7 Broader Classes of Groups

11/28:

1. Suppose that $\mathbb{Z}/m\mathbb{Z}$ is a subgroup of S_n for some $n, m > 2$. Prove that D_{2m} is also a subgroup of S_n.

2. Let $G = \mathrm{SL}_2(\mathbb{F}_3)$. Prove that the subgroup

$$H = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \right\rangle$$

is isomorphic to the quaternion group Q (where i, j, k map to the given matrices). Deduce that $\mathrm{SL}_2(\mathbb{F}_3)$ and S_4 are not isomorphic.

3. Let G be a group, and let $N \subset G$ be the subgroup generated by the elements $xyx^{-1}y^{-1}$ for all pairs $x, y \in G$. Prove that N is a normal subgroup, and that G/N is abelian.

4. Compute the order of the following groups as well as a set of generators.
   (a) The centralizer of (12345) in A_7.
   (b) The centralizer of $((12), (123))$ in $S_5 \times S_5$.
   (c) The normalizer of $H = \langle (12), (34), (56), (78) \rangle$ in S_8.

5. **Projective Linear Groups Over Finite Fields.** Let p be prime, and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Note that one can add and multiply elements of $\mathbb{F}_p$. Let $\mathrm{GL}_2(\mathbb{F}_p)$ be the group of 2×2 invertible matrices over $\mathbb{F}_p$, and let $\mathrm{SL}_2(\mathbb{F}_p) \subset \mathrm{GL}_2(\mathbb{F}_p)$ denote the subgroup of matrices of determinant one.
   (a) There are $p^2 - 1$ non-zero vectors $v \in \mathbb{F}_p^2$. Let a "line" $\ell = [v] \subset \mathbb{F}_p^2$ denote the scalar multiples λv of a non-zero vector v. Prove that the set X of lines has cardinality $|X| = p + 1$.
   (b) Prove that $\mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{GL}_2(\mathbb{F}_p)$ act naturally on X by $g \cdot [v] = [g \cdot v]$.
   (c) Prove that this action is transitive for both $\mathrm{GL}_2(\mathbb{F}_p)$ and $\mathrm{SL}_2(\mathbb{F}_p)$.
   (d) Prove that the kernel of the action consists precisely of the scalar matrices λI in either $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{GL}_2(\mathbb{F}_p)$.
   (e) Let $\mathrm{PGL}_2(\mathbb{F}_p)$ and $\mathrm{PSL}_2(\mathbb{F}_p)$ denote the quotient of G and H by the subgroup of scalar matrices. Prove that $|\mathrm{PGL}_2(\mathbb{F}_p)| = (p^2 - 1)p$ and $|\mathrm{PSL}_2(\mathbb{F}_p)| = 6$ if $p = 2$ and $\frac{1}{2}(p^2 - 1)p$ otherwise.
   (f) Prove that $\mathrm{PGL}_2(\mathbb{F}_2) = \mathrm{PSL}_2(\mathbb{F}_2) = S_3$.
   (g) Prove that $\mathrm{PGL}_2(\mathbb{F}_3) = S_4$ and $\mathrm{PSL}_2(\mathbb{F}_3) = A_4$. (Compare with Question 2.)
   (h) Prove that $\mathrm{PSL}_2(\mathbb{F}_5) = A_5$ and $\mathrm{PGL}_2(\mathbb{F}_5) = S_5$. (Hint: Using that A_6 is simple, prove that any index 6 subgroup of A_6 or S_6 is A_5 or S_5, respectively.)