2 Cycles, Cubes, and the Dodecahedron

10/10: 1. If σ is an element of S_n, then σ has a cycle decomposition into disjoint cycles of various lengths (let us include 1-cycles). Since disjoint cycles commute, the shape of the element is determined by the lengths of the various cycles, which we can assume are put in decreasing order. Any two elements with the same cycle shape are conjugate, so the conjugacy classes are determined by writing n ($= 52$, say) as a sum of decreasing integers.

(a) Find the conjugacy class in S_{52} with the largest number of elements.

Proof. Let $52 = \sum_{i=1}^{k} c_i p_i$, where $p_1, \dots, p_k$ is a decreasing sequence of natural numbers describing the cycle lengths present in the conjugacy class and the $c_i \in \mathbb{N}$ are their multiplicities.

There are 52! permutations of the numbers $1, \dots, 52$. We can partition every permutation up into p_i-cycles, but in doing so, we will realize that we have overcounted in two ways.

First off, every p_i-cycle can be written in p_i equivalent ways. Thus, for every permutation $a_1, \dots, a_{52}$, there are p_i permutations written differently that mean the same thing, so we need to divide through by p_i. Doing this for all p_i (and counting multiplicities), we need to divide through by $\prod_{i=1}^{k} p_i^{c_i}$.

Additionally, disjoint cycles commute. This means that the order in which we write the c_i p_i-cycles doesn't matter. Since there are $c_i!$ orders in which we can write the c_i p_i-cycles, we also need to divide through by $\prod_{i=1}^{k} c_i!$.

Therefore, the total number of elements in the conjugacy class $\sum_{i=1}^{k} c_i p_i$ is

$$\frac{n!}{\prod_{i=1}^{k} p_i^{c_i} \cdot c_i!}$$

This is the functional whose value we want to maximize.

To maximize the above functional, we can seek to minimize its denominator. To do so, we'll justify a couple of rules.

First, note that if $p, c \geq 2$, then

$$p^c \cdot c! > cp \cdot 1!$$

We can prove this by inducting on p and c in turn, keeping the other fixed. This rule tells us that if we want to minimize the above functional, it is to our benefit to reduce all multiplicities to 1 by combining cycles of the same length (as long as that length is greater than 1).

We are now down to only classes of the form $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} c_i p_i$. Thus, the problem becomes one of minimizing one of the two equations below, depending on whether or not $p_k = 1$ (remember that $p_1, \dots, p_k$ is *decreasing*, so 1, if present, will be p_k).

$$\prod_{i=1}^{k} c_i p_i = \prod_{i=1}^{k} x_i \qquad\qquad \prod_{i=1}^{k-1} c_i p_i \cdot 1^{c_k} c_k! = c_k! \prod_{i=1}^{k-1} x_i$$

With respect to this kind of product, we can note that if $a, b \geq 2$, then

$$ab \geq a + b$$

Thus, it is to our benefit to combine all cycles of length greater than 1. Thus, we have reduced to the cases

$$52 \qquad\qquad c_k!(52 - c_k)$$

respectively from the above. Since the right equation above is minimized for $c_k = 1$ and, with this value, evaluates to $51 < 52$, we know that the conjugacy class in S_{52} with the largest number of elements is:

$$\boxed{\text{The conjugacy class } 52 = 51 + 1.}$$

$\square$

(b) Find the conjugacy class in S_{52} which contains the element of largest order. (This question is somewhat computational, so an explanation of your strategy plus the answer is sufficient.)

Proof. Let $52 = \sum_{i=1}^{k} a_i$. By Exercise 1.3.15 of Dummit and Foote (2004) (and from class), the order of an element of S_n equals the least common multiple (lcm) of the lengths of the cycles in its cycle decomposition. Thus, all elements in a conjugacy class have the same order.

We now must optimize $\operatorname{lcm}(a_1, \dots, a_k)$ over all such decompositions. To do so, we will start with a guess based on some observations and then progressively refine according to two rules.

Observations:

(1) Rely (primarily) on relatively prime numbers. For example, the list $2, 4, 8$ has lcm $= 8$, but the list $2, 3, 5$ has lcm $= 30$, and a smaller sum.

(2) 1 should not be in the list because it does not contribute anything to the lcm but does add to the sum.

(3) Rely (primarily) on small numbers — remember the $ab \geq a + b$ rule for $a, b \geq 2$ from part (a). This means that it is often beneficial to split larger numbers into smaller numbers.

With these observations in hand, we'll use as our starting list

$$2, 3, 5, 7, 11, 13, 11$$

where we include the last 11 because $2 + \cdots + 13 = 41$ and $52 - 41 = 11$, i.e., we cannot include the first six numbers and the next prime (17) without the sum exceeding 52.

We now give the two rules for progressive refinement of the above list. The first one is that if $a_1, \dots, a_k$ is the final list, then

$$\operatorname{lcm}(a_1, \dots, a_k) \geq \operatorname{lcm}(a_1, \dots, a_{i-1}, n, (a_i - n), a_{i+1}, \dots, a_k)$$

for all $n < a_i$ and all $i \leq k$. The second one is that if $a_1, \dots, a_k$ is the final list, then

$$\operatorname{lcm}(a_1, \dots, a_k) \geq \operatorname{lcm}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_k, a_i + a_j)$$

for all $i \neq j \leq k$. In particular, if we ever come across a case in which either of the above two inequalities is not satisfied, then we should redefine our list on the LHS with the list on the RHS.

Using these rules, we will first attack the second 11 in the above list. We can compute that

$$\operatorname{lcm}(2, 3, 5, 7, 11, 13, 11) = 30030$$

and that

$$\operatorname{lcm}(2, 3, 5, 7, 11, 13, 1, 10) = 30030$$

but that

$$\operatorname{lcm}(2, 3, 5, 7, 11, 13, 2, 9) = 90090$$

Thus, we redefine our list to be $2, 2, 3, 5, 7, 9, 11, 13$. If we run through and check all of the cases by the first rule, we will find that there is no more splitting we can do to increase the value of this list. However, by the second rule, there is some combining: If we combine $2, 2 \mapsto 4$, then

$$\operatorname{lcm}(3, 4, 5, 7, 9, 11, 13) = 180180$$

Running both rules, we will find that we cannot progressively refine any further from here. Therefore, the conjugacy class in S_{52} which contains the element of the largest order is:

$$\boxed{\text{The conjugacy class } 52 = 13 + 11 + 9 + 7 + 5 + 4 + 3.}$$

□

2. Let $k \leq n$ be even. Prove that every element in S_n can be written as a product of k-cycles.

Proof. Every element in S_n can be written in terms of elementary transpositions. Thus, the problem becomes one of showing that every elementary transposition can be written as a product of k-cycles.

Let $(i, i+1) \in S_n$ be an elementary transposition. We will prove that

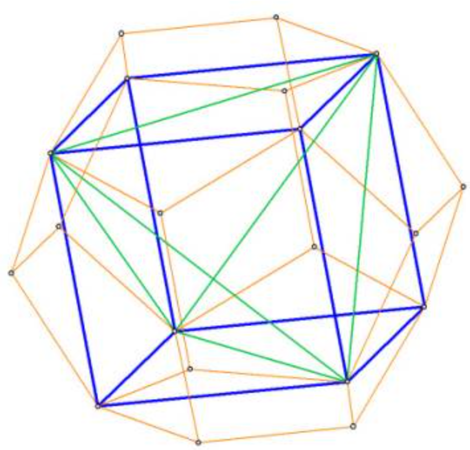
$$\begin{aligned}(i, i+1) = {} & (i, i +_n 1, \dots, i +_n (k-1))^2 \\ & \cdot (i, i +_n (k-1), i +_n (k-3), \dots, i +_n 3, i+1, i +_n (k-2), i +_n (k-4), \dots, i +_n 2)\end{aligned}$$

where $+_n$ denotes addition modulo n[1]. Indeed, we have that

$$\begin{aligned} & (i, i +_n 1, \dots, i +_n (k-1)) \\ & \cdot (i, i +_n 1, \dots, i +_n (k-1)) \\ & \cdot (i, i +_n (k-1), i +_n (k-3), \dots, i +_n 3, i+1, i +_n (k-2), i +_n (k-4), \dots, i +_n 2) \\ = {} & (i, i +_n 1, \dots, i +_n (k-1)) \\ & \cdot (i)(i+1, i +_n (k-1), i +_n (k-2), \dots, i +_n 2) \\ = {} & (i, i +_n 1, \dots, i +_n (k-1)) \\ & \cdot (i+1, i +_n (k-1), i +_n (k-2), \dots, i +_n 2) \\ = {} & (i, i+1)\end{aligned}$$

as desired. $\square$

3. Let D be a regular dodecahedron. You may assume for this question that it is possible to inscribe a cube C on the vertices of D as shown below.



Remember the following distinction: An object X in $\mathbb{R}^3$ is **fixed pointwise** by g if every point on X is fixed by g, that is, if $gx = x$ for all $x \in X$. An object $X \in \mathbb{R}^3$ is **preserved** by g if every point on X maps to another (possibly different) point on X, i.e., for all $x \in X$, there exists $y \in X$ such that $gx = y$. As an example, the circle centered at the origin is preserved by any rotation through the origin, but is not fixed pointwise unless the rotation is trivial.

- If F is a face, call a line between two vertices of F an **internal line** if the vertices are not adjacent. That is, an internal line is a line between two vertices of a pentagonal face which is not an edge of the pentagon.

[1]Motivation: We have, for example, that $(1,2) \in S_n$ with $k = 8$ can be given by $(1,2,3,4,5,6,7,8)^2(1,8,6,4,2,7,5,3)$. Essentially, what we are doing here is sending $1 \mapsto 8$ and $2 \mapsto 7$ so that when we rotate all of the numbers twice (with $(1,\dots,8)^2$), 1 and 2 land in the 2 and 1 positions. The "decreasing by 2 at a time" part is a necessary consequence of writing an 8-cycle that sends $1 \mapsto 8$ and $2 \mapsto 7$ and can be more easily understood by drawing out a function diagram and tracing the cycle.

- Observe that the cube C has 12 edges, and that each edge lies on exactly one of the 12 faces of D as an internal line.
- Choose a face F of D and let g be the symmetry of D of order 5 which is a rotation by $2\pi/5$ through the line passing through the middle of F and the middle of the opposite face $-F$.
- Label the vertices of a face F from 1 to 5. Suppose that $C = C_{(1,3)}$ intersects F in the internal edge from 1 to 3.

(b) Show that for any such g, the five cubes $C_{(1,3)}$, $C_{(2,4)}$, $C_{(3,5)}$, $C_{(1,4)}$, and $C_{(2,5)}$ obtained by applying the powers of g to each cube are distinct because they intersect F in different internal lines (which are the lines between vertices indicated by the notation).

Proof. Let g be arbitrary. Choose the z-axis to be the axis about which g rotates the dodecahedron/cube. Adopt a cylindrical coordinate system (r, θ, z). Orient the remaining coordinate axes so that vertex 1 of face F lies at $(r, 0, z)$; it follows that vertices 2-5 lie at $(r, 2\pi/5, z)$, $(r, 4\pi/5, z)$, $(r, 6\pi/5, z)$, and $(r, 8\pi/5, z)$, respectively. In this coordinate system, g^n is the orthogonal transformation that sends

$$(r, \theta, z) \mapsto \left(r, \theta + \frac{2\pi n}{5}, z\right)$$

Consider $C_{(1,3)}$, which intersects F at the internal line from vertex 1 to vertex 3. Applying the powers of g sends

$$\begin{aligned}
g(1) &= g(r,0,z) = (r,2\pi/5,z) = 2 & g(3) &= g(r,4\pi/5,z) = (r,6\pi/5,z) = 4 \\
g^2(1) &= g^2(r,0,z) = (r,4\pi/5,z) = 3 & g^2(3) &= g^2(r,4\pi/5,z) = (r,8\pi/5,z) = 5 \\
g^3(1) &= g^3(r,0,z) = (r,6\pi/5,z) = 4 & g^3(3) &= g^3(r,4\pi/5,z) = (r,2\pi,z) = (r,0,z) = 1 \\
g^4(1) &= g^4(r,0,z) = (r,8\pi/5,z) = 5 & g^4(3) &= g^4(r,4\pi/5,z) = (r,2\pi/5,z) = 2 \\
g^5(1) &= e(r,0,z) = 1 & g^5(3) &= e(r,4\pi/5,z) = 3
\end{aligned}$$

Thus, we know that the cube $g(C_{(1,3)})$ — remember that g, as an orthogonal transformation, preserves lengths, angles, and lines, so the image of a cube under g will still be a cube — intersects F at the internal line from vertex 2 to vertex 4, the cube $g^2(C_{(1,3)})$ intersects F at the internal line from vertex 3 to vertex 5, the cube $g^3(C_{(1,3)})$ intersects F at the internal line from vertex 4 to vertex 1, the cube $g^4(C_{(1,3)})$ intersects F at the internal line from vertex 5 to vertex 2, and the cube $g^5(C_{(1,3)}) = e(C_{(1,3)}) = C_{(1,3)}$ since g is of order 5 by definition. Naturally, continuing onto higher natural numbers will just get us back to these same cubes. It follows that these cubes — which are equal to $C_{(2,4)}$, $C_{(3,5)}$, $C_{(4,1)} = C_{(1,4)}$, $C_{(5,2)} = C_{(2,5)}$, and $C_{(1,3)}$, respectively — are all distinct because they intersect F in different internal lines. ☐

(c) Show that *any* symmetry of D takes C to one of these five cubes. Hint: Any pair of cubes share two vertices $\mathbf{v}, \mathbf{w}$ on F lying on an internal line of F which are connected by an edge of the cube. Given a cube centered at the origin with vertices $\mathbf{v}, \mathbf{w}$ and $|\mathbf{v}| = |\mathbf{w}|$ connected by an edge, show that the eight vertices of the cube are

$$\pm\mathbf{v},\ \pm\mathbf{w},\ \pm\mathbf{u} \pm \left(\frac{\mathbf{v} - \mathbf{w}}{2}\right)$$

where $\mathbf{u}$ is the (unique up to a $\pm$ sign) vector with $3|\mathbf{u}|^2 = 2|\mathbf{v}|^2 = 2|\mathbf{w}|^2$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0$.

Proof.

Setup: Let C be an arbitrary cube inscribed on the vertices of D, and let F be a face of D. By the second bullet point above, C intersects F at exactly one of its internal lines. Let $\mathbf{v}, \mathbf{w}$ be the vertices of F which are connected by said internal line. Define

$$\mathbf{u} = \sqrt{\frac{2}{3}}|\mathbf{v}| \cdot \frac{\mathbf{v} \times \mathbf{w}}{|\mathbf{v} \times \mathbf{w}|}$$

By the definition of the cross product, $\mathbf{u}$ is orthogonal to $\mathbf{v},\mathbf{w}$. Additionally, the way it is defined guarantees that it satisfies the magnitude relation.

Proving the hint: We now prove that the eight vertices of C are

$$\pm\mathbf{v},\ \pm\mathbf{w},\ \pm\mathbf{u}\pm\left(\frac{\mathbf{v}-\mathbf{w}}{2}\right)$$

Let A be the determinant 1, orthogonal transformation which sends $\mathbf{v}\mapsto(a,a,a)$ and $\mathbf{w}\mapsto(-a,a,a)$ for some $a\in\mathbb{R}$. We know that such a transformation exists since it is equivalent to redrawing the basis of $\mathbb{R}^3$ such that the three axes go through the center of three adjacent faces of the cube. Since orthogonal transformations preserve the cross product, we know that

$$\begin{aligned}
A\mathbf{u} &= \frac{\sqrt{2/3}|\mathbf{v}|}{|\mathbf{v}\times\mathbf{w}|}\cdot A\mathbf{v}\times A\mathbf{w}\\
&= \frac{\sqrt{2/3}|A\mathbf{v}|}{|A\mathbf{v}\times A\mathbf{w}|}\cdot\begin{pmatrix}0\\-2a^2\\2a^2\end{pmatrix}\\
&= \frac{\sqrt{2/3}\cdot\sqrt{3a^2}}{\sqrt{8a^4}}\cdot\begin{pmatrix}0\\-2a^2\\2a^2\end{pmatrix}\\
&= \frac{1}{2a}\cdot\begin{pmatrix}0\\-2a^2\\2a^2\end{pmatrix}\\
&= \begin{pmatrix}0\\-a\\a\end{pmatrix}
\end{aligned}$$

It follows that the full set of vertices of this cube can be expressed in terms of $\mathbf{u},\mathbf{v},\mathbf{w}$ as follows.

$$\begin{aligned}
(a,a,a) &= A(\mathbf{v})\\
(-a,-a,-a) &= A(-\mathbf{v})\\
(-a,a,a) &= A(\mathbf{w})\\
(a,-a,-a) &= A(-\mathbf{w})\\
(a,-a,a) &= (0,-a,a)+\left(\frac{a-(-a)}{2},a-a,a-a\right) = A\left(\mathbf{u}+\frac{\mathbf{v}-\mathbf{w}}{2}\right)\\
(-a,-a,a) &= A\left(\mathbf{u}-\frac{\mathbf{v}-\mathbf{w}}{2}\right)\\
(a,a,-a) &= A\left(-\mathbf{u}+\frac{\mathbf{v}-\mathbf{w}}{2}\right)\\
(-a,a,-a) &= A\left(-\mathbf{u}-\frac{\mathbf{v}-\mathbf{w}}{2}\right)
\end{aligned}$$

Thus, the vertices of C are given by the arguments of A, above, as desired.

Proving the claim: To prove that any symmetry of D takes C to one of the five cubes from part (b), we will let h be an arbitrary symmetry of D and prove that h maps the eight vertices of C to the eight vertices of $C_{(1,3)}$, $C_{(2,4)}$, $C_{(3,5)}$, $C_{(1,4)}$, or $C_{(2,5)}$. Per the hint, we know that $\mathbf{v},\mathbf{w}$ uniquely determine the remainder of the vertices of an inscribed cube. In particular, for each of the five cubes just listed, they are the unique cube which intersects F at the internal line that they do. Thus, since h will send C to some other inscribed cube, which must by observation 2 intersect F in one of the above internal lines, we know that h sends C to one of the five desired cubes. $\square$

(d) Let $\mathbf{v}_i$ indicate the vector corresponding to vertex i of F. Deduce that there are exactly two cubes which have $\mathbf{v}_i$ as a vertex, and that the only vertices that these two cubes have in common are $\pm\mathbf{v}_i$.

Proof. We will first prove that exactly two cubes have $\mathbf{v}_i$ as a vertex. By parts (b-c), there are exactly 5 distinct cubes inscribed in D: $C_{(1,3)}$, $C_{(2,4)}$, $C_{(3,5)}$, $C_{(1,4)}$, and $C_{(2,5)}$. Since each vertex from 1-5 appears exactly twice and in exactly two different cubes according to the above list, we have the desired result for all i.

We now prove that two cubes that both have $\mathbf{v}_i$ as a vertex only share $\pm\mathbf{v}_i$. In particular, we will prove the claim for $\mathbf{v}_1$; the argument is analogous for $\mathbf{v}_2$-$\mathbf{v}_5$. Let's begin. We know that $C_{(1,3)}$ and $C_{(1,4)}$ both have $\mathbf{v}_1$ as a vertex. We also know that the two vertices these cubes have on F are $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{v}_1, \mathbf{v}_4$, respectively. Thus, we have by part (c) that the eight vertices of the respective cubes are

$$\pm\mathbf{v}_1,\ \pm\mathbf{v}_3,\ \pm\mathbf{u}_{13} \pm \left(\frac{\mathbf{v}_1 - \mathbf{v}_3}{2}\right) \qquad\qquad \pm\mathbf{v}_1,\ \pm\mathbf{v}_4,\ \pm\mathbf{u}_{14} \pm \left(\frac{\mathbf{v}_1 - \mathbf{v}_4}{2}\right)$$

Evidently, the only overlap is at $\pm\mathbf{v}_1$ for $\mathbf{v}_3, \mathbf{v}_4$ distinct, as desired. ☐

(e) (*) Show that any rigid motion of D (i.e., any element of SO(3) preserving D) permutes the 5 cubes. Hint: Show that if a symmetry σ preserves the two cubes passing through $\mathbf{v}_i$, then it preserves their intersection and deduce that

$$\sigma\mathbf{v}_i = \pm\mathbf{v}_i$$

Deduce that this identity must hold for every i, and use this (and HW1) to show that this implies that σ is the identity.

Proof. We will first prove the hint. Let's begin.

Suppose σ preserves the two cubes C, C' passing through $\mathbf{v}_i$. To prove that σ preserves $C \cap C'$, it will suffice to show that σ maps every element in that set to another element of that set. Since $C \cap C' = \{\pm\mathbf{v}_i\}$ by part (d), we confirm this with two cases. For $\mathbf{v}_i$, since $\mathbf{v}_i \in C$ and σ preserves C, we know that $\sigma\mathbf{v}_i \in C$. Similarly, we know that $\sigma\mathbf{v}_i \in C'$. Thus, by the definition of a set union, $\sigma\mathbf{v}_i \in C \cap C'$, as desired. An analogous argument treats the other case.

It follows from the above that $\sigma\mathbf{v}_i \in \{\pm\mathbf{v}_i\}$. Therefore,

$$\sigma\mathbf{v}_i = \pm\mathbf{v}_i$$

as desired.

Now suppose for the sake of contradiction that $\sigma\mathbf{v}_1 = -\mathbf{v}_1$. Then for σ to be orthogonal, we must necessarily have $\sigma\mathbf{v}_i = -\mathbf{v}_i$ for all i. But then σ is an inversion with determinant -1, and is thus not a rigid motion, a contradiction. Therefore, we must have that $\sigma\mathbf{v}_i = \mathbf{v}_i$ for all i. It follows by HW1, Q2f since σ fixes (at least) two linearly independent vectors that σ is the identity. ☐

(f) Deduce that the symmetry group of the dodecahedron is a subgroup of S_5 of order 60.

Proof. By part (f), any rigid motion of D permutes the 5 cubes, and is thus an element of S_5. Moreover, said rigid motion must correspond to a positive-determinant matrix element of SO(3). Thus, since half of S_5 maps to SO(3) and the other half maps to $\text{O}(3) \setminus \text{SO}(3)$, and $|S_5| = 120$, we know that the symmetry group of the dodecahedron is a subgroup (like $\text{SO}(3) \leq \text{O}(3)$) of S_5 of order $120/2 = 60$. ☐

4. Embed the cube inside $\mathbb{R}^3$ so that the centers of each face are at

$$A = (1,0,0) \quad B = (-1,0,0) \quad C = (0,1,0) \quad D = (0,-1,0) \quad E = (0,0,1) \quad F = (0,0,-1)$$

Considering the symmetry group of C as a subgroup of SO(3), write down the matrix of SO(3) corresponding to the following elements.

(a) $\sigma = (A, C, E)(B, D, F)$.

Proof. To send $A, B \mapsto C, D \mapsto E, F \mapsto A, B$, we need to move the nonzero index in the matrix of the vector "down" by one each time. Thus, a permutation matrix will accomplish the job.

$$\boxed{\mathcal{M}(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

□

(b) $\tau = (C, E, D, F)$.

Proof. Here, we need to (between the two indices that change) move the nonzero index down, and then up and flip the sign, and then move it down, and then up and flip the sign again. The following matrix accomplishes this.

$$\boxed{\mathcal{M}(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}$$

□

(c) $\sigma\tau = (A, C, E)(B, D, F)(C, E, D, F) = (A, C)(B, D)(E, F)$.

Proof. Taking the product $\mathcal{M}(\sigma) \circ \mathcal{M}(\tau)$ gives us the desired matrix.

$$\boxed{\mathcal{M}(\sigma\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

□