

# MATH 25700 (Honors Basic Algebra I) Notes

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# Week 1

## Motivating Group Theory

### 1.1 Groups as Shuffles

- 9/28:
- Office hours will be pooled between the two sections.
    - Our section's TA is Abhijit Mudigonda (abhijitm@uchicago.edu). His office hours will always be in JCL 267<sup>[1]</sup>. The times are...
      - Monday: 12:30-2:00 (OH).
      - Wednesday: 1:30-2:30 (PS).
      - Thursday: 12:30-2:00 (OH).
    - The other section's TA is Ray Li (rayli@uchicago.edu). His office hours will always be in Eck 17<sup>[2]</sup>. The times are...
      - Tuesday: 5:00-7:00 (OH).
      - Thursday: 4:00-5:00 (OH).
      - Thursday: 5:00-6:00 (PS).
  - Textbook: Abstract Algebra. Download the PDF from LibGen.
  - Weekly HW due on Monday at the beginning of class. Submit online or in person. There is a webpage w/ all the homeworks, but don't do them all at once because they're subject to change.
  - Notes on math and math pedagogy.
    - There's a tendency to say here's an object, here's its properties, etc.
    - But this is not historically accurate or motivated. Calegari really gets it! Math is motivated by abstracting examples.
    - Let's not just define a group, but start with an example. This week, we will give examples of groups. In later weeks, we will establish the axiomatic framework that is really only there to understand these examples.
    - Don't stare at the page blankly waiting for inspiration when doing homework; think of examples first and test out your intuition on them to actually understand what the question means.
    - There are some hard problems; work with each other, but acknowledge our collaborators.
    - In-class midterm; final will be take-home. Calegari doesn't like timed exams.
  - Today's example: Shuffling.
    - 52 cards; can be shuffled.

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<sup>1</sup>JCL is John Crerar Library.

<sup>2</sup>Eckhart basement.

- Number of shuffles:

$$|\text{shuffles}| = 52! \approx 8 \times 10^{67}$$

- Properties of shuffles.

- **Distinguished shuffle:**  $e$ , the identity shuffle, where you do nothing.
- Shuffle once; shuffle again. The composition of two shuffles is another shuffle.
- If you repeat the *same* shuffle enough times, the cards will come back to the same order.
  - Let  $\sigma$  be a shuffle, and  $n \in \mathbb{N}$ . Does there exist  $n$  such that

$$\sigma^n = \underbrace{\sigma \circ \cdots \circ \sigma}_{n \text{ times}} = e$$

- Proving this: By the pigeonhole principle, if you have  $\sigma^1, \dots, \sigma^{52!+1}$ , then we have repeats  $a, b$  with  $52! + 1 \geq a > b \geq 1$  such that  $\sigma^a = \sigma^b$ . This statement is weaker than we want, though.
- We need more tools. A shuffle is a bijection/permutation. Thus, for every  $\sigma$ , there exists  $\sigma^{-1}$ . This allows us to do this:

$$\begin{aligned}\sigma^a &= \sigma^b \\ \sigma^{-b} \circ \sigma^a &= \sigma^{-b} \circ \sigma^b \\ \sigma^{a-b} &= e\end{aligned}$$

- This implies a bound! We get that  $n \leq 52!$ , so  $a - b \leq 52!$ .

- Define two shuffles:  $A$  and  $B$ .

- $A$  splits the deck into two halves (cards 1-26 and 27-52) and stacks (from the top down) the first card off of the 1-26 pile, then the first card off of the 27-52 pile, then the second card off of the 1-26 pile, then the second card off of the 27-52 pile, etc. The final order is 1, 27, 2, 28,  $\dots$ , 26, 52.
- $B$  does the same thing as  $A$  but with the first card off of the 27-52 pile. The final order is 27, 1, 28, 2,  $\dots$ , 52, 26.

- Computation shows that  $A^8 = e$  and  $B^{52} = e$ .

- For  $A$ ,  $2 \rightarrow 3 \rightarrow 5 \rightarrow 9 \rightarrow 17 \rightarrow 33 \rightarrow 14 \rightarrow 27 \rightarrow 2$ .
- For  $B$ , we can do the same thing but obviously the cycle is much longer.

- We shouldn't necessarily have an intuition for this right now, but in doing more examples, Calegari certainly believes we can develop it.
- First HW problem (due Friday). Can, just by using combinations of  $A$  and  $B$ , we generate any possible shuffle? Hint: Develop your intuition on a smaller value of 52.

- I really like Calegari. Very nice, relatable, not demeaning.

- **Binary operation** (on  $G$ ): A map from  $G \times G \rightarrow G$ .

- **Group:** A mathematical object consisting of a set  $G$  and a binary operation  $*$  on  $G$  satisfying the following properties.

1. There exists an identity element  $e \in G$  such that  $e \times g = g \times e = g$  for all  $g \in G$ .
2. For any  $g \in G$ , there exists  $h \in G$  such that  $h * g = g * h = e$ .
3. (Associativity) For any  $g_1, g_2, g_3 \in G$ ,  $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$ .

Denoted by  $(G, *)$ .

- In the cards example, the elements of  $G$  are the shuffles and  $*$  is the composition operation between two shuffles.

- Aside on shuffles: For bijections,  $h(g(x)) = x$  implies  $g(h(y)) = y$ .
  - Proof: Let  $x = h(y)$  — we can do this since  $h$  is a bijection. Then since  $h(g(h(y))) = h(y)$  and  $h$  is injective,  $g(h(y)) = y$ . This works for all  $y$ .
- The set of shuffles, together with composition, does form a group.
- Theorem: If  $G$  is a group such that  $|G| < \infty$ , then any  $g \in G$  has finite **order**, i.e., there exists  $n$  such that  $g^n = e$ .
- Lemma:
  1. The identity  $e$  is unique.
    - Let  $e_1, e_2$  be identities. Then
 
$$e_1 = e_1 * e_2 = e_2$$
  2. Inverses are unique.
    - Let  $h, h'$  be inverses of  $g$ . Then
 
$$h = e * h = (h' * g) * h = h' * (g * h) = h' * e = h'$$
- Proving examples is easier, but these aren't that hard.
- If you understand everything about  $S_5$ , you'll understand everything about this course.

## 1.2 The Cube Group

9/30:

- Can't download .tex file for homework?
  - Calegari will check it.
- Detail on the homework?
  - Up to your level of confidence in what you think is clear to be true.
  - The problem is not about doing linear algebra; it's about finding some facts about linearly algebraic objects.
  - Concentrate on the new geometry of the situation.
  - Project confidence to the grader that you know what you're doing.
- The symmetries of the cube.
  - Rotational symmetries.
  - Rigid transformation.
  - Preserves lengths, angles, and lines.
  - A map from the cube to itself, i.e.,  $\phi : \text{cube} \rightarrow \text{cube}$ .
  - No scaling allowed.
  - Reflectional symmetries are *not* going to be allowed for today; we will insist that the orientation is also preserved for now.
  - We want the set of all rotations and compositions of rotations. (Are compositions of rotations also rotations? We'll answer later. Yes they are.)
- Symmetries should be composable: If you compose two symmetries, you should get a third one.
  - In other words, we want the symmetries to form a group.
- We want to fix the center of the cube at the origin. Thus, a symmetry can be a linear map  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

- We want it to preserve angles, i.e., orthogonality. Thus, we should assert  $MM^T = I$ .
- We also want it to preserve orientation. Then we should have  $\det(M) = 1$ .
- **Cu**: The cube group.
- Does the permutation of faces determine  $M$ ?
  - Yes.
  - Furthermore, if we know where  $e_1, e_2$  go, then the fact that orientation and orthogonality are preserved implies that we know where  $e_3$  goes. Thus,  $M$  is determined by two (adjacent) faces.
- An upper bound on  $|\text{Cu}|$ .
  - Send  $e_1$  to one of 6 faces and send  $e_2$  to one of the 5 remaining faces (so  $|\text{Cu}| \leq 6 \cdot 5 = 30$ ).
  - Send  $e_1$  to one of 6 faces and send  $e_2$  to one of the four remaining *adjacent* faces (so  $|\text{Cu}| \leq 6 \cdot 4 = 24$ ).
  - And, in fact,  $|\text{Cu}| = 24$ .
- Moreover, since the rotations of the cube are determined by permutations of the faces, we can map  $\text{Cu} \hookrightarrow S_6$ . Additionally, composing any permutations of the faces is the same as composing any permutations of  $S_6$ , i.e.,  $\phi$  is an **injective homomorphism** to a **subgroup** of  $S_6$ .
- We can also think about permuting the vertices.
  - 3 vertices (chosen correctly) form a basis of  $\mathbb{R}^3$ .
  - Thus, since there are 8 vertices, we have another map from  $\text{Cu} \hookrightarrow S_8$ .
  - Since we can map the first vertex to any of eight and the second to only one of three adjacent vertices, the order is  $8 \cdot 3 = 24^{[3]}$ .
- We now have both  $\text{Cu}$  and  $S_4$  with order 24. Are they isomorphic?
  - One characteristic of a cube that numbers four are its four diagonals. This induces a function from  $\text{Cu} \rightarrow S_4$ . We now just need to prove it's bijective.
  - Let  $v_1, v_2, v_3, v_4$  be the vertexes of one face. Then  $-v_1, \dots, -v_4$  are the vertexes of the opposite face, and the line from each  $v_i$  to  $-v_i$  is a diagonal of the cube. To prove that the function is bijective, we will show that different elements of  $\text{Cu}$  map to different elements of  $S_4$ .
  - Let  $A$  and  $B$  be actions on the cube group such that
 
$$\begin{aligned} Bv_1 &= \pm Av_1 \\ Bv_2 &= \pm Av_2 \\ Bv_3 &= \pm Av_3 \\ Bv_4 &= \pm Av_4 \end{aligned}$$
  - Taking  $C = A^{-1}B$  means that
 
$$\begin{aligned} Cv_1 &= \pm v_1 \\ Cv_2 &= \pm v_2 \\ Cv_3 &= \pm v_3 \\ Cv_4 &= \pm v_4 \end{aligned}$$
  - If  $Cv_1 = v_1$ , it implies that  $Cv_i = v_i$  for  $i = 2, 3, 4$ .
  - Thus,  $A$  and  $B$  are distinct?

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<sup>3</sup>We have gotten the order a different way. Deep connection to prime factorization? Edges would be  $2 \cdot 12!$



## Week 2

# Group Theory Foundations

## 2.1 Groups of Low Order

- 10/3:
- Calegari: Nothing in particular to know for missing Friday; Adi will get me notes.
  - Having explored examples, today, we're coming back down to earth to flex our axiomatic muscles.
  - Distinguishing sets and binary operations.

Group	$G$	$*$	?
$S_n$	shuffles	composition	cards
$O(n)$ and $SO(n)$	(sp) orthogonal matrices	composition	vectors?
$\mathbb{Z}$	integers	addition	
$\mathbb{Z}/n\mathbb{Z}$	$\{0, 1, \dots, n-1\}$	addition modulo $n$	

Table 2.1: Elements of a group.

- Be careful not to confuse the shuffles and the cards; the cards are something else curious but are *not* the elements of the group.
- Notice that  $\mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z}$  are **commutative** groups, but the shuffles (for  $n > 1$ ) and  $O(n)$  are not.
- Note that  $S_2$ ,  $O(1)$ , and  $\mathbb{Z}/2\mathbb{Z}$  are all isomorphic groups.
- **Commutative** (group): A group such that for all  $x, y \in G$ ,  $x * y = y * x$ . Also known as **Abelian**.
- Lemma (Cancellation Lemma): Let  $x, y, z \in G$ . Then  $xy = xz$  implies  $y = z$  and  $yx = zx$  implies  $y = z$ .

*Proof.* We have that

$$\begin{aligned}
 x * y &= x * z \\
 x^{-1} * (x * y) &= x^{-1} * (x * z) && \text{Inverses exist} \\
 (x^{-1} * x) * y &= (x^{-1} * x) * z && \text{Associativity} \\
 e * y &= e * z \\
 y &= z
 \end{aligned}$$

as desired.

The proof of the second statement is symmetric. □

- This will be Calegari's only proof from the axioms directly.

- **Multiplication table** (for  $G$ ): A table with all elements of  $G$  on the top and the side, and all binary products in it.
  - The total number of binary operations is  $n^2$ ?
  - To check that a group is a group, we can write out its multiplication table and confirm pointwise that the group axioms are satisfied. However, there are also many ways to speed this process up.
  - An example of a multiplication table can be found on the right in Figure 2.1.
- **Trivial group**: The only group with  $|G| = 1$ , i.e.,  $G = \{e\}$ .
- A group of  $|G| = 2$  has the form  $G = \{e, x\}$  where we must have  $x = x^{-1}$ .
  - We can find this by inspection or invoke the **Sudoku Lemma**.
  - Thus, all groups of order 2 are isomorphic.
- Lemma (Sudoku Lemma): Fix  $x \in G$ . Then

$$\{xg \mid g \in G\} = G = \{gx \mid g \in G\}$$

*Proof.* There exists  $g$  such that  $xg = y$  for  $x, y$  fixed: Choose  $g = x^{-1}y$ .

$y$  only occurs once: If  $xg = y$  and  $xg' = y$ , transitivity and the cancellation lemma imply  $g = g'$ .  $\square$

- In layman's terms, in every row and column of the multiplication table, each element of  $G$  occurs exactly once.
- Playing Sudoku, we can show that all groups of order 3 are isomorphic.

	$e$	$x$	$y$
$e$	$e$	$x$	$y$
$x$	$x$		
$y$	$y$		

 $\longrightarrow$ 

	$e$	$x$	$y$
$e$	$e$	$x$	$y$
$x$	$x$	$y$	$e$
$y$	$y$	$e$	$x$

Figure 2.1: Playing Sudoku for  $|G| = 3$ .

- Start from the left table above.
- Notice that row 3 has a  $y$  and column 2 has an  $x$ , so by the Sudoku Lemma,  $e$  must be the element in row 3, column 2.
- Then column 2 has  $e, x$  in it, so the entry in row 2, column 2 must be  $y$ .
- Then row 2 has  $x, y$  in it, so the entry in row 2, column 3 must be  $e$ .
- Then row/column 3 both have  $e, y$  in them, so the entry in row 3, column 3 must be  $x$ .
- However, we cannot play Sudoku in the same way with groups of order 4. In fact, there are multiple groups of order 4.
  - Two cases: (1)  $x^2 \neq e$  so WLOG let  $x^2 = y$ , and (2)  $a^2 = e$  for  $a = x, y, z$ .
    - Case 1 is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ .
    - Case 2 is isomorphic to the **direct product** of  $\mathbb{Z}/2\mathbb{Z}$  with itself, also known as the **Klein 4-group**.
  - This should not come as a surprise: We've already encountered the very different groups  $S_4$  and  $\mathbb{Z}/24\mathbb{Z}$  of order 24.

- **Direct product:** The group whose set is the Cartesian product of the sets of groups  $A = (A, *_A), B = (B, *_B)$ , and whose operation is coordinate-wise multiplication. *Given by*

$$G = A \times B \qquad (a, b) *_G (a', b') = (a *_A a', b *_B b')$$

- We can prove that  $e = (e_A, e_B)$ , that  $(a, b)^{-1} = (a^{-1}, b^{-1})$ , and that associativity holds.
- We have that

$$|G| = |A| \cdot |B|$$

- There is only one group of order 5.
- Examples of groups of order 6:  $S_3, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}), (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ .
  - Are there any two groups which are distinct?
    - $S_3$  is not commutative, but the others are, so it is distinct from them.
    - $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$  and  $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$  are the same because order doesn't matter in the construction of the direct product.
    - $\mathbb{Z}/6\mathbb{Z}$  and the two direct products are the same because they both have elements of order 6 (i.e., a one-element generator). The cycles are:

$1^1 = 1$	$= 1$	$(1, 1)^1 = (1, 1)$	$= (1, 1)$
$1^2 = 1 + 1 = 2$		$(1, 1)^2 = (1 + 1, 1 + 1) = (2, 0)$	
$1^3 = 2 + 1 = 3$		$(1, 1)^3 = (2 + 1, 0 + 1) = (0, 1)$	
$1^4 = 3 + 1 = 4$		$(1, 1)^4 = (0 + 1, 1 + 1) = (1, 0)$	
$1^5 = 4 + 1 = 5$		$(1, 1)^5 = (1 + 1, 0 + 1) = (2, 1)$	
$1^6 = 5 + 1 = 0$		$(1, 1)^6 = (2 + 1, 1 + 1) = (0, 0)$	
$1^7 = 0 + 1 = 1$		$(1, 1)^3 = (0 + 1, 0 + 1) = (1, 1)$	

- These are the only two groups of order 6.
- Continuing on, there is only 1 group with  $|G| = 2047$  (which is “mostly prime” — connection between primes and number of groups?), but there are 1,774,274,116,992,170 groups of  $|G| = 2048 = 2^{11}$ .
- Conclusion: The arithmetic of  $|G|$  has an impact on the structure of  $G$ .

## 2.2 Characteristics of the Symmetric Group

10/5:

- **Symmetric group** (on  $n$  letters): The set of all bijections from the set of numbers  $\{1, \dots, n\}$  to itself, whose operation is function composition. *Denoted by  $S_n$ .*
  - Convention: Denote elements of  $S_n$  not by  $f$  but by  $\sigma, \tau$ .
  - $\sigma\tau$  means do  $\tau$  first and then  $\sigma$ .
  - $|S_n| = n!$ .
- One of the first challenges we encounter when defining new objects is a notational one.
  - We could define a function with a table, but cycle notation is easier.
- **$k$ -cycle:** The bijection

$$m \mapsto \begin{cases} a_{i+1} & m = a_i, i \neq k \\ a_1 & m = a_k \\ m & m \neq a_i \end{cases}$$

in  $S_n$ , where  $a_1, \dots, a_k$  are distinct elements of  $[n]$ . *Denoted by  $(a_1, a_2, \dots, a_k)$ .*

- If  $\sigma$  is a  $k$ -cycle, then the order of  $\sigma$  is  $k$ .
- There are  $k$  ways to write down the same  $k$ -cycle.
  - For example,  $(i, j) = (j, i)$  and  $(a, b, c) = (b, c, a) = (c, a, b)$ .
- All 1-cycles are the identity  $e$ .
- Combinatorics: How many  $k$ -cycles are there in  $S_n$ ?
  - $k = 1$ : Just one – ( $e$ ).
  - $k = 2$ :  $\binom{n}{2}$ .
  - $k = 3$ :  $\binom{n}{3} \cdot 2$ .
    - We must first choose 3 of the  $n$  possible elements to be manipulated by the  $k$ -cycle.
    - But then we can send  $a_1$  to  $a_2$  or  $a_3$ , so that's an additional two choices beyond just a selection of 3 elements. Once we send  $a_1$  to  $a_2$  or  $a_3$ , the rest of the cycle is determined, so we need not augment any more.
  - $k$ :  $\binom{n}{k} \cdot (k-1)! = \frac{n!}{(n-k)!k}$ .
    - As before, we must choose  $k$  of the  $n$  possible elements to be manipulated by the  $k$ -cycle.
    - However, here, there are  $k-1$  possibilities to which we can send  $a_1$ , so we need to multiply by that. Once we've determined  $\sigma(a_1)$ , there are  $k-2$  possibilities to which we can send  $\sigma(a_1)$ . This pattern naturally continues, and we end up needing to correct  $\binom{n}{k}$  by  $(k-1)!$ .
- Proposition: Every  $\sigma \in S_n$  can be written as a product/composition of disjoint cycles. Moreover, disjoint cycles commute.

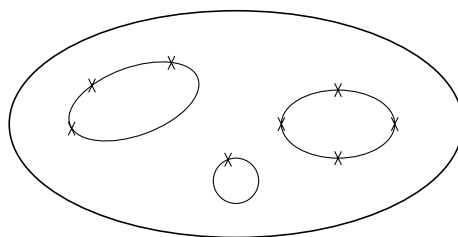


Figure 2.2: Decomposing  $\sigma$  into disjoint cycles.

- The idea behind this proposition is that every element will cycle back to itself eventually, and you can't get to elements of one cycle if you're not in the cycle (so all cycles are disjoint).
- Every permutation can be visualized by ordering the  $n$  letters in a set in  $\mathbb{R}^2$  and connecting all disjoint cycles (think a circle full of oriented circles/loops/cycles).
- Composing cycles. See what the right one does and then the left one. Canonically, start with 1.
- Proposition: The cycle decomposition of  $\sigma$  is unique up to...
  - The ordering of the disjoint cycles;
  - Cycle permutations of each cycle;
  - Include/exclude 1-cycles.

Moreover,  $|\sigma|$  is the least common multiple of the cycle lengths.

- How many elements in  $S_6$  have a cycle shape that looks like  $(x, x)(x, x)(x, x)$ ?
  - It is

$$\frac{6!}{2^3 \cdot 3!} = 15$$

- Rationale: See PSet 2, Q1a.

- The cycle decompositions of all elements in  $S_4$ .

(1, 2, 3, 4)	(1, 2, 3)	(1, 2)	(1, 2)(3, 4)	e
(1, 2, 4, 3)	(1, 3, 2)	(1, 3)	(1, 3)(2, 4)	
(1, 3, 2, 4)	(1, 2, 4)	(1, 4)	(1, 4)(2, 3)	
(1, 3, 4, 2)	(1, 4, 2)	(2, 3)		
(1, 4, 2, 3)	(1, 3, 4)	(2, 4)		
(1, 4, 3, 2)	(1, 4, 3)			
	(2, 3, 4)			
	(2, 4, 3)			

Table 2.2:  $S_4$  cycle decompositions.

- **Conjugate** (elements  $x, y$ ): Two elements  $x, y \in G$  a group for which there exists  $g \in G$  such that  $y = g \cdot x \cdot g^{-1}$ . Denoted by  $x \sim y$ .
- Lemma: Conjugacy is an equivalence relation.

(I)  $x \sim x$ .

*Proof.*  $x = exe^{-1}$ . □

(II) If  $y \sim x$ , then  $x \sim y$ .

*Proof.* Take

$$y = gxg^{-1}$$

$$g^{-1}y(g^{-1})^{-1} = x$$

□

(III) If  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ .

*Proof.* Suppose  $y = gxg^{-1}$  and  $z = hyh^{-1}$ . Then

$$z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$$

□

- **Conjugacy class** (of  $x$ ): A subset of  $G$  containing all  $g \in G$  which are conjugate to a certain  $x \in G$ . Given by

$$\{g \in G \mid g \sim x\}$$

- Straightforward: Not necessarily obvious, but there's nothing really tricky going on.
  - The joke about the mathematician who says something is obvious, someone asks why?, he thinks for 20 minutes, and then says it's obvious.
- Why is conjugacy important?
  - In linear algebra, we've seen it with similar matrices.
    - Same linear map in a different basis is the same as conjugating the matrix of the map in one basis with the change of basis matrix.
  - Conjugacy tells us that a set of objects are, in some way, the same.

## 2.3 Conjugacy

10/7:

- You can request one extension per quarter on homework (possibly more if you have a really good reason) for sickness, etc., no questions asked. Email your TA to secure this extension.
- Last time, we began covering conjugacy.
  - Conjugacy classes.
  - Conjugacy defines an equivalence relation on  $G$ .
  - $G = \bigsqcup$  conjugacy classes<sup>[1]</sup>.
- More on conjugacy today.
- The conjugacy class of  $e$  is  $\{e\}$ .
- If  $y = gxg^{-1}$ , then  $y^k = gx^k g^{-1}$ .
- Proposition:  $y \sim x$  implies  $|y| = |x|$ .

*Proof.* Suppose  $|y| = k$ , i.e.,  $y^k = e$ . By the above statement, we know that  $y^k \sim x^k$ . Since  $y^k = e$ , it follows that  $e \sim x^k$ . Thus,  $x^k$  is in the conjugacy class of  $e$ . But since the conjugacy class of  $e$  is  $\{e\}$ , this means that  $x^k = e$ , as desired.  $\square$

- Conjugacy in  $S_n$ ,  $n \geq 2$ .
  - Each  $x \in S^n$  has a cycle decomposition

$$x = (a_1, \dots, a_k)(b_1, \dots, b_m)(c_1, \dots) \cdots$$

- We want to investigate the properties of  $gxg^{-1}$  for an arbitrary  $g \in S_n$ . Ideally, we'd like to express it in a form related to  $x$ .
- Trick: Apply  $gxg^{-1}$  to  $g(a_1)$ . Then

$$gxg^{-1}(g(a_1)) = gx(a_1) = g(a_2)$$

- It follows by induction that

$$gxg^{-1} = (g(a_1), \dots, g(a_k))(g(b_1), \dots, g(b_m))(g(c_1), \dots) \cdots$$

- Now suppose that  $m \neq g(a_i), g(b_j), g(c_k), \dots$ . Then

$$g^{-1}(m) \notin \{a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots\}$$

It follows since  $x$  is the identity on such elements that  $x(g^{-1}(m)) = g^{-1}(m)$ . Therefore, since all functions involved are bijections,

$$[gxg^{-1}](m) = g[x(g^{-1}(m))] = g(g^{-1}(m)) = m$$

- It follows that  $gxg^{-1}$  has the same cycle **shape**.
- **Shape** (of  $g \in S_n$ ): The partition of  $n$  given by the lengths of the cycles in the cycle decomposition of  $g$  in decreasing order. *Also known as cycle shape.*

$S_4$	4-cycle	3-cycle	Product of 2-cycles	1-cycles
Cycle decomposition	$(x, x, x, x)$	$(x, x, x)(x)$	$(x, x)(x, x)$	$(x)(x)(x)(x)$
Shape	4	3 + 1	2 + 2	1 + 1 + 1 + 1

Table 2.3: Shape of elements in  $S_4$ .

<sup>1</sup> $\bigsqcup$  denotes a **disjoint union**. Think of the *disjoint* union of sets as a union of sets that happen to be disjoint, the same way a *direct* sum of subspaces is a sum of subspaces that happen to be linearly independent.

- Claim:  $x, y \in S_n$  are conjugate iff they have the same cycle shape.

*Proof.* We will do a proof by example that illustrates the idea of the generalized proof.

Let

$$x = (1, 2, 3)(4, 5, 6)(7, 10) \qquad y = (2, 3)(4, 1, 5)(6, 9, 10)$$

Note that both have the same cycle shape:  $3 + 3 + 2 + 1 + 1$ . We now use a two-step process to define a  $g$  such that  $y = gxg^{-1}$ .

Step 1: Including 1-cycles, line both  $x$  and  $y$  up so they “match.”

$x$	( 1      2      3 ) ( 4      5      6 ) ( 7      10 ) ( 8 ) ( 9 )
$y$	( 4      1      5 ) ( 6      9      10 ) ( 2      3 ) ( 7 ) ( 8 )
$gxg^{-1}$	( $g(1)$ $g(2)$ $g(3)$ ) ( $g(4)$ $g(5)$ $g(6)$ ) ( $g(7)$ $g(10)$ ) ( $g(8)$ ) ( $g(9)$ )

Step 2: We want  $y = gxg^{-1}$ . Thus, take  $g$  to be the map which sends every entry in  $gxg^{-1}$  to the entry of  $y$  directly above it. For example, we want  $g(1) = 4$ ,  $g(2) = 1$ ,  $g(3) = 5$ ,  $\dots$ . Noting that  $g(1) = 4$ ,  $g(4) = 6$ ,  $g(6) = 10$ ,  $\dots$ , we realize that  $g$  can actually be written as the following cycle.

$$g = (1, 4, 6, 10, 3, 5, 9, 8, 7, 2)$$

□

- Follow ups.
  - How many different  $g$ 's satisfy  $y = gxg^{-1}$ ?
    - Depends on the number of ways  $y$  can be matched up with  $x$ .
    - The above manner obviously works.
    - However, we can rotate the elements in both 3-cycles three ways, and the elements of the 2-cycle two ways, so that's  $3 \cdot 3 \cdot 2 = 18$   $g$ 's right there.
    - Additionally, we can swap the place of the 3-cycles and the 1-cycles entirely, so that's an additional  $2 \cdot 2$  times as many ways.
    - All told, there are  $3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 72$  possible  $g$ 's.
    - See HW2, Q1a for a treatment of an analogous problem.
  - Counting the size of conjugacy classes.
    - Suppose  $G$  is an abelian group. Then if  $y = gxg^{-1}$ ,  $y = gg^{-1}x = x$ , so the size of the conjugacy class of any  $x \in G$  is 1.
    - For this reason, the elements of  $\mathbb{Z}/n\mathbb{Z}$  and of  $\text{SO}(2)$  are conjugate only to themselves.
    - However, we get something different for  $\text{O}(2)$ . Here, we can prove that the conjugacy class of every rotation  $r$  is  $\{r, r^{-1}\}$ , and that all reflections are in the same conjugacy class<sup>[2]</sup>.
      - > Let  $r$  denote a rotation, and  $s$  denote a reflection.
      - > Suppose  $x = r$ . Then

$$\begin{aligned} e &= (sr)^2 \\ &= sr sr \\ r^{-1} &= sr s \\ &= sr s^{-1} \end{aligned}$$

where we have HW1, Q2d(i) to justify the first equality and the fact that every reflection is its own inverse<sup>[3]</sup> to justify the last equality.

<sup>2</sup>This is fundamentally related to the structure of point groups in inorganic chemistry! Remember that in  $C_{5v}$ , for instance,  $C_5, C_5^4$  are conjugate,  $C_5^2, C_5^3$  are conjugate, and all reflections get lumped together.

<sup>3</sup>Intuitively, applying any reflection twice yields the original object.

➤ On the other hand, suppose  $x = s$ . Then if  $r$  is any rotation,

$$\begin{aligned} sr sr &= e \\ r sr &= s^{-1} \\ r sr r^{-2} &= s^{-1} r^{-2} \\ r sr^{-1} &= sr' \end{aligned}$$

where  $r'$  denotes  $r^{-2}$  to express the main takeaway: that  $s$  is conjugate to itself times any rotation (for  $r'$  arbitrary, we may choose  $r = (r')^2$ ). In other words, since all reflections are related by some rotation, all reflections are, indeed, in the same conjugacy class.

- Generators of  $S_n$ ,  $n \geq 3$ .
- Lemma: The set of 2-cycles generates  $S_n$ .

*Proof.* It only requires  $n - 1$  swaps between pairs of elements to get to any permutation. For example, to get to

$$\begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & \mapsto 2 \\ 4 & 1 \end{array}$$

we can swap 1 and 3 (so  $1 \mapsto 3$ ), then 2 and 4 (so  $2 \mapsto 4$ ), then “3” and “4” (so  $3 \mapsto 2$  and  $4 \mapsto 1$ ). More graphically,

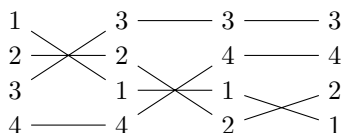


Figure 2.3: Generating  $S_n$  with 2-cycles.

The idea is we fix the first element and then work down the list. □

- $S_n$  is also generated by

$$\{(1, 2), (2, 3), (3, 4), \dots, (n - 1, n)\} \qquad \{(1, 2), (1, 3), (1, 4), \dots, (1, n)\}$$

– Both of these sets have cardinality  $n - 1$ .

- As we can see from the above...

- If we generate  $S_n$  with all 2-cycles, the generator set has cardinality  $\frac{n}{2}(n - 1)$ ;
- If we generate  $S_n$  with all elementary 2-cycles, the generator set has cardinality  $n - 1$ .

- But we can do even better: Let  $\sigma = (1, 2)$  and  $\tau = (1, 2, \dots, n)$ . Then any

$$(k, k + 1) = \tau^{k-1} \sigma \tau^{-(k-1)}$$

- Indeed, we can see that using the RHS above,  $k \mapsto 1 \mapsto 2 \mapsto k + 1$  and  $k + 1 \mapsto 2 \mapsto 1 \mapsto k$ . Every other element receives the identity treatment, as we can confirm.



# Week 3

???

## 3.1 Introduction to Subgroups and Generators

10/10:

- Defining **subgroups**.
  - Let  $G = (G, *)$  be a group, and let  $H \subseteq G$  be a subset.
  - What properties do we want  $H$  to satisfy to consider it a “subgroup?”
    - $H$  should inherit the binary operation from  $G$ .
    - $H$  should be closed under multiplication using said binary operation.
    - $H$  should be nonempty.
    - $H$  should contain the inverses of every element — this is automatic if  $G$  is finite since the inverse of an element  $g$  of order  $n$  is  $g^{n-1}$  and  $g^{n-1} \in H$  by closure under multiplication.
    - $H$  should also be associative; we also inherit this for free from  $G$ .
- Easy way to construct a subgroup.
  - Let  $G$  be a group, and let  $x_1, x_2, \dots \in G$ . We can let  $H = \langle x_1, x_2, \dots \rangle$ , i.e.,  $H$  is the group **generated** by  $x_1, x_2, \dots$ . In other words,  $H$  is the set of all finite products  $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots$ .
  - This construction does give you all possible subgroups, but when you write it down, it’s very hard to say what group you get.
- Example: If you have  $H \subset G$  a subgroup, then  $H = \langle h |_{h \in H} \rangle$ .
- **Cyclic** (group): A group  $G$  for which there exists  $g \in G$  such that  $G = \langle g \rangle$ .
- Examples:
  - If  $1 < n < \infty$ , then  $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$ .
  - However, the generator isn’t always unique —  $\mathbb{Z}/7\mathbb{Z} = \langle 3 \rangle$ .
  - If  $G$  is generated by an element, it’s also generated by its inverse. For example,  $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ .
- Proposition: Let  $G$  be a cyclic group. It follows that
  1. If  $|G| = \infty$ , then  $G$  is isomorphic to  $\mathbb{Z}$ ;
  2. If  $|G| = n < \infty$ , then  $G$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ .

*Proof.* Assertion 1: Let  $G = \langle g \rangle$ . Then

$$G = \{\dots, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots\}$$

Now suppose for the sake of contradiction that  $g^a = g^b$  for some  $a, b \in \mathbb{Z}$ . Then  $g^{a-b} = e$ , so  $|G| \leq a-b$ , a contradiction. Therefore,  $G = \{G^{\mathbb{Z}}\}$ . In particular, we may define  $\phi : \mathbb{Z} \rightarrow G$  by  $k \mapsto g^k$ . This map has the property that  $a + b \mapsto g^a g^b$ , i.e.,  $\phi(a)\phi(b) = \phi(ab)^{[1]}$ .

Assertion 2: Let  $G = \langle g \rangle$ . Then

$$G = \{e, g, g^2, \dots, g^{n-1}\}$$

Now suppose for the sake of contradiction that  $g^a = g^b$ . Then  $g^{a-b} = e$ , so  $|G| \leq a-b < n$ , a contradiction. Therefore, we may once again define  $\phi : \mathbb{Z}/n\mathbb{Z} \rightarrow G$  as above. Note that  $a + b \mapsto g^{(a+b) \bmod n}$ . This is still a homomorphism, though.  $\square$

- Claim: Any subgroup of a cyclic group is also cyclic.
- Example:  $G = \mathbb{Z}$ ,  $H = \langle 2002, 686 \rangle$ .
  - $H = \{2002x + 686y \mid x, y \in \mathbb{Z}\}$ .
  - To say that  $H$  is cyclic is to say that it is equal to the integer multiples of some  $d \in \mathbb{Z}$ , i.e., there exists  $d$  such that  $G = \{zd \mid z \in \mathbb{Z}\}$ .
  - We can take  $d = \gcd(2002, 686)$ .
  - (Nonconstructive) proof: Let  $d$  be the smallest positive integer in  $H$ . Suppose for the sake of contradiction that  $md + k$  is in the group for some  $1 \leq k < d$ . Then adding  $-d$   $m$  times, we get that  $k \in H$ , a contradiction since we assumed  $d$  was the smallest positive integer in  $H$ .
- Let  $G = \langle x, y \rangle$  be a group that is generated by two elements. Find a subgroup  $H \subset G$  such that  $H$  *must* be generated by more than 2 elements.
  - Let's work with  $S_n = \langle (1, 2, \dots, n), (1, 2) \rangle$ .
  - The subgroup  $H = \langle (1, 2), (3, 4), (5, 6) \rangle$  will work.
    - $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .
    - Suppose  $H = \langle a, b \rangle$ . We can get  $e, a, b, ab$ . But because everything commutes, we can rearrange any product to  $a^i b^j$  and cancel.
- When you want to answer questions like, “Is  $\mathbb{Z}/180180\mathbb{Z}$  a subgroup of  $S_n$  for some  $n$ ,” you need some more information on the structure of  $S_n$ .
- Group **presentations** allow us to write and describe a group really easily.
  - Seems useful at first, but isn't really that useful once you see it more.

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<sup>1</sup>We all know that this is a **homomorphism**; Calegari just doesn't want to call it that yet.