Week 6

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6.1 Examples of Group Actions

10/31:

- Today: A number of interesting group actions.
- **Left action** (of G on X): A group action of the form $g \cdot x$ (as opposed to $x \cdot g$).
- Let G be a group, and let $X = G$. Take $g \cdot x = gx$.
  - Axiom confirmation.
    1. $e \cdot x = ex = x$.
    2. $g \cdot (h \cdot x) = ghx = gh \cdot x$.
  - Let $e \in X$. Then $\text{Orb}(e) = X$. In particular, this means that the action is transitive.
  - $\text{Stab}(x) = \{g \in G \mid gx = x\} = \{e\}$ for $x \in X$ arbitrary, in general.
  - $\ker = \{e\}$. This also follows from the above. Thus, the action is faithful.
- Corollary: Let G be a finite group. Then G is isomorphic to a subgroup of S_n for some n. We may take $n = |G|$.
  - Construction: We invoke the proposition from last lecture. In particular, we know that the action $G \circlearrowright G$ implies the existence of a homomorphism $\phi : G \to S_G$ defined by $g \mapsto \psi_g$.
  - The map in the above construction has trivial kernel. By the FIT, $G/\ker \cong \operatorname{im}\phi$. Combining these results, we obtain $G \cong G/\ker \cong \operatorname{im}\phi \leq S_n$.
  - Applying this construction to S_3, we deduce that $S_3 \leq S_6$.
- $\text{SO}(2) \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}^\infty$.
  - In infinite cases, you usually want to consider some other topological things that disappear in the finite case.
- Let G be a group and take $X = G$ again. We can also consider $g \cdot x = gxg^{-1}$.
  - Axioms.
    1. $e \cdot x = exe^{-1} = x$.
    2. $g \cdot (h \cdot x) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = gh \cdot x$.
  - $\text{Orb}(e) = \{e\}$; not transitive if $|G| > 1$.
  - Let $x \in X$. Then $\text{Orb}(x)$ is the conjugacy class of x.
  - $\text{Stab}(x) = C_G(x)$.
  - $\ker = Z(G)$. Thus, the group action is faithful iff the center is trivial. Abelian implies not faithful.

- A nice thing about these constructions is that they cast other constructions we've encountered in the more general language of group actions.
- **Right actions** are even nastier than left cosets and right cosets, so Calegari will not mention them again.
  - $g \cdot x = x \cdot g^{-1}$ and $g \cdot (h \cdot x) = (x \cdot h^{-1}) \cdot g^{-1}$.
- Let $G = G$, X be the subgroups of G. $g \cdot H = gHg^{-1}$.
  - Note that $H \leq G$ does indeed imply that $gHg^{-1} \leq G$. In particular, ...
    - H is nonempty (contains at least e), so $gHg^{-1} \supset \{geg^{-1}\}$ is nonempty;
    - $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$ imply that $gh_1g^{-1}gh_2g^{-1} = g(h_1h_2)g^{-1} \in gHg^{-1}$;
    - $ghg^{-1} \in gHg^{-1}$ has inverse $gh^{-1}g^{-1} \in gHg^{-1}$.
  - Axioms (entirely analogous to the last example).
  - $\text{Orb}(H)$ is the "conjugates" of H.
  - $\text{Stab}(H) = N_G(H)$.
  - ker =?. We know that $Z(G) \subset \ker$. The conclusion is that there is not a nice definition for the kernel other than the intersections of the stabilizers/normalizers.
    - ...
    - If any $H \triangleleft G$ is normal, and $x \in G$ had order 2, then $\langle x \rangle \triangleleft G$, meaning that $gxg^{-1} \in \langle x \rangle$, i.e., $x \in Z(G)$, so this rules out D_8??
- Fix G and $H \leq G$. Let $X = G/H$ (not assuming $H \triangleleft G$, so we know that G/H is the set of left cosets but it is not a group in general). Define $g \cdot xH = gxH$.
  - We have $g \cdot xhH = gxhH$.
  - Orbit: $\text{Orb}(eH) = X$.
  - Stabilizer: $\text{Stab}(eH) = H$.
    - $\text{Stab}(gH) = gHg^{-1}$.
    - This is because $(ghg^{-1})gH = ghH = gH$.
    - Go to the more general case $G \circlearrowright X$, $\text{Stab}(x) = H$. Then $gHg^{-1} \subset \text{Stab}(g \cdot x)$??
  - Transitive: Yes (see orbits).
  - Faithful: If H is normal, no. If H contains a normal subgroup, no. Maybe yes.
  - Kernel: If H is normal, then $\ker = H$. In general, $\ker = \bigcap_{g \in G} gHg^{-1}$ (the largest normal subgroup of H).
- Takeaway: General constructions allow us to see things we've already done.
- Next time: The most useful theorem of the course, that provides lots of information on relations between objects.

6.2 Orbit-Stabilizer Theorem

11/2:

- We will have a take-home open-book final. Should take you a couple hours or a little more to do, but we'll have more time than that. Don't Google answers or collaborate. We'll have more practice problems (and 50% of the exam will be on that sheet); if we do every problem on the sheet, we'll certainly get an A.
- We will cover all theoretical material by Thanksgiving and then spend the rest of the time exploring applications.
- Today: The most fundamental theorem of the class.

- Let G be a group acting on a set X.
- Theorem (Orbit-Stabilizer Theorem): Let $x \in X$ be arbitrary. Then

$$|G| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$$

Proof. We will break up G and count it in two different ways.

$$G = \bigsqcup_{y \in \operatorname{Orb}(x)} \{g \mid g \cdot x = y\}$$

Each of these sets is equal to $g \cdot \operatorname{Stab}(x)$ (the left coset of the stabilizer by g).

Thus,

$$|G| = \sum_{\operatorname{Orb}(x)} |g \cdot \operatorname{Stab}(x)| = \sum_{\operatorname{Orb}(x)} |\operatorname{Stab}(x)| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$$

as desired. □

- Examples:
  - Let $H \leq G$, $X = G/H$. Then G acts on X by left multiplication. Taking $x = H$ in particular, we have that

$$|G| = |G/H| \cdot |H|$$

  - $G = S_n$, $X = [n]$.
    - Then $S_n = \{\sigma(1) = 1\} \cup \{\sigma(1) = 2\} \cup \cdots \cup \{\sigma(1) = n\}$. This is analogous to the proof strategy decomposition.
  - G acts on G by conjugation.
    - Take $g \in G$. Then $\operatorname{Orb}(g) = \{g\}$, i.e., the conjugacy class of g, and $\operatorname{Stab}(g) = C_G(g)$. Therefore, we have the below corollary.
  - $G = S_n$.
    - Let $g = (1, \dots, k)$ for $2 \leq k \leq n$. Recall that $|g| = n!/(n-k)!k$. Thus, $|C_{S_n}(g)| = (n-k)! \cdot k$.
    - Alternatively, we can derive the order of this centralizer directly: $C_{S_n}(g) = \langle g \rangle \times S_{n-k}$, i.e., all powers of the k-cycle and everything that's disjoint. $\times$ denotes the direct product.
  - $G = S_4$, $g = (12)(34)$.
    - $|\{g\}| = 3$, so $|C_G(g)| = 8$.
    - Here $C_G(g) = D_8$. Visualize a square with vertices clockwise (1,4,2,3).
  - $G = S_6$, $g = (16)(25)(34)$.
    - We have that $|\{g\}| = 6!/2^3 \cdot 3! = 15$, so $|C_{S_6}(g)| = 48$. The centralizer is the set of all elements satisfying $\sigma(i) + \sigma(7-i) = 7$.
    - Moreover, there is an injective homomorphism from $\widetilde{\text{Cu}} \hookrightarrow S_6$ whose image is exactly the centralizer of $(16)(25)(34)$. Moreover, it follows that $C_{S_6}(g) \cong S_4 \times S_2$.
    - Let $h = (16)$. Then $|\{h\}| = |\{g\}| = 15$. Does there exist an automorphism of S_6 to S_6 which sends $h \to g$? No: $S_2 \times S_4 \cong C_{S_6}(h)$ and $C_{S_6}(g) \cong S_2 \times S_4$.
- Corollary: We have that

$$|G| = |\{g\}| \cdot |C_G(g)|$$

- $\widetilde{\textbf{Cu}}$: The set of all orthogonal symmetries of the cube (i.e., including reflections).
  - There is an isomorphism between $\text{Cu} \times \mathbb{Z}/2\mathbb{Z}$ and $\widetilde{\text{Cu}}$ defined by $(g, 1) \mapsto g$ and $(g, -1) \mapsto -g$. The reverse function is $g \mapsto (g \cdot \deg g, \deg g)$.
  - $\widetilde{\text{Cu}}$ acts on 6 faces.
- The pace will be this fast through Thanksgiving.