4 Types of Subgroups

10/24: 1. Let H and K be normal subgroups of G such that $H \cap K$ is trivial. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. (Exercise 3.1.42 of Dummit and Foote (2004).)

2. Show that S_4 does not have a normal subgroup of order 3 or order 8.

3. If H is a subgroup of G, define the **normalizer** of H to be

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

   (a) Prove that $N_G(H) = G$ if and only if H is normal.
   (b) Prove that $N_G(H)$ contains H.
   (c) Prove that H is a **normal** subgroup of $N_G(H)$.
   (d) Compute $N_G(H)$ for the following pairs (G, H).
      i. $(S_4, \langle(1,2,3,4)\rangle)$.
      ii. $(S_5, \langle(1,2,3,4,5)\rangle)$.

4. Prove that the subgroup N generated by elements of the form $x^{-1}y^{-1}xy$ for all $x, y \in G$ is normal. (Exercise 3.1.41 of Dummit and Foote (2004).)

5. Prove that if $G/Z(G)$ is cyclic, then G is abelian. (For a hint, see Exercise 3.1.36 of Dummit and Foote (2004).)

6. Let G be a finite group, and let $H \subset G$ be a subgroup of index two — i.e., $|G|/|H| = 2$. Prove that H is normal.

7. Let G be a finite group, and let $H \subset G$ be a subgroup of index three — i.e., $|G|/|H| = 3$. Show that H is not necessarily normal.

8. **Automorphism Groups**. Define an automorphism of a group G to be an isomorphism $\phi : G \to G$ from G to itself. (See §4.4 of Dummit and Foote (2004).)

   (a) Prove that the identity map is an automorphism.
   (b) Prove that the composition of two automorphisms is an automorphism.
   (c) Prove that the set of automorphisms forms a group under composition. We will call this group $\mathrm{Aut}(G)$.
   (d) If $g \in G$ is a fixed element, prove that the map $\phi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$ is an isomorphism.
   (e) Prove that the map $\psi : G \to \mathrm{Aut}(G)$ given by $\psi(g) = \phi_g$ (sending the element g to the automorphism ϕ_g) is a homomorphism of groups.
   (f) Prove that the kernel of the map $\psi : G \to \mathrm{Aut}(G)$ is the center

$$Z(G) = \{g \in G \mid gx = xg, \ \forall x \in G\}$$

   (g) Define the inner automorphism group $\mathrm{Inn}(G)$ of G to be the subgroup of $\mathrm{Aut}(G)$ given by the image of G under ψ. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
   (h) Show that if G is abelian, then $\mathrm{Inn}(G)$ is trivial.
   (i) Let $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. Prove that...
      i. $\mathrm{Aut}(\mathbb{Z}/3\mathbb{Z}) = \mathrm{Out}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$;
      ii. $\mathrm{Out}(S_3) = \{1\}$;
      iii. $\mathrm{Aut}(K) \cong \mathrm{Out}(K) \cong S_3$, where $K = (\mathbb{Z}/2\mathbb{Z})^2$ is the Klein 4-group.

9. Let p be an odd prime number. Prove that there are no surjective homomorphisms from S_n to $\mathbb{Z}/p\mathbb{Z}$ for any prime p. (Hint: Consider the image of the two-cycles).