Week 6

Fundamentals of Group Actions

6.1 Examples of Group Actions

10/31:

- Today: A number of interesting group actions.
- **Left action** (of G on X): A group action of the form $g \cdot x$ (as opposed to $x \cdot g$).
- Let G be a group, and let $X = G$. Take $g \cdot x = gx$.
  - Axiom confirmation.
    1. $e \cdot x = ex = x$.
    2. $g \cdot (h \cdot x) = ghx = gh \cdot x$.
  - Let $e \in X$. Then $\text{Orb}(e) = X$. In particular, this means that the action is transitive.
  - $\text{Stab}(x) = \{g \in G \mid gx = x\} = \{e\}$ for $x \in X$ arbitrary, in general.
  - $\ker = \{e\}$. This also follows from the above. Thus, the action is faithful.
- Corollary: Let G be a finite group. Then G is isomorphic to a subgroup of S_n for some n. We may take $n = |G|$.
  - Construction: We invoke the proposition from last lecture. In particular, we know that the action $G \circlearrowright G$ implies the existence of a homomorphism $\phi : G \to S_G$ defined by $g \mapsto \psi_g$.
  - The map in the above construction has trivial kernel. By the FIT, $G/\ker \cong \operatorname{im}\phi$. Combining these results, we obtain $G \cong G/\ker \cong \operatorname{im}\phi \leq S_n$.
  - Applying this construction to S_3, we deduce that $S_3 \leq S_6$.
- $\text{SO}(2) \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}^\infty$.
  - In infinite cases, you usually want to consider some other topological things that disappear in the finite case.
- Let G be a group and take $X = G$ again. We can also consider $g \cdot x = gxg^{-1}$.
  - Axioms.
    1. $e \cdot x = exe^{-1} = x$.
    2. $g \cdot (h \cdot x) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = gh \cdot x$.
  - $\text{Orb}(e) = \{e\}$; not transitive if $|G| > 1$.
  - Let $x \in X$. Then $\text{Orb}(x)$ is the conjugacy class of x.
  - $\text{Stab}(x) = C_G(x)$.
  - $\ker = Z(G)$. Thus, the group action is faithful iff the center is trivial. Abelian implies not faithful.

- A nice thing about these constructions is that they cast other constructions we've encountered in the more general language of group actions.
- **Right actions** are even nastier than left cosets and right cosets, so Calegari will not mention them again.
  - $g \cdot x = x \cdot g^{-1}$ and $g \cdot (h \cdot x) = (x \cdot h^{-1}) \cdot g^{-1}$.
- Let $G = G$, X be the subgroups of G. $g \cdot H = gHg^{-1}$.
  - Note that $H \leq G$ does indeed imply that $gHg^{-1} \leq G$. In particular, ...
    - H is nonempty (contains at least e), so $gHg^{-1} \supset \{geg^{-1}\}$ is nonempty;
    - $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$ imply that $gh_1g^{-1}gh_2g^{-1} = g(h_1h_2)g^{-1} \in gHg^{-1}$;
    - $ghg^{-1} \in gHg^{-1}$ has inverse $gh^{-1}g^{-1} \in gHg^{-1}$.
  - Axioms (entirely analogous to the last example).
  - Orb(H) is the "conjugates" of H.
  - Stab(H) $= N_G(H)$.
  - ker =?. We know that $Z(G) \subset$ ker. The conclusion is that there is not a nice definition for the kernel other than the intersections of the stabilizers/normalizers.
    - ...
    - If any $H \triangleleft G$ is normal, and $x \in G$ had order 2, then $\langle x \rangle \triangleleft G$, meaning that $gxg^{-1} \in \langle x \rangle$, i.e., $x \in Z(G)$, so this rules out D_8??
- Fix G and $H \leq G$. Let $X = G/H$ (not assuming $H \triangleleft G$, so we know that G/H is the set of left cosets but it is not a group in general). Define $g \cdot xH = gxH$.
  - We have $g \cdot xhH = gxhH$.
  - Orbit: Orb(eH) $= X$.
  - Stabilizer: Stab(eH) $= H$.
    - Stab(gH) $= gHg^{-1}$.
    - This is because $(ghg^{-1})gH = ghH = gH$.
    - Go to the more general case $G \circlearrowright X$, Stab(x) $= H$. Then $gHg^{-1} \subset$ Stab($g \cdot x$)??
  - Transitive: Yes (see orbits).
  - Faithful: If H is normal, no. If H contains a normal subgroup, no. Maybe yes.
  - Kernel: If H is normal, then ker $= H$. In general, ker $= \bigcap_{g \in G} gHg^{-1}$ (the largest normal subgroup of H).
- Takeaway: General constructions allow us to see things we've already done.
- Next time: The most useful theorem of the course, that provides lots of information on relations between objects.

6.2 Orbit-Stabilizer Theorem

11/2:

- We will have a take-home open-book final. Should take you a couple hours or a little more to do, but we'll have more time than that. Don't Google answers or collaborate. We'll have more practice problems (and 50% of the exam will be on that sheet); if we do every problem on the sheet, we'll certainly get an A.
- We will cover all theoretical material by Thanksgiving and then spend the rest of the time exploring applications.
- Today: The most fundamental theorem of the class.

- Let G be a group acting on a set X.
- Theorem (Orbit-Stabilizer Theorem): Let $x \in X$ be arbitrary. Then

$$|G| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$$

Proof. We will break up G and count it in two different ways. Let $x \in X$ be arbitrary and consider $\operatorname{Orb}(x)$. By definition, $\operatorname{Orb}(x)$ is the set of all y such that $g \cdot x = y$ for some $g \in G$. Equivalently, every $g \in G$ maps x to some $y \in \operatorname{Orb}(x)$. Thus, we can partition G into sets of g that map x to a particular y, knowing that every g must send it to some y. Symbolically,

$$G = \bigsqcup_{y \in \operatorname{Orb}(x)} \{g \mid g \cdot x = y\}$$

Each of the sets over which we sum above is equal to $g \cdot \operatorname{Stab}(x)$ (the left coset of the stabilizer by g). Thus, for each $y \in \operatorname{Orb}(x)$, we contribute $|g \cdot \operatorname{Stab}(x)|$ to $|G|$. Symbolically,

$$|G| = \sum_{\operatorname{Orb}(x)} |g \cdot \operatorname{Stab}(x)| = \sum_{\operatorname{Orb}(x)} |\operatorname{Stab}(x)| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$$

as desired. ☐

- Examples:
  - Let $H \leq G$, $X = G/H$. Then G acts on X by left multiplication. Taking $x = H$ in particular, we have that
  $$|G| = |G/H| \cdot |H|$$
  and we recover Lagrange's theorem as a special case of the O-S theorem.
  - $G = S_n$, $X = [n]$.
    - Then $S_n = \{\sigma(1) = 1\} \cup \{\sigma(1) = 2\} \cup \cdots \cup \{\sigma(1) = n\}$. This is analogous to the proof strategy decomposition.
  - G acts on G by conjugation.
    - Take $g \in G$. Then $\operatorname{Orb}(g) = \{g\}$, i.e., the conjugacy class of g, and $\operatorname{Stab}(g) = C_G(g)$. Therefore, we have the below corollary.
  - $G = S_n$.
    - Let $g = (1, \dots, k)$ for $2 \leq k \leq n$. Recall that $|\{g\}| = n!/(n-k)!k$. Thus, $|C_{S_n}(g)| = (n-k)! \cdot k$.
    - Alternatively, we can derive the order of this centralizer directly: $C_{S_n}(g) = \langle g \rangle \times S_{n-k}$, i.e., all powers of the k-cycle and everything that's disjoint. $\times$ denotes the direct product.
  - $G = S_4$, $g = (12)(34)$.
    - $|\{g\}| = 3$, so $|C_G(g)| = 8$.
    - Here $C_G(g) = D_8$. Visualize a square with vertices clockwise (1,4,2,3).
  - $G = S_6$, $g = (16)(25)(34)$.
    - We have that $|\{g\}| = 6!/2^3 \cdot 3! = 15$, so $|C_{S_6}(g)| = 48$. The centralizer is the set of all elements satisfying $\sigma(i) + \sigma(7-i) = 7$.
    - Moreover, there is an injective homomorphism from $\widetilde{\mathrm{Cu}} \hookrightarrow S_6$ whose image is exactly the centralizer of $(16)(25)(34)$. Moreover, it follows that $C_{S_6}(g) \cong S_4 \times S_2$.
    - Let $h = (16)$. Then $|\{h\}| = |\{g\}| = 15$. Does there exist an automorphism of S_6 to S_6 which sends $h \to g$? No: $S_2 \times S_4 \cong C_{S_6}(h)$ and $C_{S_6}(g) \cong S_2 \times S_4$.
- Corollary: We have that
$$|G| = |\{g\}| \cdot |C_G(g)|$$

- $\widetilde{\textbf{Cu}}$: The set of all orthogonal symmetries of the cube (i.e., including reflections).
  - There is an isomorphism between $\text{Cu} \times \mathbb{Z}/2\mathbb{Z}$ and $\widetilde{\text{Cu}}$ defined by $(g, 1) \mapsto g$ and $(g, -1) \mapsto -g$. The reverse function is $g \mapsto (g \cdot \deg g, \deg g)$.
  - $\widetilde{\text{Cu}}$ acts on 6 faces.
- The pace will be this fast through Thanksgiving.

6.3 Blog Post: The Orbit-Stabilizer Theorem, Cayley's Theorem

From Calegari (2022).

11/13:
- Lemma: Let $G \circlearrowright X$ and let $x \in X$. Let $y \in \text{Orb}(x)$, i.e., let there exist $\sigma \in G$ such that $y = \sigma \cdot x$. Then
  1. $\text{Stab}(y) = \sigma \cdot \text{Stab}(x) \cdot \sigma^{-1}$.

     Proof. Let $H := \text{Stab}(x)$. We use a bidirectional inclusion argument.
     Suppose first that $\sigma h \sigma^{-1} \in \sigma H \sigma^{-1}$. Then

     $$\sigma h \sigma^{-1} \cdot y = \sigma h \cdot (\sigma^{-1} \cdot y) = \sigma h \cdot x = \sigma \cdot (h \cdot x) = \sigma \cdot x = y$$

     so $\sigma h \sigma^{-1} \in \text{Stab}(y)$, as desired.
     Now suppose that $g \in \text{Stab}(y)$. An analogous argument to the above shows that $\sigma^{-1} g \sigma \in \text{Stab}(x)$, so $g = \sigma h \sigma^{-1} \in \sigma H \sigma^{-1}$, as desired. □
  2. The set of elements $g \in G$ such that $g \cdot x = y$ is exactly the coset $\sigma \cdot \text{Stab}(x)$.

     Proof. As before, let $H := \text{Stab}(x)$ and proceed via a bidirectional inclusion argument.
     Suppose first that $\sigma h \in \sigma H$. Then

     $$\sigma h \cdot x = \sigma \cdot (h \cdot x) = \sigma \cdot x = y$$

     so σh is in the first set, as desired.
     Now suppose that $g \cdot x = y$. Since $\sigma \cdot x = y$ as well by hypothesis, it follows by transitivity that

     $$\begin{aligned} g \cdot x &= \sigma \cdot x \\ \sigma^{-1} \cdot (g \cdot x) &= \sigma^{-1} \cdot (\sigma \cdot x) \\ \sigma^{-1} g \cdot x &= x \end{aligned}$$

     This implies that $\sigma^{-1} g \in H$, i.e., that $g \in \sigma H$, as desired. □
- This lemma further justifies the following step we took when proving the Orbit-Stabilizer Theorem in class: Equating each $\{g \mid g \cdot x = y\} = \sigma \, \text{Stab}(x)$.
- Further comments on $G \circlearrowright G/H$ (H a subgroup).
  - Why the action is well-defined.
    - $g \cdot xhH = gxhH = gxH = g \cdot xH$.
    - What saves the day here is that we're combining an unambiguous term (g) with our ambiguous term (xH) instead of trying to combine two ambiguous terms (e.g., xH and yH).
  - An example where the action is faithful.
    - Let $G = S_n$ and $H = \{\sigma \mid \sigma(1) = 1\} \cong S_{n-1}$.
    - Note that if $\sigma \in H$, then $(1, k)\sigma(1, k)^{-1}$ sends $\sigma(k) = k$.

  - Thus,
$$\ker = \bigcap_{g\in G} gHg^{-1} \subset \bigcap_{k=1,\dots,n} (1,k)H(1,k)^{-1} = \{e\}$$
so the action is faithful, here.
- When $H = \{e\}$, $G \circlearrowright G/H$ is entirely analogous to left multiplication within the group: $g \cdot x = gx$.

- Lemma: $G \circlearrowright G$ by left multiplication is faithful.

  Proof. To prove this result, we will actually prove the stronger result that $\text{Stab}(x) = \{e\}$ for all $x \in G$, from which it will follow that $\ker = \bigcap_{x\in G} \text{Stab}(x) = \{e\}$. We have this stronger result by the cancellation lemma since
$$\begin{aligned} g \cdot x &= x \\ gx &= ex \\ g &= e \end{aligned}$$
for all $g \in \text{Stab}(x)$. $\square$

- Corollary (Cayley's Theorem): If $|G| = n$, then $G \leq S_n$.

  Proof. From the construction $G \circlearrowright G$ via left multiplication, we get a homomorphism $\phi : G \to S_G$ as per the Proposition in Lecture 5.2. Since this action is faithful (by the lemma), this homomorphism is an injection. This implies that $G \cong \operatorname{im}\phi \leq S_G \cong S_n$, as desired. $\square$

- Implication: Even without knowing anything about G, we can get useful information by considering its actions on a set.

- More on $G \circlearrowright G$ by conjugation.
  - Since $|G| = |\{g\}| \cdot |C_G(g)|$, we can calculate the orders of centralizers. From the order, we can often get even more specific information.
  - Consider $G = S_n$.
    - If $g = (1, 2, \dots, n)$, then $|\{g\}| = (n-1)!$ and $C_G(g) = n!/(n-1)! = n$. This combined with the fact that g commutes with g implies that
$$C_{S_n}((1, 2, \dots, n)) = \langle (1, 2, \dots, n) \rangle$$
    - If $g = (1, 2, \dots, k)$, then $|\{g\}| = n!/k(n-k)!$ so $|C_{S_n}(g)| = k \cdot (n-k)!$. Naturally, $g \in C_{S_n}(g)$, but so are all elements which fix $1, 2, \dots, k$ and shuffle $k+1, k+2, \dots, n$. Thus,
$$C_{S_n}((1, 2, \dots, k)) = \mathbb{Z}/k\mathbb{Z} \times S_{n-k}$$
    - Let g have cycle shape corresponding to the partition $a_1n_1 + a_2n_2 + \cdots$ where $n_1 > n_2 > \cdots$ denote cycle lengths and the a_i denote the corresponding multiplicity. We can deduce that the centralizer has order $\prod n_i^{a_i} a_i!$.
It follows from the fact that disjoint cycles commute that g commutes with each component cycle, i.e., if $g = \cdots (a_1, \dots, a_k) \cdots$, then g and $(a_1, \dots, a_k)$ commute. g therefore also commutes with all powers of each component cycle. Going even further, g commutes with all products of all powers of each component cycle, i.e., if $g = (a_1, \dots, a_k)(b_1, \dots, b_\ell)(c_1, c_2, \dots) \cdots$, then
$$C_{S_n}(g) \supset \langle (a_1, \dots, a_k), (b_1, \dots, b_\ell), (c_1, c_2, \dots), \dots \rangle$$
The group on the right above is isomorphic to $\prod (\mathbb{Z}/n_i\mathbb{Z})^{a_i}$ and thus has order $\prod n_i^{a_i}$.
What are the other elements in the centralizer that account for the $\prod a_i!$ term?? Is it the products of the powers of the cycles??

- How many elements $g \in G$ make $g \cdot x = y$ true?
  - Equivalent to asking how many $g \in G$ make $gxg^{-1} = y$.
  - Relating to before, this will be a coset of the centralizer (we need a particular solution, and then we can compose it with all homogeneous solutions).

- More on $G \circlearrowright X$ (X is the set of subsets of G).
  - Let H be a subgroup. Since $\text{Orb}(H)$ is the conjugates of H and $\text{Stab}(H) = N_G(H)$, we have by the Orbit-Stabilizer Theorem that the number of subgroups of G conjugate to H is equal to $|G|/|N_G(H)| = [G : N_G(H)]$.

6.4 Group Actions on the Quotient Group

11/4:

- Let $G \supset H$ and $X = G/H$. Consider a group action $G \circlearrowright X$ defined by $g \cdot xH = gxH$ that is transitive.
- Recall that $xH = yH$ iff $x = yh$ for some $h \in H$ iff $y^{-1}x \in H$.
- Example: Consider $G = S_4$ and $H = D_8 = \langle (1234), (13) \rangle$.
- Let $A = H$, $B = (123)H$, $C = (123)^2H$ be the three elements of $X = G/H = S_4/D_8$.
- We define a homomorphism $\phi : S_4 \to S_X = S_{\{A,B,C\}}$ by

$$\phi(\sigma) = \begin{cases} A & \mapsto \sigma A \\ B & \mapsto \sigma B \\ C & \mapsto \sigma C \end{cases}$$

  - Example: $\phi(123) = (ABC)$.
  - Example: $\phi(1234)$ is the element of $S_{\{A,B,C\}}$ that sends $A \mapsto (1234)H = H = A$, $B \mapsto (1234)(123)H = (1324)H = C$, and $C \mapsto (1234)(132)H = (14)H = B$. Thus, $\phi(1234) = (BC)$.
  - Let $x = (14)$ and $y = (123)$. Then $y^{-1}x = (321)(14) = (1432) = (1234)^{-1} \in H$, so $xH = yH$.

- Investigating $\ker \phi$.
  - $\phi((13)(24)) = (BC)^2 = e$. Thus, $(13)(24) \in \ker$ and it follows that everything conjugate to it is as well.
  - By the FIT, $S_4/\ker \phi \cong S_3$ so $|\ker \phi| = 4$.
  - Thus, $\ker \phi = \{e, (12)(34), (13)(24), (14)(23)\}$.

- Investigating the stabilizers on X.
  - $\text{Stab}(A) = H$.
    - Naturally, every $h \in H$ makes $hH = H$.
  - $\text{Stab}(B) = \text{Stab}((123)H) = (123)H(123)^{-1}$.
    - This is because any $(123)h(123)^{-1} \in (123)H(123)^{-1}$ makes

$$(123)h(123)^{-1}(123)H = (123)hH = (123)H$$

  - It follows by similar logic that $\text{Stab}(C) = (132)H(132)^{-1}$.

- Is something about H special in determining this action?
  - Suppose you take $H' = (123)H(123)^{-1}$. Is $G \circlearrowright G/H'$ the same action? The cosets of H' are $(123)H'$ and $(132)H'$. Let $A' = (132)H'$, $B' = H'$, and $C' = (123)H'$.

  - It follows that $A' = (132)(123)H(123)^{-1} = A(123)^{-1}$, $B' = (123)H(123)^{-1} = B(123)^{-1}$ and $C' = (123)(123)H(123)^{-1} = C(123)^{-1}$.
  - Conclusion: Take H, gHg^{-1}. Let A be a left coset of H. Then Ag^{-1} is a left coset of gHg^{-1}.
  - First, a coset (like A) is the set of all elements that send x to y.
  - Suppose $g \cdot x = z$. Then the coset is Ag^{-1}??

- Take G and $H = \{e\}$, $G \circlearrowright G$ the left matrices??
- Another example: Let $G = S_3 = \{e, (123), (123)^2, (12), (12)(123), (12)(123)^2\}$.
- Again, we can define a homomorphism $\phi : G \to S_G$. Call the above elements of S_3 A-F, respectively, as listed above.
  - Example: $\phi(123) = (ABC)(DFE)$.
  - Example: $\phi(12) = (AD)(BE)(CF)$.
- Let $|g| = k$, e.g., $g^{k=1}$ is distinct.
  - x, gx and $g^{k-1}x$ all distinct.
  - The cycle class of $\phi(g)$ is all k-cycles where $k = |g|\big||G|$.
  - The remark here is that if $|g| = k$, not only are $e, \dots, g^{k-1}$ distinct, but $x, \dots, g^{k-1}x$ are distinct.
- Exotic automorphism of S_6.
- Take S_5, and let X be the set of subgroups of S_5 of order 5. We may also call this the subgroups generated by 5-cycles.
- Let S_5 act on X by conjugation.
- The action is transitive.
- $|X| = 24/4 = 6$.
  - There are $\binom{5}{5}(5-1)! = 24$ elements of order 5, i.e., 5-cycles in S_5.
  - Each subgroup of S_5 of order 5 contains 4 distinct 5-cycles and e.
  - These remarks imply the above result.
- Therefore, we get a map $\phi : S_5 \to S_X$.
- Take $P = \langle (12345) \rangle$.
  - We have
  $$\text{Stab}(P) = \{g \in G \mid g \cdot P = P\} = \{g \in G \mid gPg^{-1} = P\} = N_{S_5}(P)$$
  - Since the action is transitive, $\text{Orb}(P) = X$. Thus, by the Orbit-Stabilizer theorem,
  $$|N_{S_5}(P)| = \frac{|G|}{|X|} = \frac{120}{6} = 20$$
- $\ker \phi = \{\{e\}, A_5, S_5\}$.
- By the FIT, $\{S_5, \mathbb{Z}/2\mathbb{Z}, e\}$. We can't have order ?? so we eliminate e, we can't have order 5 so we eliminate $\mathbb{Z}/2\mathbb{Z}$. Thus, the only thing is S_5. It's doing too many interesting things to have such a small image.
- We obtain an injective map from S_5 to S_6. Why do it in such a strange way? Because it also has the property that its image acts transitively on six points.

  - Remark: You can restrict to $A_5 \to S_6$, and we've seen this before where $A_5 \cong$ Do and S_6 is the pairs of opposite faces.

- So what we say is that we have an **exotic** subgroup S_5 inside S_6.
- Let's call S_5, H now. $[S_6 : H] = 6$. Thus, we have $S_6 \circlearrowright S_6/H$ by left multiplication. This action is transitive. $\mathrm{Stab}(H) = H$.
- $\psi : S_6 \to S_{S_6/H}$.
- $\ker\psi = \{1, A_6, S_6\}$, $\operatorname{im}\psi = \{S_6, \mathbb{Z}/2\mathbb{Z}, e\}$ where we know once again that the latter two can't happen.
- So we get $\psi : S_6 \to S_{S_6/H} \cong S_6$ is exotic??
  - H under this map maps to a boring S_5.
  - We know that we're sending a whole bunch of shit around (see picture).
- There will be a blog post on all of this nonsense.
- Future: Groups of order 5, groups of prime order, the Sylow theorems, and simple groups.