MATH 25700 (Honors Basic Algebra I) Notes

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Weeks

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Week 1

Motivating Group Theory

1.1 Groups as Shuffles

9/28:

- Office hours will be pooled between the two sections.
  - Our section's TA is Abhijit Mudigonda (abjihitm@uchicago.edu). His office hours will always be in JCL 267[1]. The times are...
    - Monday: 12:30-2:00 (OH).
    - Wednesday: 1:30-2:30 (PS).
    - Thursday: 12:30-2:00 (OH).
  - The other section's TA is Ray Li (rayli@uchicago.edu). His office hours will always be in Eck 17[2]. The times are...
    - Tuesday: 5:00-7:00 (OH).
    - Thursday: 4:00-5:00 (OH).
    - Thursday: 5:00-6:00 (PS).
- Textbook: Abstract Algebra. Download the PDF from LibGen.
- Weekly HW due on Monday at the beginning of class. Submit online or in person. There is a webpage w/ all the homeworks, but don't do them all at once because they're subject to change.
- Notes on math and math pedagogy.
  - There's a tendency to say here's an object, here's its properties, etc.
  - But this is not historically accurate or motivated. Calegari really gets it! Math is motivated by abstracting examples.
  - Let's not just define a group, but start with an example. This week, we will give examples of groups. In later weeks, we will establish the axiomatic framework that is really only there to understand these examples.
  - Don't stare at the page blankly waiting for inspiration when doing homework; think of examples first and test out your intuition on them to actually understand what the question means.
  - There are some hard problems; work with each other, but acknowledge our collaborators.
  - In-class midterm; final will be take-home. Calegari doesn't like timed exams.
- Today's example: Shuffling.
  - 52 cards; can be shuffled.

[1] JCL is John Crerar Library.
[2] Eckhart basement.

- Number of shuffles:
$$|\text{shuffles}| = 52! \approx 8 \times 10^{67}$$
- Properties of shuffles.
  - **Distinguished shuffle**: e, the identity shuffle, where you do nothing.
  - Shuffle once; shuffle again. The composition of two shuffles is another shuffle.
  - If you repeat the *same* shuffle enough times, the cards will come back to the same order.
    - Let σ be a shuffle, and $n \in \mathbb{N}$. Does there exist n such that
$$\sigma^n = \underbrace{\sigma \circ \cdots \circ \sigma}_{n \text{ times}} = e$$
    - Proving this: By the piegeonhole principle, if you have $\sigma^1, \ldots, \sigma^{52!+1}$, then we have repeats a, b with $52! + 1 \geq a > b \geq 1$ such that $\sigma^a = \sigma^b$. This statement is weaker than we want, though.
    - We need more tools. A shuffle is a bijection/permutation. Thus, for every σ, there exists σ^{-1}. This allows us to do this:
$$\begin{aligned} \sigma^a &= \sigma^b \\ \sigma^{-b} \circ \sigma^a &= \sigma^{-b} \circ \sigma^b \\ \sigma^{a-b} &= e \end{aligned}$$
    - This implies a bound! We get that $n \leq 52!$, so $a - b \leq 52!$.
- Define two shuffles: A and B.
  - A splits the deck into two halves (cards 1-26 and 27-52) and stacks (from the top down) the first card off of the 1-26 pile, then the first card off of the 27-52 pile, then the second card off of the 1-26 pile, then the second card off of the 27-52 pile, etc. The final order is $1, 27, 2, 28, \ldots, 26, 52$.
  - B does the same thing as A but with the first card off of the 27-52 pile. The final order is $27, 1, 28, 2, \ldots, 52, 26$.
- Computation shows that $A^8 = e$ and $B^{52} = e$.
  - For A, $2 \to 3 \to 5 \to 9 \to 17 \to 33 \to 14 \to 27 \to 2$.
  - For B, we can do the same thing but obviously the cycle is much longer.
- We shouldn't necessarily have an intuition for this right now, but in doing more examples, Calegari certainly believes we can develop it.
- First HW problem (due Friday). Can, just by using combinations of A and B, we generate any possible shuffle? Hint: Develop your intuition on a smaller value of 52.

- I really like Calegari. Very nice, relatable, not demeaning.
- **Binary operation** (on G): A map from $G \times G \to G$.
- **Group**: A mathematical object consisting of a set G and a binary operation $*$ on G satisfying the following properties.
  1. There exists an identity element $e \in G$ such that $e \times g = g \times e = g$ for all $g \in G$.
  2. For any $g \in G$, there exists $h \in G$ such that $h * g = g * h = e$.
  3. (Associativity) For any $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$.

  Denoted by $\boldsymbol{(G, *)}$.
- In the cards example, the elements of G are the shuffles and $*$ is the composition operation between two shuffles.

- Aside on shuffles: For bijections, $h(g(x)) = x$ implies $g(h(y)) = y$.
  - Proof: Let $x = h(y)$ — we can do this since h is a bijection. Then since $h(g(h(y))) = h(y)$ and h is injective, $g(h(y)) = y$. This works for all y.
- The set of shuffles, together with composition, does form a group.
- Theorem: If G is a group such that $|G| < \infty$, then any $g \in G$ has finite **order**, i.e., there exists n such that $g^n = e$.
- Lemma:
  1. The identity e is unique.
     - Let e_1, e_2 be identities. Then
     $$e_1 = e_1 * e_2 = e_2$$
  2. Inverses are unique.
     - Let h, h' be inverses of g. Then
     $$h = e * h = (h' * g) * h = h' * (g * h) = h' * e = h'$$
- Proving examples is easier, but these aren't that hard.
- If you understand everything about S_5, you'll understand everything about this course.

1.2 The Cube Group

9/30:
- Can't download `.tex` file for homework?
  - Calegari will check it.
- Detail on the homework?
  - Up to your level of confidence in what you think is clear to be true.
  - The problem is not about doing linear algebra; it's about finding some facts about linearly algebraic objects.
  - Concentrate on the new geometry of the situation.
  - Project confidence to the grader that you know what you're doing.
- The symmetries of the cube.
  - Rotational symmetries.
  - Rigid transformation.
  - Preserves lengths, angles, and lines.
  - A map from the cube to itself, i.e., ϕ : cube $\to$ cube.
  - No scaling allowed.
  - Reflectional symmetries are *not* going to be allowed for today; we will insist that the orientation is also preserved for now.
  - We want the set of all rotations and compositions of rotations. (Are compositions of rotations also rotations? We'll answer later. Yes they are.)
- Symmetries should be composable: If you compose two symmetries, you should get a third one.
  - In other words, we want the symmetries to form a group.
- We want to fix the center of the cube at the origin. Thus, a symmetry can be a linear map $M : \mathbb{R}^3 \to \mathbb{R}^3$.

  - We want it to preserve angles, i.e., orthogonality. Thus, we should assert $MM^T = I$.
  - We also want it to preserve orientation. Then we should have $\det(M) = 1$.

- **Cu**: The cube group.
- Does the permutation of faces determine M?
  - Yes.
  - Furthermore, if we know where e_1, e_2 go, then the fact that orientation and orthogonality are preserved implies that we know where e_3 goes. Thus, M is determined by two (adjacent) faces.
- An upper bound on $|\text{Cu}|$.
  - Send e_1 to one of 6 faces and send e_2 to one of the 5 remaining faces (so $|\text{Cu}| \leq 6 \cdot 5 = 30$).
  - Send e_1 to one of 6 faces and send e_2 to one of the four remaining *adjacent* faces (so $|\text{Cu}| \leq 6 \cdot 4 = 24$).
  - And, in fact, $|\text{Cu}| = 24$.
- Moreover, since the rotations of the cube are determined by permutations of the faces, we can map $\text{Cu} \hookrightarrow S_6$. Additionally, composing any permutations of the faces is the same as composing any permutations of S_6, i.e., ϕ is an **injective homomorphism** to a **subgroup** of S_6.
- We can also think about permuting the vertices.
  - 3 vertices (chosen correctly) form a basis of $\mathbb{R}^3$.
  - Thus, since there are 8 vertices, we have another map from $\text{Cu} \hookrightarrow S_8$.
  - Since we can map the first vertex to any of eight and the second to only one of three adjacent vertices, the order is $8 \cdot 3 = 24$[3].
- We now have both Cu and S_4 with order 24. Are they isomorphic?
  - One characteristic of a cube that numbers four are its four diagonals. This induces a function from $\text{Cu} \to S_4$. We now just need to prove it's bijective.
  - Let v_1, v_2, v_3, v_4 be the vertexes of one face. Then $-v_1, \ldots, -v_4$ are the vertexes of the opposite face, and the line from each v_i to $-v_i$ is a diagonal of the cube. To prove that the function is bijective, we will show that different elements of Cu map to different elements of S_4.
  - Let A and B be actions on the cube group such that

$$\begin{aligned} Bv_1 &= \pm Av_1 \\ Bv_2 &= \pm Av_2 \\ Bv_3 &= \pm Av_3 \\ Bv_4 &= \pm Av_4 \end{aligned}$$

  - Taking $C = A^{-1}B$ means that

$$\begin{aligned} Cv_1 &= \pm v_1 \\ Cv_2 &= \pm v_2 \\ Cv_3 &= \pm v_3 \\ Cv_4 &= \pm v_4 \end{aligned}$$

  - If $Cv_1 = v_1$, it implies that $Cv_i = v_i$ for $i = 2, 3, 4$.
  - Thus, A and B are distinct?

[3] We have gotten the order a different way. Deep connection to prime factorization? Edges would be $2 \cdot 12$!

Week 2

???

2.1 Groups of Low Order

10/3:

- Calegari: Nothing in particular to know for missing Friday; Adi will get me notes.
- Having explored examples, today, we're coming back down to earth to flex our axiomatic muscles.
- Distinguishing sets and binary operations.

Group	G	$*$	?
S_n	shuffles	composition	cards
$\mathrm{O}(n)$ and $\mathrm{SO}(n)$	(sp) orthogonal matrices	composition	vectors?
$\mathbb{Z}$	integers	addition	
$\mathbb{Z}/n\mathbb{Z}$	$\{0, 1, \ldots, n-1\}$	addition modulo n	

Table 2.1: Elements of a group.

  - Be careful not to confuse the shuffles and the cards; the cards are something else curious but are *not* the elements of the group.
  - Notice that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are **commutative** groups, but the shuffles (for $n > 1$) and $\mathrm{O}(n)$ are not.
  - Note that S_2, $\mathrm{O}(1)$, and $\mathbb{Z}/2\mathbb{Z}$ are all isomorphic groups.
- **Commutative** (group): A group such that for all $x, y \in G$, $x * y = y * x$. *Also known as* **Abelian**.
- Lemma (Cancellation Lemma): Let $x, y, z \in G$. Then $xy = xz$ implies $y = z$ and $yx = zx$ implies $y = z$.

  Proof. We have that

$$\begin{aligned} x * y &= x * z \\ x^{-1} * (x * y) &= x^{-1} * (x * z) && \text{Inverses exist} \\ (x^{-1} * x) * y &= (x^{-1} * x) * z && \text{Associativity} \\ e * y &= e * z \\ y &= z \end{aligned}$$

  as desired.

  The proof of the second statement is symmetric. □

  - This will be Calegari's only proof from the axioms directly.

- **Multiplication table** (for G): A table with all elements of G on the top and the side, and all binary products in it.
  - The total number of binary operations is n^{n^2}?
  - To check that a group is a group, we can write out its multiplication table and confirm pointwise that the group axioms are satisfied. However, there are also many ways to speed this process up.
  - An example of a multiplication table can be found on the right in Figure 2.1.
- **Trivial group**: The only group with $|G| = 1$, i.e., $G = \{e\}$.
- A group of $|G| = 2$ has the form $G = \{e, x\}$ where we must have $x = x^{-1}$.
  - We can find this by inspection or invoke the **Sudoku Lemma**.
  - Thus, all groups of order 2 are isomorphic.
- Lemma (Sudoku Lemma): Fix $x \in G$. Then

$$\{xg \mid g \in G\} = G = \{gx \mid g \in G\}$$

  Proof. There exists g such that $xg = y$ for x, y fixed: Choose $g = x^{-1}y$.
  y only occurs once: If $xg = y$ and $xg' = y$, transitivity and the cancellation lemma imply $g = g'$. □

  - In layman's terms, in every row and column of the multiplication table, each element of G occurs exactly once.
- Playing Sudoku, we can show that all groups of order 3 are isomorphic.

	e	x	y
e	e	x	y
x	x		
y	y		

$\longrightarrow$

	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

Figure 2.1: Playing Sudoku for $|G| = 3$.

  - Start from the left table above.
  - Notice that row 3 has a y and column 2 has an x, so by the Sudoku Lemma, e must be the element in row 3, column 2.
  - Then column 2 has e, x in it, so the entry in row 2, column 2 must by y.
  - Then row 2 has x, y in it, so the entry in row 2, column 3 must be e.
  - Then row/column 3 both have e, y in them, so the entry in row 3, column 3 must be x.
- However, we cannot play Sudoku in the same way with groups of order 4. In fact, there are multiple groups of order 4.
  - Two cases: (1) $x^2 \neq e$ so WLOG let $x^2 = y$, and (2) $a^2 = e$ for $a = x, y, z$.
    - Case 1 is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
    - Case 2 is isomorphic to the **direct product** of $\mathbb{Z}/2\mathbb{Z}$ with itself, also known as the **Klein 4-group**.
  - This should not come as a surprise: We've already encountered the very different groups S_4 and $\mathbb{Z}/24\mathbb{Z}$ of order 24.

- **Direct product**: The group whose set is the Cartesian product of the sets of groups $A = (A, *_A), B = (B, *_B)$, and whose operation is coordinate-wise multiplication. *Given by*

$$G = A \times B \qquad (a, b) *_G (a', b') = (a *_A a', b *_B b')$$

  - We can prove that $e = (e_A, e_B)$, that $(a, b)^{-1} = (a^{-1}, b^{-1})$, and that associativity holds.
  - We have that
$$|G| = |A| \cdot |B|$$

- There is only one group of order 5.
- Examples of groups of order 6: S_3, $\mathbb{Z}/6\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
  - Are there any two groups which are distinct?
    - S_3 is not commutative, but the others are, so it is distinct from them.
    - $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ are the same because order doesn't matter in the construction of the direct product.
    - $\mathbb{Z}/6\mathbb{Z}$ and the two direct products are the same because they both have elements of order 6 (i.e., a one-element generator). The cycles are:

$$\begin{aligned}
1^1 &= 1 &&= 1 & (1,1)^1 &= (1,1) &&= (1,1)\\
1^2 &= 1+1 &&= 2 & (1,1)^2 &= (1+1,1+1) &&= (2,0)\\
1^3 &= 2+1 &&= 3 & (1,1)^3 &= (2+1,0+1) &&= (0,1)\\
1^4 &= 3+1 &&= 4 & (1,1)^4 &= (0+1,1+1) &&= (1,0)\\
1^5 &= 4+1 &&= 5 & (1,1)^5 &= (1+1,0+1) &&= (2,1)\\
1^6 &= 5+1 &&= 0 & (1,1)^6 &= (2+1,1+1) &&= (0,0)\\
1^7 &= 0+1 &&= 1 & (1,1)^3 &= (0+1,0+1) &&= (1,1)
\end{aligned}$$

  - These are the only two groups of order 6.
- Continuing on, there is only 1 group with $|G| = 2047$ (which is "mostly prime" — connection between primes and number of groups?), but there are 1,774,274,116,992,170 groups of $|G| = 2048 = 2^{11}$.
- Conclusion: The arithmetic of $|G|$ has an impact on the structure of G.

2.2 Characteristics of the Symmetric Group

10/5:

- **Symmetric group** (on n letters): The set of all bijections from the set of numbers $\{1, \ldots, n\}$ to itself, whose operation is function composition. *Denoted by* $\boldsymbol{S_n}$.
  - Convention: Denote elements of S_n not by f but by σ, τ.
  - $\sigma\tau$ means do τ first and then σ.
  - $|S_n| = n!$.
- One of the first challenges we encounter when defining new objects is a notational one.
  - We could define a function with a table, but cycle notation is easier.
- ***k*-cycle**: The bijection

$$m \mapsto \begin{cases} a_{i+1} & m = a_i,\ i \neq k \\ a_1 & m = a_k \\ m & m \neq a_i \end{cases}$$

  in S_n, where $a_1, \ldots, a_k$ are distinct elements of $[n]$. *Denoted by* $\boldsymbol{(a_1, a_2, \ldots, a_k)}$.

- If σ is a k-cycle, then the order of σ is k.
- There are k ways to write down the same k-cycle.
  - For example, $(i,j) = (j,i)$ and $(a,b,c) = (b,c,a) = (c,a,b)$.
- All 1-cycles are the identity e.
- Combinatorics: How many k-cycles are there in S_n?
  - $k = 1$: Just one – (e).
  - $k = 2$: $\binom{n}{2}$.
  - $k = 3$: $\binom{n}{3} \cdot 2$.
    - ➢ We must first choose 3 of the n possible elements to be manipulated by the k-cycle.
    - ➢ But then we can send a_1 to a_2 or a_3, so that's an additional two choices beyond just a selection of 3 elements. Once we send a_1 to a_2 or a_3, the rest of the cycle is determined, so we need not augment any more.
  - k: $\binom{n}{k} \cdot (k-1)! = \frac{n!}{(n-k)!k}$.
    - ➢ As before, we must choose k of the n possible elements to be manipulated by the k-cycle.
    - ➢ However, here, there are $k-1$ possibilities to which we can send a_1, so we need to multiply by that. Once we've determined $\sigma(a_1)$, there are $k-2$ possibilities to which we can send $\sigma(a_1)$. This pattern naturally continues, and we end up needing to correct $\binom{n}{k}$ by $(k-1)!$.

- Proposition: Every $\sigma \in S_n$ can be written as a product/composition of disjoint cycles. Moreover, disjoint cycles commute.

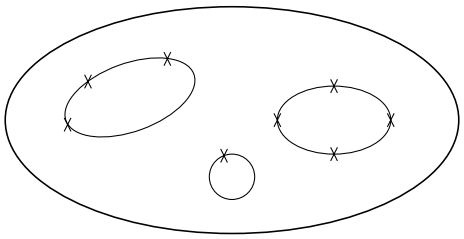

Figure 2.2: Decomposing σ into disjoint cycles.

  - The idea behind this proposition is that every element will cycle back to itself eventually, and you can't get to elements of one cycle if you're not in the cycle (so all cycles are disjoint).
  - Every permutation can be visualized by ordering the n letters in a set in $\mathbb{R}^2$ and connecting all disjoint cycles (think a circle full of oriented circles/loops/cycles).

- Composing cycles. See what the right one does and then the left one. Canonically, start with 1.

- Proposition: The cycle decomposition of σ is unique up to...
  - The ordering of the disjoint cycles;
  - Cycle permutations of each cycle;
  - Include/exclude 1-cycles.

  Moreover, $|\sigma|$ is the least common multiple of the cycle lengths.

- How many elements in S_6 have a cycle shape that looks like $(x,x)(x,x)(x,x)$?
  - It is
$$\frac{6!}{2^3 \cdot 3!} = 15$$
  - Rationale: See PSet 2, Q1a.

- The cycle decompositions of all elements in S_4.

$(1,2,3,4)$	$(1,2,3)$	$(1,2)$	$(1,2)(3,4)$	e
$(1,2,4,3)$	$(1,3,2)$	$(1,3)$	$(1,3)(2,4)$	
$(1,3,2,4)$	$(1,2,4)$	$(1,4)$	$(1,4)(2,3)$	
$(1,3,4,2)$	$(1,4,2)$	$(2,3)$		
$(1,4,2,3)$	$(1,3,4)$	$(2,4)$		
$(1,4,3,2)$	$(1,4,3)$			
	$(2,3,4)$			
	$(2,4,3)$			

Table 2.2: S_4 cycle decompositions.

- **Conjugate** (elements x, y): Two elements $x, y \in G$ a group for which there exists $g \in G$ such that $y = g \cdot x \cdot g^{-1}$. *Denoted by* $\boldsymbol{x \sim y}$.
- Lemma: Conjugacy is an equivalence relation.

  (I) $x \sim x$.

  Proof. $x = exe^{-1}$. □

  (II) If $y \sim x$, then $x \sim y$.

  Proof. Take

$$\begin{aligned} y &= gxg^{-1} \\ g^{-1}y(g^{-1})^{-1} &= x \end{aligned}$$

  □

  (III) If $x \sim y$ and $y \sim z$, then $x \sim z$.

  Proof. Suppose $y = gxg^{-1}$ and $z = hyh^{-1}$. Then

$$z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$$

  □

- **Conjugacy class**: A subset of G containing all $g \in G$ which are conjugate to a certain $x \in G$.
- Straightforward: Not necessarily obvious, but there's nothing really tricky going on.
  - The joke about the mathematician who says something is obvious, someone asks why?, he thinks for 20 minutes, and then says it's obvious.
- Why is conjugacy important?
  - In linear algebra, we've seen it with similar matrices.
    - Same linear map in a different basis is the same as conjugating the matrix of the map in one basis with the change of basis matrix.
  - Conjugacy tells us that a set of objects are, in some way, the same.