9 Final Exam

12/8: **1.** $(40 = 20 + 20$ points) **For this question: no working required**

Compute the orders of the following sets:

1. The centralizer of $(123)(456)(789) \in S_9$.

Proof. Let $g = (123)(456)(789)$, and let $\{g\}$ denote the conjugacy class of g in S_n. We build up to applying the orbit-stabilizer theorem.

Since the conjugacy class of g is the set of all elements with the same cycle shape, and we have a formula for calculating the number of elements in the symmetric group of order n given a certain cycle shape, we apply the formula to learn that

$$|\{g\}| = \frac{n!}{\prod_{i=1}^{k} p_i^{c_i} \cdot c_i!} = \frac{9!}{3^3 \cdot 3!} = 2240$$

Let $S_9 \circlearrowright S_9$ by conjugation. Then $\text{Orb}(g) = \{g\}$ and $\text{Stab}(g) = C_{S_9}(g)$, so we have by the orbit-stabilizer theorem and the above that

$$\begin{aligned} |\text{Orb}(g)| \cdot |\text{Stab}(g)| &= |S_9| \\ |C_{S_9}(g)| &= \frac{9!}{2240} \\ \boxed{|C_{S_9}(g)| = 162} \end{aligned}$$

□

2. The normalizer of $H = \langle (12345) \rangle \subset S_6$.

Proof. Let X be the set of subgroups of S_6 of order 5. Every subgroup in X is generated by a 5-cycle. In particular, there are

$$\binom{6}{5} \cdot (5-1)! = 144$$

5-cycles in S_6. Additionally, each such subgroup contains 4 distinct 5-cycles and e, so

$$|X| = \frac{144}{4} = 36$$

Let $S_6 \circlearrowright X$ by conjugation. Since all 5-cycles are conjugate in S_6, the action is transitive. By the definition of the stabilizer and normalizer, we have that

$$\text{Stab}(H) = \{\sigma \in S_6 \mid \sigma \cdot H = H\} = \{\sigma \in S_6 \mid \sigma H \sigma^{-1} = H\} = N_{S_6}(H)$$

It follows by the orbit-stabilizer theorem that

$$\begin{aligned} |\text{Orb}(H)| \cdot |\text{Stab}(H)| &= |S_6| \\ |N_{S_6}(H)| &= \frac{6!}{36} \\ \boxed{|N_{S_6}(H)| = 20} \end{aligned}$$

□

2. $(40 = 20+20$ points) For each of the following groups G, find the smallest n such that G is isomorphic to a subgroup of S_n. Justify your answers.

1. The group $G = S_5 \times \mathbb{Z}/4\mathbb{Z}$.

Proof. We know that

$$C_{S_n}((1,\dots,k)) \cong S_{n-k} \times \mathbb{Z}/k\mathbb{Z}$$

Thus, solving $k = 4$ and $n - k = 5$, we find

$$\boxed{n = 9}$$

□

2. The dihedral group D_{72} of order 72.

   Proof. We know that D_{72} contains an element of order $72/2 = 36$. The order of an element of S_n is equal to the least common multiple of its constituent cycle lengths. Thus, we know that the product of a disjoint 4-cycle and 9-cycle in S_{13} satisfies

   $$|(1,2,3,4)(5,6,7,8,9,10,11,12,13)| = 36$$

   and that S_{13} is the smallest symmetric group to contain such an element. Thus, we can map

   $$r \mapsto (1,2,3,4)(5,6,7,8,9,10,11,12,13)$$

   Additionally, we can geometrically picture separate rotations of a 4-gon and a 9-gon to motivate choosing the following as a reflection element.

   $$s \mapsto (2,4)(6,13)(7,12)(8,11)(9,10)$$

   Together, these elements satisfy the relations

   $$r^{36} = s^2 = e \qquad\qquad rs = sr^{-1}$$

   which characterize D_{72}. Therefore, we take

   $$\boxed{n = 13}$$

   □

3. Let A be a finite abelian group. Suppose that A acts transitively and faithfully on a set X. Prove that $|A| = |X|$.

Proof. We approach this proof from the perspective of the Orbit-Stabilizer Theorem. According to it,

$$|A| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$$

for all $x \in X$. Since $A \circlearrowright X$ is transitive, $\operatorname{Orb}(x) = X$, and we can further refine the above to

$$|A| = |X| \cdot |\operatorname{Stab}(x)|$$

Thus, to prove that $|A| = |X|$, it will suffice to show that $|\operatorname{Stab}(x)| = 1$ for all $x \in X$. To do so, we will show that $\operatorname{Stab}(x) = \operatorname{Stab}(y)$ for all $x, y \in X$, from which it will follow by the faithfulness of the action that

$$\operatorname{Stab}(x) = \bigcap_{y \in X} \operatorname{Stab}(y) = \ker = \{e\}$$

for all $x \in X$, as desired. Let $x, y \in X$ be arbitrary. Since A acts transitively, there exists $g \in A$ such that $g \cdot x = y$. Now suppose $h \in \operatorname{Stab}(y)$. Then since A is abelian,

$$\begin{aligned} g \cdot x &= y \\ &= h \cdot y \\ &= h \cdot (g \cdot x) \\ &= hg \cdot x \\ &= gh \cdot x \\ &= g \cdot (h \cdot x) \end{aligned}$$

It follows by the faithfulness of the action that $h \cdot x = x$, i.e., $h \in \operatorname{Stab}(x)$. Having shown that an arbitrary element of one stabilizer is necessarily in another, we know that all stabilizers are equal, and thus have the desired result. □

4. (45 = 15 + 15 + 15 points) Determine whether the following statements are true or false. Justify your answer in each case.

1. There are finitely many groups up to isomorphism which act faithfully on 5 points.

   Proof. Let X be a set with $|X| = 5$, and let G be a group such that $G \circlearrowright X$ faithfully. This group action induces a homomorphism $\phi : G \to S_5$ which, due to the faithfulness of the action, must have $\ker \phi = \{e\}$. In other words, ϕ is an injection. Therefore, by the first isomorphism theorem, there exists a bijection
$$\tilde{\phi} : G \to \operatorname{im} \phi \leq S_5$$
   i.e., we must have that G is isomorphic to a subgroup of S_5. But since there are only finitely many subgroups of S_5 up to isomorphism, the statement is

$$\boxed{\text{True.}}$$

   □

2. Any finite group is a subgroup of A_n for some integer n.

   Proof. Let G be an arbitrary finite group, and let $|G| = m$. By Cayley's theorem, we know that $G \leq S_m$. Thus, if we can prove that $S_m \leq A_n$ for some n, we are done. Let $n = m+2$, and define $\phi : S_m \to A_n$ by
$$\sigma \mapsto \begin{cases} \sigma & \sigma = \tau_1 \cdots \tau_{2k} \\ \sigma(n+1, n+2) & \sigma = \tau_1 \cdots \tau_{2k+1} \end{cases}$$
   Obviously, ϕ is well-defined and outputs only even permutations. To prove that ϕ is a homomorphism, it will suffice to show that $\phi(\sigma\sigma') = \phi(\sigma)\phi(\sigma')$ for all $\sigma, \sigma' \in S_m$. We divide into four cases (σ, σ' even, σ even & σ' odd, σ odd & σ' even, and σ, σ' odd). For case 1, we have that $\sigma\sigma'$ is even as well, and hence
$$\phi(\sigma\sigma') = \sigma\sigma' = \phi(\sigma)\phi(\sigma')$$
   For case 2, we have that $\sigma\sigma'$ is odd, and hence
$$\phi(\sigma\sigma') = \sigma\sigma'(n+1, n+2) = \phi(\sigma)\phi(\sigma')$$
   For case 3, we have that $\sigma\sigma'$ is odd, and hence
$$\phi(\sigma\sigma') = \sigma\sigma'(n+1, n+2) = \sigma(n+1, n+2)\sigma' = \phi(\sigma)\phi(\sigma')$$
   where we have used the fact that disjoint cycles commute in the next to last step (this is why it's important to go up by +2). Lastly, for case 4, we have that $\sigma\sigma'$ is even, and hence
$$\phi(\sigma\sigma') = \sigma\sigma' = \sigma\sigma'(n+1, n+2)^2 = \sigma(n+1, n+2)\sigma'(n+1, n+2) = \phi(\sigma)\phi(\sigma')$$
   Therefore, ϕ is a homomorphism.
   We can also see that ϕ is injective since distinct σ, σ' will map to distinct outputs σ, σ' (and the presence or absence of an appended disjoint cycle will not change the distinctness of the original permutations). Thus, $S_m \leq A_{m+2}$, as desired.
   Therefore, $G \leq A_{|G|+2}$, and we have proven that the statement is

$$\boxed{\text{True.}}$$

   □

3. A group of order $2688 = 2^7 \cdot 3 \cdot 7$ has a transitive action on a set X with $|X| = 21$.

Proof. Leg G be a group of order $|G| = 2688$. By Sylow I, G has a 2-Sylow subgroup P. By Lagrange's theorem,

$$|G/P| = |G|/|P| = 21$$

We know that $G \circlearrowright G/P$ transitively for P a subgroup, so take $X = G/P$. Therefore, we have proven that the statement is

$$\boxed{\text{True.}}$$

□

5. (30 points) Let G be a simple group of order 168. Determine the number of elements of G of order 7.

Proof. Since 7 is a prime number (and one that only appears once in the prime factorization of 168), we know that the number of elements in G of order 7 will be equal to the number n_7 of 7-Sylows. By Sylow III, we know that $n_7 \equiv 1 \bmod 7$. Additionally, we have by Lagrange's theorem that $n_7 \mid 168$; in particular, since $168 = 24 \cdot 7$ and $n_7 \not\equiv 0 \bmod 7$, we know that $n_7 \nmid 7$ so it must be that $n_7 \mid 24$.

The only two natural numbers that are both congruent to 1 mod 7 and divide 24 are $n_7 \in \{1, 8\}$. But if $n_7 = 1$, then by Sylow II, the sole 7-Sylow is normal in G, contradicting its simplicity. Therefore, we must have that

$$\boxed{n_7 = 8}$$

□

6. ($20 = 10 + 10$ points) Let $n > 1$ be an integer and let G be a group of order n. The left action of G on itself induces an injective map $\psi : G \to S_n$.

1. Prove that if $g \in G$ has order 2, then n is even and the cycle decomposition of $\psi(g)$ consists of $n/2$ disjoint 2-cycles.

   Proof. Since g has order 2, Lagrange's theorem implies that $2 \mid |G|$, so n must be an even number, as desired.

   Let $G \circlearrowright G$ be the described left action. Let $h \in G$ be arbitrary. Since $|g| = 2$ and hence $g \neq e$, we have by the Sudoku lemma that $gh \neq h$. It follows by the group action axioms that

   $$g \cdot h = gh \qquad\qquad g \cdot gh = h$$

   Thus, to every $h \in G$, there corresponds a unique matching element gh. By the faithfulness of the group action, there is no overlap between the pairs h, gh and hence G can be partitioned into pairs of elements h, gh. Moreover, we have by the above that for any $h \in G$,

   $$\psi(g)(h) = g \cdot h = gh \qquad\qquad \psi(g)(gh) = g \cdot gh = h$$

   so every pair h, gh presents as a 2-cycle in the cycle decomposition of $\psi(g) \in S_G \cong S_n$, as desired. □

2. Prove that if $n \equiv 2 \bmod 4$ and $n > 2$, then G is not a simple group.

   Proof. Suppose for the sake of contradiction that G is a group of order n. An equivalent formulation to $n \equiv 2 \bmod 4$ is stating that $n = 2m$ where m is an odd number greater than or equal to 3. Since $|G| = 2 \cdot m$ where $m \nmid 2$, Sylow I implies that G contains a 2-Sylow of order 2. In particular, there exists $g \in G$ of order 2 (take the nontrivial element of the 2-Sylow).

   Applying part (1), we learn that $\psi(g) \in S_n$ consists of $n/2 = m$ (an odd number) of 2-cycles. Thus, $\psi(g) \notin A_n$.

   However, as a simple group with a transitive action on a set of $n \geq 2$ points, Lemma 11 asserts that either $G \hookrightarrow A_n$ or $|G| = 2$. By the above, we know that $G \not\hookrightarrow A_n$, but by hypothesis, $n \neq 2$ either, a contradiction. □

7. Let G be a subgroup of A_8 which is simple of order 504. Prove that the action of G is 2-transitive, that is, for any pairs $\{a, b\}$ and $\{c, d\}$ of two distinct elements of $\{1, 2, 3, \ldots, 8\}$, there is an element g such that

$$g(a) = c \qquad\qquad g(b) = d$$

Proof. $504 = 2^3 \cdot 3^2 \cdot 7$. $\square$