Week 6

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6.1   Examples of Group Actions

- Today: A number of interesting group actions.

- **Left action** (of G on X): A group action of the form $g \cdot x$ (as opposed to $x \cdot g$).

- Let G be a group, and let $X = G$. Take $g \cdot x = gx$.

  - Axiom confirmation.
    1. $e \cdot x = ex = x$.
    2. $g \cdot (h \cdot x) = ghx = gh \cdot x$.
  - Let $e \in X$. Then $\mathrm{Orb}(e) = X$. In particular, this means that the action is transitive.
  - $\mathrm{Stab}(x) = \{g \in G \mid gx = x\} = \{e\}$ for $x \in X$ arbitrary, in general.
  - $\ker = \{e\}$. This also follows from the above. Thus, the action is faithful.

- Corollary: Let G be a finite group. Then G is isomorphic to a subgroup of S_n for some n. We may take $n = |G|$.

  - Construction: We invoke the proposition from last lecture. In particular, we know that the action $G \curvearrowright G$ implies the existence of a homomorphism $\phi : G \to S_G$ defined by $g \mapsto \psi_g$.
  - The map in the above construction has trivial kernel. By the FIT, $G/\ker \cong \mathrm{im}\,\phi$. Combining these results, we obtain $G \cong G/\ker \cong \mathrm{im}\,\phi \leq S_n$.
  - Applying this construction to S_3, we deduce that $S_3 \leq S_6$.

- $\mathrm{SO}(2) \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}^\infty$.

  - In infinite cases, you usually want to consider some other topological things that disappear in the finite case.

- Let G be a group and take $X = G$ again. We can also consider $g \cdot x = gxg^{-1}$.

  - Axioms.
    1. $e \cdot x = exe^{-1} = x$.
    2. $g \cdot (h \cdot x) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = gh \cdot x$.
  - $\mathrm{Orb}(e) = \{e\}$; not transitive if $|G| > 1$.
  - Let $x \in X$. Then $\mathrm{Orb}(x)$ is the conjugacy class of x.
  - $\mathrm{Stab}(x) = C_G(x)$.
  - $\ker = Z(G)$. Thus, the group action is faithful iff the center is trivial. Abelian implies not faithful.

- A nice thing about these constructions is that they cast other constructions we've encountered in the more general language of group actions.

- **Right actions** are even nastier than left cosets and right cosets, so Calegari will not mention them again.

    - $g \cdot x = x \cdot g^{-1}$ and $g \cdot (h \cdot x) = (x \cdot h^{-1}) \cdot g^{-1}$.

- Let $G = G$, X be the subgroups of G. $g \cdot H = gHg^{-1}$.

    - Note that $H \leq G$ does indeed imply that $gHg^{-1} \leq G$. In particular, . . .
        - H is nonempty (contains at least e), so $gHg^{-1} \supset \{geg^{-1}\}$ is nonempty;
        - $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$ imply that $gh_1g^{-1}gh_2g^{-1} = g(h_1h_2)g^{-1} \in gHg^{-1}$;
        - $ghg^{-1} \in gHg^{-1}$ has inverse $gh^{-1}g^{-1} \in gHg^{-1}$.
    - Axioms (entirely analogous to the last example).
    - $\mathrm{Orb}(H)$ is the "conjugates" of H.
    - $\mathrm{Stab}(H) = N_G(H)$.
    - ker =?. We know that $Z(G) \subset$ ker. The conclusion is that there is not a nice definition for the kernel other than the intersections of the stabilizers/normalizers.
        - ...
        - If any $H \triangleleft G$ is normal, and $x \in G$ had order 2, then $\langle x \rangle \triangleleft G$, meaning that $gxg^{-1} \in \langle x \rangle$, i.e., $x \in Z(G)$, so this rules out D_8??

- Fix G and $H \leq G$. Let $X = G/H$ (not assuming $H \triangleleft G$, so we know that G/H is the set of left cosets but it is not a group in general). Define $g \cdot xH = gxH$.

    - We have $g \cdot xhH = gxhH$.
    - Orbit: $\mathrm{Orb}(eH) = X$.
    - Stabilizer: $\mathrm{Stab}(eH) = H$.
        - $\mathrm{Stab}(gH) = gHg^{-1}$.
        - This is because $(ghg^{-1})gH = ghH = gH$.
        - Go to the more general case $G \curvearrowright X$, $\mathrm{Stab}(x) = H$. Then $gHg^{-1} \subset \mathrm{Stab}(g \cdot x)$??
    - Transitive: Yes (see orbits).
    - Faithful: If H is normal, no. If H contains a normal subgroup, no. Maybe yes.
    - Kernel: If H is normal, then ker $= H$. In general, ker $= \bigcap_{g \in G} gHg^{-1}$ (the largest normal subgroup of H).

- Takeaway: General constructions allow us to see things we've already done.

- Next time: The most useful theorem of the course, that provides lots of information on relations between objects.