1(x)) \end{aligned}$$

[1] The symbol $\oint$ denotes a line integral over a closed curve C.

It follows since M is continuous, and therefore integrable, that

$$\begin{aligned}
\iint_R -\frac{\partial M}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b -\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_a^b [M(x, f_1(x)) - M(x, f_2(x))]\,\mathrm{d}x \\
&= \int_a^b [M(x, f_1(x))\,\mathrm{d}x + \int_b^a M(x, f_2(x))]\,\mathrm{d}x \\
&= \int_{C_1} M\,\mathrm{d}x + \int_{C_2} M\,\mathrm{d}x \\
&= \oint_C M\,\mathrm{d}x
\end{aligned}$$

as desired.

It follows by a symmetric argument that

$$\iint_R \frac{\partial N}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \oint_C N\,\mathrm{d}y$$

Therefore, we have by addition that

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$ [2]

as desired. □

- Green's theorem provides an easy to calculate the area enclosed by a curve.

Corollary 17.1. *If C is a simple closed curve such that a line parallel to either axis cuts it in at most two points, them the area enclosed by C is equal to*

$$\frac{1}{2}\oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$$

Proof. From Section 16.2, we have that

$$\begin{aligned}
A &= \iint_R 1\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{2}\right)\right)\mathrm{d}x\,\mathrm{d}y \\
&= \oint_C \left(-\frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y\right) && \text{Theorem 17.1} \\
&= \frac{1}{2}\oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x)
\end{aligned}$$

as desired. □

- Note that Green's theorem also applies to a number of shapes that don't fit the theorem statement's direct criteria.
  - For instance, we can prove that it holds for a rectangle in the xy-plane with sides parallel to the x- or y-axes.

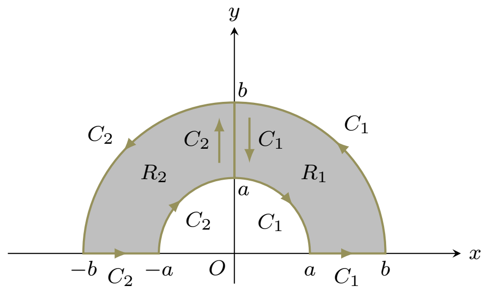

Figure 17.3: Creating composite regions that satisfy Green's theorem.

- Additionally, we can add together regions that satisfy Green's theorem individually to form bigger regions that satisfy it (the line integrals in the overlapping part of Figure 17.3 cancel).
- In fact, we can add together any finite number of subregions that satisfy Green's theorem.

- **Curl** (of a vector $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$): The cross product of the del operator and $\mathbf{F}$. *Given by*

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \mathbf{j}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \mathbf{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)
\end{aligned}$$

- It follows that Green's theorem in vector form is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{R} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}$$

where $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, $\mathrm{d}\mathbf{R} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y$, and $\mathrm{d}\mathbf{A} = \mathbf{k}\,\mathrm{d}x\,\mathrm{d}y$.

  - "In words, Green's theorem states that the integral around C of the tangential component of $\mathbf{F}$ is equal to the integral, over the region R bounded by C, of the component of $\operatorname{curl}\mathbf{F}$ that is normal to R; this integral, specifically, is the flux through R of $\operatorname{curl}\mathbf{F}$" (Thomas, 1972, p. 604).

- **Divergence** (of a vector $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$): The dot product of the del operator and $\mathbf{F}$. *Given by*

$$\begin{aligned}
\operatorname{div}\mathbf{G} &= \nabla \cdot \mathbf{G} \\
&= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x} + \frac{\partial P}{\partial x}
\end{aligned}$$

- If $\mathbf{F} = \mathbf{i}M(x,y) + \mathbf{j}N(x,y)$ is a field and $\mathbf{G} = \mathbf{i}N(x,y) - \mathbf{j}M(x,y)$ is the orthogonal field, then an alternate vector formulation of Green's theorem is

$$\int_C \mathbf{G} \cdot \mathbf{n}\,\mathrm{d}s = \iint_R \nabla \cdot \mathbf{G}\,\mathrm{d}x\,\mathrm{d}y$$

  - "In words, [this] says that the line integral of the normal compoment of any vector field $\mathbf{G}$ around the boundary of a region R in which $\mathbf{G}$ is continuous and has continuous partial derivatives is equal to the double integral of the divergence of $\mathbf{G}$ over R" (Thomas, 1972, p. 604).

17.6 Divergence Theorem

9/4: • **Divergence theorem**: Let $\mathbf{F} = \mathbf{i}M + \mathbf{j}N + \mathbf{k}P$, where M, N, P are continuous functions of (x, y, z) that have continuous first-order partial derivatives $\operatorname{div}\mathbf{F} = \partial M/\partial x + \partial N/\partial y + \partial P/\partial z$. Let $\mathrm{d}\boldsymbol{\sigma} = \mathbf{n}\,\mathrm{d}\sigma$ be a vector element of surface area directed along the unit outer normal vector $\mathbf{n}$. Let Σ be the surface enclosing the region D, where D is some convex region with no holes, Σ is a piecewise smooth surface, the projection of D into the xy-plane is a simply connected region R_{xy}, and any line perpendicular to the xy-plane at an interior point of R_{xy} intersects the surface Σ in at most two points, producing surfaces $\Sigma_1 : z_1 = f_1(x, y)$ and $\Sigma_2 : z_2 = f_2(x, y)$, $(x, y) \in R_{xy}$, $z_1 \le z_2$. Then

$$\iiint_D \operatorname{div}\mathbf{F}\,\mathrm{d}V = \iint_\Sigma \mathbf{F}\cdot\mathrm{d}\boldsymbol{\sigma}$$

Alternatively, if we let $\mathbf{n} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$, then we can rewrite the divergence theorem as

$$\iiint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_\Sigma (M\cos\alpha + N\cos\beta + P\cos\gamma)\,\mathrm{d}\sigma$$

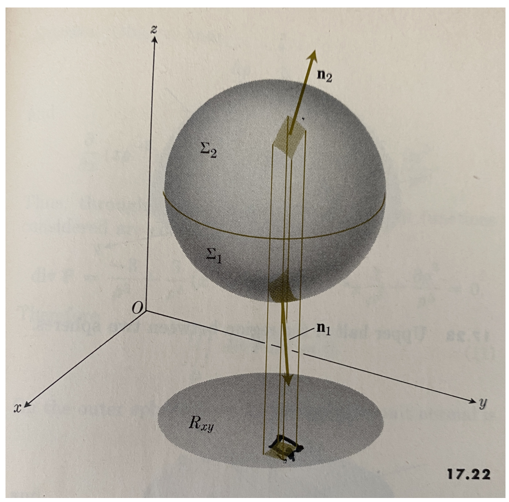

Figure 17.4: Divergence theorem.

- We will not rigorously prove the divergence theorem, but we will justify $\iiint_D \partial P/\partial z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_\Sigma P\cos\gamma\,\mathrm{d}\sigma$.
- From Figure 17.4, we can see that the outer normal on Σ_2 has a positive $\mathbf{k}$-component, and so $\cos\gamma_2\,\mathrm{d}\sigma_2 = \mathrm{d}x\,\mathrm{d}y$.
- On the other hand, the outer normal on Σ_1 has a negative $\mathbf{k}$-component, so $\cos\gamma_1\,\mathrm{d}\sigma_1 = -\,\mathrm{d}x\,\mathrm{d}y$.
- Therefore,

$$\begin{aligned}
\iint_\Sigma P\cos\gamma\,\mathrm{d}\sigma &= \iint_{\Sigma_2} P_2\cos\gamma_2\,\mathrm{d}\sigma_2 + \iint_{\Sigma_1} P_1\cos\gamma_1\,\mathrm{d}\sigma_1\\
&= \iint_{R_{xy}} P(x, y, z_2)\,\mathrm{d}x\,\mathrm{d}y - \iint_{R_{xy}} P(x, y, z_1)\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{R_{xy}} [P(x, y, z_2) - P(x, y, z_1)]\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{R_{xy}} \int_{z_1}^{z_2}\left[\frac{\partial P}{\partial z}\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y\\
&= \iiint_D \frac{\partial P}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

- "Verify [the divergence theorem] for the sphere $x^2 + y^2 + z^2 = a^2$ if $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$" (Thomas, 1972, p. 607).
  - We will take this one piece at a time and then assemble all the pieces. We do this for both integrals in the divergence theorem.
  - Left integral:
    - First off,
$$\operatorname{div}\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$
    - Therefore,
$$\iiint_D \operatorname{div}\mathbf{F}\,\mathrm{d}V = \iiint_D 3\,\mathrm{d}V = 3\iiint_D \mathrm{d}V = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$
  - Right integral:
    - To define $\mathbf{n}$, we use the fact that $\mathbf{n} = \pm\nabla f/|\nabla f|$ where $f(x,y,z) = x^2+y^2+z^2-a^2$, meaning that the outer unit normal is
$$\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$$
    - It follows that
$$\mathbf{F}\cdot\mathrm{d}\boldsymbol{\sigma} = \frac{x^2+y^2+z^2}{a}\,\mathrm{d}\sigma = \frac{a^2}{a}\,\mathrm{d}\sigma = a\,\mathrm{d}\sigma$$
    - Therefore,
$$\iint_\Sigma \mathbf{F}\cdot\mathrm{d}\boldsymbol{\sigma} = \iint_\Sigma a\,\mathrm{d}\sigma = a(4\pi a^2) = 4\pi a^3$$
  - Clearly, the two integrals are equal, as desired.
- As with Green's theorem, the divergence theorem can be extended to more complex regions, as long as those regions can be split up into a finite number of simple regions of the type originally discussed.
- Thomas (1972) proves that the divergence of the velocity of an incompressible fluid is zero in a region where there are no sources or sinks.

17.7 Stokes's Theorem

- The following is a formal statement of **Stokes's theorem**.

  Theorem 17.2 (Stokes's theorem)**.** Let Σ be a smooth, simply connected, orientable surface bounded by a simple closed curve C. Let $\mathbf{F} = \mathbf{i}M + \mathbf{j}N + \mathbf{k}P$, where M, N, P are continuous functions of (x,y,z), together with their first-order partial derivatives throughout a region D containing Σ and C in its interior. Let $\mathbf{n}$ be a positive unit vector normal to Σ, and let the positive direction around C be the one induced by the positive orientation of Σ. Then
$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{R} = \iint_\Sigma \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\boldsymbol{\sigma}$$
  where $\mathrm{d}\mathbf{R} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z = \mathbf{T}\,\mathrm{d}s$ and $\mathrm{d}\boldsymbol{\sigma} = \mathbf{n}\,\mathrm{d}\sigma = (\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma)\,\mathrm{d}\sigma$.
  - Thomas (1972) proves Stokes's theorem from Green's theorem for a polyhedral surface consisting of a finite number of plane regions, much the same way he built up larger surfaces from Green's theorem earlier on.
  - A rigorous proof of Stokes's theorem for more general surfaces is beyond the level of an intro calculus course. More intuitively, we can think about taking the limit of a finite construction for more and more subdivision of a general surface.

- **Orientable** (surface): A surface Σ such that it is possible to consistently assign a unique direction, called positive, at each point of Σ. As we move the normal over Σ without touching its boundary, the direction cosines of the unit vector $\mathbf{n}$ should vary continuously. Also, when we return to the starting position, $\mathbf{n}$ should return to its original direction.
  - Recall that Möbius strips are non-orientable surfaces.
- Defining the positive direction around C consistently with the positive direction on Σ.

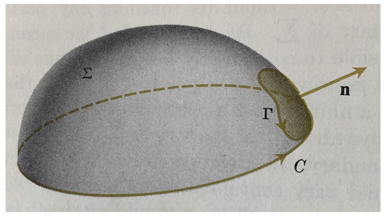

Figure 17.5: Orienting a surface and its boundary consistently.

  - "Imagine a simple closed curve Γ on Σ, near the boundary C, and let $\mathbf{n}$ be normal to σ at some point inside Γ. We then assign to Γ a positive direction, the counterclockwise direction as viewed by an observer who is at the end of $\mathbf{n}$ and looking down. (Note that such a direction keeps the interior of Γ on the observer's left as he progresses around Γ. We could equally well have specified $\mathbf{n}$'s direction by this condition.) Now we move Γ about Σ until it touches and is tangent to C. The direction of the positive tangent to Γ at this point of common tangency we shall take to be the positive direction along C" (Thomas, 1972, p. 612).
  - Note that it is a consequence of the orientability of Σ that a consistent assignment of positive direction along C is induced by this process (it would not hold for a Möbius strip, for example).
- "Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. Let C be their curve of intersection, and let $\mathbf{F} = \mathbf{i}(z-y) + \mathbf{j}(z+x) - \mathbf{k}(x+y)$. Compute $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{R}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma}$ and compare" (Thomas, 1972, p. 614).
  - Naturally, C is defined by $z = 0 = 4 - x^2 + y^2$, or $x^2 + y^2 = 4$.
  - With the substitutions $x = 2\cos\theta$, $y = 2\sin\theta$, and $z = 0$, we get the following.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{R} &= \oint_C (z-y)\,\mathrm{d}x + (z+x)\,\mathrm{d}y - (x+y)\,\mathrm{d}z \\
&= \int_0^{2\pi} (0 - 2\sin\theta)(-2\sin\theta\,\mathrm{d}\theta) + (0 + 2\cos\theta)(2\cos\theta\,\mathrm{d}\theta) + (2\cos\theta + 2\sin\theta)(0) \\
&= \int_0^{2\pi} 4(\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta \\
&= 4\int_0^{2\pi} \mathrm{d}\theta \\
&= 8\pi
\end{aligned}$$

    - Note that we can also evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{R}$ more directly with

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{R} = &\left[\int_2^{-2} \left(0 - \sqrt{4-x^2}\right)\mathrm{d}x + \int_{-2}^{2} \left(0 - \left(-\sqrt{4-x^2}\right)\right)\mathrm{d}x\right] \\
&+ \left[\int_2^{-2} \left(0 - \sqrt{4-y^2}\right)\mathrm{d}y + \int_{-2}^{2} \left(0 - \left(-\sqrt{4-y^2}\right)\right)\mathrm{d}y\right]
\end{aligned}$$

  - As to the other integral, we'll do some setup first.

- First off, we have

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & z+x & -x-y \end{vmatrix} \\ &= -2\mathbf{i}+2\mathbf{j}+2\mathbf{k}\end{aligned}$$

- Next up, if $f(x,y,z) = z-4+x^2+y^2$, we have

$$\begin{aligned}\mathbf{n} &= \frac{\nabla f}{|\nabla f|} \\ &= \frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}\end{aligned}$$

- To evaluate this surface integral, we'll project it onto the xy-plane. Its shadow is $x^2+y^2 \leq 4$.
- And to account for this change of domain, we must use

$$\begin{aligned}\mathrm{d}\sigma &= \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sqrt{4x^2+4y^2+1}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

- Putting everything together, we have

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\sigma &= \iint_{x^2+y^2\leq 4} (-2\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot\frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}\sqrt{4x^2+4y^2+1}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{x^2+y^2\leq 4} (-4x+4y+2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{x^2+y^2\leq 4} 2\,\mathrm{d}x\,\mathrm{d}y \\ &= 2(\pi\cdot 2^2) \\ &= 8\pi\end{aligned}$$

  - Note that we can remove $-4x+4y$ from the integrand because odd powers of x or y integrate to 0 over the interior of the circle.

- Stokes's theorem can also be extended to a surface with finitely many holes in a manner analogously to how we did so with Green's theorem.

- Note that Stokes's theorem interprets curl as equating the circulation of fluid around the closed curve C with the flus of the curl though the surface spanning C.