Week 4

???

4.1 Quotient Groups

10/17: - Notational confusion regarding $\mathbb{Z}/10\mathbb{Z}$.
  - Let $G = \mathbb{Z}$ and $H = 10\mathbb{Z}$ (the multiples of 10).
  - A few of the cosets are as follows:

$$\begin{aligned} H &= \{\dots, -20, -10, 0, 10, 20, 30, \dots\} \\ 1 + H &= \{\dots, -19, -9, 1, 11, 21, 31, \dots\} \\ 2 + H &= \{\dots, -18, -8, 2, 12, 22, 32, \dots\} \end{aligned}$$

  - Evidently, $|\mathbb{Z}/10\mathbb{Z}| = 10$.
  - Yet $\mathbb{Z}/10\mathbb{Z}$ is also the notation for the cyclic group of order 10.
  - This notation is not an error, but reveals something deep: We can make the set of cosets into a group and define addition by

$$(a + 10\mathbb{Z}) + (b + 10\mathbb{Z}) = (a + b + 10\mathbb{Z})$$

    More specifically, we can define an isomorphism between the two definitions of $\mathbb{Z}/10\mathbb{Z}$ via $a+H \mapsto a$ for $a = 0, \dots, 9$.

- This example motivates the following goal.
- Goal: Make G/H, which is a set, into a group.
  - This set needs a binary operation. It makes natural sense to define the binary operation as follows.

$$xH * yH = xyH$$

  - We then need an identity coset, inverse cosets, and associativity.
    - The identity is H.
    - The inverse of xH is $x^{-1}H$.
    - Associativity of G/H follows from the associativity of G (which tells us that $(ab)c = a(bc)$). More specifically,

$$\begin{aligned} aH *_H (bH *_H cH) &= aH *_H (b *_G c)H \\ &= a *_G (b *_G c)H \\ &= (a *_G b) *_g cH \\ &= (a *_G b)H *_H cH \\ &= (aH *_H bH) *_H cH \end{aligned}$$

- Calegari's impromptu explanation of associativity drives home that he really is very good at drilling down to the core of an idea and working with it. He really has a very similar mind to mine.
- Something else we need to investigate: Equivalence classes, and defining functions on equivalence classes.
  - We need to make sure that functions are defined the same regardless of how you label the equivalence classes.
  - Consider the set of names.
    - Say we define equivalency classes based on all names which share the same first letter.
    - Then we define a function F on the equivalency classes based on the last letter.
    - But then [Frank] = [Fen] will be mapped to two different elements of the alphabet, so F is not well-defined.
  - Thus, for our example, we need to guarantee that if $x, x' \in xH$, then $xH * yH = x'H * yH$.
- Check: Independence of choice.
  - Suppose we relabel $x \mapsto xh$ and $y \mapsto yh$. We need

    $$xhyh' = xyh''$$

    for some $h'' \in H$.
    - Note that x, y, h, h' are all fixed; h'' is the only free thing (i.e., is what we're looking for).
  - Algebraically manipulating the above implies that we want

    $$h'' = y^{-1}hyh'$$

  - Thus, we know that $h'' \in G$, but we need to make sure that $h'' \in H$. Alternatively, we want $y^{-1}hy = h''(h')^{-1} \in H$.
  - An example where $y^{-1}hy$ is not in H: $G = S_3$, $H = \langle(1,2)\rangle$, $h = (1,2)$, $y = (1,3)$, $yhy^{-1} = (2,3)$.
- Why did $\mathbb{Z}/10\mathbb{Z}$ work? Because it was abelian, so conjugacy cancelled $y^{-1}hy = y^{-1}yh = h$.
  - We could restrict ourselves entirely to abelian groups, but can we be more general?
- What should we require of G/H?
  - The cananonical map of sets $\phi : G \to G/H$ is given by $\phi(x) = xH$.
  - We should require that ϕ is a homomorphism (i.e., that the group structure of G is preserved for G/H).
  - See how $xH * yH = xyH$ is analogous to $\phi(x)\phi(y) = \phi(xy)$.
- Let's suppose $\phi : G \to G/H$ is a homomorphism.
  - Then $\phi(g) = eH$ implies that $g \in H$, i.e., $\ker\phi = H$.
  - Realization: An alternate way to do HW3, Q2b would have been in terms of quotient groups: In that case, $G/H \cong S_{26}$, and the following proposition would give us the surjectivity and kernel requirements.
- Lemma: Let ϕ be a homomorphism from G to another group. Let $K = \ker\phi \subset G$. Then K has the following property, which is not true for all subgroups but is for kernels: If $x \in K$ and $g \in G$, then $gxg^{-1} \in K$.

  Proof. Since $\phi(x) = e$, we have that

  $$\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = e$$

  □

- **Normal** (subgroup): A subgroup H of G such that for all $x \in H$ and $g \in G$, $gxg^{-1} \in H$. *Denoted by* $\boldsymbol{H \trianglelefteq G}$, $\boldsymbol{H \triangleleft G}$.
  - We often write gHg^{-1}.
- Example: As per the lemma, $\ker\phi$ is a normal subgroup.
- Example: If G be abelian, then every $H \trianglelefteq G$.
- Lemma: A subset $H \subset G$ is normal iff
  1. H is a subgroup.
  2. H is a union of some number of conjugacy classes.
- Proposition: Let G be a group and $H \triangleleft G$. Then G/H is a group under the multiplication

$$xH * yH = xyH$$

and the map $\phi : G \to G/H$ is a surjective homomorphism with kernel H.

Proof. We want to show that $xhyh' = xyh''h'$. We can do so via multiplying the following by x on the left and h' on the right:

$$\begin{aligned} hy &= (yy^{-1})hy \\ &= y(y^{-1}hy) \\ &= yh'' \end{aligned}$$

Note that we get from the second to the third line above because H is a normal subgroup, i.e., conjugates of its elements are elements of it. This implies the desired result. □

- Example: Let $G = \mathbb{Z}$, $H = 10\mathbb{Z}$, and $G/H = \mathbb{Z}/10\mathbb{Z}$.
- Example: Let $G = G$ and $H = \{e\}$.
  - H is normal since it's a subgroup and it's a union of conjugacy classes.
  - In this case, $G/H \cong G$.
- Example: $G = \mathrm{O}(2)$ and $H = \mathrm{SO}(2)$.
  - G is not abelian here.
  - From HW1, the cosets are $H = \{\text{rotations}\}$ and $\{\text{reflections}\}$.
  - The cosets are H and sH for some reflection $s \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$.
  - What the group structure tells us here is that rotation $\circ$ reflection is like even $\times$ odd numbers.
  - $G/H \cong \mathbb{Z}/2\mathbb{Z}$ here.
- An equivalent formulation of normality.
- Proposition: $H \triangleleft G$ iff the left cosets coincide with the right cosets, i.e.,

$$gH = Hg$$

Proof. Suppose first that $H \triangleleft G$. Use a bidirectional inclusion argument. Let $gh \in gH$. Then

$$gh = ghg^{-1}g = h'g \in Hg$$

where h' may or may not equal h, but we know it is an element of H by the definition of normal subgroups. The argument is symmetric in the other direction.

Now suppose $gH = Hg$. Let $h \in H$. Then there exist $h, h' \in H$ such that $gh = h'g$. Therefore, $ghg^{-1} = h' \in H$. □

- This is a nice resolution of left and right cosets.
  - It tells us when they're the same, and when they're different.
- Implication: If $H \triangleleft G$, then

$$xH \cdot yH = x(Hy)H = x(yH)H = xyHH = xyH$$

- Midterm next week.

4.2 First Isomorphism Theorem

10/19:
- Last time:
  - If $K \triangleleft G$, then the map $\phi : G \to G/K$ defined by $g \mapsto gK$ is a surjective homomorphism with kernel K.
- Today: Understand a general surjective homomorphism $\phi : G \twoheadrightarrow H$ with kernel $K \triangleleft G$.

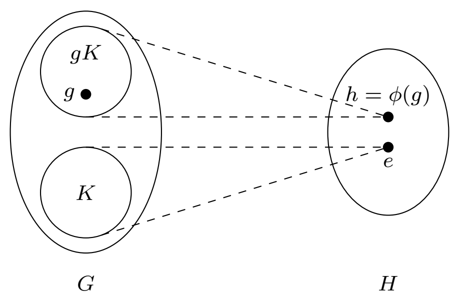

Figure 4.1: Visualizing a surjective homomorphism.

  - In general, we know that $K \mapsto \{e\}$.
  - Since ϕ is surjective, every $h \in H$ equals $\phi(g)$ for some $g \in G$.
  - More broadly, $gK \mapsto \{h\}$.
  - Can you get more elements than those in gK that map to h? Perhaps elements of Kg or KgK? Well since K is normal, $kg = gk$.
  - Thus, all surjective homomorphisms have the same general structure.
    - In particular, they all map disjoint cosets to single elements.
    - Alternatively, we can take the perspective that they send every element to their coset with the kernel.
- Lemma: If $\phi : G \to H$ is a surjective homomorphism, $h \in H$, $\phi(g) = h$, and $K = \ker \phi$, then $\phi^{-1}(h) = gK$.

  Proof. Suppose $g' \in \phi^{-1}(h)$. Suppose $g' = gx$ (we do know that such an x exists in G; in particular, choose $x = g^{-1}g'$). Then

  $$\phi(g') = \phi(gx) = \phi(g)\phi(x)$$

  Since $\phi(g') = h = \phi(g)$, we have by the cancellation lemma that

  $$e = \phi(x)$$

  i.e., $x \in K$. Therefore, $g' \in gK$, as desired. $\square$

- We can define a bijection $\tilde{\phi} : G/K \mapsto H$ defined by $gK \mapsto \phi(g)$.
- Claim: $\tilde{\phi}$ is an isomorphism of groups.

  Proof. Need to check that $\tilde{\phi}$ is a homomorphism, surjective, and injective. We also need to check that it is well-defined (we did this with our picture).

  Surjective: Let $h \in H$ be arbitrary. Then $h = \phi(g)$. It follows that $h = \tilde{\phi}(gK)$.

  Injective: Show that $\ker \tilde{\phi} = \{eK\}$. Let $gK \in \ker \tilde{\phi}$. Then $\phi(g) = \tilde{\phi}(gK) = e$. Thus, $g \in K$. Therefore, $gK = eK$, as desired.

  Homomorphism: Check $\tilde{\phi}(xK)\tilde{\phi}(yK) = \tilde{\phi}(xyK)$. Since $\tilde{\phi}(zK) = \phi(z)$, we have the desired property. Explicitly,

  $$\tilde{\phi}(xyK) = \phi(xy) = \phi(x)\phi(y) = \tilde{\phi}(xK)\tilde{\phi}(yK)$$

  $\square$

- Takeaway: All surjective homomorphisms are somewhat the same.
- Generalize:
- Let $\phi : G \to H$ be a homomorphism.
  - We know that $G \twoheadrightarrow \operatorname{im}\phi \hookrightarrow H$. Essentially, we can break up any homomorphism into the composition of a surjective homomorphism onto the image and an injective homomorphism into H.
- Theorem (FIT: First Isomorphism Theorem): To every homomorphism ϕ there corresponds an isomorphism $\tilde{\phi} : G/\ker\phi \to \operatorname{im}\phi$ such that

  $$\tilde{\phi}(g \cdot \ker\phi) = \phi(g)$$

$$\begin{tikzcd}
 & g \arrow[r, mapsto] & \phi(g) & \\
g \arrow[ddr, mapsto] & G \arrow[r, two heads, "\phi"] \arrow[d] & \operatorname{im}\phi \arrow[r, hook] & H \\
 & G/\ker\phi \arrow[ur, "\tilde{\phi}"', "\wr"] & \phi(g) & \\
 & g \cdot \ker\phi \arrow[ur, mapsto] & &
\end{tikzcd}$$

Figure 4.2: First isomorphism theorem.

  - The triangle is **commutative**. This means that sending g along both paths gets you to the same result.
  - The way to understand normal subgroups is to understand the homomorphisms.
- $N \subset G$ is normal if
  1. N is a subgroup.
  2. N is normal, i.e., N is a union of conjugacy classes.
  3. $e \in N$.
  4. $|h|\big||G|$ (Lagrange).

- 3-4 both follow from 1. They are not sufficient conditions for normality, but they can put restrictions on what is normal and make the computation easier.
- Examples.
  - Let $\phi : \mathbb{Z} \to H$ send $1 \mapsto h$ and $k \mapsto h^k$.

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, two heads] & H \\
\mathbb{Z}/n\mathbb{Z} \arrow[r, "\sim"] & \langle h \rangle \arrow[u, hook] \\
k + n\mathbb{Z} \arrow[r, mapsto] & h^k
\end{tikzcd}
$$

Figure 4.3: An example of the FIT.

    - $\operatorname{im}\phi = \langle h \rangle$.
    - $\ker\phi = n\mathbb{Z}$ where $|h| = n$; if $|h| = \infty$, then $\ker\phi = \{0\}$.
    - The FIT tells us that there is a map from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ to $\langle h \rangle$ to H. The first map sends $k \mapsto k + n\mathbb{Z}$ and the second sends $k + n\mathbb{Z} \mapsto h^k$.
  - Let $G = S_3$.
    - The conjugacy classes are

$$\{e\} \qquad \{(1,2),(1,3),(2,3)\} \qquad \{(1,2,3),(1,3,2)\}$$

    - Thus, the only possible normal subgroup N is

$$H = \{e\} \cup (xxx) = \langle (1,2,3) \rangle$$

      - $e \in N$ eliminates union 2,3; Lagrange eliminates union 1,2 (which has order 4).
  - Let $G = S_4$.
    - The conjugacy classes are

$$e \qquad (xx) \qquad (xxx) \qquad (xxxx) \qquad (xx)(xx)$$

    - The number of elements of the above form is

$$1 \qquad 6 \qquad 8 \qquad 6 \qquad 3$$

    - The divisors of $|S_4| = 24$ are 1,2,3,4,6,8,12,24.
      - 1 is possible; no way to get 2,3; 4 is possible; 6,8 are impossible; 12,24 are possible.
      - The 4 example is

$$K = \langle e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) \rangle$$

- $S_3 / \langle (1,2,3) \rangle \cong \mathbb{Z}/2\mathbb{Z}$.
- S_4/K is a group of order 6.
- The first instance corresponds to some map from $S_3 \to S_2$.
  - You can get an isomorphism from S_3 to D_6.
  - The surjective map sends rotations to the identity and reflections to the nonidentity element.
  - By the FIT, $S_3 / \langle (1,2,3) \rangle \cong S_2$.
    - Yes, if you know enough about the quotient group, you can think about its properties. But it's easier to use the FIT.

- We constructed a map $S_4 \to \text{Cu} \to S_3$. If $N = \ker$, by the FIT, $S_4/N \cong S_3$.
  - As per the above example, we need to take $N = K$ here.
- Example: $G = \text{O}(2)$.
  - The normal subgroups of $\text{O}(2)$ are $\{e\}$, $\{r, r^{-1}\}$, and $\{\text{reflections}\}$.
  - If $N \triangleleft \text{O}(2)$ contains a reflection, then $N = \text{O}(2)$.
  - Let $N \subset \text{SO}(2)$ be such that $|N| = k$, i.e., N is generated by the rotation of $2\pi k/N$. What is $\text{O}(2)/N$? You can think of $\text{SO}(2)$ as a rotation in $\mathbb{R}$. Thus, $\mathbb{R}/2\pi\mathbb{Z} \cong \text{O}(2)$. Thus, $\text{SO}(2)/N \cong \text{SO}(2)$.
- Next time: Replace S_4 with S_5.
- The midterm is most likely Wednesday next week.
  - The midterm will not be on Monday, but it could test stuff covered next Monday.
- Read the blog post on dihedral groups and the other blog posts I've missed!