Week 2

???

2.1 Groups of Low Order

10/3:

- Calegari: Nothing in particular to know for missing Friday; Adi will get me notes.
- Having explored examples, today, we're coming back down to earth to flex our axiomatic muscles.
- Distinguishing sets and binary operations.

Group	G	$*$	?
S_n	shuffles	composition	cards
$\mathrm{O}(n)$ and $\mathrm{SO}(n)$	(sp) orthogonal matrices	composition	vectors?
$\mathbb{Z}$	integers	addition	
$\mathbb{Z}/n\mathbb{Z}$	$\{0, 1, \dots, n-1\}$	addition modulo n	

Table 2.1: Elements of a group.

  - Be careful not to confuse the shuffles and the cards; the cards are something else curious but are *not* the elements of the group.
  - Notice that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are **commutative** groups, but the shuffles (for $n > 1$) and $\mathrm{O}(n)$ are not.
  - Note that S_2, $\mathrm{O}(1)$, and $\mathbb{Z}/2\mathbb{Z}$ are all isomorphic groups.
- **Commutative** (group): A group such that for all $x, y \in G$, $x * y = y * x$. *Also known as* **Abelian**.
- Lemma (Cancellation Lemma): Let $x, y, z \in G$. Then $xy = xz$ implies $y = z$ and $yx = zx$ implies $y = z$.

Proof. We have that

$$\begin{aligned}
x * y &= x * z \\
x^{-1} * (x * y) &= x^{-1} * (x * z) && \text{Inverses exist} \\
(x^{-1} * x) * y &= (x^{-1} * x) * z && \text{Associativity} \\
e * y &= e * z \\
y &= z
\end{aligned}$$

as desired.

The proof of the second statement is symmetric. ∎

  - This will be Calegari's only proof from the axioms directly.

- **Multiplication table** (for G): A table with all elements of G on the top and the side, and all binary products in it.
  - The total number of binary operations is n^{n^2}?
  - To check that a group is a group, we can write out its multiplication table and confirm pointwise that the group axioms are satisfied. However, there are also many ways to speed this process up.
  - An example of a multiplication table can be found on the right in Figure 2.1.
- **Trivial group**: The only group with $|G| = 1$, i.e., $G = \{e\}$.
- A group of $|G| = 2$ has the form $G = \{e, x\}$ where we must have $x = x^{-1}$.
  - We can find this by inspection or invoke the **Sudoku Lemma**.
  - Thus, all groups of order 2 are isomorphic.
- Lemma (Sudoku Lemma): Fix $x \in G$. Then

  $$\{xg \mid g \in G\} = G = \{gx \mid g \in G\}$$

  Proof. There exists g such that $xg = y$ for x, y fixed: Choose $g = x^{-1}y$.
  y only occurs once: If $xg = y$ and $xg' = y$, transitivity and the cancellation lemma imply $g = g'$. $\square$

  - In layman's terms, in every row and column of the multiplication table, each element of G occurs exactly once.
- Playing Sudoku, we can show that all groups of order 3 are isomorphic.

	e	x	y
e	e	x	y
x	x		
y	y		

$\longrightarrow$

	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

Figure 2.1: Playing Sudoku for $|G| = 3$.

  - Start from the left table above.
  - Notice that row 3 has a y and column 2 has an x, so by the Sudoku Lemma, e must be the element in row 3, column 2.
  - Then column 2 has e, x in it, so the entry in row 2, column 2 must by y.
  - Then row 2 has x, y in it, so the entry in row 2, column 3 must be e.
  - Then row/column 3 both have e, y in them, so the entry in row 3, column 3 must be x.
- However, we cannot play Sudoku in the same way with groups of order 4. In fact, there are multiple groups of order 4.
  - Two cases: (1) $x^2 \neq e$ so WLOG let $x^2 = y$, and (2) $a^2 = e$ for $a = x, y, z$.
    - Case 1 is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
    - Case 2 is isomorphic to the **direct product** of $\mathbb{Z}/2\mathbb{Z}$ with itself, also known as the **Klein 4-group**.
  - This should not come as a surprise: We've already encountered the very different groups S_4 and $\mathbb{Z}/24\mathbb{Z}$ of order 24.

- **Direct product**: The group whose set is the Cartesian product of the sets of groups $A = (A, *_A)$, $B = (B, *_B)$, and whose operation is coordinate-wise multiplication. *Given by*

$$G = A \times B \qquad\qquad (a, b) *_G (a', b') = (a *_A a', b *_B b')$$

  - We can prove that $e = (e_A, e_B)$, that $(a, b)^{-1} = (a^{-1}, b^{-1})$, and that associativity holds.
  - We have that

$$|G| = |A| \cdot |B|$$

- There is only one group of order 5.
- Examples of groups of order 6: S_3, $\mathbb{Z}/6\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
  - Are there any two groups which are distinct?
    - S_3 is not commutative, but the others are, so it is distinct from them.
    - $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ are the same because order doesn't matter in the construction of the direct product.
    - $\mathbb{Z}/6\mathbb{Z}$ and the two direct products are the same because they both have elements of order 6 (i.e., a one-element generator). The cycles are:

$$\begin{aligned}
1^1 &= 1 &&= 1 & (1,1)^1 &= (1,1) &&= (1,1)\\
1^2 &= 1+1 &&= 2 & (1,1)^2 &= (1+1, 1+1) &&= (2,0)\\
1^3 &= 2+1 &&= 3 & (1,1)^3 &= (2+1, 0+1) &&= (0,1)\\
1^4 &= 3+1 &&= 4 & (1,1)^4 &= (0+1, 1+1) &&= (1,0)\\
1^5 &= 4+1 &&= 5 & (1,1)^5 &= (1+1, 0+1) &&= (2,1)\\
1^6 &= 5+1 &&= 0 & (1,1)^6 &= (2+1, 1+1) &&= (0,0)\\
1^7 &= 0+1 &&= 1 & (1,1)^3 &= (0+1, 0+1) &&= (1,1)
\end{aligned}$$

  - These are the only two groups of order 6.
- Continuing on, there is only 1 group with $|G| = 2047$ (which is "mostly prime" — connection between primes and number of groups?), but there are 1,774,274,116,992,170 groups of $|G| = 2048 = 2^{11}$.
- Conclusion: The arithmetic of $|G|$ has an impact on the structure of G.