Week 5

???

5.1 Special Normal Subgroups

10/24:

- Last time: If $H \triangleleft S_n$, $n \neq 4$, then $H = \{e\}, S_n, A_n$. If $n = 4$, H can also equal $\{e\} \cup \{(xx)(xx)\}$.
- Theorem: Let $n \neq 4$. Then the only normal subgroups of A_n are the identity and A_n.
  - Let $H \triangleleft A_n \triangleleft S_n$. From this, you could propose concluding that since we know all normal subgroups of S_n, and H is less than or equal to A_n, we know that $H = \{e\}, A_n$.
  - Issue: $\triangleleft$ is not transitive. Conjugacy classes change depending on where you're sitting.
  - Consider A, B, C: If $A \triangleleft B \triangleleft C$, then is $A \triangleleft C$?
  - This theorem is on HW5.
  - Counterexample:

$$A = \langle (1,2)(3,4) \rangle \qquad B = \{e\} \cup \{(xx)(xx)\} \qquad C = S_4$$

    - This is not so far from the simplest example.
  - Calegari reemphasizes that, "if you understand everything about S_4, then you understand everything in this class."
- We know that if $H \leq A_4$, then $|H| \mid 12$ by Lagrange's theorem.
- Claim: A_4 has no subgroups of order 6.
  - If H has index 2, then $H \triangleleft A_4$. This was a HW problem.
  - Thus, we can try to understand conjugacy classes in A_n. Whereas in S_n, we have a beautifully simple way to characterize all conjugacy classes, we do not have that in A_n. For example, $(1,2,3)$ and $(1,3,2)$ are not conjugate. $(2,3)(1,2,3)(2,3) = (1,3,2)$. $(1,2)(1,2,3)(1,2) = (2,1,3)$. $(1,3)(1,2,3)(1,3) = (3,2,1)$. But none of these transpositions are in A_n.
  - There are four conjugacy classes in A_4.

$$\{e\} \quad \{(12)(34), (13)(24), (14)(23)\} \quad \{(123), (243), (134), (142)\} \quad \{(132), (234), (143), (124)\}$$

    - Note that if $x, y \in A_4$ are of order 3, either $x \sim y$ or $x \sim y^{-1}$.
  - In A_5, all 3-cycles are conjugate; in A_4, they're not.
  - The sizes of the conjugacy classes in A_4 are $1 + 4 + 4 + 3$. That's enough to prove that there is no subgroup of order 6.
  - Alternate proof.

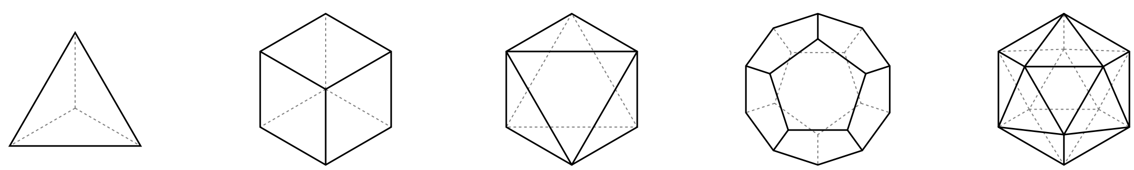
Proof. Suppose for the sake of contradiction that A_4 has a normal subgroup H of index 2. Then by the proposition from Lecture 4.1, there exists a surjective homomorphism from $A_4 \to A_4/H$. Additionally, since $|A_4/H| = 2$ and there is only one group of order 2, $A_4/H \cong \mathbb{Z}/2\mathbb{Z}$. Thus, there exists a surjective homomorphism $\phi : A_4 \to \mathbb{Z}/2\mathbb{Z}$.

We know that every alternating group (including A_4) is generated by 3-cycles. Let σ be an arbitrary 3-cycle generator of A_4. We know that $\phi(\sigma) = 0$ or $\phi(\sigma) = 1$. If $\phi(\sigma) = 1$, then

$$0 = \phi(e) = \phi(\sigma^3) = 3\phi(\sigma) = 1 +_2 1 +_2 1 = 1$$

which clearly cannot happen. Thus, $\phi(\sigma) = 0$. Consequently, the image of all of the generators of A_4 under ϕ is 0. But this implies that $\phi(A_4) = \{0\} \subsetneq \mathbb{Z}/2\mathbb{Z}$, contradicting our hypothesis that ϕ is surjective. $\square$

- Here ends the material that will be covered on the midterm.
- We now move on to something we will come back to later.
- A_n in nature for $n = 4, 5$.



(a) Tetrahedron. (b) Cube. (c) Octahedron. (d) Dodecahedron. (e) Icosahedron.

Figure 5.1: The platonic solids.

  - Recall the cube group Cu.
  - The cube is an example of a **platonic solid**.
  - Other examples: Tetrahedron, octahedron, icosahedron, and dodecahedron. We define corresponding symmetry groups Te, Oc, Do, and Ic.
  - Consider the tetrahedral group to start.
    - Since any rigid motion permutes the vertices, we have a map $\text{Te} \hookrightarrow S_4$. Moving 2 vertices fixes the rest. Thus, $\text{Te} \leq S_4$. Therefore, $|\text{Te}| = 12$ so $\text{Te} \cong A_4$.
  - We determined in HW2 that...
    - $\text{Do} \hookrightarrow S_5$ and $|\text{Do}| = 60$. Thus, $\text{Do} \cong A_5$.
  - Consider the octahedron.

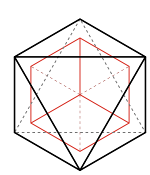


Figure 5.2: Inscribing a cube in an octahedron.

    - $|\text{Oc}| = 6 \cdot 4 = 24$. Rationale: Fix one vertex anywhere and then fix another (the other one can only take on the four adjacent positions, though); the positions of the rest are determined from these two.
    - Let's look at fixing opposite faces. This does give an injective map to S_4, and it follows that $\text{Oc} \cong S_4$.

        - Relation between Oc and Cu. We can inscribe a cube in the octahedron by connecting each vertex of the cube to the midpoint of one of the faces of the octahedron and vice versa. Thus, we get maps Oc $\to$ Cu, leading to Oc $\cong$ Cu.
    - We can similarly inscribe a dodecahedron in an icosahedron.
    - Thus, the cube and the octahedron have the same symmetry, and the dodecahedron and icosahedron have the same symmetry.
- **Platonic solid**: A solid geometric shape in three dimensions for which the faces, edges, and vertices are all indistinguishable.
    - We will study the platonic solids in more depth later.
- Problem: What symmetries can objects in $\mathbb{R}^3$ have?
    - Rephrase: What are the finite subgroups of SO(3)?
    - An octagon is Calegari's favorite polygon.
    - An octagonal prism has much the same symmetry in $\mathbb{R}^3$ as an octagon does in $\mathbb{R}^2$. This leads to $D_{2n} \leq \text{SO}(3)$.
        - Recall the map from the blog post.
    - We also have $\mathbb{Z}/n\mathbb{Z} \leq D_{2n}$.
- It follows that the groups $\mathbb{Z}/n\mathbb{Z}$, D_{2n}, A_4, S_4, and S_5 occur as finite subgroups of SO(3).
- Theorem: All finite subgroups of SO(3) are on this list. Moreover, all related versions are conjugate.
    - This is a companion theorem to the theorem that there are only five platonic solids.
    - Neither theorem implies the other, but they are related.
- Infinite subgroups of SO(3): O(2), SO(3), SO(2).
- This theorem will be completely evident by the end of the course.
- You can either use this theorem to understand A_4, A_5, or use an understanding of A_4, A_5 to rationalize this theorem.
- Points to the main focus of the class: Understanding groups not just based on writing down elements but by their action on a certain set. This is the focus of the second half of the course.
- Midterm: 50 mins, closed book, Wednesday. Final exam must be in-person by department rules, but Calegari is fighting for us. Calegari is hoping that the midterm should not be a speed test.
    - Trying to test our skills, not our ability to memorize stuff.
    - How to do well: Learn group theory.
- The quaternion group.
    - A 4D vector space where you define a noncommutative product. If you just take 8 specific quaternions, the group of order 8 is distinct from D_8 but related.