

9/13: **2.8** The details of several steps in the particle-in-a-box model in this chapter have been omitted. Work out the details of the following steps:

- a. Show that if $\Psi = A \sin rx + B \cos sx$ (A , B , r , and s are constants) is a solution to the wave equation for the one-dimensional box, then

$$r = s = \sqrt{2mE} \left(\frac{2\pi}{h} \right)$$

Solution.

$$\begin{aligned} \frac{-h^2}{8\pi^2m} \cdot \frac{\partial^2 \Psi(x)}{\partial x^2} &= E\Psi(x) \\ \frac{-h^2}{8\pi^2m} \cdot \frac{\partial^2}{\partial x^2} (A \sin rx + B \cos sx) &= E(A \sin rx + B \cos sx) \\ \frac{-h^2}{8\pi^2m} \cdot \frac{\partial}{\partial x} (Ar \cos rx - Bs \sin sx) &= E(A \sin rx + B \cos sx) \\ \frac{-h^2}{8\pi^2m} \cdot (-Ar^2 \sin rx - Bs^2 \cos sx) &= E(A \sin rx + B \cos sx) \\ \frac{Ar^2 h^2}{8\pi^2m} \sin rx + \frac{Bs^2 h^2}{8\pi^2m} \cos sx &= AE \sin rx + BE \cos sx \\ 0 &= \left(\frac{Ar^2 h^2}{8\pi^2m} - AE \right) \sin rx + \left(\frac{Bs^2 h^2}{8\pi^2m} - BE \right) \cos sx \end{aligned}$$

Choose $x = 0$.

$$\begin{aligned} &= \frac{Bs^2 h^2}{8\pi^2m} - BE \\ E &= \frac{s^2 h^2}{8\pi^2m} \\ \frac{8\pi^2mE}{h^2} &= s^2 \\ s &= \sqrt{\frac{8\pi^2mE}{h^2}} \\ \boxed{s = \sqrt{2mE} \frac{2\pi}{h}} \end{aligned}$$

With this result ...

$$\begin{aligned} 0 &= \left(\frac{Ar^2 h^2}{8\pi^2m} - AE \right) \sin rx + \left(\frac{Bs^2 h^2}{8\pi^2m} - BE \right) \cos sx \\ &= \left(\frac{Ar^2 h^2}{8\pi^2m} - AE \right) \sin rx + \left(B \left(\frac{s^2 h^2}{8\pi^2m} \right) - BE \right) \cos sx \\ &= \left(\frac{Ar^2 h^2}{8\pi^2m} - AE \right) \sin rx + (BE - BE) \cos sx \\ &= \left(\frac{Ar^2 h^2}{8\pi^2m} - AE \right) \sin rx \end{aligned}$$

Choose $x = \frac{\pi}{2r}$.

$$\begin{aligned} &= \frac{Ar^2 h^2}{8\pi^2m} - AE \\ \boxed{r = \sqrt{2mE} \frac{2\pi}{h}} \end{aligned}$$

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- d. Show that substituting the value of r given in part c into $\Psi = A \sin rx$ and applying the normalizing requirement gives $A = \sqrt{2/a}$.

Solution.

$$\begin{aligned} 1 &= \int_{\text{all space}} \Psi \Psi^* d\tau \\ &= \int_0^a \left(A \sin \frac{n\pi x}{a} \right) \left(A \sin \frac{n\pi x}{a} \right) dx \\ &= \int_0^a A^2 \sin^2 \frac{n\pi x}{a} dx \end{aligned}$$

Use $\sin^2 u = \frac{1 - \cos 2u}{2}$.

$$\begin{aligned} &= A^2 \int_0^a \frac{1 - \cos \frac{2n\pi x}{a}}{2} dx \\ &= \frac{A^2}{2} \left(\int_0^a dx - \int_0^a \cos \frac{2n\pi x}{a} dx \right) \\ &= \frac{A^2}{2} \left([x]_0^a - \left[\frac{a}{2n\pi} \sin \frac{2n\pi x}{a} \right]_0^a \right) \\ &= \frac{A^2}{2} \left((a - 0) - \left(\frac{a}{2n\pi} \sin 2n\pi - \frac{a}{2n\pi} \sin 0 \right) \right) \\ &= \frac{A^2}{2} \left(a - \left(\frac{a}{2n\pi} \sin 2n\pi \right) \right) \end{aligned}$$

Since n is an integer, $\sin 2n\pi = 0$.

$$\begin{aligned} &= \frac{aA^2}{2} \\ \frac{2}{a} &= A^2 \\ \boxed{A} &= \sqrt{\frac{2}{a}} \end{aligned}$$

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