Week 7

Group Action Applications: A_5 and the Sylow Theorems

7.1 Actions of A_5

11/7:

- Classifying subgroups of $G = A_5 \cong$ Do.
- Let $H \leq G$. We must have $|H|\,\big|\,|G|$ by Lagrange's theorem.
  - Thus, if $H \leq A_5$, we must have

$$|H| \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

- A good place to start is with orders of H that correspond to cyclic subsets.
- In particular, let's start with subgroups of the form $\langle(**)(**)\rangle$, which all have order 2.
  - Are such groups conjugate?
  - To prove that two groups of the form $\langle(**)(**)\rangle$ are conjugate, it will suffice to show that their generators are conjugate (since the only other element — the identity — will naturally be conjugate to itself).
  - Let $x, y \in A_5$ be arbitrary elements of the form $(**)(**)$. Then there exists $g \in S_5$ such that $gxg^{-1} = y$.
  - But is $g \in A_5$? If $g \in A_5$, then we are done. If $g \notin A_5$, then can we find an element $g' \in A_5$ such that $g'xg'^{-1} = y$?
  - First, note that if $gxg^{-1} = y = g'xg'^{-1}$, then

$$\begin{aligned} g^{-1}(gxg^{-1})g' &= g^{-1}(g'xg'^{-1})g' \\ x(g^{-1}g') &= (g^{-1}g')x \end{aligned}$$

    Thus, $g^{-1}g' \in C_{S_5}(x)$, or $g' = gh$ for some $h \in C_{S_5}(x)$.
    - If $g \notin A_5$ and we want $g' \in A_5$, then we must have $h \notin A_5$.
      - Intuitively, this means that if g is the product of an odd number of permutations and we want $g' = gh$ to be the product of an even number of permutations, h had better be a product of an odd number of permutations as well.
      - More formally, consider G/A_5. If $g \in gA_5 \neq A_5$ and we want $g' \in g'A_5 = A_5$, then by homomorphically mapping gA_5 to $1 \in \mathbb{Z}/2\mathbb{Z}$ and A_5 to $0 \in \mathbb{Z}/2\mathbb{Z}$, we must have $h \in gA_5$ to get $gh \in A_5$.
    - Regardless, this example motivates the following two propositions, which we can use to resolve the original conjugacy question.

  - By Proposition 1, since $x \sim y$ in S_5 and $C_{S_5}(x) \not\subset A_5$ (take the first transposition in $(**)(**)$; for example, know that (12) commutes with $(12)(34)$), we know that $x \sim y$ in A_5.
  - Therefore, there are 15 subgroups of the form $\langle (**)(**) \rangle$, all of which are conjugate in A_5.

- Proposition 1: Let $x \sim y$ in S_n. Then if $C_{S_n}(x) \not\subset A_n$, then $x \sim y$ in A_n.

  Proof. Since $x \sim y$ in S_n, there exists $g \in S_n$ such that $gxg^{-1} = y$. If $g \in A_n$, then we are done. Now suppose $g \notin A_n$. Since $C_{S_n}(x) \not\subset A_n$, there exists $h \in C_{S_n}(x)$ such that $hxh^{-1} = x$ and $h \notin A_n$. Since $g, h \notin A_n$, we have that $gh \in A_n$. Additionally, we have that

  $$(gh)x(gh)^{-1} = g(hxh^{-1})g^{-1} = gxg^{-1} = y$$

  Therefore, $x \sim y$ in A_n, as desired. □

- Proposition 2: If $C_{S_n}(x) \subset A_n$ and $\sigma x \sigma^{-1} = y$, then $x \sim y$ in A_n iff $\sigma \in A_n$.

  Proof. Suppose first that $x \sim y$ in A_n. Then $gxg^{-1} = y$ for some $g \in A_n$. Then as per the above, $gxg^{-1} = \sigma x \sigma^{-1}$ implies that $g^{-1}\sigma \in C_{S_n}(x)$. Thus, $\sigma = gh$ for some $h \in C_{S_n}(x) \subset A_n$. But since $g, h \in A_n$, we must have $\sigma \in A_n$, too.

  Now suppose that $\sigma \in A_n$. Then since $\sigma x \sigma^{-1} = y$, $x \sim y$ in A_n as desired. □

- Now we discuss subgroups of the form $\langle (***) \rangle$.
  - Let x be an arbitrary element of A_5 of the form $(***)$. In particular, suppose $x = (abc)$ for $a, b, c \in [5]$.
  - Then $(de) \in C_{S_5}(x)$, where $d, e \in [5]$ are the other two elements that are not already represented by a, b, c.
  - Moreover, (de) will be in the centralizers of both x and x^2.
  - There are $\binom{5}{2} = 10$ subgroups of the form we're discussing (20 generators/elements of the form $(***)$, though).
  - Suppose we have two subgroups $\langle x \rangle, \langle y \rangle$ of the form being discussed. We know that $\langle x \rangle, \langle y \rangle$ are conjugate in S_5. But since $C_{S_5}(x) \not\subset A_5$ again as per the above, we know the groups are conjugate in A_5.
  - Therefore, there are 10 subgroups of the form $\langle (***) \rangle$, all of which are conjugate in A_5.

- Now we discuss subgroups of the form $\langle (*****) \rangle$.
  - We know that $|C_{S_5}((12345))| \cdot |\{(12345)\}| = 120$. Additionally, only a power of (12345) commutes with it in this case, so the first term is 5. Thus, the second must be 24.
    - In sum, we have showed that there are 24 elements conjugate to (12345) in S_5.
    - Another way we could show this is by counting all of the 5-cycles and knowing that they are all conjugate as 5-cycles. Indeed, there are $4! = 24$ 5-cycles.
  - Claim: In A_5, $|x| = 5$ implies $x \sim x$, $x \nsim x^2$, $x \nsim x^3$, and $x \sim x^4 = x^{-1}$.

    Proof. We know that $|x| = 5$. Thus, let $x = (abcde)$.

    By the above statements on $C_{S_5}((12345))$, we know that $C_{S_5}(x) \subset A_5$. Thus, by proposition 2, $gxg^{-1} = x'$ iff $g \in A_n$. Thus,

    $$\begin{aligned} exe^{-1} = x &\quad\Longrightarrow\quad x \sim x \\ [(bc)(cd)(de)]x[(bc)(cd)(de)]^{-1} = (bced)(abcde)(bced)^{-1} = (acebd) &\quad\Longrightarrow\quad x \nsim x^2 \\ (bdec)(abcde)(bdec)^{-1} = (adbec) &\quad\Longrightarrow\quad x \nsim x^3 \\ [(be)(cd)](abcde)[(be)(cd)]^{-1} = (aedcb) &\quad\Longrightarrow\quad x \sim x^4 = x^{-1} \end{aligned}$$

    as desired. □

  - $x^2 \sim x^3$ in A_5 as well.
  - $(abced)$ and $(acebd)$ are conjugate by $(bce) \in A_5$.
  - Six subgroups, all conjugate.
  - All of the subgroups are conjugate, but not all of the elements are conjugate?

- Consider $K = \{e, (12)(34), (13)(24), (14)(23)\} \triangleleft A_4 \subset A_5$.
- Consider a transitive group action from A_5 to $X = \{\text{cong of } K\}$.
- $\text{Stab}(K) = N_{A_5}(K) \supset A_4$.
- By O.S. trm, $X = |A_5|/|A_4| = 5$.
- Let $H \subset A_5$ have $|H| = 4$.
- We want to show that H fixes a point. Equivalently, we want to find $x \in \{1, 2, 3, 4, 5\}$ such that $|\operatorname{Orb}(x)| = 1$.
- Since $4 = |H| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$ and $5 \equiv 1 \mod 2$. Thus, there is a fixed point.
- Thus, there are 15 cyclic subgroups of order 4 like K, and they are all conjugate.
- $H \leq A_5$ has index d iff there is a transitive action and puts A_5/H. Induces a map from $A_5 \to S_d$?? As A_5 has no normal subgroups. If $d = 2, 3, 4, \ldots$?? If $d = 5$, then $A_5 \to S_5 \to S_5/A_5$. But really $A_5 \to S_5 \to S_5/A_5 \cong \mathbb{Z}/2\mathbb{Z}$.
- The hard ones are 6, 10, or 12.
- Consider a subgroup of A_5 of order 6. Must be $\mathbb{Z}/6\mathbb{Z}$ or S_3. These groups have subgroups of order 3. If we have this, it must be a subgroup of $S_3 \times S_2 \cap A_5$. Important: $\langle (1, 2, 3) \rangle$ and $(1, 2)(4, 5)$.
- Same analysis for subgroups of order 10. Subsets of order 1,2,5,10. (12) orbits include...
- Table with sets.
- If we spend a couple of hours understanding this example in complete detail, that will be very helpful for the final.