6 Theory of Group Actions

11/14: You should think about and try to solve the starred questions, but several of them are quite messy and some are difficult, so only submit the ones without stars.

1. Exercises 4.1.7-4.1.8 of Dummit and Foote (2004).

   7. Let G be a transitive permutation group on the finite set A. A **block** is a nonempty subset B of A such that for all $\sigma \in G$, either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$ (here $\sigma(B)$ is the set $\{\sigma(b) \mid b \in B\}$).

   (a) Prove that if B is a block containing the element $a \in A$, then the set G_B defined by $G_B = \{\sigma \in G \mid \sigma(B) = B\}$ is a subgroup of G containing G_a.

   (b) Show that if B is a block and $\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B)$ are all the distinct images of B under the elements of G, then these form a partition of A.

   (c) A (transitive) group G on a set A is said to be **primitive** if the only blocks in A are the trivial ones: The sets of size 1 and A itself. Show that S_4 is primitive on $A = \{1, 2, 3, 4\}$. Show that D_8 is not primitive as a permutation group on the four vertices of a square.

   (d) Prove that the transitive group G is primitive on A if and only if for each $a \in A$, the only subgroups of G containing G_a are G_a and G (i.e., G_a is a **maximal** subgroup of G). *Hint.* See Exercise 2.4.16. Use part (a).

   8. A transitive permutation group G on a set A is called **doubly transitive** if for any (hence all) $a \in A$, the subgroup G_a is transitive on the set $A \setminus \{a\}$.

   (a) Prove that S_n is doubly transitive on $\{1, 2, \ldots, n\}$ for all $n \geq 2$.

   (b) Prove that a doubly transitive group is primitive. Deduce that D_8 is not doubly transitive in its action on the four vertices of a square.

2. Exercise 4.2.9 of Dummit and Foote (2004).

   9. Prove that if p is a prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.

3. Suppose that G acts transitively and faithfully on a finite set X, and that G is abelian. Prove that $|G| = |X|$. Show that the equality need not hold if G is not abelian.

4. Let G be a finite group and let H be any subgroup.

   (a) Prove that the left action of G on the coset space G/H has kernel $N = \bigcap_{g \in G} gHg^{-1}$.

   (b) Prove that $N = \bigcap_{g \in G} gHg^{-1}$ is the largest normal subgroup of G contained in H.

5. **The Quaternions.** Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be a 4-dimensional vector space over $\mathbb{R}$. Define a non-commutative associative multiplication structure on $\mathbb{H}$ by the formulae

$$ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \qquad i^2 = j^2 = k^2 = -1$$

   (a) ($\star$) Show that there is a map $\phi : \mathbb{H} \to M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the vector space of 2×2 matrices over $\mathbb{C}$, defined by sending

$$i \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \qquad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad k \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

   for which

   i. ϕ is injective as a map of vector spaces over $\mathbb{R}$.

   ii. ϕ respects multiplication; if q_1, q_2 are two quaternions, then $\phi(q_1q_2) = \phi(q_1)\phi(q_2)$. This should reduce easily enough to the case where q_i, q_j are elements of the set $\phi(1), \phi(i), \phi(j), \phi(k)$. The map ϕ is not a group homomorphism since 0 is not an invertible quaternion, but we shall see below in part (c) that non-zero quaternions form a group, so ϕ restricted to $\mathbb{H}^\times$ is actually a homomorphism from $\mathbb{H}^\times$ to $\mathrm{GL}_2(\mathbb{C})$.

(b) Define the conjugate of a quaternion $q = a + bi + cj + dk$ by $\bar{q} := a - bi - cj - dk$. Prove that $N(q) := q\bar{q} = a^2 + b^2 + c^2 + d^2$.

(c) Prove that non-zero quaternions $\mathbb{H}^\times$ form a group under multiplication.

(d) Let $Q = \langle i, j \rangle$ be the subgroup of $\mathbb{H}^\times$ generated by i, j. Prove that Q is a group of order 8. (Q is known as the "quaternion group.")

(e) Prove that every subgroup of Q is normal.

(f) Let $N = \pm 1 \subset Q$. Prove that $Q/N \cong (\mathbb{Z}/2\mathbb{Z})^2$ and that Q/N is not isomorphic to a subgroup of Q.

(g) ($\star$) Let Γ be the subgroup of $\mathbb{H}^\times$ generated by the elements of Q together with $\frac{1}{2}(1 + i + j + k)$. Prove that Γ is a group of order 24.

(h) Prove that Γ is *not* isomorphic to S_4, and Q is *not* isomorphic to D_8. In fact, $\Gamma = \mathrm{SL}_2(\mathbb{F}_3)$.

(i) ($\star$) Construct a surjective homomorphism from Γ to A_4.

(j) Prove that the subgroup $\mathbb{H}^1$ of quaternions q with $N(q) = 1$ is a subgroup of $\mathbb{H}^\times$. Deduce that the 3-sphere $S^3 \subset \mathbb{R}^4$ defined by $a^2 + b^2 + c^2 + d^2 = 1$ has a natural structure of a group. Note that S^1 also has a natural group structure given by rotations in $\mathrm{SO}(2)$. It turns out that S^n has a natural (i.e., continuous) group structure only for $n = 1$ and $n = 3$.

(k) ($\star$) Say that a quaternion is **pure** if it is of the form $bi + cj + dk$, i.e., $a = 0$. We may identify pure quaternions with $\mathbb{R}^3$. Show that if u is a pure quaternion, then quq^{-1} is still a pure quaternion for any $q \in \mathbb{H}^\times$.

(l) ($\star$) Prove that the action of q on $\mathbb{R}^3$ by $q \cdot u = quq^{-1}$ is via elements of $\mathrm{SO}(3)$, and deduce that there is a homomorphism $\mathbb{H}^\times \to \mathrm{SO}(3)$.

(m) ($\star$) Prove that the restriction of this homomorphism to $\mathbb{H}^1 \to \mathrm{SO}(3)$ is surjective and has kernel of order 2.