8 p-Sylows and Simple Groups

12/2: 1. Show that the 2-Sylow subgroups of S_4 and S_5 are isomorphic to D_8, and the 2-Sylow subgroups of A_4 and A_5 are isomorphic to the Klein 4-group.

2. Let H be the subset of $\mathrm{GL}_3(\mathbb{F}_p)$ of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

   (a) Prove that H is a p-Sylow subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$.

   (b) Prove that H is not normal.

   (c) Determine the number n_p of p-Sylow subgroups of $\mathrm{GL}_3(\mathbb{F}_p)$.

   (d) Determine the normalizer of H.

3. Suppose that P is a normal p-Sylow subgroup of G. Suppose that H is a subgroup of G. Prove that $P \cap H$ is the unique p-Sylow subgroup of H. (Exercise 4.5.33 of Dummit and Foote (2004).)

4. Prove that if $n < p^2$, the p-Sylow subgroup of S_n is abelian. Prove that if $n \geq p^2$, the p-Sylow subgroup of S_n is *not* abelian.

5. Let N be a normal subgroup of G, and suppose that the largest power of p dividing $|N|$ is equal to the largest power of p dividing $|G|$. Prove that the p-Sylow subgroups of G are precisely the p-Sylow subgroups of N.

6. Prove that there do not exist any simple groups of order p^2q for distinct primes p, q. (*Hint*: Consider the congruence restrictions from Sylow III.)

7. Prove that there do not exist any simple groups of the following orders. (Warning: Not in order of difficulty.)

   (a) (*) 336.

   (b) 1176.

   (c) 2907.

   (d) 6545.