Week 5

Applications and Generalizations

5.1 Special Normal Subgroups

10/24:

- Last time: If $H \lhd S_n$, $n \neq 4$, then $H = \{e\}, S_n, A_n$. If $n = 4$, H can also equal $\{e\} \cup \{(xx)(xx)\}$.
- Theorem: Let $n \neq 4$. Then the only normal subgroups of A_n are the identity and A_n.
  - Let $H \lhd A_n \lhd S_n$. From this, you could propose concluding that since we know all normal subgroups of S_n, and H is less than or equal to A_n, we know that $H = \{e\}, A_n$.
  - Issue: $\lhd$ is not transitive. Conjugacy classes change depending on where you're sitting.
  - Consider A, B, C: If $A \lhd B \lhd C$, then is $A \lhd C$?
  - This theorem is on HW5.
  - Counterexample:

$$A = \langle (1,2)(3,4) \rangle \qquad B = \{e\} \cup \{(xx)(xx)\} \qquad C = S_4$$

    - This is not so far from the simplest example.
  - Calegari reemphasizes that, "if you understand everything about S_4, then you understand everything in this class."
- We know that if $H \leq A_4$, then $|H| \big| 12$ by Lagrange's theorem.
- Claim: A_4 has no subgroups of order 6.
  - If H has index 2, then $H \lhd A_4$. This was a HW problem.
  - Thus, we can try to understand conjugacy classes in A_n. Whereas in S_n, we have a beautifully simple way to characterize all conjugacy classes, we do not have that in A_n. For example, $(1,2,3)$ and $(1,3,2)$ are not conjugate. $(2,3)(1,2,3)(2,3) = (1,3,2)$. $(1,2)(1,2,3)(1,2) = (2,1,3)$. $(1,3)(1,2,3)(1,3) = (3,2,1)$. But none of these transpositions are in A_n.
  - There are four conjugacy classes in A_4.

$$\{e\} \quad \{(12)(34), (13)(24), (14)(23)\} \quad \{(123), (243), (134), (142)\} \quad \{(132), (234), (143), (124)\}$$

    - Note that if $x, y \in A_4$ are of order 3, either $x \sim y$ or $x \sim y^{-1}$.
  - In A_5, all 3-cycles are conjugate; in A_4, they're not.
  - The sizes of the conjugacy classes in A_4 are $1 + 4 + 4 + 3$. That's enough to prove that there is no subgroup of order 6.
  - Alternate proof.

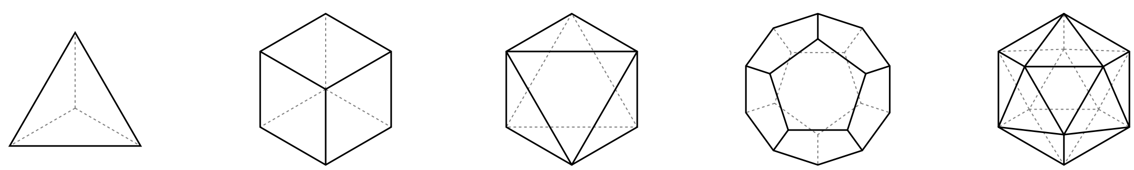
Proof. Suppose for the sake of contradiction that A_4 has a normal subgroup H of index 2. Then by the proposition from Lecture 4.1, there exists a surjective homomorphism from $A_4 \to A_4/H$. Additionally, since $|A_4/H| = 2$ and there is only one group of order 2, $A_4/H \cong \mathbb{Z}/2\mathbb{Z}$. Thus, there exists a surjective homomorphism $\phi : A_4 \to \mathbb{Z}/2\mathbb{Z}$.

We know that every alternating group (including A_4) is generated by 3-cycles. Let σ be an arbitrary 3-cycle generator of A_4. We know that $\phi(\sigma) = 0$ or $\phi(\sigma) = 1$. If $\phi(\sigma) = 1$, then

$$0 = \phi(e) = \phi(\sigma^3) = 3\phi(\sigma) = 1 +_2 1 +_2 1 = 1$$

which clearly cannot happen. Thus, $\phi(\sigma) = 0$. Consequently, the image of all of the generators of A_4 under ϕ is 0. But this implies that $\phi(A_4) = \{0\} \subsetneq \mathbb{Z}/2\mathbb{Z}$, contradicting our hypothesis that ϕ is surjective. ☐

- Here ends the material that will be covered on the midterm.
- We now move on to something we will come back to later.
- A_n in nature for $n = 4, 5$.



(a) Tetrahedron. (b) Cube. (c) Octahedron. (d) Dodecahedron. (e) Icosahedron.

Figure 5.1: The platonic solids.

  - Recall the cube group Cu.
  - The cube is an example of a **platonic solid**.
  - Other examples: Tetrahedron, octahedron, icosahedron, and dodecahedron. We define corresponding symmetry groups Te, Oc, Do, and Ic.
  - Consider the tetrahedral group to start.
    - Since any rigid motion permutes the vertices, we have a map $\text{Te} \hookrightarrow S_4$. Moving 2 vertices fixes the rest. Thus, $\text{Te} \leq S_4$. Therefore, $|\text{Te}| = 12$ so $\text{Te} \cong A_4$.
  - We determined in HW2 that...
    - $\text{Do} \hookrightarrow S_5$ and $|\text{Do}| = 60$. Thus, $\text{Do} \cong A_5$.
  - Consider the octahedron.

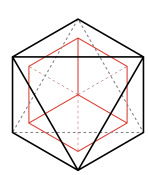


Figure 5.2: Inscribing a cube in an octahedron.

    - $|\text{Oc}| = 6 \cdot 4 = 24$. Rationale: Fix one vertex anywhere and then fix another (the other one can only take on the four adjacent positions, though); the positions of the rest are determined from these two.
    - Let's look at fixing opposite faces. This does give an injective map to S_4, and it follows that $\text{Oc} \cong S_4$.

    - Relation between Oc and Cu. We can inscribe a cube in the octahedron by connecting each vertex of the cube to the midpoint of one of the faces of the octahedron and vice versa. Thus, we get maps Oc $\to$ Cu, leading to Oc $\cong$ Cu.
  - We can similarly inscribe a dodecahedron in an icosahedron.
  - Thus, the cube and the octahedron have the same symmetry, and the dodecahedron and icosahedron have the same symmetry.

- **Platonic solid**: A solid geometric shape in three dimensions for which the faces, edges, and vertices are all indistinguishable.
  - We will study the platonic solids in more depth later.
- Problem: What symmetries can objects in $\mathbb{R}^3$ have?
  - Rephrase: What are the finite subgroups of SO(3)?
  - An octagon is Calegari's favorite polygon.
  - An octagonal prism has much the same symmetry in $\mathbb{R}^3$ as an octagon does in $\mathbb{R}^2$. This leads to $D_{2n} \leq \mathrm{SO}(3)$.
    - Recall the map from the blog post.
  - We also have $\mathbb{Z}/n\mathbb{Z} \leq D_{2n}$.
- It follows that the groups $\mathbb{Z}/n\mathbb{Z}$, D_{2n}, A_4, S_4, and S_5 occur as finite subgroups of SO(3).
- Theorem: All finite subgroups of SO(3) are on this list. Moreover, all related versions are conjugate.
  - This is a companion theorem to the theorem that there are only five platonic solids.
  - Neither theorem implies the other, but they are related.
- Infinite subgroups of SO(3): O(2), SO(3), SO(2).
- This theorem will be completely evident by the end of the course.
- You can either use this theorem to understand A_4, A_5, or use an understanding of A_4, A_5 to rationalize this theorem.
- Points to the main focus of the class: Understanding groups not just based on writing down elements but by their action on a certain set. This is the focus of the second half of the course.
- Midterm: 50 mins, closed book, Wednesday. Final exam must be in-person by department rules, but Calegari is fighting for us. Calegari is hoping that the midterm should not be a speed test.
  - Trying to test our skills, not our ability to memorize stuff.
  - How to do well: Learn group theory.
- The quaternion group.
  - A 4D vector space where you define a noncommutative product. If you just take 8 specific quaternions, the group of order 8 is distinct from D_8 but related.

5.2 Group Actions

10/28:

- Let G be a group and X be a set.
- **Group action** (of G on X): A map $\cdot : G \times X \to X$ satisfying the following. *Denoted by* $\boldsymbol{G \circlearrowright X}$.
  1. For all $g, h \in G$ and $x \in X$, $g \cdot (h \cdot x) = gh \cdot x$.
  2. For all $x \in X$, $e \cdot x = x$.

- Note that condition 2 does not follow from condition 1, and an "inverse condition" follows from both.
  - In particular, condition 1 relates certain elements of the domain of the group action but does not relate any elements of the domain to elements of X (as condition 2 does).
  - The inverse condition $g^{-1} \cdot (g \cdot x) = g \cdot (g^{-1} \cdot x) = x$ follows from conditions 1-2 via

$$g^{-1} \cdot (g \cdot x) = g^{-1} g \cdot x = e \cdot x = x = e \cdot x = gg^{-1} \cdot x = g \cdot (g^{-1} \cdot x)$$

- Example: If G is any group and X is any set, we may define a group action by $g \cdot x = x$ for all $g \in G$ and $x \in X$.
- Lemma: Let $G \circlearrowright X$ and $g \in G$. Define $\psi_g : X \to X$ by $x \mapsto g \cdot x$. Then ψ_g is a bijection.

  Proof. Injectivity:

$$\begin{aligned} \psi_g(x) &= \psi_g(y) \\ g \cdot x &= g \cdot y \\ g^{-1} \cdot (g \cdot x) &= g^{-1} \cdot (g \cdot y) \\ e \cdot x &= e \cdot y \\ x &= y \end{aligned}$$

  Surjectivity: Given $x \in X$, we want y such that $\psi_g(y) = x$. Choose $y = g^{-1} \cdot x$. $\square$

- This allows us to recast group actions into the following equivalent form.
- Let X be a set and S_X be the set of all bijections from $X \to X$ under composition. Note that if $|X| = n$, then $S_X \cong S_n$.
- Proposition: An action G on the set X is equivalent to a homomorphism from G to S_X defined by $g \mapsto \psi_g$.

  Proof. A statement of the proposition that makes it more clear what exactly it is we want to prove is, "there exists an action $\cdot : G \times X \to X$ iff there exists a homomorphism $\phi : G \to X$ defined by $g \mapsto \psi_g$." Let's begin.

  Suppose first that $\cdot : G \times X \to X$ is group action of G on X. Define $\phi : G \to S_X$ by $g \mapsto \psi_g$. To prove that ϕ is a homomorphism, it will suffice to show that $\phi(gh) = \phi(g) \circ \phi(h)$ for all $g, h \in G$. Let $g, h \in G$ be arbitrary. Then by condition 1, we have for any and all $x \in X$ that

$$\begin{aligned} g \cdot (h \cdot x) &= gh \cdot x \\ \psi_g(\psi_h(x)) &= \psi_{gh}(x) \\ [\psi_g \circ \psi_h](x) &= \psi_{gh}(x) \\ [\phi(g) \circ \phi(h)](x) &= [\phi(gh)](x) \end{aligned}$$

  Therefore, $\phi(gh) = \phi(g) \circ \phi(h)$, as desired.

  Now suppose that $\phi : G \to S_X$ is a homomorphism defined by $g \mapsto \psi_g$. Define $\cdot : G \times X \to X$ by $g \cdot x = [\phi(g)](x)$. To prove that $\cdot$ is a group action, it will suffice to show that for all $g, h \in G$ and $x \in X$, $g \cdot (h \cdot x) = gh \cdot x$ and $e \cdot x = x$. Let $g, h \in G$ and $x \in X$ be arbitrary. Then

$$g \cdot (h \cdot x) = g \cdot \psi_h(x) = [\psi_g \circ \psi_h](x) = [\phi(g) \circ \phi(h)](x) = [\phi(gh)](x) = \psi_{gh}(x) = gh \cdot x$$

  and

$$e \cdot x = \psi_e(x) = x$$

  as desired. $\square$

- You need to be careful with what the set is and what the group is; $x \cdot y$ probably doesn't make any sense (unless you start to get into cases where X is a group, too).
- **Kernel** (of a group action): The set of all $g \in G$ such that $g \cdot x = x$ for all $x \in X$.
  - The kernel is a (normal) subgroup of G.
  - We know this since it is equivalent to the kernel of the homomorphism described by the above proposition.
- **Faithful** (group action): A group action for which the kernel is trivial, i.e., $\ker = \{e\}$.
  - Such a group action is "faithful" because it is telling the whole story, i.e., not leaving out any information, i.e., mapping everything to everything.
  - The trivial group action is an example of a group action that isn't faithful.
- **Orbit** (of $x \in X$): The set of $g \cdot x$ for all $g \in G$. *Denoted by* $\mathbf{Orb}(\boldsymbol{x})$.
  - A subset of X.
  - Everywhere you can get to from your starting point x.
- **Transitive** (group action): A group action for which $\text{Orb}(x) = X$ for some (any) $x \in X$.
  - In what way is a transitive group action *transitive*??
- **Stabilizer** (of $x \in X$): The set of all $g \in G$ for which $g \cdot x = x$. *Denoted by* $\mathbf{Stab}(\boldsymbol{x})$.
  - A subgroup of G.
- The kernel is a subgroup of the stabilizer. More specifically,

$$\ker = \bigcap_{x \in X} \text{Stab}(x)$$

  - This is because the elements of the stabilizer fix *some* $x \in X$, whereas the elements of the kernel fix *all* $x \in X$.
- Orbits are equivalence relations, i.e., $x \in \text{Orb}(x)$ and $x \in \text{Orb}(y)$ imply that $\text{Orb}(x) = \text{Orb}(y)$.
  - In particular,

$$X = \bigsqcup \text{Orbits}$$

- Let $G = S_n$ and $X = [n]$.
- Let $G = \text{Cu}$.
- Examples.

G	X	$\lvert X\rvert$	Transitive	Faithful	Kernel	$\text{Stab}(x)$	$\lvert\text{Stab}(x)\rvert$
	Faces	6	✓	✓	$\{e\}$	$\mathbb{Z}/4\mathbb{Z}$ (rotations by $90°$)	4
	Vertices	8	✓	✓	$\{e\}$	$\mathbb{Z}/3\mathbb{Z}$	3
	Edges	12	✓	✓	$\{e\}$	$\mathbb{Z}/2\mathbb{Z}$	2
Cu	Diagonals	4	✓	✓	$\{e\}$	$S_3 \cong D_6$	6
	Pairs of opposite faces	3	✓	X	Rotations by $180°$, in particular, $\lvert K\rvert = 4$	D_8	8
	Inscribed tetrahedra	2	✓	X	A_4	A_4	12
	$\text{Ed} \cup \text{Fac}$	18	X	✓	$\{e\}$	$\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$	4 or 2

Table 5.1: Examples of group actions.

  - The last two rows we filled out by first asserting transitivity, A_4, and 12; and edges union faces, not transitive.

- On Monday, we will look into group actions where the geometry is not so convenient.