

# MATH 25700 (Honors Basic Algebra I) Notes

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# Week 1

## Motivating Group Theory

### 1.1 Groups as Shuffles

- 9/28:
- Office hours will be pooled between the two sections.
    - Our section's TA is Abhijit Mudigonda (abhijitm@uchicago.edu). His office hours will always be in JCL 267<sup>[1]</sup>. The times are...
      - Monday: 12:30-2:00 (OH).
      - Wednesday: 1:30-2:30 (PS).
      - Thursday: 12:30-2:00 (OH).
    - The other section's TA is Ray Li (rayli@uchicago.edu). His office hours will always be in Eck 17<sup>[2]</sup>. The times are...
      - Tuesday: 5:00-7:00 (OH).
      - Thursday: 4:00-5:00 (OH).
      - Thursday: 5:00-6:00 (PS).
  - Textbook: Abstract Algebra. Download the PDF from LibGen.
  - Weekly HW due on Monday at the beginning of class. Submit online or in person. There is a webpage w/ all the homeworks, but don't do them all at once because they're subject to change.
  - Notes on math and math pedagogy.
    - There's a tendency to say here's an object, here's its properties, etc.
    - But this is not historically accurate or motivated. Calegari really gets it! Math is motivated by abstracting examples.
    - Let's not just define a group, but start with an example. This week, we will give examples of groups. In later weeks, we will establish the axiomatic framework that is really only there to understand these examples.
    - Don't stare at the page blankly waiting for inspiration when doing homework; think of examples first and test out your intuition on them to actually understand what the question means.
    - There are some hard problems; work with each other, but acknowledge our collaborators.
    - In-class midterm; final will be take-home. Calegari doesn't like timed exams.
  - Today's example: Shuffling.
    - 52 cards; can be shuffled.

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<sup>1</sup>JCL is John Crerar Library.

<sup>2</sup>Eckhart basement.

- Number of shuffles:

$$|\text{shuffles}| = 52! \approx 8 \times 10^{67}$$

- Properties of shuffles.

- **Distinguished shuffle:**  $e$ , the identity shuffle, where you do nothing.
- Shuffle once; shuffle again. The composition of two shuffles is another shuffle.
- If you repeat the *same* shuffle enough times, the cards will come back to the same order.
  - Let  $\sigma$  be a shuffle, and  $n \in \mathbb{N}$ . Does there exist  $n$  such that

$$\sigma^n = \underbrace{\sigma \circ \cdots \circ \sigma}_{n \text{ times}} = e$$

- Proving this: By the pigeonhole principle, if you have  $\sigma^1, \dots, \sigma^{52!+1}$ , then we have repeats  $a, b$  with  $52! + 1 \geq a > b \geq 1$  such that  $\sigma^a = \sigma^b$ . This statement is weaker than we want, though.
- We need more tools. A shuffle is a bijection/permutation. Thus, for every  $\sigma$ , there exists  $\sigma^{-1}$ . This allows us to do this:

$$\begin{aligned}\sigma^a &= \sigma^b \\ \sigma^{-b} \circ \sigma^a &= \sigma^{-b} \circ \sigma^b \\ \sigma^{a-b} &= e\end{aligned}$$

- This implies a bound! We get that  $n \leq 52!$ , so  $a - b \leq 52!$ .

- Define two shuffles:  $A$  and  $B$ .

- $A$  splits the deck into two halves (cards 1-26 and 27-52) and stacks (from the top down) the first card off of the 1-26 pile, then the first card off of the 27-52 pile, then the second card off of the 1-26 pile, then the second card off of the 27-52 pile, etc. The final order is 1, 27, 2, 28,  $\dots$ , 26, 52.
- $B$  does the same thing as  $A$  but with the first card off of the 27-52 pile. The final order is 27, 1, 28, 2,  $\dots$ , 52, 26.

- Computation shows that  $A^8 = e$  and  $B^{52} = e$ .

- For  $A$ ,  $2 \rightarrow 3 \rightarrow 5 \rightarrow 9 \rightarrow 17 \rightarrow 33 \rightarrow 14 \rightarrow 27 \rightarrow 2$ .
- For  $B$ , we can do the same thing but obviously the cycle is much longer.

- We shouldn't necessarily have an intuition for this right now, but in doing more examples, Calegari certainly believes we can develop it.
- First HW problem (due Friday). Can, just by using combinations of  $A$  and  $B$ , we generate any possible shuffle? Hint: Develop your intuition on a smaller value of 52.

- I really like Calegari. Very nice, relatable, not demeaning.

- **Binary operation** (on  $G$ ): A map from  $G \times G \rightarrow G$ .

- **Group:** A mathematical object consisting of a set  $G$  and a binary operation  $*$  on  $G$  satisfying the following properties.

1. There exists an identity element  $e \in G$  such that  $e \times g = g \times e = g$  for all  $g \in G$ .
2. For any  $g \in G$ , there exists  $h \in G$  such that  $h * g = g * h = e$ .
3. (Associativity) For any  $g_1, g_2, g_3 \in G$ ,  $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$ .

Denoted by  $(G, *)$ .

- In the cards example, the elements of  $G$  are the shuffles and  $*$  is the composition operation between two shuffles.

- Aside on shuffles: For bijections,  $h(g(x)) = x$  implies  $g(h(y)) = y$ .
  - Proof: Let  $x = h(y)$  — we can do this since  $h$  is a bijection. Then since  $h(g(h(y))) = h(y)$  and  $h$  is injective,  $g(h(y)) = y$ . This works for all  $y$ .
- The set of shuffles, together with composition, does form a group.
- Theorem: If  $G$  is a group such that  $|G| < \infty$ , then any  $g \in G$  has finite **order**, i.e., there exists  $n$  such that  $g^n = e$ .
- Lemma:
  1. The identity  $e$  is unique.
    - Let  $e_1, e_2$  be identities. Then
 
$$e_1 = e_1 * e_2 = e_2$$
  2. Inverses are unique.
    - Let  $h, h'$  be inverses of  $g$ . Then
 
$$h = e * h = (h' * g) * h = h' * (g * h) = h' * e = h'$$
- Proving examples is easier, but these aren't that hard.
- If you understand everything about  $S_5$ , you'll understand everything about this course.

## 1.2 The Cube Group

9/30:

- Can't download .tex file for homework?
  - Calegari will check it.
- Detail on the homework?
  - Up to your level of confidence in what you think is clear to be true.
  - The problem is not about doing linear algebra; it's about finding some facts about linearly algebraic objects.
  - Concentrate on the new geometry of the situation.
  - Project confidence to the grader that you know what you're doing.
- The symmetries of the cube.
  - Rotational symmetries.
  - Rigid transformation.
  - Preserves lengths, angles, and lines.
  - A map from the cube to itself, i.e.,  $\phi : \text{cube} \rightarrow \text{cube}$ .
  - No scaling allowed.
  - Reflectional symmetries are *not* going to be allowed for today; we will insist that the orientation is also preserved for now.
  - We want the set of all rotations and compositions of rotations. (Are compositions of rotations also rotations? We'll answer later. Yes they are.)
- Symmetries should be composable: If you compose two symmetries, you should get a third one.
  - In other words, we want the symmetries to form a group.
- We want to fix the center of the cube at the origin. Thus, a symmetry can be a linear map  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

- We want it to preserve angles, i.e., orthogonality. Thus, we should assert  $MM^T = I$ .
- We also want it to preserve orientation. Then we should have  $\det(M) = 1$ .
- **Cu**: The cube group.
- Does the permutation of faces determine  $M$ ?
  - Yes.
  - Furthermore, if we know where  $e_1, e_2$  go, then the fact that orientation and orthogonality are preserved implies that we know where  $e_3$  goes. Thus,  $M$  is determined by two (adjacent) faces.
- An upper bound on  $|\text{Cu}|$ .
  - Send  $e_1$  to one of 6 faces and send  $e_2$  to one of the 5 remaining faces (so  $|\text{Cu}| \leq 6 \cdot 5 = 30$ ).
  - Send  $e_1$  to one of 6 faces and send  $e_2$  to one of the four remaining *adjacent* faces (so  $|\text{Cu}| \leq 6 \cdot 4 = 24$ ).
  - And, in fact,  $|\text{Cu}| = 24$ .
- Moreover, since the rotations of the cube are determined by permutations of the faces, we can map  $\text{Cu} \hookrightarrow S_6$ . Additionally, composing any permutations of the faces is the same as composing any permutations of  $S_6$ , i.e.,  $\phi$  is an **injective homomorphism** to a **subgroup** of  $S_6$ .
- We can also think about permuting the vertices.
  - 3 vertices (chosen correctly) form a basis of  $\mathbb{R}^3$ .
  - Thus, since there are 8 vertices, we have another map from  $\text{Cu} \hookrightarrow S_8$ .
  - Since we can map the first vertex to any of eight and the second to only one of three adjacent vertices, the order is  $8 \cdot 3 = 24^{[3]}$ .
- We now have both  $\text{Cu}$  and  $S_4$  with order 24. Are they isomorphic?
  - One characteristic of a cube that numbers four are its four diagonals. This induces a function from  $\text{Cu} \rightarrow S_4$ . We now just need to prove it's bijective.
  - Let  $v_1, v_2, v_3, v_4$  be the vertexes of one face. Then  $-v_1, \dots, -v_4$  are the vertexes of the opposite face, and the line from each  $v_i$  to  $-v_i$  is a diagonal of the cube. To prove that the function is bijective, we will show that different elements of  $\text{Cu}$  map to different elements of  $S_4$ .
  - Let  $A$  and  $B$  be actions on the cube group such that
 
$$\begin{aligned} Bv_1 &= \pm Av_1 \\ Bv_2 &= \pm Av_2 \\ Bv_3 &= \pm Av_3 \\ Bv_4 &= \pm Av_4 \end{aligned}$$
  - Taking  $C = A^{-1}B$  means that
 
$$\begin{aligned} Cv_1 &= \pm v_1 \\ Cv_2 &= \pm v_2 \\ Cv_3 &= \pm v_3 \\ Cv_4 &= \pm v_4 \end{aligned}$$
  - If  $Cv_1 = v_1$ , it implies that  $Cv_i = v_i$  for  $i = 2, 3, 4$ .
  - Thus,  $A$  and  $B$  are distinct?

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<sup>3</sup>We have gotten the order a different way. Deep connection to prime factorization? Edges would be  $2 \cdot 12!$



## Week 2

# Group Theory Foundations

## 2.1 Groups of Low Order

- 10/3:
- Calegari: Nothing in particular to know for missing Friday; Adi will get me notes.
  - Having explored examples, today, we're coming back down to earth to flex our axiomatic muscles.
  - Distinguishing sets and binary operations.

Group	$G$	$*$	?
$S_n$	shuffles	composition	cards
$O(n)$ and $SO(n)$	(sp) orthogonal matrices	composition	vectors?
$\mathbb{Z}$	integers	addition	
$\mathbb{Z}/n\mathbb{Z}$	$\{0, 1, \dots, n-1\}$	addition modulo $n$	

Table 2.1: Elements of a group.

- Be careful not to confuse the shuffles and the cards; the cards are something else curious but are *not* the elements of the group.
- Notice that  $\mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z}$  are **commutative** groups, but the shuffles (for  $n > 1$ ) and  $O(n)$  are not.
- Note that  $S_2$ ,  $O(1)$ , and  $\mathbb{Z}/2\mathbb{Z}$  are all isomorphic groups.
- **Commutative** (group): A group such that for all  $x, y \in G$ ,  $x * y = y * x$ . Also known as **Abelian**.
- Lemma (Cancellation Lemma): Let  $x, y, z \in G$ . Then  $xy = xz$  implies  $y = z$  and  $yx = zx$  implies  $y = z$ .

*Proof.* We have that

$$\begin{aligned}
 x * y &= x * z \\
 x^{-1} * (x * y) &= x^{-1} * (x * z) && \text{Inverses exist} \\
 (x^{-1} * x) * y &= (x^{-1} * x) * z && \text{Associativity} \\
 e * y &= e * z \\
 y &= z
 \end{aligned}$$

as desired.

The proof of the second statement is symmetric. □

- This will be Calegari's only proof from the axioms directly.

- **Multiplication table** (for  $G$ ): A table with all elements of  $G$  on the top and the side, and all binary products in it.
  - The total number of binary operations is  $n^2$ ?
  - To check that a group is a group, we can write out its multiplication table and confirm pointwise that the group axioms are satisfied. However, there are also many ways to speed this process up.
  - An example of a multiplication table can be found on the right in Figure 2.1.
- **Trivial group**: The only group with  $|G| = 1$ , i.e.,  $G = \{e\}$ .
- A group of  $|G| = 2$  has the form  $G = \{e, x\}$  where we must have  $x = x^{-1}$ .
  - We can find this by inspection or invoke the **Sudoku Lemma**.
  - Thus, all groups of order 2 are isomorphic.
- Lemma (Sudoku Lemma): Fix  $x \in G$ . Then

$$\{xg \mid g \in G\} = G = \{gx \mid g \in G\}$$

*Proof.* There exists  $g$  such that  $xg = y$  for  $x, y$  fixed: Choose  $g = x^{-1}y$ .

$y$  only occurs once: If  $xg = y$  and  $xg' = y$ , transitivity and the cancellation lemma imply  $g = g'$ .  $\square$

- In layman's terms, in every row and column of the multiplication table, each element of  $G$  occurs exactly once.
- Playing Sudoku, we can show that all groups of order 3 are isomorphic.

	$e$	$x$	$y$
$e$	$e$	$x$	$y$
$x$	$x$		
$y$	$y$		

→

	$e$	$x$	$y$
$e$	$e$	$x$	$y$
$x$	$x$	$y$	$e$
$y$	$y$	$e$	$x$

Figure 2.1: Playing Sudoku for  $|G| = 3$ .

- Start from the left table above.
- Notice that row 3 has a  $y$  and column 2 has an  $x$ , so by the Sudoku Lemma,  $e$  must be the element in row 3, column 2.
- Then column 2 has  $e, x$  in it, so the entry in row 2, column 2 must be  $y$ .
- Then row 2 has  $x, y$  in it, so the entry in row 2, column 3 must be  $e$ .
- Then row/column 3 both have  $e, y$  in them, so the entry in row 3, column 3 must be  $x$ .
- However, we cannot play Sudoku in the same way with groups of order 4. In fact, there are multiple groups of order 4.
  - Two cases: (1)  $x^2 \neq e$  so WLOG let  $x^2 = y$ , and (2)  $a^2 = e$  for  $a = x, y, z$ .
    - Case 1 is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ .
    - Case 2 is isomorphic to the **direct product** of  $\mathbb{Z}/2\mathbb{Z}$  with itself, also known as the **Klein 4-group**.
  - This should not come as a surprise: We've already encountered the very different groups  $S_4$  and  $\mathbb{Z}/24\mathbb{Z}$  of order 24.

- **Direct product:** The group whose set is the Cartesian product of the sets of groups  $A = (A, *_A), B = (B, *_B)$ , and whose operation is coordinate-wise multiplication. *Given by*

$$G = A \times B \qquad (a, b) *_G (a', b') = (a *_A a', b *_B b')$$

- We can prove that  $e = (e_A, e_B)$ , that  $(a, b)^{-1} = (a^{-1}, b^{-1})$ , and that associativity holds.
- We have that

$$|G| = |A| \cdot |B|$$

- There is only one group of order 5.
- Examples of groups of order 6:  $S_3, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}), (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ .
  - Are there any two groups which are distinct?
    - $S_3$  is not commutative, but the others are, so it is distinct from them.
    - $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$  and  $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$  are the same because order doesn't matter in the construction of the direct product.
    - $\mathbb{Z}/6\mathbb{Z}$  and the two direct products are the same because they both have elements of order 6 (i.e., a one-element generator). The cycles are:

$1^1 = 1$	$= 1$	$(1, 1)^1 = (1, 1)$	$= (1, 1)$
$1^2 = 1 + 1 = 2$		$(1, 1)^2 = (1 + 1, 1 + 1) = (2, 0)$	
$1^3 = 2 + 1 = 3$		$(1, 1)^3 = (2 + 1, 0 + 1) = (0, 1)$	
$1^4 = 3 + 1 = 4$		$(1, 1)^4 = (0 + 1, 1 + 1) = (1, 0)$	
$1^5 = 4 + 1 = 5$		$(1, 1)^5 = (1 + 1, 0 + 1) = (2, 1)$	
$1^6 = 5 + 1 = 0$		$(1, 1)^6 = (2 + 1, 1 + 1) = (0, 0)$	
$1^7 = 0 + 1 = 1$		$(1, 1)^3 = (0 + 1, 0 + 1) = (1, 1)$	

- These are the only two groups of order 6.
- Continuing on, there is only 1 group with  $|G| = 2047$  (which is “mostly prime” — connection between primes and number of groups?), but there are 1,774,274,116,992,170 groups of  $|G| = 2048 = 2^{11}$ .
- Conclusion: The arithmetic of  $|G|$  has an impact on the structure of  $G$ .

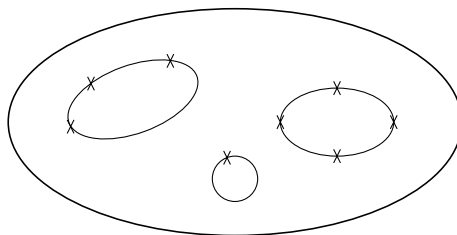
## 2.2 The Symmetric Group

- 10/5:
- **Symmetric group** (on  $n$  letters): The set of all bijections from the set of numbers  $\{1, \dots, n\}$  to itself, whose operation is function composition. *Denoted by  $S_n$ .*
    - Convention: Denote elements of  $S_n$  not by  $f$  but by  $\sigma, \tau$ .
    - $\sigma\tau$  means do  $\tau$  first and then  $\sigma$ .
    - $|S_n| = n!$ .
  - One of the first challenges we encounter when defining new objects is a notational one.
    - We could define a function with a table, but cycle notation is easier.
  - **$k$ -cycle:** The bijection

$$m \mapsto \begin{cases} a_{i+1} & m = a_i, i \neq k \\ a_1 & m = a_k \\ m & m \neq a_i \end{cases}$$

in  $S_n$ , where  $a_1, \dots, a_k$  are distinct elements of  $[n]$ . *Denoted by  $(a_1, a_2, \dots, a_k)$ .*

- If  $\sigma$  is a  $k$ -cycle, then the order of  $\sigma$  is  $k$ .
- There are  $k$  ways to write down the same  $k$ -cycle.
  - For example,  $(i, j) = (j, i)$  and  $(a, b, c) = (b, c, a) = (c, a, b)$ .
- All 1-cycles are the identity  $e$ .
- Combinatorics: How many  $k$ -cycles are there in  $S_n$ ?
  - $k = 1$ : Just one – ( $e$ ).
  - $k = 2$ :  $\binom{n}{2}$ .
  - $k = 3$ :  $\binom{n}{3} \cdot 2$ .
    - We must first choose 3 of the  $n$  possible elements to be manipulated by the  $k$ -cycle.
    - But then we can send  $a_1$  to  $a_2$  or  $a_3$ , so that's an additional two choices beyond just a selection of 3 elements. Once we send  $a_1$  to  $a_2$  or  $a_3$ , the rest of the cycle is determined, so we need not augment any more.
  - $k$ :  $\binom{n}{k} \cdot (k-1)! = \frac{n!}{(n-k)!k}$ .
    - As before, we must choose  $k$  of the  $n$  possible elements to be manipulated by the  $k$ -cycle.
    - However, here, there are  $k-1$  possibilities to which we can send  $a_1$ , so we need to multiply by that. Once we've determined  $\sigma(a_1)$ , there are  $k-2$  possibilities to which we can send  $\sigma(a_1)$ . This pattern naturally continues, and we end up needing to correct  $\binom{n}{k}$  by  $(k-1)!$ .
- Proposition: Every  $\sigma \in S_n$  can be written as a product/composition of disjoint cycles. Moreover, disjoint cycles commute.

Figure 2.2: Decomposing  $\sigma$  into disjoint cycles.

- The idea behind this proposition is that every element will cycle back to itself eventually, and you can't get to elements of one cycle if you're not in the cycle (so all cycles are disjoint).
- Every permutation can be visualized by ordering the  $n$  letters in a set in  $\mathbb{R}^2$  and connecting all disjoint cycles (think a circle full of oriented circles/loops/cycles).
- Composing cycles. See what the right one does and then the left one. Canonically, start with 1.
- Proposition: The cycle decomposition of  $\sigma$  is unique up to...
  - The ordering of the disjoint cycles;
  - Cycle permutations of each cycle;
  - Include/exclude 1-cycles.

Moreover,  $|\sigma|$  is the least common multiple of the cycle lengths.

- How many elements in  $S_6$  have a cycle shape that looks like  $(x, x)(x, x)(x, x)$ ?
  - It is

$$\frac{6!}{2^3 \cdot 3!} = 15$$

- Rationale: See PSet 2, Q1a.

- The cycle decompositions of all elements in  $S_4$ .

(1, 2, 3, 4)	(1, 2, 3)	(1, 2)	(1, 2)(3, 4)	e
(1, 2, 4, 3)	(1, 3, 2)	(1, 3)	(1, 3)(2, 4)	
(1, 3, 2, 4)	(1, 2, 4)	(1, 4)	(1, 4)(2, 3)	
(1, 3, 4, 2)	(1, 4, 2)	(2, 3)		
(1, 4, 2, 3)	(1, 3, 4)	(2, 4)		
(1, 4, 3, 2)	(1, 4, 3)			
	(2, 3, 4)			
	(2, 4, 3)			

Table 2.2:  $S_4$  cycle decompositions.

- **Conjugate** (elements  $x, y$ ): Two elements  $x, y \in G$  a group for which there exists  $g \in G$  such that  $y = g \cdot x \cdot g^{-1}$ . Denoted by  $x \sim y$ .
- Lemma: Conjugacy is an equivalence relation.

(I)  $x \sim x$ .

*Proof.*  $x = exe^{-1}$ . □

(II) If  $y \sim x$ , then  $x \sim y$ .

*Proof.* Take

$$y = gxg^{-1}$$

$$g^{-1}y(g^{-1})^{-1} = x$$

□

(III) If  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ .

*Proof.* Suppose  $y = gxg^{-1}$  and  $z = hyh^{-1}$ . Then

$$z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$$

□

- **Conjugacy class** (of  $x$ ): A subset of  $G$  containing all  $g \in G$  which are conjugate to a certain  $x \in G$ . Given by

$$\{g \in G \mid g \sim x\}$$

- Straightforward: Not necessarily obvious, but there's nothing really tricky going on.
  - The joke about the mathematician who says something is obvious, someone asks why?, he thinks for 20 minutes, and then says it's obvious.
- Why is conjugacy important?
  - In linear algebra, we've seen it with similar matrices.
    - Same linear map in a different basis is the same as conjugating the matrix of the map in one basis with the change of basis matrix.
  - Conjugacy tells us that a set of objects are, in some way, the same.

## 2.3 Conjugacy

10/7:

- You can request one extension per quarter on homework (possibly more if you have a really good reason) for sickness, etc., no questions asked. Email your TA to secure this extension.
- Last time, we began covering conjugacy.
  - Conjugacy classes.
  - Conjugacy defines an equivalence relation on  $G$ .
  - $G = \bigsqcup$  conjugacy classes<sup>[1]</sup>.
- More on conjugacy today.
- The conjugacy class of  $e$  is  $\{e\}$ .
- If  $y = gxg^{-1}$ , then  $y^k = gx^k g^{-1}$ .
- Proposition:  $y \sim x$  implies  $|y| = |x|$ .

*Proof.* Suppose  $|y| = k$ , i.e.,  $y^k = e$ . By the above statement, we know that  $y^k \sim x^k$ . Since  $y^k = e$ , it follows that  $e \sim x^k$ . Thus,  $x^k$  is in the conjugacy class of  $e$ . But since the conjugacy class of  $e$  is  $\{e\}$ , this means that  $x^k = e$ , as desired.  $\square$

- Conjugacy in  $S_n$ ,  $n \geq 2$ .
  - Each  $x \in S^n$  has a cycle decomposition

$$x = (a_1, \dots, a_k)(b_1, \dots, b_m)(c_1, \dots) \cdots$$

- We want to investigate the properties of  $gxg^{-1}$  for an arbitrary  $g \in S_n$ . Ideally, we'd like to express it in a form related to  $x$ .
- Trick: Apply  $gxg^{-1}$  to  $g(a_1)$ . Then

$$gxg^{-1}(g(a_1)) = gx(a_1) = g(a_2)$$

- It follows by induction that

$$gxg^{-1} = (g(a_1), \dots, g(a_k))(g(b_1), \dots, g(b_m))(g(c_1), \dots) \cdots$$

- Now suppose that  $m \neq g(a_i), g(b_j), g(c_k), \dots$ . Then

$$g^{-1}(m) \notin \{a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots\}$$

It follows since  $x$  is the identity on such elements that  $x(g^{-1}(m)) = g^{-1}(m)$ . Therefore, since all functions involved are bijections,

$$[gxg^{-1}](m) = g[x(g^{-1}(m))] = g(g^{-1}(m)) = m$$

- It follows that  $gxg^{-1}$  has the same cycle **shape**.
- **Shape** (of  $g \in S_n$ ): The partition of  $n$  given by the lengths of the cycles in the cycle decomposition of  $g$  in decreasing order. *Also known as cycle shape.*

$S_4$	4-cycle	3-cycle	Product of 2-cycles	1-cycles
Cycle decomposition	$(x, x, x, x)$	$(x, x, x)(x)$	$(x, x)(x, x)$	$(x)(x)(x)(x)$
Shape	4	3 + 1	2 + 2	1 + 1 + 1 + 1

Table 2.3: Shape of elements in  $S_4$ .

<sup>1</sup> $\bigsqcup$  denotes a **disjoint union**. Think of the *disjoint* union of sets as a union of sets that happen to be disjoint, the same way a *direct* sum of subspaces is a sum of subspaces that happen to be linearly independent.

- Claim:  $x, y \in S_n$  are conjugate iff they have the same cycle shape.

*Proof.* We will do a proof by example that illustrates the idea of the generalized proof.

Let

$$x = (1, 2, 3)(4, 5, 6)(7, 10) \qquad y = (2, 3)(4, 1, 5)(6, 9, 10)$$

Note that both have the same cycle shape:  $3 + 3 + 2 + 1 + 1$ . We now use a two-step process to define a  $g$  such that  $y = gxg^{-1}$ .

Step 1: Including 1-cycles, line both  $x$  and  $y$  up so they “match.”

$x$	( 1      2      3 )	( 4      5      6 )	( 7      10 )	( 8 )	( 9 )
$y$	( 4      1      5 )	( 6      9      10 )	( 2      3 )	( 7 )	( 8 )
$gxg^{-1}$	$(g(1) \ g(2) \ g(3))$	$(g(4) \ g(5) \ g(6))$	$(g(7) \ g(10))$	$(g(8))$	$(g(9))$

Step 2: We want  $y = gxg^{-1}$ . Thus, take  $g$  to be the map which sends every entry in  $gxg^{-1}$  to the entry of  $y$  directly above it. For example, we want  $g(1) = 4$ ,  $g(2) = 1$ ,  $g(3) = 5$ ,  $\dots$ . Noting that  $g(1) = 4$ ,  $g(4) = 6$ ,  $g(6) = 10$ ,  $\dots$ , we realize that  $g$  can actually be written as the following cycle.

$$g = (1, 4, 6, 10, 3, 5, 9, 8, 7, 2)$$

□

- Follow ups.
  - How many different  $g$ 's satisfy  $y = gxg^{-1}$ ?
    - Depends on the number of ways  $y$  can be matched up with  $x$ .
    - The above manner obviously works.
    - However, we can rotate the elements in both 3-cycles three ways, and the elements of the 2-cycle two ways, so that's  $3 \cdot 3 \cdot 2 = 18$   $g$ 's right there.
    - Additionally, we can swap the place of the 3-cycles and the 1-cycles entirely, so that's an additional  $2 \cdot 2$  times as many ways.
    - All told, there are  $3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 72$  possible  $g$ 's.
    - See HW2, Q1a for a treatment of an analogous problem.
  - Counting the size of conjugacy classes.
    - Suppose  $G$  is an abelian group. Then if  $y = gxg^{-1}$ ,  $y = gg^{-1}x = x$ , so the size of the conjugacy class of any  $x \in G$  is 1.
    - For this reason, the elements of  $\mathbb{Z}/n\mathbb{Z}$  and of  $\text{SO}(2)$  are conjugate only to themselves.
    - However, we get something different for  $\text{O}(2)$ . Here, we can prove that the conjugacy class of every rotation  $r$  is  $\{r, r^{-1}\}$ , and that all reflections are in the same conjugacy class<sup>[2]</sup>.
      - > Let  $r$  denote a rotation, and  $s$  denote a reflection.
      - > Suppose  $x = r$ . Then

$$\begin{aligned} e &= (sr)^2 \\ &= sr sr \\ r^{-1} &= sr s \\ &= sr s^{-1} \end{aligned}$$

where we have HW1, Q2d(i) to justify the first equality and the fact that every reflection is its own inverse<sup>[3]</sup> to justify the last equality.

<sup>2</sup>This is fundamentally related to the structure of point groups in inorganic chemistry! Remember that in  $C_{5v}$ , for instance,  $C_5, C_5^4$  are conjugate,  $C_5^2, C_5^3$  are conjugate, and all reflections get lumped together.

<sup>3</sup>Intuitively, applying any reflection twice yields the original object.

➤ On the other hand, suppose  $x = s$ . Then if  $r$  is any rotation,

$$\begin{aligned} sr sr &= e \\ r sr &= s^{-1} \\ r sr r^{-2} &= s^{-1} r^{-2} \\ r sr^{-1} &= sr' \end{aligned}$$

where  $r'$  denotes  $r^{-2}$  to express the main takeaway: that  $s$  is conjugate to itself times any rotation (for  $r'$  arbitrary, we may choose  $r = (r')^2$ ). In other words, since all reflections are related by some rotation, all reflections are, indeed, in the same conjugacy class.

- Generators of  $S_n$ ,  $n \geq 3$ .
- Lemma: The set of 2-cycles generates  $S_n$ .

*Proof.* It only requires  $n - 1$  swaps between pairs of elements to get to any permutation. For example, to get to

$$\begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & \mapsto 2 \\ 4 & 1 \end{array}$$

we can swap 1 and 3 (so  $1 \mapsto 3$ ), then 2 and 4 (so  $2 \mapsto 4$ ), then “3” and “4” (so  $3 \mapsto 2$  and  $4 \mapsto 1$ ). More graphically,

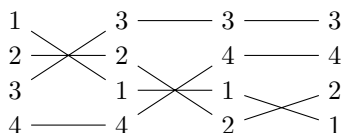


Figure 2.3: Generating  $S_n$  with 2-cycles.

The idea is we fix the first element and then work down the list. □

- $S_n$  is also generated by

$$\{(1, 2), (2, 3), (3, 4), \dots, (n - 1, n)\} \qquad \{(1, 2), (1, 3), (1, 4), \dots, (1, n)\}$$

– Both of these sets have cardinality  $n - 1$ .

- As we can see from the above...

- If we generate  $S_n$  with all 2-cycles, the generator set has cardinality  $\frac{n}{2}(n - 1)$ ;
- If we generate  $S_n$  with all elementary 2-cycles, the generator set has cardinality  $n - 1$ .

- But we can do even better: Let  $\sigma = (1, 2)$  and  $\tau = (1, 2, \dots, n)$ . Then any

$$(k, k + 1) = \tau^{k-1} \sigma \tau^{-(k-1)}$$

- Indeed, we can see that using the RHS above,  $k \mapsto 1 \mapsto 2 \mapsto k + 1$  and  $k + 1 \mapsto 2 \mapsto 1 \mapsto k$ . Every other element receives the identity treatment, as we can confirm.



## Week 3

# Types of Subgroups and Group Functions

### 3.1 Subgroups and Generators

10/10:

- Defining **subgroups**.
  - Let  $G = (G, *)$  be a group, and let  $H \subseteq G$  be a subset.
  - What properties do we want  $H$  to satisfy to consider it a “subgroup?”
    - $H$  should inherit the binary operation from  $G$ .
    - $H$  should be closed under multiplication using said binary operation.
    - $H$  should be nonempty.
    - $H$  should contain the inverses of every element — this is automatic if  $G$  is finite since the inverse of an element  $g$  of order  $n$  is  $g^{n-1}$  and  $g^{n-1} \in H$  by closure under multiplication.
    - $H$  should also be associative; we also inherit this for free from  $G$ .
- Easy way to construct a subgroup.
  - Let  $G$  be a group, and let  $x_1, x_2, \dots \in G$ . We can let  $H = \langle x_1, x_2, \dots \rangle$ , i.e.,  $H$  is the group **generated** by  $x_1, x_2, \dots$ . In other words,  $H$  is the set of all finite products  $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots$ .
  - This construction does give you all possible subgroups, but when you write it down, it’s very hard to say what group you get.
- Example: If you have  $H \subset G$  a subgroup, then  $H = \langle h |_{h \in H} \rangle$ .
- **Cyclic** (group): A group  $G$  for which there exists  $g \in G$  such that  $G = \langle g \rangle$ .
- Examples:
  - If  $1 < n < \infty$ , then  $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$ .
  - However, the generator isn’t always unique —  $\mathbb{Z}/7\mathbb{Z} = \langle 3 \rangle$ .
  - If  $G$  is generated by an element, it’s also generated by its inverse. For example,  $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ .
- Proposition: Let  $G$  be a cyclic group. It follows that
  1. If  $|G| = \infty$ , then  $G$  is isomorphic to  $\mathbb{Z}$ ;
  2. If  $|G| = n < \infty$ , then  $G$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ .

*Proof.* Assertion 1: Let  $G = \langle g \rangle$ . Then

$$G = \{\dots, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots\}$$

Now suppose for the sake of contradiction that  $g^a = g^b$  for some  $a, b \in \mathbb{Z}$ . Then  $g^{a-b} = e$ , so  $|G| \leq a-b$ , a contradiction. Therefore,  $G = \{G^{\mathbb{Z}}\}$ . In particular, we may define  $\phi : \mathbb{Z} \rightarrow G$  by  $k \mapsto g^k$ . This map has the property that  $a + b \mapsto g^a g^b$ , i.e.,  $\phi(a)\phi(b) = \phi(ab)^{[1]}$ .

Assertion 2: Let  $G = \langle g \rangle$ . Then

$$G = \{e, g, g^2, \dots, g^{n-1}\}$$

Now suppose for the sake of contradiction that  $g^a = g^b$ . Then  $g^{a-b} = e$ , so  $|G| \leq a-b < n$ , a contradiction. Therefore, we may once again define  $\phi : \mathbb{Z}/n\mathbb{Z} \rightarrow G$  as above. Note that  $a + b \mapsto g^{(a+b) \bmod n}$ . This is still a homomorphism, though.  $\square$

- Claim: Any subgroup of a cyclic group is also cyclic.
- Example:  $G = \mathbb{Z}$ ,  $H = \langle 2002, 686 \rangle$ .
  - $H = \{2002x + 686y \mid x, y \in \mathbb{Z}\}$ .
  - To say that  $H$  is cyclic is to say that it is equal to the integer multiples of some  $d \in \mathbb{Z}$ , i.e., there exists  $d$  such that  $G = \{zd \mid z \in \mathbb{Z}\}$ .
  - We can take  $d = \gcd(2002, 686)$ .
  - (Nonconstructive) proof: Let  $d$  be the smallest positive integer in  $H$ . Suppose for the sake of contradiction that  $md + k$  is in the group for some  $1 \leq k < d$ . Then adding  $-d$   $m$  times, we get that  $k \in H$ , a contradiction since we assumed  $d$  was the smallest positive integer in  $H$ .
- Let  $G = \langle x, y \rangle$  be a group that is generated by two elements. Find a subgroup  $H \subset G$  such that  $H$  must be generated by more than 2 elements.
  - Let's work with  $S_n = \langle (1, 2, \dots, n), (1, 2) \rangle$ .
  - The subgroup  $H = \langle (1, 2), (3, 4), (5, 6) \rangle$  will work.
    - $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .
    - Suppose  $H = \langle a, b \rangle$ . We can get  $e, a, b, ab$ . But because everything commutes, we can rearrange any product to  $a^i b^j$  and cancel.
- When you want to answer questions like, “Is  $\mathbb{Z}/180180\mathbb{Z}$  a subgroup of  $S_n$  for some  $n$ ,” you need some more information on the structure of  $S_n$ .
- Group **presentations** allow us to write and describe a group really easily.
  - Seems useful at first, but isn't really that useful once you see it more.

## 3.2 Homomorphisms

- 10/12:
- We've studied groups a lot at this point. But as with vector spaces, we don't have a complete theory of groups until we consider maps between them.
  - Today: Homomorphisms.
  - Let  $H, G$  be groups.
  - What qualities do we want a map of groups to have?
    - Maps between vector spaces preserve linearity, so maps between groups should probably preserve the group operation.

---

<sup>1</sup>We all know that this is a **homomorphism**; Calegari just doesn't want to call it that yet.

- Bijection? As with linear maps, the bijective case is interesting, but we don't want to be this restrictive.
- In fact, that first quality is the only one we want.

• **Homomorphism:** A map  $\phi : H \rightarrow G$  of sets such that  $\phi(x *_H y) = \phi(x) *_G \phi(y)$ .

• **Lemma:** Let  $\phi : H \rightarrow G$  be a homomorphism. Then...

1.  $\phi(e_H) = e_G$ .
2.  $\phi(x^{-1}) = \phi(x)^{-1}$ .

*Proof.* Claim 1:

$$\begin{aligned} e_G \phi(x) &= \phi(x) = \phi(x e_H) = \phi(x) \phi(e_H) \\ e_G &= \phi(e_H) \end{aligned}$$

Claim 2:

$$e_G = \phi(e_H) = \phi(x x^{-1}) = \phi(x) \phi(x^{-1})$$

□

- **Image** (of  $\phi$ ): The subset of  $G$  such that for all  $h \in H$ ,  $\phi(h) = g$ . Denoted by **im**  $\phi$ .
- **Kernel** (of  $\phi$ ): The subset of  $H$  containing all  $h \in H$  such that  $\phi(h) = e_G$ . Denoted by **ker**  $\phi$ .
- **Lemma:**

1. **im**  $\phi \subset G$  is a subgroup.
2. **ker**  $\phi \subset H$  is a subgroup.

*Proof.* Claim 1: We know that  $\phi(e_H) = e_G$ , so

$$\text{im } \phi \neq \emptyset$$

as desired. Next, let  $g_1, g_2 \in \text{im } \phi$ . Suppose  $g_1 = \phi(h_1)$  and  $g_2 = \phi(h_2)$ . Then since  $H$  is closed under multiplication as a subgroup,  $h_1 h_2 \in H$ . It follows that

$$g_1 g_2 = \phi(h_1) \phi(h_2) = \phi(h_1 h_2) \in \text{im } \phi$$

as desired. Lastly, let  $g \in \text{im } \phi$ . Suppose  $g = \phi(h)$ . Then since  $H$  is closed under inverses as a subgroup,  $h^{-1} \in H$ . It follows that

$$g^{-1} = \phi(h)^{-1} = \phi(h^{-1}) \in \text{im } \phi$$

as desired.

Claim 2: We know that  $\phi(e_H) = e_G$ , so

$$\text{ker } \phi \neq \emptyset$$

as desired. Next, let  $g_1, g_2 \in \text{ker } \phi$ . Then

$$e_G = e_G e_G = \phi(g_1) \phi(g_2) = \phi(g_1 g_2)$$

so  $g_1 g_2 \in \text{ker } \phi$ , as desired. Lastly, let  $g \in \text{ker } \phi$ . Then

$$e_G = \phi(e_H) = \phi(g g^{-1}) = \phi(g) \phi(g^{-1}) = e_G \phi(g^{-1}) = \phi(g^{-1})$$

□

- Examples:

$H$	$G$	$\phi$	$\text{im } \phi$	$\ker \phi$
$H$	$G$	$\phi(h) = e$	$\{e\}$	$H$
$H \leq G$	$G$	inclusion	$H$	$\{e\}$
$\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$k \mapsto k \bmod n$	$\mathbb{Z}/n\mathbb{Z}$	$n\mathbb{Z}$
$O(n)$	$\mathbb{R}^*$	$\det$	$\{\pm 1\}$	$SO(n)$
$GL_n \mathbb{R}$	$\mathbb{R}^*$	$\det$	$\mathbb{R}^*$	$SL_n \mathbb{R}$

Table 3.1: Examples of images and kernels.

- The first example shows that there is always at least one homomorphism between two groups.
- $\mathbb{R}^*$  is the group of nonzero real numbers with multiplication as the group operation.
- The  $O(n)$  example expresses the fact that  $\det(AB) = \det(A)\det(B)$ , i.e., that the determinant is a homomorphism.
  - The kernel is  $SO(n)$  since 1 is the multiplicative identity of  $\mathbb{R}^*$  and all matrices in  $SO(n) \subset O(n)$  get mapped to 1 by the determinant.
- $GL_n \mathbb{R}$  is the set of all  $n \times n$  invertible matrices over the field  $\mathbb{R}$ .
- **Isomorphism:** A bijective homomorphism from  $H \rightarrow G$ .
  - If an isomorphism exists between  $H$  and  $G$ , we say, “ $H$  is isomorphic to  $G$ .”
- Lemma:  $H$  is isomorphic to  $G$  implies  $G$  is isomorphic to  $H$ .

*Proof.*  $\phi : H \rightarrow G$  a bijection implies the existence of  $\phi^{-1} : G \rightarrow H$ . Claim: This is an isomorphism. We can formalize the notion, or just think of  $\phi$  as relabeling elements of  $H$  and  $\phi^{-1}$  as unrelabeling them.  $\square$

- Lemma: A homomorphism  $\phi : H \rightarrow G$  is **injective** iff  $\ker \phi = \{e_H\}$ .

*Proof.* Suppose  $\phi$  is injective. We know that  $\phi(e_H) = e_G$  from a previous lemma; this implies that  $e_H \in \ker \phi$ . Now let  $x \in \ker \phi$  be arbitrary. Then  $\phi(x) = e_G = \phi(e_H)$ . But since  $\phi$  is injective, we have that  $x = e_H$ . Thus, we have proven that  $e_H \in \ker \phi$ , and any  $x \in \ker \phi$  is equal to  $e_H$ ; hence, we know that  $\ker \phi = \{e_H\}$ , as desired.

Now suppose that  $\ker \phi = \{e_H\}$ . Let  $\phi(x) = \phi(y)$ . It follows that

$$\phi(xy^{-1}) = \phi(x)\phi(y^{-1}) = \phi(x)\phi(y)^{-1} = \phi(x)\phi(x)^{-1} = e_G$$

But this implies that

$$\begin{aligned} xy^{-1} &= e_H \\ x &= y \end{aligned}$$

as desired.  $\square$

- Problem: Is there a surjective homomorphism  $\phi : S_5 \rightarrow S_4$ ?
  - Proposal 1: Send 5-cycles to the identity and everything else to itself.
  - Proposal 2: “Drop 5”  $(1, 2)(3, 4, 5) \mapsto (1, 2)(3, 4)$ .
    - Counterexample:  $(1, 2, 3, 4, 5) \mapsto (1, 2, 3, 4)$ .
  - Proposal 3: If it doesn’t do something to everything, send it to  $e$ .

- Lemma: Let  $\phi : H \mapsto G$  be a homomorphism. If  $|h| = n$ , then  $|\phi(h)|$  divides  $n$ , i.e.,  $n$  is a multiple of  $|\phi(h)|$ .

*Proof.* If  $h^n = e$ , then  $\phi(h^n) = e = \phi(h)^n$ . □

- Equipped with this lemma, let's return to the previous problem.
  - Suppose for the sake of contradiction that such a surjective homomorphism  $\phi$  exists.
  - Consider a 5-cycle  $h \in S_5$ ; obviously,  $|h| = 5$ .
  - It follows by the lemma that  $\phi(h) \in S_4$  has order which divides 5. But since the maximum order of an element in  $S_4$  is 4, this means that  $|\phi(h)| = 1$ , so  $\phi(h) = e$ .
- If one 5-cycle maps to the identity, then all of their products must, too.
- What can map to an order 3 element in  $S_4$ ?
- If  $\psi(g) = (1, 2, 3)$ , then  $|g|$  is divisible by 3.
- In fact, no surjective map exists!
- In order for homomorphisms to exist, there must be some reason. If there aren't any (nontrivial ones), proving this can be easy.
- Now consider  $S_4 \mapsto S_3$ .
  - 4-cycles to  $e$  or 2-cycles.
  - 3-cycles to 3-cycles.
- Idea:  $S_4 \cong \text{Cu} \cong S_3$ .
  - 3 pairs of opposite faces and 4 diagonals.

### 3.3 Cosets

10/14:

- Asking, “what’s the intuition for this question?” in OH.
  - Calegari: Intuition is borne of experience. You get intuition from grubby computations, and then you finally recognize the structure. If you don't know what's going on, it's good to struggle. Start with the simplest possible example and then struggle until you develop intuition.
- Last time, we discussed the fact that there is no surjective homomorphism from  $S_5 \rightarrow S_4$ , but there is a surjective homomorphism from  $S_4 \rightarrow S_3$ . How about the case  $S_{n+1} \rightarrow S_n$  for arbitrary  $n$ ?
- Teaser theorem: Let  $n > m$  and  $\phi : S_n \twoheadrightarrow S_m$ . Then
  1.  $m = 1$ .
  2.  $m = 2$ .
  3.  $m = 3$ .
- Think about the problem of maps from  $G \rightarrow \Gamma$ , where  $\Gamma$  is another group. What we know:
  - Let  $K = \ker \phi$ . Recall that  $\phi$  is injective iff  $\ker \phi = \{e\}$ . But there is some additional structure: If  $\phi(g) = x$ , then  $\phi(gK) = x$  where  $gK = \{gk \in G \mid k \in K\}$ . Another way of phrasing this: If  $\phi(g') = x$ , then  $g' = gk$  for some  $k \in K$ .
  - This motivates the following definition.

- **Left coset:** The set defined as follows, where  $g \in G$  and  $H$  is a subgroup of  $G$ . Denoted by  $gH$ . Given by

$$gH = \{gh \mid h \in H\}$$

- You can define cosets for  $H$  a subset (not a subgroup) of  $G$ , but we will not be interested in these cases.
- Claim:  $gH \subset G$  means that  $gh = gh'$  implies  $h = h'$ ??
- Example:  $G = S_3$ ,  $H = \langle e, (1, 2) \rangle$ .

$g$	$gH$
$e$	$\{e, (1, 2)\}$
$(1, 2)$	$\{e, (1, 2)\}$
$(1, 3)$	$\{(1, 3), (1, 2, 3)\}$
$(1, 2, 3)$	$\{(1, 3), (1, 2, 3)\}$
$(2, 3)$	$\{(2, 3), (1, 3, 2)\}$
$(1, 3, 2)$	$\{(2, 3), (1, 3, 2)\}$

Table 3.2: Cosets of  $\langle e, (1, 2) \rangle$  in  $S_3$ .

- Observations: Cosets are pairwise disjoint.  $x \in gH$  implies  $xH = gH$ .
- $G/H$ : The set of all left cosets of  $H$  in  $G$ .
- Proposition:
  1. Any two cosets in  $G/H$  are either (i) the same or (ii) disjoint.
  2. All  $g \in G$  lie in a unique coset (in particular,  $gH$ ).
  3.  $|gH| = |H|$ .

*Proof.* Claim 1: Let  $C_1, C_2 \in G/H$ . We divide into two cases ( $C_1 \cap C_2 = \emptyset$  and  $C_1 \cap C_2 \neq \emptyset$ ). In the first case,  $C_1, C_2$  are disjoint, as desired. In the latter case, they are not disjoint, so we need to prove that they are the same. Suppose  $g \in C_1 \cap C_2$ . Let  $C_1 = \gamma H$ . We will prove that  $gH = \gamma H$  via a bidirectional inclusion argument. It will follow by similar logic that  $gH = C_2$ , from which transitivity will imply that  $C_1 = gH = C_2$ , as desired. Let's begin. Let  $x \in gH$ . Then  $x = gh$  for some  $h \in H$ . Additionally, we know that  $g \in \gamma H$  by hypothesis, so  $g = \gamma h'$  for some  $h' \in H$ . It follows by combining the last two equations that  $x = \gamma h'h$ . But since  $h'h \in H$ ,  $x \in \gamma H$  as desired. A symmetric argument works in the other direction.

Claim 2: We know that  $g \in gH$  since  $e \in H$  and  $g = ge$ . Additionally, if  $g \in \gamma H$ , we have by part (1) that  $\gamma H = gH$ , so  $g$  does lie in a *unique* coset.

Claim 3: Suppose there exist  $h, h' \in H$  such that  $gh = gh'$ . Then  $h = h'$  by the cancellation lemma. Thus, every distinct  $h \in H$  induces a distinct  $gh \in gH$ . Therefore,  $|gH| = |H|$ , as desired.  $\square$

- Notice that so far, general statements we've made about groups have been very easy to prove; it's only in particular instances that things become tricky.
- Decomposition of a group into equivalence classes: Cosets and conjugacy both do this.
- Corollary: Let  $H$  be a subgroup of  $G$ . Then

$$|G| = |G/H| \cdot |H|$$

*Proof.* Sketch: Partition  $G$  into cosets, each of order  $|H|$ . But there are  $|G/H|$  of these. Thus, the number of elements in  $G$  is  $|G/H| \cdot |H|$ .  $\square$

- **Index** (of  $H$  in  $G$ ): The number of cosets into which  $H$  partitions  $G$ . Denoted by  $[G : H]$ . Given by

$$[G : H] = |G/H|$$

- If  $|G| < \infty$ , then  $[G : H] = |G|/|H|$ . If  $|G| = \infty$ , then we can still define the concept  $|G/H|$ , but we don't have a nice formula for it.
- Example: Let  $G = \mathbb{Z}$  and  $H = 2\mathbb{Z}$  (i.e.,  $H$  is the set of even integers). Then the orbits are all even and all odd numbers. The index is 2??
- Theorem (Lagrange):
  1. Let  $G$  be a finite group,  $H \subset G$ . Then  $|H|$  divides  $|G|$ .
  2. Let  $G$  be a finite group. Let  $g \in G$ . Then  $|g|$  divides  $|G|$ .
- Example: Let  $p$  be prime. If  $|G| = p$ , then  $G \cong \mathbb{Z}/p\mathbb{Z}$ .

*Proof.* Take  $g \in G$  such that  $g \neq e$ . By Lagrange's theorem,  $|g|$  divides  $p$ . But this means that  $|g| = 1$  or  $|g| = p$ . But it's not the first case because  $g \neq e$ . Thus,  $G = \langle g \rangle \cong \mathbb{Z}/p\mathbb{Z}$ , as desired.  $\square$

- **Right coset:** The set defined as follows, where  $g \in G$  and  $H$  is a subgroup of  $G$ . Denoted by  $Hg$ . Given by

$$Hg = \{hg \mid h \in H\}$$

- $H/G$ : The set of all right cosets of  $H$  in  $G$ .
- The theories of left and right cosets are very similar, but they are not entirely equivalent.
  - For example,  $H = \langle e, (1, 2) \rangle$  implies

$$(1, 3)H = \{(1, 3), (1, 2, 3)\}$$

$$H(1, 3) = \{(1, 3), (1, 3, 2)\}$$

# Week 4

???

## 4.1 Quotient Groups

10/17: • Notational confusion regarding  $\mathbb{Z}/10\mathbb{Z}$ .

- Let  $G = \mathbb{Z}$  and  $H = 10\mathbb{Z}$  (the multiples of 10).
- A few of the cosets are as follows:

$$\begin{aligned}H &= \{\dots, -20, -10, 0, 10, 20, 30, \dots\} \\1 + H &= \{\dots, -19, -9, 1, 11, 21, 31, \dots\} \\2 + H &= \{\dots, -18, -8, 2, 12, 22, 32, \dots\}\end{aligned}$$

- Evidently,  $|\mathbb{Z}/10\mathbb{Z}| = 10$ .
- Yet  $\mathbb{Z}/10\mathbb{Z}$  is also the notation for the cyclic group of order 10.
- This notation is not an error, but reveals something deep: We can make the set of cosets into a group and define addition by

$$(a + 10\mathbb{Z}) + (b + 10\mathbb{Z}) = (a + b + 10\mathbb{Z})$$

More specifically, we can define an isomorphism between the two definitions of  $\mathbb{Z}/10\mathbb{Z}$  via  $a + H \mapsto a$  for  $a = 0, \dots, 9$ .

- This example motivates the following goal.
- Goal: Make  $G/H$ , which is a set, into a group.
  - This set needs a binary operation. It makes natural sense to define the binary operation as follows.

$$xH * yH = xyH$$

- We then need an identity coset, inverse cosets, and associativity.
  - The identity is  $H$ .
  - The inverse of  $xH$  is  $x^{-1}H$ .
  - Associativity of  $G/H$  follows from the associativity of  $G$  (which tells us that  $(ab)c = a(bc)$ ). More specifically,

$$\begin{aligned}aH *_H (bH *_H cH) &= aH *_H (b *_G c)H \\&= a *_G (b *_G c)H \\&= (a *_G b) *_G cH \\&= (a *_G b)H *_H cH \\&= (aH *_H bH) *_H cH\end{aligned}$$



- Calegari's impromptu explanation of associativity drives home that he really is very good at drilling down to the core of an idea and working with it. He really has a very similar mind to mine.
- Something else we need to investigate: Equivalence classes, and defining functions on equivalence classes.
  - We need to make sure that functions are defined the same regardless of how you label the equivalence classes.
  - Consider the set of names.
    - Say we define equivalency classes based on all names which share the same first letter.
    - Then we define a function  $F$  on the equivalency classes based on the last letter.
    - But then  $[\text{Frank}] = [\text{Fen}]$  will be mapped to two different elements of the alphabet, so  $F$  is not well-defined.
  - Thus, for our example, we need to guarantee that if  $x, x' \in xH$ , then  $xH * yH = x'H * yH$ .
- Check: Independence of choice.
  - Suppose we relabel  $x \mapsto xh$  and  $y \mapsto yh$ . We need

$$xhyh' = xyh''$$

for some  $h'' \in H$ .

- Note that  $x, y, h, h'$  are all fixed;  $h''$  is the only free thing (i.e., is what we're looking for).
- Algebraically manipulating the above implies that we want
 
$$h'' = y^{-1}hyh'$$
  - Thus, we know that  $h'' \in G$ , but we need to make sure that  $h'' \in H$ . Alternatively, we want  $y^{-1}hy = h''(h')^{-1} \in H$ .
  - An example where  $y^{-1}hy$  is not in  $H$ :  $G = S_3$ ,  $H = \langle (1, 2) \rangle$ ,  $h = (1, 2)$ ,  $y = (1, 3)$ ,  $yhy^{-1} = (2, 3)$ .
- Why did  $\mathbb{Z}/10\mathbb{Z}$  work? Because it was abelian, so conjugacy cancelled  $y^{-1}hy = y^{-1}yh = h$ .
  - We could restrict ourselves entirely to abelian groups, but can we be more general?
- What should we require of  $G/H$ ?
  - The canonical map of sets  $\phi : G \rightarrow G/H$  is given by  $\phi(x) = xH$ .
  - We should require that  $\phi$  is a homomorphism (i.e., that the group structure of  $G$  is preserved for  $G/H$ ).
  - See how  $xH * yH = xyH$  is analogous to  $\phi(x)\phi(y) = \phi(xy)$ .
- Let's suppose  $\phi : G \rightarrow G/H$  is a homomorphism.
  - Then  $\phi(g) = eH$  implies that  $g \in H$ , i.e.,  $\ker \phi = H$ .
  - Realization: An alternate way to do HW3, Q2b would have been in terms of quotient groups: In that case,  $G/H \cong S_{26}$ , and the following proposition would give us the surjectivity and kernel requirements.
- Lemma: Let  $\phi$  be a homomorphism from  $G$  to another group. Let  $K = \ker \phi \subset G$ . Then  $K$  has the following property, which is not true for all subgroups but is for kernels: If  $x \in K$  and  $g \in G$ , then  $gxg^{-1} \in K$ .

*Proof.* Since  $\phi(x) = e$ , we have that

$$\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = e$$

□

- **Normal (subgroup):** A subgroup  $H$  of  $G$  such that for all  $x \in H$  and  $g \in G$ ,  $gxg^{-1} \in H$ . Denoted by  $H \trianglelefteq G$ ,  $H \triangleleft G$ .

– We often write  $gHg^{-1}$ .

- Example: As per the lemma,  $\ker \phi$  is a normal subgroup.
- Example: If  $G$  be abelian, then every  $H \trianglelefteq G$ .
- Lemma: A subset  $H \subset G$  is normal iff
  1.  $H$  is a subgroup.
  2.  $H$  is a union of some number of conjugacy classes.
- Proposition: Let  $G$  be a group and  $H \triangleleft G$ . Then  $G/H$  is a group under the multiplication

$$xH * yH = xyH$$

and the map  $\phi : G \rightarrow G/H$  is a surjective homomorphism with kernel  $H$ .

*Proof.* We want to show that  $xhyh' = xyh''h'$ . We can do so via multiplying the following by  $x$  on the left and  $h'$  on the right:

$$\begin{aligned} hy &= (yy^{-1})hy \\ &= y(y^{-1}hy) \\ &= yh'' \end{aligned}$$

Note that we get from the second to the third line above because  $H$  is a normal subgroup, i.e., conjugates of its elements are elements of it. This implies the desired result.  $\square$

- Example: Let  $G = \mathbb{Z}$ ,  $H = 10\mathbb{Z}$ , and  $G/H = \mathbb{Z}/10\mathbb{Z}$ .
- Example: Let  $G = G$  and  $H = \{e\}$ .
  - $H$  is normal since it's a subgroup and it's a union of conjugacy classes.
  - In this case,  $G/H \cong G$ .
- Example:  $G = O(2)$  and  $H = SO(2)$ .
  - $G$  is not abelian here.
  - From HW1, the cosets are  $H = \{\text{rotations}\}$  and  $\{\text{reflections}\}$ .
  - The cosets are  $H$  and  $sH$  for some reflection  $s \in O(2) \setminus SO(2)$ .
  - What the group structure tells us here is that rotation  $\circ$  reflection is like even  $\times$  odd numbers.
  - $G/H \cong \mathbb{Z}/2\mathbb{Z}$  here.
- An equivalent formulation of normality.
- Proposition:  $H \triangleleft G$  iff the left cosets coincide with the right cosets, i.e.,

$$gH = Hg$$

*Proof.* Suppose first that  $H \triangleleft G$ . Use a bidirectional inclusion argument. Let  $gh \in gH$ . Then

$$gh = ghg^{-1}g = h'g \in Hg$$

where  $h'$  may or may not equal  $h$ , but we know it is an element of  $H$  by the definition of normal subgroups. The argument is symmetric in the other direction.

Now suppose  $gH = Hg$ . Let  $h \in H$ . Then there exist  $h, h' \in H$  such that  $gh = h'g$ . Therefore,  $ghg^{-1} = h' \in H$ .  $\square$

- This is a nice resolution of left and right cosets.
  - It tells us when they're the same, and when they're different.
- Implication: If  $H \triangleleft G$ , then

$$xH \cdot yH = x(Hy)H = x(yH)H = xyHH = xyH$$

- Midterm next week.

## 4.2 First Isomorphism Theorem

10/19:

- Last time:
  - If  $K \triangleleft G$ , then the map  $\phi : G \rightarrow G/K$  defined by  $g \mapsto gK$  is a surjective homomorphism with kernel  $K$ .
- Today: Understand a general surjective homomorphism  $\phi : G \rightarrow H$  with kernel  $K \triangleleft G$ .

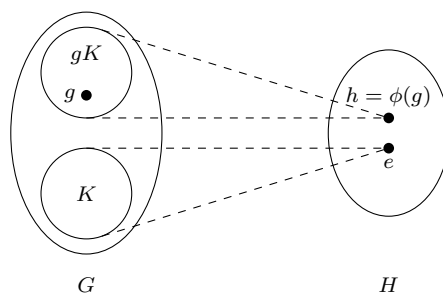


Figure 4.1: Visualizing a surjective homomorphism.

- In general, we know that  $K \mapsto \{e\}$ .
- Since  $\phi$  is surjective, every  $h \in H$  equals  $\phi(g)$  for some  $g \in G$ .
- More broadly,  $gK \mapsto \{h\}$ .
- Can you get more elements than those in  $gK$  that map to  $h$ ? Perhaps elements of  $Kg$  or  $KgK$ ? Well since  $K$  is normal,  $kg = gk$ .
- Thus, all surjective homomorphisms have the same general structure.
  - In particular, they all map disjoint cosets to single elements.
  - Alternatively, we can take the perspective that they send every element to their coset with the kernel.
- Lemma: If  $\phi : G \rightarrow H$  is a surjective homomorphism,  $h \in H$ ,  $\phi(g) = h$ , and  $K = \ker \phi$ , then  $\phi^{-1}(h) = gK$ .

*Proof.* Suppose  $g' \in \phi^{-1}(h)$ . Suppose  $g' = gx$  (we do know that such an  $x$  exists in  $G$ ; in particular, choose  $x = g^{-1}g'$ ). Then

$$\phi(g') = \phi(gx) = \phi(g)\phi(x)$$

Since  $\phi(g') = h = \phi(g)$ , we have by the cancellation lemma that

$$e = \phi(x)$$

i.e.,  $x \in K$ . Therefore,  $g' \in gK$ , as desired. □

- We can define a bijection  $\tilde{\phi} : G/K \mapsto H$  defined by  $gK \mapsto \phi(g)$ .
- Claim:  $\tilde{\phi}$  is an isomorphism of groups.

*Proof.* Need to check that  $\tilde{\phi}$  is a homomorphism, surjective, and injective. We also need to check that it is well-defined (we did this with our picture).

Surjective: Let  $h \in H$  be arbitrary. Then  $h = \phi(g)$ . It follows that  $h = \tilde{\phi}(gK)$ .

Injective: Show that  $\ker \tilde{\phi} = \{eK\}$ . Let  $gK \in \ker \tilde{\phi}$ . Then  $\phi(g) = \tilde{\phi}(gK) = e$ . Thus,  $g \in K$ . Therefore,  $gK = eK$ , as desired.

Homomorphism: Check  $\tilde{\phi}(xK)\tilde{\phi}(yK) = \tilde{\phi}(xyK)$ . Since  $\tilde{\phi}(zK) = \phi(z)$ , we have the desired property. Explicitly,

$$\tilde{\phi}(xyK) = \phi(xy) = \phi(x)\phi(y) = \tilde{\phi}(xK)\tilde{\phi}(yK)$$

□

- Takeaway: All surjective homomorphisms are somewhat the same.
- Generalize:
- Let  $\phi : G \rightarrow H$  be a homomorphism.
  - We know that  $G \twoheadrightarrow \text{im } \phi \hookrightarrow H$ . Essentially, we can break up any homomorphism into the composition of a surjective homomorphism onto the image and an injective homomorphism into  $H$ .
- Theorem (FIT: First Isomorphism Theorem): To every homomorphism  $\phi$  there corresponds an isomorphism  $\tilde{\phi} : G/\ker \phi \rightarrow \text{im } \phi$  such that

$$\tilde{\phi}(g \cdot \ker \phi) = \phi(g)$$

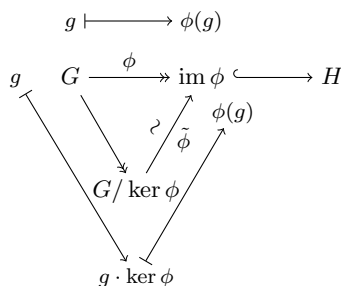


Figure 4.2: First isomorphism theorem.

- The triangle is **commutative**. This means that sending  $g$  along both paths gets you to the same result.
- The way to understand normal subgroups is to understand the homomorphisms.
- $N \subset G$  is normal if
  1.  $N$  is a subgroup.
  2.  $N$  is normal, i.e.,  $N$  is a union of conjugacy classes.
  3.  $e \in N$ .
  4.  $|h|||G|$  (Lagrange).

- 3-4 both follow from 1. They are not sufficient conditions for normality, but they can put restrictions on what is normal and make the computation easier.
- Examples.
  - Let  $\phi: \mathbb{Z} \rightarrow H$  send  $1 \mapsto h$  and  $k \mapsto h^k$ .

$$\begin{array}{ccc}
 \mathbb{Z} & \longrightarrow & H \\
 \downarrow & & \uparrow \\
 \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\sim} & \langle h \rangle \\
 k + n\mathbb{Z} & \longmapsto & h^k
 \end{array}$$

Figure 4.3: An example of the FIT.

- $\text{im } \phi = \langle h \rangle$ .
- $\ker \phi = n\mathbb{Z}$  where  $|h| = n$ ; if  $|h| = \infty$ , then  $\ker \phi = \{0\}$ .
- The FIT tells us that there is a map from  $\mathbb{Z}$  to  $\mathbb{Z}/n\mathbb{Z}$  to  $\langle h \rangle$  to  $H$ . The first map sends  $k \mapsto k + n\mathbb{Z}$  and the second sends  $k + n\mathbb{Z} \mapsto h^k$ .
- Let  $G = S_3$ .
  - The conjugacy classes are

$$\{e\} \qquad \{(1, 2), (1, 3), (2, 3)\} \qquad \{(1, 2, 3), (1, 3, 2)\}$$

- Thus, the only possible normal subgroup  $N$  is

$$H = \{e\} \cup (xxx) = \langle (1, 2, 3) \rangle$$

➤  $e \in N$  eliminates union 2,3; Lagrange eliminates union 1,2 (which has order 4).

- Let  $G = S_4$ .
  - The conjugacy classes are

$$e \qquad (xx) \qquad (xxx) \qquad (xxxx) \qquad (xx)(xx)$$

- The number of elements of the above form is

$$1 \qquad 6 \qquad 8 \qquad 6 \qquad 3$$

- The divisors of  $|S_4| = 24$  are 1,2,3,4,6,8,12,24.
  - 1 is possible; no way to get 2,3; 4 is possible; 6,8 are impossible; 12,24 are possible.
  - The 4 example is

$$K = \langle e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3) \rangle$$

- $S_3 / \langle (1, 2, 3) \rangle \cong \mathbb{Z}/2\mathbb{Z}$ .
- $S_4 / K$  is a group of order 6.
- The first instance corresponds to some map from  $S_3 \rightarrow S_2$ .
  - You can get an isomorphism from  $S_3$  to  $D_6$ .
  - The surjective map sends rotations to the identity and reflections to the nonidentity element.
  - By the FIT,  $S_3 / \langle (1, 2, 3) \rangle \cong S_2$ .
    - Yes, if you know enough about the quotient group, you can think about its properties. But it's easier to use the FIT.

- We constructed a map  $S_4 \rightarrow \text{Cu} \rightarrow S_3$ . If  $N = \ker$ , by the FIT,  $S_4/N \cong S_3$ .
  - As per the above example, we need to take  $N = K$  here.
- Example:  $G = \text{O}(2)$ .
  - The normal subgroups of  $\text{O}(2)$  are  $\{e\}$ ,  $\{r, r^{-1}\}$ , and  $\{\text{reflections}\}$ .
  - If  $N \triangleleft \text{O}(2)$  contains a reflection, then  $N = \text{O}(2)$ .
  - Let  $N \subset \text{SO}(2)$  be such that  $|N| = k$ , i.e.,  $N$  is generated by the rotation of  $2\pi k/N$ . What is  $\text{O}(2)/N$ ? You can think of  $\text{SO}(2)$  as a rotation in  $\mathbb{R}$ . Thus,  $\mathbb{R}/2\pi\mathbb{Z} \cong \text{O}(2)$ . Thus,  $\text{SO}(2)/N \cong \text{SO}(2)$ .
- Next time: Replace  $S_4$  with  $S_5$ .
- The midterm is most likely Wednesday next week.
  - The midterm will not be on Monday, but it could test stuff covered next Monday.
- Read the blog post on dihedral groups and the other blog posts I've missed!