# 5 Group Structure Applications

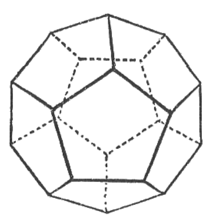
11/7: 1. Automorphisms of $S_n$.

   (a) Let $\psi : G \to G$ be an isomorphism. If $\{c\}$ is a conjugacy class of $G$, prove that the image $\psi(\{c\})$ of $\{c\}$ under $\psi$ is the conjugacy class $\{\psi(c)\}$.
   (b) Deduce that $|\{c\}| = |\{\psi(c)\}|$.
   (c) Let $G = S_n$. Prove that $|\{(1,2)\}| = n(n-1)/2$.
   (d) If $n \neq 6$ and $\sigma \in S_n$ has order 2, prove that $|\{\sigma\}| = |\{(1,2)\}|$ if and only if $\sigma$ is a 2-cycle.
   (e) Deduce that if $\psi : S_n \to S_n$ is an isomorphism and $n \neq 6$, then $\psi$ takes 2-cycles to 2-cycles.
   (f) Suppose that $\psi[(1,2)] = (i,j)$. Prove that, after possibly swapping $i$ and $j$, $\psi[(1,3)] = (i,k)$ for some $k \notin \{i,j\}$.
   (g) Let $g \in S_n$ denote any element with $g(i) = 1$, $g(j) = 2$, and $g(k) = 3$. Let $\phi_g$ be the (inner) automorphism of $S_n$ given by conjugation by $g$. After replacing $\psi$ by $\phi_g \circ \psi$, deduce that one can assume that $\psi[(1,2)] = (1,2)$ and $\psi[(1,3)] = (1,3)$.
   (h) Assume that $\psi(1,i) = (1,i)$ for all $i < k$ with $k > 3$. Prove that $\psi(1,k) = (1,j)$ for some $j \geq k$. As in part (g), show that after replacing $\psi$ by $\phi_h \circ \psi$ for some $h$, one can assume in addition that $\psi[(1,k)] = (1,k)$.
   (i) Deduce that $\psi$ is the identity, and hence that any automorphism of $S_n$ (for $n \neq 6$) is given by conjugation, i.e., $\mathrm{Out}(S_n) = 1$ for $n \neq 6$.

2. Let $H$ be a finite subgroup of $G$ of index $n$. Let $A$ be the set of left cosets $G/H$, and consider the left action of $G$ on $A$. (See Exercise 4.2.8 of Dummit and Foote (2004).)

   (a) Let $n = |G/H|$, and consider the associated homomorphism $G \to S_{G/H} \cong S_n$. Prove that the kernel of this map is a subgroup of $H$.
   (b) By considering the kernel of the map $G \to S_n$, deduce that $G$ contains a normal subgroup $N$ contained in $H$ of index dividing $n!$ and divisible by $n$.

3. Let Do $\cong A_5$ denote the symmetry group of the dodecahedron. Fill out the missing entries in the table below for various sets $X$ on which Do acts transitively. Since the action of Do is transitive for each $X$, all stabilizers $S$ for any point $x \in X$ are conjugate to the stabilizers of any other point. Hence, they are isomorphic as subgroups; simply list a group (from our known list of groups: symmetric, alternating, dihedral, cyclic, quaternion, etc.) isomorphic to any of the stabilizers.



Dodecahedron.

*Proof.*

| $X$ | $\lvert X\rvert$ | Faithful? | Stab$(x)$ | $\lvert S\rvert$ |
|---|---|---|---|---|
| Dodecahedra | 1 | No | $S = \mathrm{Do} \cong A_5$ | 60 |
| Inscribed cubes | | | | |
| Pairs of opposite faces | | | | |
| Faces | | | | |
| Vertices | | | | |
| Edges | | | | |

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