Week 4

???

4.1 Quotient Groups

10/17:

- Notational confusion regarding $\mathbb{Z}/10\mathbb{Z}$.
  - Let $G = \mathbb{Z}$ and $H = 10\mathbb{Z}$ (the multiples of 10).
  - A few of the cosets are as follows:

$$\begin{aligned} H &= \{\dots,-20,-10,0,10,20,30,\dots\} \\ 1+H &= \{\dots,-19,-9,1,11,21,31,\dots\} \\ 2+H &= \{\dots,-18,-8,2,12,22,32,\dots\} \end{aligned}$$

  - Evidently, $|\mathbb{Z}/10\mathbb{Z}| = 10$.
  - Yet $\mathbb{Z}/10\mathbb{Z}$ is also the notation for the cyclic group of order 10.
  - This notation is not an error, but reveals something deep: We can make the set of cosets into a group and define addition by

$$(a+10\mathbb{Z}) + (b+10\mathbb{Z}) = (a+b+10\mathbb{Z})$$

    More specifically, we can define an isomorphism between the two definitions of $\mathbb{Z}/10\mathbb{Z}$ via $a+H \mapsto a$ for $a = 0,\dots,9$.

- This example motivates the following goal.
- Goal: Make G/H, which is a set, into a group.
  - This set needs a binary operation. It makes natural sense to define the binary operation as follows.

$$xH * yH = xyH$$

  - We then need an identity coset, inverse cosets, and associativity.
    - The identity is H.
    - The inverse of xH is $x^{-1}H$.
    - Associativity of G/H follows from the associativity of G (which tells us that $(ab)c = a(bc)$). More specifically,

$$\begin{aligned} aH *_H (bH *_H cH) &= aH *_H (b *_G c)H \\ &= a *_G (b *_G c)H \\ &= (a *_G b) *_g cH \\ &= (a *_G b)H *_H cH \\ &= (aH *_H bH) *_H cH \end{aligned}$$

- Calegari's impromptu explanation of associativity drives home that he really is very good at drilling down to the core of an idea and working with it. He really has a very similar mind to mine.
- Something else we need to investigate: Equivalence classes, and defining functions on equivalence classes.
  - We need to make sure that functions are defined the same regardless of how you label the equivalence classes.
  - Consider the set of names.
    - Say we define equivalency classes based on all names which share the same first letter.
    - Then we define a function F on the equivalency classes based on the last letter.
    - But then [Frank] = [Fen] will be mapped to two different elements of the alphabet, so F is not well-defined.
  - Thus, for our example, we need to guarantee that if $x, x' \in xH$, then $xH * yH = x'H * yH$.
- Check: Independence of choice.
  - Suppose we relabel $x \mapsto xh$ and $y \mapsto yh$. We need
$$xhyh' = xyh''$$
for some $h'' \in H$.
    - Note that x, y, h, h' are all fixed; h'' is the only free thing (i.e., is what we're looking for).
  - Algebraically manipulating the above implies that we want
$$h'' = y^{-1}hyh'$$
  - Thus, we know that $h'' \in G$, but we need to make sure that $h'' \in H$. Alternatively, we want $y^{-1}hy = h''(h')^{-1} \in H$.
  - An example where $y^{-1}hy$ is not in H: $G = S_3$, $H = \langle(1,2)\rangle$, $h = (1,2)$, $y = (1,3)$, $yhy^{-1} = (2,3)$.
- Why did $\mathbb{Z}/10\mathbb{Z}$ work? Because it was abelian, so conjugacy cancelled $y^{-1}hy = y^{-1}yh = h$.
  - We could restrict ourselves entirely to abelian groups, but can we be more general?
- What should we require of G/H?
  - The cananonical map of sets $\phi : G \to G/H$ is given by $\phi(x) = xH$.
  - We should require that ϕ is a homomorphism (i.e., that the group structure of G is preserved for G/H).
  - See how $xH * yH = xyH$ is analogous to $\phi(x)\phi(y) = \phi(xy)$.
- Let's suppose $\phi : G \to G/H$ is a homomorphism.
  - Then $\phi(g) = eH$ implies that $g \in H$, i.e., $\ker\phi = H$.
  - Realization: An alternate way to do HW3, Q2b would have been in terms of quotient groups: In that case, $G/H \cong S_{26}$, and the following proposition would give us the surjectivity and kernel requirements.
- Lemma: Let ϕ be a homomorphism from G to another group. Let $K = \ker\phi \subset G$. Then K has the following property, which is not true for all subgroups but is for kernels: If $x \in K$ and $g \in G$, then $gxg^{-1} \in K$.

  Proof. Since $\phi(x) = e$, we have that
$$\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = e$$
□

- **Normal** (subgroup): A subgroup H of G such that for all $x \in H$ and $g \in G$, $gxg^{-1} \in H$. *Denoted by* $\boldsymbol{H \trianglelefteq G}$, $\boldsymbol{H \triangleleft G}$.
  - We often write gHg^{-1}.
- Example: As per the lemma, $\ker\phi$ is a normal subgroup.
- Example: If G be abelian, then every $H \trianglelefteq G$.
- Lemma: A subset $H \subset G$ is normal iff
  1. H is a subgroup.
  2. H is a union of some number of conjugacy classes.
- Proposition: Let G be a group and $H \triangleleft G$. Then G/H is a group under the multiplication

$$xH * yH = xyH$$

  and the map $\phi : G \to G/H$ is a surjective homomorphism with kernel H.

  Proof. Recall that we want $xhyh' = xyh''h'$. Apply the cancellation lemma. Then

$$\begin{aligned} hy &= yy^{-1}hy \\ &= y(y^{-1}hy) \\ &= yh'' \end{aligned}$$

  where we get from the second to the third line above because H is a normal subgroup, i.e., conjugates of its elements are elements of it. This implies the desired result. $\square$

- Example: Let $G = \mathbb{Z}$, $H = 10\mathbb{Z}$, and $G/H = \mathbb{Z}/10\mathbb{Z}$.
- Example: Let $G = G$ and $H = \{e\}$.
  - H is normal since it's a subgroup and it's a union of conjugacy classes.
  - In this case, $G/H \cong G$.
- Example: $G = \mathrm{O}(2)$ and $H = \mathrm{SO}(2)$.
  - G is not abelian here.
  - From HW1, the cosets are $H = \{\text{rotations}\}$ and $\{\text{reflections}\}$.
  - The cosets are H and sH for some reflection $s \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$.
  - What the group structure tells us here is that rotation $\circ$ reflection is like even $\times$ odd numbers.
  - $G/H \cong \mathbb{Z}/2\mathbb{Z}$ here.
- An equivalent formulation of normality.
- Proposition: $H \triangleleft G$ iff the left cosets coincide with the right cosets, i.e.,

$$gH = Hg$$

  Proof. Suppose first that $H \triangleleft G$. Use a bidirectional inclusion argument. Let $gh \in gH$. Then

$$gh = ghg^{-1}g = h'g \in Hg$$

  where h' may or may not equal h, but we know it is an element of H by the definition of normal subgroups. The argument is symmetric in the other direction.

  Now suppose $gH = Hg$. Let $h \in H$. Then there exist g, h' such that $gh = h'g$. Therefore, $ghg^{-1} = h' \in H$. $\square$

- This is a nice resolution of left and right cosets.
  - It tells us when they're the same, and when they're different.
- Implication: If $H \triangleleft G$, then

$$xH \cdot yH = x(Hy)H = x(yH)H = xyHH = xyH$$

- Midterm next week.