Week 5

Applications and Generalizations

5.1 Special Normal Subgroups

10/24:

- Last time: If $H \lhd S_n$, $n \neq 4$, then $H = \{e\}, S_n, A_n$. If $n = 4$, H can also equal $\{e\} \cup \{(xx)(xx)\}$.
- Theorem: Let $n \neq 4$. Then the only normal subgroups of A_n are the identity and A_n.
  - Let $H \lhd A_n \lhd S_n$. From this, you could propose concluding that since we know all normal subgroups of S_n, and H is less than or equal to A_n, we know that $H = \{e\}, A_n$.
  - Issue: $\lhd$ is not transitive. Conjugacy classes change depending on where you're sitting.
  - Consider A, B, C: If $A \lhd B \lhd C$, then is $A \lhd C$?
  - This theorem is on HW5.
  - Counterexample:

$$A = \langle (1,2)(3,4) \rangle \qquad B = \{e\} \cup \{(xx)(xx)\} \qquad C = S_4$$

    - This is not so far from the simplest example.
  - Calegari reemphasizes that, "if you understand everything about S_4, then you understand everything in this class."
- We know that if $H \leq A_4$, then $|H| \big| 12$ by Lagrange's theorem.
- Claim: A_4 has no subgroups of order 6.
  - If H has index 2, then $H \lhd A_4$. This was a HW problem.
  - Thus, we can try to understand conjugacy classes in A_n. Whereas in S_n, we have a beautifully simple way to characterize all conjugacy classes, we do not have that in A_n. For example, $(1,2,3)$ and $(1,3,2)$ are not conjugate. $(2,3)(1,2,3)(2,3) = (1,3,2)$. $(1,2)(1,2,3)(1,2) = (2,1,3)$. $(1,3)(1,2,3)(1,3) = (3,2,1)$. But none of these transpositions are in A_n.
  - There are four conjugacy classes in A_4.

$$\{e\} \quad \{(12)(34), (13)(24), (14)(23)\} \quad \{(123), (243), (134), (142)\} \quad \{(132), (234), (143), (124)\}$$

    - Note that if $x, y \in A_4$ are of order 3, either $x \sim y$ or $x \sim y^{-1}$.
  - In A_5, all 3-cycles are conjugate; in A_4, they're not.
  - The sizes of the conjugacy classes in A_4 are $1 + 4 + 4 + 3$. That's enough to prove that there is no subgroup of order 6.
  - Alternate proof.

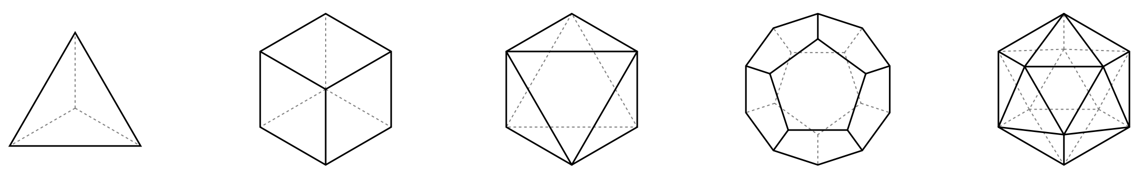
Proof. Suppose for the sake of contradiction that A_4 has a normal subgroup H of index 2. Then by the proposition from Lecture 4.1, there exists a surjective homomorphism from $A_4 \to A_4/H$. Additionally, since $|A_4/H| = 2$ and there is only one group of order 2, $A_4/H \cong \mathbb{Z}/2\mathbb{Z}$. Thus, there exists a surjective homomorphism $\phi : A_4 \to \mathbb{Z}/2\mathbb{Z}$.

We know that every alternating group (including A_4) is generated by 3-cycles. Let σ be an arbitrary 3-cycle generator of A_4. We know that $\phi(\sigma) = 0$ or $\phi(\sigma) = 1$. If $\phi(\sigma) = 1$, then

$$0 = \phi(e) = \phi(\sigma^3) = 3\phi(\sigma) = 1 +_2 1 +_2 1 = 1$$

which clearly cannot happen. Thus, $\phi(\sigma) = 0$. Consequently, the image of all of the generators of A_4 under ϕ is 0. But this implies that $\phi(A_4) = \{0\} \subsetneq \mathbb{Z}/2\mathbb{Z}$, contradicting our hypothesis that ϕ is surjective. $\square$

- Here ends the material that will be covered on the midterm.
- We now move on to something we will come back to later.
- A_n in nature for $n = 4, 5$.



(a) Tetrahedron. (b) Cube. (c) Octahedron. (d) Dodecahedron. (e) Icosahedron.

Figure 5.1: The platonic solids.

  - Recall the cube group Cu.
  - The cube is an example of a **platonic solid**.
  - Other examples: Tetrahedron, octahedron, icosahedron, and dodecahedron. We define corresponding symmetry groups Te, Oc, Do, and Ic.
  - Consider the tetrahedral group to start.
    - Since any rigid motion permutes the vertices, we have a map $\text{Te} \hookrightarrow S_4$. Moving 2 vertices fixes the rest. Thus, $\text{Te} \leq S_4$. Therefore, $|\text{Te}| = 12$ so $\text{Te} \cong A_4$.
  - We determined in HW2 that...
    - $\text{Do} \hookrightarrow S_5$ and $|\text{Do}| = 60$. Thus, $\text{Do} \cong A_5$.
  - Consider the octahedron.

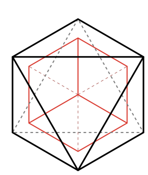


Figure 5.2: Inscribing a cube in an octahedron.

    - $|\text{Oc}| = 6 \cdot 4 = 24$. Rationale: Fix one vertex anywhere and then fix another (the other one can only take on the four adjacent positions, though); the positions of the rest are determined from these two.
    - Let's look at fixing opposite faces. This does give an injective map to S_4, and it follows that $\text{Oc} \cong S_4$.

        - Relation between Oc and Cu. We can inscribe a cube in the octahedron by connecting each vertex of the cube to the midpoint of one of the faces of the octahedron and vice versa. Thus, we get maps $\text{Oc} \to \text{Cu}$, leading to $\text{Oc} \cong \text{Cu}$.
    - We can similarly inscribe a dodecahedron in an icosahedron.
    - Thus, the cube and the octahedron have the same symmetry, and the dodecahedron and icosahedron have the same symmetry.
- **Platonic solid**: A solid geometric shape in three dimensions for which the faces, edges, and vertices are all indistinguishable.
    - We will study the platonic solids in more depth later.
- Problem: What symmetries can objects in $\mathbb{R}^3$ have?
    - Rephrase: What are the finite subgroups of SO(3)?
    - An octagon is Calegari's favorite polygon.
    - An octagonal prism has much the same symmetry in $\mathbb{R}^3$ as an octagon does in $\mathbb{R}^2$. This leads to $D_{2n} \leq \text{SO}(3)$.
        - Recall the map from the blog post.
    - We also have $\mathbb{Z}/n\mathbb{Z} \leq D_{2n}$.
- It follows that the groups $\mathbb{Z}/n\mathbb{Z}$, D_{2n}, A_4, S_4, and S_5 occur as finite subgroups of SO(3).
- Theorem: All finite subgroups of SO(3) are on this list. Moreover, all related versions are conjugate.
    - This is a companion theorem to the theorem that there are only five platonic solids.
    - Neither theorem implies the other, but they are related.
- Infinite subgroups of SO(3): O(2), SO(3), SO(2).
- This theorem will be completely evident by the end of the course.
- You can either use this theorem to understand A_4, A_5, or use an understanding of A_4, A_5 to rationalize this theorem.
- Points to the main focus of the class: Understanding groups not just based on writing down elements but by their action on a certain set. This is the focus of the second half of the course.
- Midterm: 50 mins, closed book, Wednesday. Final exam must be in-person by department rules, but Calegari is fighting for us. Calegari is hoping that the midterm should not be a speed test.
    - Trying to test our skills, not our ability to memorize stuff.
    - How to do well: Learn group theory.
- The quaternion group.
    - A 4D vector space where you define a noncommutative product. If you just take 8 specific quaternions, the group of order 8 is distinct from D_8 but related.

5.2 Group Actions

10/28:

- Let G be a group and X be a set.
- **Group action** (of G on X): A map $\cdot : G \times X \to X$ satisfying the following. *Denoted by* $\boldsymbol{G \circlearrowright X}$.
    1. For all $g, h \in G$ and $x \in X$, $g \cdot (h \cdot x) = gh \cdot x$.
    2. For all $x \in X$, $e \cdot x = x$.

- Note that condition 2 does not follow from condition 1, and an "inverse condition" follows from both.
  - In particular, condition 1 relates certain elements of the domain of the group action but does not relate any elements of the domain to elements of X (as condition 2 does).
  - The inverse condition $g^{-1} \cdot (g \cdot x) = g \cdot (g^{-1} \cdot x) = x$ follows from conditions 1-2 via

$$g^{-1} \cdot (g \cdot x) = g^{-1}g \cdot x = e \cdot x = x = e \cdot x = gg^{-1} \cdot x = g \cdot (g^{-1} \cdot x)$$

- Example: If G is any group and X is any set, we may define a group action by $g \cdot x = x$ for all $g \in G$ and $x \in X$.
- Lemma: Let $G \circlearrowright X$ and $g \in G$. Define $\psi_g : X \to X$ by $x \mapsto g \cdot x$. Then ψ_g is a bijection.

  Proof. Injectivity:

$$\begin{aligned}\psi_g(x) &= \psi_g(y)\\ g \cdot x &= g \cdot y\\ g^{-1} \cdot (g \cdot x) &= g^{-1} \cdot (g \cdot y)\\ e \cdot x &= e \cdot y\\ x &= y\end{aligned}$$

  Surjectivity: Given $x \in X$, we want y such that $\psi_g(y) = x$. Choose $y = g^{-1} \cdot x$. $\square$

- This allows us to recast group actions into the following equivalent form.
- Let X be a set and S_X be the set of all bijections from $X \to X$ under composition. Note that if $|X| = n$, then $S_X \cong S_n$.
- Proposition: An action G on the set X is equivalent to a homomorphism from G to S_X defined by $g \mapsto \psi_g$.

  Proof. A statement of the proposition that makes it more clear what exactly it is we want to prove is, "there exists an action $\cdot : G \times X \to X$ iff there exists a homomorphism $\phi : G \to X$ defined by $g \mapsto \psi_g$." Let's begin.

  Suppose first that $\cdot : G \times X \to X$ is group action of G on X. Define $\phi : G \to S_X$ by $g \mapsto \psi_g$. To prove that ϕ is a homomorphism, it will suffice to show that $\phi(gh) = \phi(g) \circ \phi(h)$ for all $g, h \in G$. Let $g, h \in G$ be arbitrary. Then by condition 1, we have for any and all $x \in X$ that

$$\begin{aligned}g \cdot (h \cdot x) &= gh \cdot x\\ \psi_g(\psi_h(x)) &= \psi_{gh}(x)\\ [\psi_g \circ \psi_h](x) &= \psi_{gh}(x)\\ [\phi(g) \circ \phi(h)](x) &= [\phi(gh)](x)\end{aligned}$$

  Therefore, $\phi(gh) = \phi(g) \circ \phi(h)$, as desired.

  Now suppose that $\phi : G \to S_X$ is a homomorphism defined by $g \mapsto \psi_g$. Define $\cdot : G \times X \to X$ by $g \cdot x = [\phi(g)](x)$. To prove that $\cdot$ is a group action, it will suffice to show that for all $g, h \in G$ and $x \in X$, $g \cdot (h \cdot x) = gh \cdot x$ and $e \cdot x = x$. Let $g, h \in G$ and $x \in X$ be arbitrary. Then

$$g \cdot (h \cdot x) = g \cdot \psi_h(x) = [\psi_g \circ \psi_h](x) = [\phi(g) \circ \phi(h)](x) = [\phi(gh)](x) = \psi_{gh}(x) = gh \cdot x$$

  and

$$e \cdot x = \psi_e(x) = x$$

  as desired. $\square$

- You need to be careful with what the set is and what the group is; $x \cdot y$ probably doesn't make any sense (unless you start to get into cases where X is a group, too).
- **Kernel** (of a group action): The set of all $g \in G$ such that $g \cdot x = x$ for all $x \in X$.
  - The kernel is a (normal) subgroup of G.
  - We know this since it is equivalent to the kernel of the homomorphism described by the above proposition.
- **Faithful** (group action): A group action for which the kernel is trivial, i.e., ker $= \{e\}$.
  - Such a group action is "faithful" because it is telling the whole story, i.e., not leaving out any information, i.e., mapping everything to everything.
  - The trivial group action is an example of a group action that isn't faithful.
- **Orbit** (of $x \in X$): The set of $g \cdot x$ for all $g \in G$. *Denoted by* $\mathbf{Orb}(\boldsymbol{x})$.
  - A subset of X.
  - Everywhere you can get to from your starting point x.
- **Transitive** (group action): A group action for which $\text{Orb}(x) = X$ for some (any) $x \in X$.
  - In what way is a transitive group action *transitive*??
- **Stabilizer** (of $x \in X$): The set of all $g \in G$ for which $g \cdot x = x$. *Denoted by* $\mathbf{Stab}(\boldsymbol{x})$.
  - A subgroup of G.
- The kernel is a subgroup of the stabilizer. More specifically,

$$\ker = \bigcap_{x \in X} \text{Stab}(x)$$

  - This is because the elements of the stabilizer fix *some* $x \in X$, whereas the elements of the kernel fix *all* $x \in X$.
- Orbits are equivalence relations, i.e., $x \in \text{Orb}(x)$ and $x \in \text{Orb}(y)$ imply that $\text{Orb}(x) = \text{Orb}(y)$.
  - In particular,

$$X = \bigsqcup \text{Orbits}$$

- Let $G = S_n$ and $X = [n]$.
- Let $G = \text{Cu}$.
- Examples.

G	X	$\lvert X\rvert$	Transitive	Faithful	Kernel	$\text{Stab}(x)$	$\lvert\text{Stab}(x)\rvert$
	Faces	6	✓	✓	$\{e\}$	$\mathbb{Z}/4\mathbb{Z}$ (rotations by 90°)	4
	Vertices	8	✓	✓	$\{e\}$	$\mathbb{Z}/3\mathbb{Z}$	3
	Edges	12	✓	✓	$\{e\}$	$\mathbb{Z}/2\mathbb{Z}$	2
Cu	Diagonals	4	✓	✓	$\{e\}$	$S_3 \cong D_6$	6
	Pairs of opposite faces	3	✓	X	Rotations by 180°, in particular, $\lvert K\rvert = 4$	D_8	8
	Inscribed tetrahedra	2	✓	X	A_4	A_4	12
	$\text{Ed} \cup \text{Fac}$	18	X	✓	$\{e\}$	$\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$	4 or 2

Table 5.1: Examples of group actions.

  - The last two rows we filled out by first asserting transitivity, A_4, and 12; and edges union faces, not transitive.
- On Monday, we will look into group actions where the geometry is not so convenient.

5.3 Blog Post: Group Actions

From Calegari (2022).

11/13:

- Motivating group actions.
  - We now address the "?" category in Table 2.1.
  - $G = S_{52}$ is an incredibly huge group, yet it seems pretty manageable since we can write down any element and multiply two or more by hand, for instance, without too much difficulty. Why is this group with 52! elements so manageable? It seems like it has something to do with the set X of 52 numbers that S_{52} "moves around."
  - Similarly, $G = \mathrm{GL}_n(\mathbb{R})$ denotes the group of $n \times n$ invertible matrices but this set is made more manageable by understanding its action on $X = \mathbb{R}^n$.
  - In both cases, G acts on X in some sense. Elements of G form bijective maps $g : X \to X$. Moreover, the group product corresponds to function composition.
  - Question: Can we find a suitable X for any group G?
  - Hint: Look at Cu. There are several sets on which Cu acts, each of which leads to a greater understanding of the group, itself.
  - Conclusion: In general, there will be many *different* X that we will want to consider, even when G is the symmetric group.
- $g \cdot (h \cdot x) = gh \cdot x$ is called a **compatibility property**.
- Covers the Lemma and Proposition from class.
- Examples of group actions.
  - The trivial group action (see class notes).
  - If $G = \mathrm{Cu}$ and X is the set of inscribed tetrahedra, pairs of opposite faces, diagonals, faces, or edges, we get homomorphisms from G to S_2, S_3, S_4, S_6, and S_{12}, respectively. Going even further, we can let X be the disjoint union of the set of edges and faces to obtain a homomorphism from G to S_{18}.
  - Consider $G = \mathrm{SO}(3)$ and $X = \mathbb{R}^3$ or $X = S^2$ (the unit 2-sphere).
  - Consider $G = S_n$ and $\sigma = \{1, 2, \dots, n\}$.